BROWNIAN MOTION WITH ASYMPTOTICALLY NORMAL REFLECTION IN UNBOUNDED DOMAINS: FROM TRANSIENCE TO STABILITY

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We quantify the asymptotic behaviour of multidimensional drifftess diffusions in domains unbounded in a single direction, with asymptotically normal reflections from the boundary. We identify the critical growth/contraction rates of the domain that separate stability, null recurrence and transience. In the stable case we prove existence and uniqueness of the invariant distribution and establish the polynomial rate of decay of its tail. We also establish matching polynomial upper and lower bounds on the rate of convergence to stationarity in total variation. All exponents are explicit in the model parameters that determine the asymptotics of the growth rate of the domain, the interior covariance, and the reflection vector field.

Proofs are probabilistic, and use upper and lower tail bounds for additive functionals up to return times to compact sets, for which we develop novel sub/supermartingale criteria, applicable to general continuous semimartingales. Narrowing domains fall outside of the standard literature, in part because boundary local time can accumulate arbitrarily rapidly. Establishing Feller continuity (essential for characterizing stability) thus requires an extension of the usual approach.

Our recurrence/transience classification extends previous work on strictly normal reflections, and expands the range of phenomena observed across all dimensions. For all recurrent cases, we provide quantitative information through upper and lower bounds on tails of return times to compact sets.

CONTENTS

MSC2020 subject classifications: Primary 60J60; secondary 60J65, 60J25.

Keywords and phrases: reflected diffusion, normal reflection, generalized parabolic domains, recurrence, transience, tails of return-times, invariant distribution, convergence rate in total variation.

1. Introduction and main results. We study the asymptotic behaviour of a multidimensional diffusion in an unbounded, generalized parabolic domain, with asymptotically normal reflection from the boundary. Our model includes Brownian motion with normal reflection as well as related models with multiplicative noise. We show that the phase transition between recurrence and transience occurs for asymptotically expanding domains. If the domain narrows asymptotically, we identify the phase transition between null and positive recurrence. In the recurrent case we characterise the asymptotic behaviour of the tails of the return times. Moreover, in the positive-recurrent case we prove the existence of the invariant distribution of the reflected diffusion and establish the polynomial rate of decay of its tail. Finally, we establish the polynomial rate of convergence to stationarity by proving matching upper and lower bounds on the total variation distance between the marginal and the invariant distribution.

All the aforementioned phenomena depend on the asymptotic behaviour of the boundary and are exhibited by a normally reflected Brownian motion. In particular, this implies that a normally reflected Brownian motion in an unbounded domain of any dimension (greater than 1) may converge in total variation to its invariant distribution, which has heavy tails.

For any $d \in \mathbb{N} := \{1, 2, \dots\}$, let $\|\cdot\|_d$ denote the usual Euclidean norm on $\mathbb{R}^d$. We will often write $\mathbb{R}_+ := [0, \infty)$. Let $\mathbb{S}^{d-1} := \{u \in \mathbb{R}^d : \|u\|_d = 1\}$ be unit sphere in $\mathbb{R}^d$ and let $e_x := (1, 0) \in \mathbb{S}^d \cap (\mathbb{R}_+ \times \mathbb{R}^d)$ denote the unit vector in the x-direction. For any $u \in \mathbb{S}^{d-1}$, define $e_u := (0, u) \in \mathbb{S}^d$. Then we can express any $z = (x, y) \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}$ as $z = xe_x + \|y\|_d e_{\hat{y}}$, where $\hat{y} := y/\|y\|_d \in \mathbb{S}^{d-1}$ for $\|y\|_d > 0$ (if $\|y\|_d = 0$, we may choose $\hat{y}$ to be any vector in $\mathbb{S}^{d-1}$). We now state our assumptions.

(D1) Let b be a continuous function on $\mathbb{R}_+$, with $b > 0$ on $(0, \infty)$ and $b(0) = 0$. Suppose that b is twice continuously differentiable on $(0, \infty)$, such that (i) $\liminf_{x\to 0}(b(x)b'(x)) > 0$, and (ii) $\lim_{x\to 0}(b''(x)/b'(x)^3)$ exists in $(-\infty, 0]$.

