ON THE GONCHAROV DEPTH CONJECTURE AND POLYLOGARITHMS OF DEPTH TWO

STEVEN CHARLTON, HERBERT GANGL, DANYLO RADCHENKO, AND DANIIL RUDENKO

Abstract. We prove the surjectivity part of Goncharov's depth conjecture. We also show that the depth conjecture implies that multiple polylogarithms of depth d and weight n can be expressed via a single function $\mathrm{Li}_{n-d+1,1,\dots,1}(a_1, a_2, \dots, a_d)$, and we prove this latter statement for $d = 2$.

1. Introduction

Multiple polylogarithms are multivalued functions of variables $a_1, \dots, a_d \in \mathbb{C}$ depending on positive integer parameters $n_1, \dots, n_d \in \mathbb{N}$. In the polydisc $|a_1|, |a_2|, \dots, |a_d| < 1$ polylogarithms are defined by power series

$$\mathrm{Li}_{n_1,n_2,\dots,n_d}(a_1, a_2, \dots, a_d) = \sum_{0<m_1<m_2<\cdots<m_d} \frac{a_1^{m_1} a_2^{m_2} \dots a_d^{m_d}}{m_1^{n_1} m_2^{n_2} \dots m_d^{n_d}} .$$

The number $n = n_1 + \cdots + n_d$ is called the weight of the multiple polylogarithm, and the number d is called its depth. Goncharov suggested an ambitious conjecture, giving a necessary and sufficient condition for a sum of polylogarithms to have certain depth. In §3 we show that the Goncharov depth conjecture would have the following remarkable corollary.

Conjecture 1. *Any multiple polylogarithm of weight $n \geq 2$ and depth $d \geq 2$ can be expressed as a linear combination of multiple polylogarithms $\mathrm{Li}_{n-d+1,1,\dots,1}$ and products of polylogarithms of lower weight.*

We expect that there exists a presentation where all the arguments are Laurent monomials in $\sqrt[N]{a_1}, \dots, \sqrt[N]{a_d}$ for sufficiently large N. We show that Conjecture 1 is true for $d = 2$.

Theorem 2. *For every $0 < k < n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that the multiple polylogarithm $\mathrm{Li}_{k,n-k}(x, y)$ can be expressed as a linear combination of functions*

$$\mathrm{Li}_{n-1,1}\big(\sqrt[N]{x}^{\,r} \sqrt[N]{y}^{\,s},\ \sqrt[N]{x}^{\,t} \sqrt[N]{y}^{\,u}\big) \quad \text{for} \ \ r, s, t, u \in \mathbb{Z}$$

and products of classical polylogarithms, where each appearance of $\sqrt[N]{z}$ denotes any Nth root of z.

Here is an example of this type of identity in weight four and depth two

$$\mathrm{Li}_{2,2}(x, y) = -4\,\mathrm{Li}_{3,1}\left(-\frac{\sqrt{x}}{\sqrt{y}}, y\right) - 4\,\mathrm{Li}_{3,1}\left(\frac{\sqrt{x}}{\sqrt{y}}, y\right) + 4\,\mathrm{Li}_{3,1}\left(-\frac{\sqrt{y}}{\sqrt{x}}, x\right) + 4\,\mathrm{Li}_{3,1}\left(\frac{\sqrt{y}}{\sqrt{x}}, x\right)$$
$$+ \mathrm{Li}_{3,1}(x, y) - \mathrm{Li}_{3,1}(y, x) - \mathrm{Li}_{3,1}\left(\frac{y}{x}, x\right) - \frac{1}{2}\,\mathrm{Li}_4(xy) + \mathrm{Li}_1(x)\,\mathrm{Li}_3(y) .$$

In this formula $\sqrt{x}$ (resp. $\sqrt{y}$) denotes some fixed square root of x (resp. y). The identity holds for all $|x|, |y| < 1$.

In §2 we give an elementary proof of Theorem 2. In §3 we recall the statement of Goncharov's depth conjecture and prove a part of it (Theorem 5). Next, we show that the depth conjecture implies Conjecture 1.

Date: August 16, 2023.

2. Proof of Theorem 2

We define $L(x, y \mid t_1, t_2)$ to be the following generating function

$$L(x, y \mid t_1, t_2) := \sum_{k,l>0} \mathrm{Li}_{k,l}(x/y, y) t_1^{k-1} t_2^{l-1} = \sum_{m,n>0} \frac{x^m y^n}{(m - t_1)(m + n - t_2)} .$$

The key observation used in the proof of Theorem 2 is the following identity.