For b as in Assumption (D1), define the closed domain

$$\mathcal{D} \coloneqq \{z = (x, y) \in \mathbb{R}_+ \times \mathbb{R}^d : \|y\|_d \le b(x)\} \tag{1}$$

and let $\partial\mathcal{D} \coloneqq \{z \in \mathcal{D} : \|y\|_d = b(x)\}$ be the boundary of $\mathcal{D}$ in $\mathbb{R}^{d+1}$. Denote by $\mathcal{M}^+_{d+1}$ the group of positive definite square matrices of dimension $(d + 1)$. For $\Sigma \in \mathcal{M}^+_{d+1}$, write $\Sigma^{1/2}$ for the symmetric square root of Σ.

(C1) Let $\Sigma : \mathcal{D} \to \mathcal{M}^+_{d+1}$ be bounded, globally Lipschitz, and uniformly elliptic, i.e., there exists $\delta_\Sigma > 0$ such that, for every $u \in \mathbb{S}^d$ and all $z \in \mathcal{D}$, we have $\langle \Sigma(z)u, u\rangle \ge \delta_\Sigma$.

(V1) Suppose $\phi : \partial\mathcal{D} \to \mathbb{R}^{d+1}$ is a C^2-vector field, satisfying $\sup_{z\in\partial\mathcal{D}} \|\phi(z)\|_{d+1} < \infty$ and

$$\inf_{x>0} \inf_{\hat{y}\in\mathbb{S}^{d-1}} \langle \phi(x, b(x)\hat{y}), n(x, b(x)\hat{y})\rangle > 0,$$

where $n(z) = n(x, b(x)\hat{y})$ is the inwards-pointing unit normal vector at $z = (x, y) \in \partial\mathcal{D}$.

REMARK 1.1. Assumption (V1) requires the vector field ϕ to be smooth, of bounded magnitude and have a uniformly positive component in the normal direction (throughout the paper, $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^{d+1}$). Assumption (D1) guarantees that the boundary $\partial\mathcal{D}$ is sufficiently regular everywhere, including the origin (see [29, Lem. 4.3] for details). Assumption (C1) ensures that the reflected process Z, defined in (2) below, is not locally constrained in any direction. Throughout we use the matrix norm $\|\Sigma(z)\|_{\text{op}} := \sup_{v\in\mathbb{S}^d} \|\Sigma(z)v\|_{d+1}$. (In particular, since $\|\Sigma^{1/2}(z)\|^2_{\text{op}} = \sup_{v\in\mathbb{S}^d}\langle\Sigma(z)v, v\rangle = \|\Sigma(z)\|_{\text{op}}$ equals the largest eigenvalue of Σ, the boundedness of Σ implies the boundedness of $\Sigma^{1/2}$.)

Let $W = (W_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion in $\mathbb{R}^{d+1}$. For Σ as in (C1) and ϕ as in (V1), let the processes $Z = (Z_t)_{t\in[0,\tau_\mathcal{E})}$ and $L = (L_t)_{t\in[0,\tau_\mathcal{E})}$, with state spaces $\mathcal{D}$ and $\mathbb{R}_+$, respectively, satisfy the stochastic differential equation (SDE)

$$\begin{aligned} Z_t &= z + \int_0^t \Sigma^{1/2}(Z_s)\mathrm{d}W_s + \int_0^t \phi(Z_s)\mathrm{d}L_s, \\ L_t &= \int_0^t \mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}L_s, \qquad t \in [0, \tau_\mathcal{E}), \end{aligned} \tag{2}$$

where $\tau_\mathcal{E} \in [0, \infty]$ is a possibly finite explosion time and L denotes the local time process of Z on the boundary $\partial\mathcal{D}$. For any starting point $Z_0 = z \in \mathcal{D}$, by [29, Thm A.1], SDE (2) has a unique strong solution (with law denoted by $\mathbb{P}_z$) under the assumptions in (C1), (D1) and (V1).

The reflected process Z is *transient* (resp. *recurrent*) if $\lim_{t\to\infty} \|Z_t\|_{d+1} = \infty$ (resp. there exists $r_0 \in \mathbb{R}_+$ satisfying $\liminf_{t\to\infty} \|Z_t\|_{d+1} \le r_0$) $\mathbb{P}_z$-a.s. The asymptotic growth of the

domain $\mathcal{D}$ is described by the parameter

$$\beta := \limsup_{x\to\infty} \frac{xb'(x)}{b(x)}, \tag{3}$$

typically equal to the limit $\lim_{x\to\infty} xb'(x)/b(x)$ (e.g. if $b(x) = ax^\beta$ for $x \geq x_0 > 0$ and some $a > 0$, or, more generally, if b is regularly varying and b' is eventually monotone [3, p. 59]).

Under assumptions (V1), (D1) and (C1), the process Z may explode with positive probability. In fact, by [29, Thm 2.2], we may have $\mathbb{P}(\tau_{\mathcal{E}} < \infty) = 1$. The following additional assumptions preclude explosion (and in fact imply $\mathbb{P}(\tau_{\mathcal{E}} = \infty) = 1$) and allow us to characterise transience and recurrence of the process Z.

(D2) Suppose that (D1) holds, $\lim_{x\to\infty} b'(x) = \lim_{x\to\infty} b''(x) = 0$, the limit in (3) exists and satisfies $\lim_{x\to\infty} xb'(x)/b(x) = \beta \in (-\infty, 1)$.

(C2) Suppose that (C1) holds and that there exist $\sigma_1^2, \sigma_2^2 \in (0,\infty)$ such that

$$\langle \Sigma(z)e_x, e_x\rangle = \sigma_1^2(1 + o_{\mathcal{D}}(1)) \quad \text{and} \quad \operatorname{Tr}\Sigma(z) - \sigma_1^2 = \sigma_2^2(1 + o_{\mathcal{D}}(1)) \quad \text{as } x \to \infty.$$

(V2) Suppose that (V1) holds and that there exist $s_0, c_0 \in (0,\infty)$ such that

$$\langle \phi(z), e_x\rangle = s_0 b'(x)(1 + o_{\partial\mathcal{D}}(1)) \quad \text{and} \quad \langle \phi(z), -e_{\hat{y}}\rangle = c_0(1 + o_{\partial\mathcal{D}}(1)) \quad \text{as } x \to \infty.$$

In Assumptions (C2) and (V2) (and throughout the paper), for any $g : \mathbb{R}_+ \to (0,\infty)$ and $\mathcal{H} \in \{\mathcal{D}, \partial\mathcal{D}\}$, $o_{\mathcal{H}}(g(x))$ as $x \to \infty$ denotes a function $f : \mathcal{H} \to \mathbb{R}$ such that the following limit $\lim_{x\to\infty} \sup_{y:(x,y)\in\mathcal{H}} |f(x,y)|/g(x) = 0$ holds. The trace of Σ is denoted by $\operatorname{Tr}\Sigma$.

REMARK 1.2. Since Assumption (D2) requires $\lim_{x\to\infty} xb'(x)/b(x) = \beta < 1$, for any $\beta' \in (\beta, 1)$ we have $b(x) < x^{\beta'}$ for all sufficiently large $x \in \mathbb{R}_+$, implying that b has sublinear growth as $x \to \infty$. Note however that, as $x \to \infty$, Assumption (D2) allows b to have any of the following limits: infinity (requiring $\beta \geq 0$), a positive finite limit (requiring $\beta = 0$) or a limit equal to 0 (requiring $\beta \leq 0$). Interestingly, in the case $\beta = 0$ the boundary function b may exhibit a variety of different behaviours at infinity. For instance, b may grow to infinity (e.g. $b(x) \approx \log x$), converge to 0 (e.g. $b(x) \approx 1/\log x$), or be asymptotically constant. Furthermore, it is also possible for the function b to oscillate, i.e. $\limsup_{x\to\infty} b(x) = \infty$ and $\liminf_{x\to\infty} b(x) = 0$. For example, any function b satisfying

$$b(x) \approx (1 + (\log\log x)^{-2} + \sin\log\log x)\log\log x \quad \text{as } x \to \infty,$$

exhibits such behaviour (see Appendix B for proof that such a function satisfies (D2)).

Condition (D2) requires that β in (3) lies in the interval $(-\infty, 1)$. By Remark 1.2, this makes the growth of $\mathcal{D}$ sublinear (and possibly asymptotically narrowing). This is to let the local time in SDE (2) influence the long-time behavior of the reflected process Z. Assumption (C2) permits Σ to vary smoothly with $z = (x, y) \in \mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^d$, but requires that as $x \to \infty$, the diagonal entry of Σ in the x-direction and the sum of the remaining diagonal entries converge to positive values σ_1^2 and σ_2^2, respectively. Assumption (C2) also ensures that the process does not stop interacting with the boundary far away from the origin. Assumption (V2) specifies linear factors $s_0, c_0 \in (0,\infty)$ that scale the projections of the vector

field ϕ in the x and normal directions so that, as $x \to \infty$, the former projection is asymptotic to $s_0 b'(x)$ while the latter converges to c_0. Since $b'(x) \to 0$ (as $\beta < 1$), we see that (V2) makes the vector field ϕ *asymptotically normal*. Strictly normal reflection has $s_0 = c_0$. The asymptotic normality ensures non-explosion.