Proposition 3. *For any integers $\alpha, \beta, \gamma > 0$ with $\gamma = \alpha + \beta$ and any x, y with $|x|, |y| < 1$ we have*

$$\sum_{X^\alpha = x, Y^\beta = y} L(X, Y \mid \alpha\beta t, 0) - \sum_{Z^\gamma = xy, Y^\beta = y} L(Z, Y \mid \gamma\beta t, 0) + \sum_{Z^\gamma = xy, X^\alpha = x} L(Z, X \mid -\gamma\alpha t, 0) \tag{1}$$
$$= L(xy, x \mid -\alpha t, \beta t) + \frac{1}{\gamma t} \sum_{k \geq 2} \mathrm{Li}_k(xy)(\beta t)^{k-1} .$$

Proof. Note that

$$\sum_{\substack{X^\alpha = x \\ Y^\beta = y}} L(X, Y \mid t_1, t_2) = \sum_{m,n>0} \frac{\alpha\beta x^m y^n}{(m\alpha - t_1)(m\alpha + n\beta - t_2)} .$$

From this we calculate that the LHS of (1) is equal to

$$\sum_{m,n>0} \Big[\frac{\beta x^m y^n}{(m - \beta t)(m\alpha + n\beta)} - \frac{\beta x^m y^{m+n}}{(m - \beta t)(m\alpha + (m+n)\beta)} + \frac{\alpha x^{m+n} y^m}{(m + \alpha t)((m+n)\alpha + m\beta)} \Big]$$
$$= \sum_{m \geq n > 0} \frac{\beta x^m y^n}{(m - \beta t)(m\alpha + n\beta)} + \sum_{m > n > 0} \frac{\alpha x^m y^n}{(n + \alpha t)(m\alpha + n\beta)}$$
$$= \frac{1}{\gamma t} \sum_{m = n > 0} \Big[\frac{(xy)^n}{n - \beta t} - \frac{(xy)^n}{n} \Big] + \sum_{m > n > 0} \frac{x^m y^n}{(m - \beta t)(n + \alpha t)}$$
$$= \frac{1}{\gamma t} \sum_{k \geq 2} \mathrm{Li}_k(xy)(\beta t)^{k-1} + L(xy, x \mid -\alpha t, \beta t) . \qquad \square$$

Proof of Theorem 2. Expanding both sides of (1) as a power series in t and comparing the coefficients of t^{n-2} we see that for any integers $\alpha, \beta > 0$ the function

$$U_n^{\alpha,\beta}(x, y) := \sum_{\substack{k+l=n, \\ k,l>0}} \mathrm{Li}_{k,l}(y, x)(-\alpha)^{k-1} \beta^{l-1}$$

is expressible in terms of $\mathrm{Li}_{n-1,1}$ and Li_n. Since the matrix $((-i)^{k-1}(n-i)^{n-d-1})_{i,k=1}^{n-1}$ is invertible (its determinant is of Vandermonde type), each individual function $\mathrm{Li}_{k,l}(y, x)$ for $k + l = n$ can be written as a rational linear combination of the functions $U_n^{1,n-1}(x, y), U_n^{2,n-2}(x, y), \ldots, U_n^{n-1,1}(x, y)$, and hence it also can be expressed in terms of $\mathrm{Li}_{n-1,1}$ and Li_n, as claimed. $\square$

3. Surjectivity part of the Goncharov depth conjecture

To state the Goncharov depth conjecture we recall the definition of the Lie coalgebra $\mathcal{L}_\bullet(\mathrm{F})$ of (formal) polylogarithms with values in a field F ([1], see also [4, §2.1]). The Lie coalgebra $\mathcal{L}_\bullet(\mathrm{F})$ is positively graded by weight; the component of weight n is generated over $\mathbb{Q}$ by formal symbols $\mathrm{Li}^{\mathcal{L}}_{n_0\,;\,n_1,\ldots,n_d}(a_1, \ldots, a_d)$ for $n_0 \in \mathbb{Z}_{\geq 0}, n_1, \ldots, n_d \in \mathbb{N}$ with $n_0 + n_1 + \cdots + n_d = n$ and $a_1, \ldots, a_d \in \mathrm{F}^\times$, which are subject to (mostly unknown) functional equations for polylogarithms. The cobracket $\Delta \colon \mathcal{L}_\bullet(\mathrm{F}) \longrightarrow \bigwedge^2 \mathcal{L}_\bullet(\mathrm{F})$ was discovered by Goncharov ([1], [2], [3]); the definition was inspired by properties of mixed Hodge structures related to multiple polylogarithms. The Lie coalgebra $\mathcal{L}_\bullet(\mathrm{F})$ is filtered by depth; denote by $\mathcal{D}_d\mathcal{L}_\bullet(\mathrm{F})$ the subspace spanned by polylogarithms of depth not greater than d; let $\mathrm{gr}^{\mathcal{D}}_d \mathcal{L}_\bullet(\mathrm{F})$ be the associated graded space. The subspace $\mathcal{D}_1\mathcal{L}_\bullet(\mathrm{F})$ spanned by classical polylogarithms $\mathrm{Li}^{\mathcal{L}}_n(a)$ is denoted by $\mathcal{B}_n(\mathrm{F})$.

Assume that $\Delta = \sum_{1\le i\le j}\Delta_{ij}$ for $\Delta_{ij}\colon \mathcal{L}_{i+j}(\mathrm{F}) \longrightarrow \mathcal{L}_i(\mathrm{F})\wedge\mathcal{L}_j(\mathrm{F})$. The truncated cobracket is a map $\overline{\Delta}\colon \mathcal{L}(\mathrm{F}) \longrightarrow \bigwedge^2\mathcal{L}(\mathrm{F})$ defined by the formula $\overline{\Delta} = \sum_{2\le i\le j}\Delta_{ij}$. In other words, $\overline{\Delta}$ is obtained from Δ by omitting the component $\mathcal{L}_1(\mathrm{F})\wedge\mathcal{L}_{n-1}(\mathrm{F})$ from the cobracket. Denote by $\mathrm{coLie}_\bullet(V)$ the cofree graded Lie coalgebra cogenerated by a graded vector space V.

By [4, Proposition 4.1], the iterated truncated cobracket $\overline{\Delta}^{[d-1]}$ vanishes on $\mathcal{D}_{d-1}\mathcal{L}_\bullet(\mathrm{F})$ and defines a map

$$\overline{\Delta}^{[d-1]}\colon \mathrm{gr}^{\mathcal{D}}_d\mathcal{L}_{\ge 2}(\mathrm{F}) \longrightarrow \mathrm{coLie}_d\left(\bigoplus_{n\ge 2}\mathcal{B}_n(\mathrm{F})\right). \tag{2}$$

Conjecture 4 (Goncharov, [3, Conjecture 7.6])**.** *A linear combination of multiple polylogarithms has depth less than or equal to d if and only if its d-th iterated truncated cobracket vanishes. Moreover, the map $\overline{\Delta}^{[d-1]}$ for $d\ge 1$ is an isomorphism.*

We prove the surjectivity part of Conjecture 4.

Theorem 5. *Assume that the field* F *is quadratically closed. Then the map*

$$\overline{\Delta}^{[d-1]}\colon \mathrm{gr}^{\mathcal{D}}_d\mathcal{L}_{\ge 2}(\mathrm{F}) \longrightarrow \mathrm{coLie}_d\left(\bigoplus_{n\ge 2}\mathcal{B}_n(\mathrm{F})\right)$$

is surjective.

Proof. It is easy to see that

$$\overline{\Delta}^{[d-1]}\left(\mathrm{Li}^{\mathcal{L}}_{n-d\,;\,1,\dots,1}(a_1,\dots,a_d)\right) = \sum_{\substack{n_1+n_2+\dots+n_d=n\\ n_i\ge 2}} \mathrm{Li}^{\mathcal{L}}_{n_1}(a_1)\otimes\dots\otimes\mathrm{Li}^{\mathcal{L}}_{n_d}(a_d).$$

Recall that if F contains all degree r roots of unity then classical polylogarithms $\mathrm{Li}_n(a)$ satisfy the following *distribution relations*:

$$\mathrm{Li}^{\mathcal{L}}_n(a^r) = r^{n-1}\sum_{\zeta^r=1}\mathrm{Li}^{\mathcal{L}}_n(\zeta a)\,.$$