With (D2), (C2), (V2) in hand, we can express the critical growth rate β_c of the boundary $\partial\mathcal{D}$ at which the global behavior of Z transitions between recurrence and transience as

$$\beta_c := \frac{c_0 \sigma_1^2}{s_0 \sigma_2^2}. \tag{4}$$

In order to understand the behavior of Z if the boundary of the domain grows at the critical rate $\beta = \beta_c$ (see (3) above for the link between the growth rate of the boundary and the exponent β), we need to quantify the limits in Assumptions (D2), (C2) and (V2).

(D2+) Assume (D1) and that there exists $\beta \in (0,1)$ such that

$$x b'(x) = \beta b(x)(1 + o(b(x)^2 x^{-2})) \quad \text{as } x \to \infty.$$

(C2+) Assume (C1) and that there exist $\sigma_1^2, \sigma_2^2 \in (0,\infty)$ and $\varepsilon > 0$ such that, as $x \to \infty$,

$$\langle \Sigma(z) e_x, e_x \rangle = \sigma_1^2 (1 + o_{\mathcal{D}}(x^{-\varepsilon})) \quad \text{and} \quad \operatorname{Tr} \Sigma(z) - \sigma_1^2 = \sigma_2^2 (1 + o_{\mathcal{D}}(x^{-\varepsilon})).$$

(V2+) Assume (V1) and that there exist $s_0, c_0 \in (0,\infty)$ such that, as $x \to \infty$,

$$\langle \phi(z), e_x \rangle = s_0 b'(x)(1 + o_{\partial\mathcal{D}}(b(x)^2 x^{-2})),$$
$$\langle \phi(x,y), -e_{\hat{y}} \rangle = c_0 (1 + o_{\partial\mathcal{D}}(b(x)^2 x^{-2})).$$

Having given all our assumptions, we now state our main results.

THEOREM 1.3. *Suppose that Assumptions* (D2)*,* (C2)*,* (V2) *hold and the process Z solves SDE* (2)*. Then the following statements hold for all starting points $z \in \mathcal{D}$:*

(a) *if $\beta < \beta_c$, then Z is recurrent;*
(b) *if $\beta > \beta_c$, then Z is transient;*
(c) *if $\beta = \beta_c$ and Assumptions* (V2+)*,* (D2+) *and* (C2+) *are satisfied, then Z is recurrent.*

Write $Z = (X, Y) \in \mathcal{D}$ in coordinates, so that $X_t \in \mathbb{R}_+$ for $t \in \mathbb{R}_+$. For any $r \in \mathbb{R}_+$, let

$$\varsigma_r := \inf\{t \in \mathbb{R}_+ : X_t \le r\} \tag{5}$$

(with convention $\inf \emptyset := \infty$) be the return time of Z to $\mathcal{D} \cap [0,r] \times \mathbb{R}^d$. Define

$$m_c := (1 - \beta/\beta_c)/2. \tag{6}$$

Note that, if $\beta < \beta_c$, then $m_c > 0$. In this case the following result implies that the return time ς_r is a.s. finite, with m_c being the critical moment exponent.

THEOREM 1.4. *Suppose that Assumptions* (D2)*,* (C2)*,* (V2) *hold and the process Z solves SDE* (2)*. Then the following statements hold.*

(a) *If $\beta \in (\beta_c, 1)$, then for any level $r \in (0,\infty)$ and starting point $z \in \mathcal{D} \cap (r,\infty) \times \mathbb{R}^d$, there is positive probability that component X does not reach level r, i.e., $\mathbb{P}_z(\varsigma_r = \infty) > 0$.*

(b) *If $\beta \in (-\infty, \beta_c)$, then for any $\varepsilon > 0$, $z = (x, y) \in \mathcal{D}$ and $r \in (0, x)$, there exist constants $C_1, C_2 \in (0, \infty)$ for which*

$$C_1 t^{-m_c - \varepsilon} \leq \mathbb{P}_z(\varsigma_r \geq t) \leq C_2 t^{-m_c + \varepsilon} \quad \textit{for all } t \in (1, \infty).$$

We say that the process Z is *positive recurrent* if $\mathbb{E}_z[\varsigma_r] < \infty$ holds for all $z \in \mathcal{D}$ and sufficiently large r. The process Z is *null recurrent* if it is recurrent but not positive recurrent. Theorem 1.4(b) implies that Z is positive (resp. null) recurrent if $\beta < -\beta_c$ (resp. $-\beta_c < \beta < \beta_c$).