It follows that for any $s\in\mathbb{N}$

$$\begin{aligned}
&\overline{\Delta}^{[d-1]}\left(\sum_{x^{2^s}=a_d}\mathrm{Li}^{\mathcal{L}}_{n-d\,;\,1,\dots,1,1}(a_1,\dots,a_{d-1},x)\right)\\
&\quad= \sum_{\substack{n_1+n_2+\dots+n_d=n\\ n_i\ge 2}} 2^{-s(n_d-1)}\,\mathrm{Li}^{\mathcal{L}}_{n_1}(a_1)\otimes\dots\otimes\mathrm{Li}^{\mathcal{L}}_{n_d}(a_d)\\
&\quad= \sum_{2\le n_d\le n-2d+2}\left(\sum_{\substack{n_1+n_2+\dots+n_{d-1}=n-n_d\\ n_i\ge 2}}\mathrm{Li}^{\mathcal{L}}_{n_1}(a_1)\otimes\dots\otimes\mathrm{Li}^{\mathcal{L}}_{n_{d-1}}(a_{d-1})\right)\otimes 2^{-s(n_d-1)}\,\mathrm{Li}^{\mathcal{L}}_{n_d}(a_d)\,.
\end{aligned}$$

From the properties of the Vandermonde determinant it follows that for every $n_d\ge 2$ the element

$$\left(\sum_{\substack{n_1+n_2+\dots+n_{d-1}=n-n_d\\ n_i\ge 2}}\mathrm{Li}^{\mathcal{L}}_{n_1}(a_1)\otimes\dots\otimes\mathrm{Li}^{\mathcal{L}}_{n_{d-1}}(a_{d-1})\right)\otimes\mathrm{Li}^{\mathcal{L}}_{n_d}(a_d)$$

lies in the image of $\overline{\Delta}^{[d-1]}$. Continuing in a similar fashion, we conclude that for every $n_1,\dots,n_d\in\mathbb{N}$ the element

$$\mathrm{Li}^{\mathcal{L}}_{n_1}(a_1)\otimes\dots\otimes\mathrm{Li}^{\mathcal{L}}_{n_d}(a_d)$$

lies in the image of $\overline{\Delta}^{[d-1]}$. From here the statement follows. □

Assume that Goncharov's depth conjecture holds. It follows from the proof of Theorem 5 that $\mathcal{L}_\bullet(\mathrm{F})$ is generated by functions $\mathrm{Li}_{n-d\,;\,1,\dots,1}(a_1,\dots,a_d)$. The shuffle antipode and stuffle antipode for multiple polylogarithms [4] (cf. also [3]), respectively, imply that

$$\begin{aligned}&\mathrm{Li}^{\mathcal{L}}_{n-d\,;\,1,\dots,1}(a_1,\dots,a_d)\\ &\equiv(-1)^{n+1}\,\mathrm{Li}^{\mathcal{L}}_{0\,;\,1,\dots,1,n-d+1}(a_{d-1}^{-1},\dots,a_2^{-1},a_1^{-1},a_1\cdots a_d)\quad(\text{mod products})\\ &\equiv(-1)^{n-d}\,\mathrm{Li}^{\mathcal{L}}_{0\,;\,n-d+1,1,\dots,1}(a_1\cdots a_d,a_1^{-1},a_2^{-1},\dots,a_{d-1}^{-1})\quad(\text{mod products, depth}<d)\,,\end{aligned}$$

where $\mathrm{Li}^{\mathcal{L}}_{0\,;\,n-d+1,1,\dots,1}$ corresponds to the function $\mathrm{Li}_{n-d+1,1,\dots,1}$, so Conjecture 4 implies Conjecture 1.

Theorem 5 has the following striking corollary.

Corollary 6. *Let* F *be a quadratically closed field. Assume that Conjecture 4 holds for* $d=1$. *Then it holds for all* $d\geq 1$ *and the Lie coalgebra* $\mathcal{L}_{\geq 2}(\mathrm{F})$ *with cobracket* $\overline{\Delta}$ *is cofree.*

Proof. First, we assume that (2) is an isomorphism. Our goal is to show that $\mathcal{L}_{\geq 2}(\mathrm{F})$ is cofree, or, equivalently, that the Chevalley-Eilenberg complex $\bigwedge^\bullet(\mathcal{L}_{\geq 2}(\mathrm{F}))$ is exact in degree at least two. The depth filtration on $\mathcal{L}_{\geq 2}(\mathrm{F})$ induces a filtration on the complex $\bigwedge^\bullet(\mathcal{L}_{\geq 2}(\mathrm{F}))$. Consider the spectral sequence of this filtered complex; its first page is the cohomology of the associated graded complex, which coincides with the Chevalley-Eilenberg complex of the Lie coalgebra $\mathrm{gr}^{\mathcal{D}}_\bullet\mathcal{L}_{\geq 2}(\mathrm{F})$. By (2), the latter coalgebra is cofree, so the spectral sequence collapses at the first page. This implies the statement.

Now, our goal is to prove that (2) is an isomorphism. We argue by induction on d; the base case $d=1$ is a tautology. Suppose that for $k\leq d-1$ the map $\overline{\Delta}^{[k-1]}$ is an isomorphism. By Theorem 5, it is sufficient to show that $\overline{\Delta}^{[d-1]}$ is injective. Consider an element $x\in\mathcal{D}_d\mathcal{L}_{\geq 2}(\mathrm{F})$ such that $\overline{\Delta}^{[d-1]}(x)=0$. The map

$$\overline{\Delta}^{[\bullet]}\colon \mathrm{gr}^{\mathcal{D}}\mathcal{L}_{\geq 2}(\mathrm{F})\longrightarrow \mathrm{coLie}\Big(\bigoplus_{n\geq 2}\mathcal{B}_n(\mathrm{F})\Big)$$

is a morphism of Lie coalgebras, so

$$\sum_{i+j=d}\overline{\Delta}^{[i-1]}\wedge\overline{\Delta}^{[j-1]}(\overline{\Delta}(x))=0. \tag{3}$$

By the induction assumption, (3) implies that $\overline{\Delta}(x)$ vanishes in $\bigwedge^2\mathrm{gr}^{\mathcal{D}}\mathcal{L}_{\geq 2}(\mathrm{F})=\mathrm{gr}^{\mathcal{D}}\left(\bigwedge^2\mathcal{L}_{\geq 2}(\mathrm{F})\right)$ so $\overline{\Delta}(x)\in\mathcal{D}_{d-1}\big(\bigwedge^2\mathcal{L}_{\geq 2}(\mathrm{F})\big)$. The same spectral sequence argument as above shows that Lie coalgebra $\mathcal{D}_{d-1}\mathcal{L}_{\geq 2}(\mathrm{F})$ with cobracket $\overline{\Delta}$ is cofree. Thus there exists $y\in\mathcal{D}_{d-1}\mathcal{L}_{\geq 2}(\mathrm{F})$ such that $\overline{\Delta}(x-y)=0$, so $x-y\in\mathcal{D}_1\mathcal{L}_{\geq 2}(\mathrm{F})$ by the assumption that Conjecture 4 holds for $d=1$. It follows that $x\in\mathcal{D}_{d-1}\mathcal{L}_{\geq 2}(\mathrm{F})$. □

Acknowledgments: We are very grateful to Alexander Goncharov for suggesting Corollary 6 as a consequence of Theorem 5. SC was supported by Deutsche Forschungsgemeinschaft Eigene Stelle grant CH 2561/1-1, for Projektnummer 442093436, while at Universität Hamburg. HG is grateful to the MPIM Bonn for excellent working conditions.

References

[1] A. B. Goncharov. Hyperlogarithms, mixed Tate motives and multiple ζ-numbers, 1993. Preprint MSRI 058-93.
[2] A. B. Goncharov. Geometry of configurations, polylogarithms, and motivic cohomology. *Adv. Math.*, 114(2):197–318, 1995.
[3] A. B. Goncharov. Multiple polylogarithms and mixed Tate motives, 2001. arXiv:math/0103059.
[4] A. Matveiakin and D. Rudenko. Cluster Polylogarithms I: Quadrangular Polylogarithms, 2022. arXiv:2208.01564.

Max Planck Institute for Mathematics, Vivatsgasse 7, Bonn 53111, Germany.
Email address: charlton@mpim-bonn.mpg.de

Department of Mathematical Sciences, Durham University, Durham DH1 3LE, United Kingdom
Email address: herbert.gangl@durham.ac.uk

Laboratoire Paul Painlevé, Université de Lille, F-59655 Villeneuve d'Ascq, France
Email address: danradchenko@gmail.com

Department of Mathematics, University of Chicago, 5801 S Ellis Ave, 60637 Chicago, IL, USA
Email address: rudenkodaniil@gmail.com