By Theorem 1.4(b), positive recurrence of Z occurs if $\beta < -\beta_c$. In this case, we study properties of the invariant distribution of Z on $\mathcal{D}$. Recall that, by [29, Thm A.1], the process Z is strong Markov. An *invariant distribution* π of Z is a probability measure on the Borel σ-algebra $\mathcal{B}(\mathcal{D})$ generated by the open subsets of $\mathcal{D}$, satisfying $\int_{\mathcal{D}} \mathbb{E}_z[f(Z_t)]\pi(\mathrm{d}z) = \int_{\mathcal{D}} f(z)\pi(\mathrm{d}z)$ for all bounded measurable functions $f : \mathcal{D} \to \mathbb{R}_+$ and all $t \in \mathbb{R}_+$. A *total variation* distance between two probability measures φ_1 and φ_2, defined on $\mathcal{B}$, is given by $\|\varphi_1 - \varphi_2\|_{\mathrm{TV}} = \sup_{B \in \mathcal{B}} |\varphi_1(B) - \varphi_2(B)|$. Define

$$M_c := -(1 + \beta/\beta_c)/2,$$

and note that, in the case $\beta < -\beta_c$, we have $M_c > 0$.

THEOREM 1.5. *Suppose* (D2), (C2), (V2) *hold, Z solves SDE* (2)*, and $\beta < -\beta_c$. Then the reflected process Z possesses a unique invariant distribution π. Moreover, for any $\varepsilon > 0$, the following statements hold:*

(a) *there exist constants $c_\pi, C_\pi \in (0, \infty)$ such that*

$$c_\pi r^{-2M_c - \varepsilon} \leq \pi\left(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}\right) \leq C_\pi r^{-2M_c + \varepsilon}, \quad \textit{for all } r \in [1, \infty);$$

(b) *for any starting point $z \in \mathcal{D}$ of Z, there exist constants $c_{\mathrm{TV}}, C_{\mathrm{TV}} \in (0, \infty)$ such that*

$$c_{\mathrm{TV}} t^{-M_c - \varepsilon} \leq \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{\mathrm{TV}} \leq C_{\mathrm{TV}} t^{-M_c + \varepsilon}, \quad \textit{for all } t \in [1, \infty).$$

Theorem 1.5 shows that, surprisingly, a reflected Brownian motion Z in $\mathcal{D} \subset \mathbb{R}^{d+1}$ with normal reflection (i.e., Z in the class of models with $\sigma_1^2 = 1$, $\sigma_2^2 = d$, and $c_0 = s_0$, so that $\beta_c = 1/d$) on an unbounded domain may be polynomially ergodic if the domain narrows sufficiently fast, i.e. $\beta < -1/d$. In this case the tail $\pi\left(\{z \in \mathcal{D} : \|z\|_{d+1} > r\}\right)$ of the invariant distribution π decays asymptotically as $r^{1+d\beta}$ when $r \to \infty$. Note that if $\beta < -1/d$, the domain $\mathcal{D}$ has finite volume in $\mathbb{R}^{d+1}$. However, by modifying either the covariance matrix or the reflection vector field so that $\beta_c < 1/d$, for any $\beta \in (-1/d, -\beta_c)$ we obtain polynomial stability of the reflected Brownian motion in a domain with infinite volume.

Theorem 1.5(a) characterises the critical moment of the invariant distribution π of the reflected process Z in a domain $\mathcal{D}$ with $\beta < -\beta_c$: for any α in $[0, 2M_c)$ (resp. $(2M_c, \infty)$), the moment $\int_{\mathcal{D}} \|z\|_{d+1}^{\alpha} \pi(\mathrm{d}z)$ is finite (resp. infinite). Moreover, by Remark 6.4 below, for every $z \in \mathcal{D}$ we have $\mathbb{P}_z(Z_t \in \partial\mathcal{D}) = 0$ for Lebesgue almost every $t \in \mathbb{R}_+$. By Theorem 1.5(b), this implies $\pi(\partial\mathcal{D}) = 0$.

The polynomial rate of convergence in total variation of Z_t to stationarity, given in Theorem 1.5(b), is half of the rate of decay of the tail of its stationary measure. Differently put, by Theorem 1.5, we have

$$\lim_{t\to\infty} \frac{\log \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{\mathrm{TV}}}{\log t} = \frac{1}{2} \lim_{r\to\infty} \frac{\log \pi(\{z \in \mathcal{D} : \|z\|_{d+1} > r\})}{\log r} = -M_c.$$

1.1. *Discussion of the main results.* The fact that ϕ is an asymptotically normal reflection implies that, in the case with $\beta > 0$, the process accumulates a positive drift in the horizontal direction when it reaches the boundary. Here, we observe phase transitions between recurrence and transience depending on the model parameters. When $\beta < 0$, the process accumulates a negative drift in horizontal direction. In this case, the process is always recurrent, and in some cases the invariant distribution exists.

We now comment on the structure of the proofs and discuss features of the theorems in the previous section. A key step in the proofs of Theorems 1.3, 1.4 and 1.5 consists of reducing the problem to certain super/submartingale conditions that can be verified. We stress that the processes involved (that turn out to be super/submartingales) in all non-critical cases, covering phenomena from transience to stability, are transformations of the reflected process in SDE (2) via a *single* parametric family of Lyapunov functions. The class of functions we use are not, *and should not be*, harmonic because the analysis of the return times and quantitative properties of the invariant distribution and rate of convergence require the presence of a sufficiently strong drift.

1.1.1. *Positive recurrence.* Theorem 1.5 provides detailed information on the ergodicity of the reflected process Z, with lower bounds matching the upper bounds. To the best of our knowledge, this is the first characterisation of the rate convergence to stationarity in the context of reflecting diffusions, including those with drift. Upper bounds abound: for example [35, 36] provide upper bounds on the rate of convergence for various reflected diffusions with drift via drift conditions in [9] (due to the presence of the drift, the upper bounds in this case are sub-exponential).

In contrast, the literature for lower bounds is scarce. Our approach to the lower bounds on the rate of convergence is purely probabilistic. It rests on novel continuous super/submartingale methods, based on Lemma 3.7 below, which provide a general setting where the full force of the idea behind [14, Thm 5.1] (see Lemma A.1 in Appendix A below) can be exploited. The approach is robust to the underlying stochastic model and appears to be applicable to a general continuous ergodic Markov processes.

The first step in the proof of Theorem 1.5 consists of establishing the Feller continuity (Theorem 6.5 below) and irreducibility (Proposition 6.1 below) for the reflected process Z. The key technical challenge in this step consists of controlling the growth of the local time (cf. Section 1.2 below), which requires establishing Feller continuity of the stopped process first (see the proof of Theorem 6.5 for details).

Once Feller continuity and irreducibility of Z have been established, the upper bounds in Theorem 1.5 are proved using supermartingale conditions together with the classical conver-

gence results in [9], applicable in the subexponential case. This yields finiteness of moments of the invariant distribution π, which is then translated into upper bounds on the tails in Theorem 1.5 via Markov's inequality.

The lower bounds in Theorem 1.5 require a lower bound on the tail of the invariant distribution π. In contrast to the upper bounds, characterising infinite moments of π alone does not yield a lower bound on the tail of π. In order to circumvent this problem, we give a sufficient condition for $\int_{\mathcal{D}} H \mathrm{d}\pi = \infty$ for any non-decreasing (not necessarily polynomial) function H. This sufficient condition relies on the lower bounds on the tails of the return times to compact sets in Lemma 3.7 below. Once established, the criterion yields lower bounds on the tail of π via elementary methods (see proof of Lemma 7.5 below). The lower bound on the rate of convergence in total variation in Theorem 1.5 follow from a supermartingale property of a transformed reflected diffusion in (2), the lower bounds on the tail of the stationary distribution π and a general result in [14] (see Lemma A.1 in Appendix A below) that converts the tails of the stationary distribution to a lower bound on the convergence rate in total variation.

Finally we note that, in Theorem 1.5, the critical case $\beta = -\beta_c$ is omitted for brevity. Its analysis would require additional assumptions and a new Lyapunov function, analogous to the ones used in the proof of the critical case of Theorem 1.3(c). We expect that, under appropriate assumptions, such analysis would yield ergodicity of the normally reflected Brownian motion with logarithmic decay in any dimension.

1.1.2. *Return times.* If the reflected process is transient, return times to compact sets are not finite almost surely (we will thus discuss Theorem 1.4(a) in Section 1.1.3 below). In the recurrent case, the upper bound on the tail of the return time in Theorem 1.4(b) is established via a supermartingale condition of a transformation of the reflected process, which (via [28]) implies the finiteness of the moments of return times. The upper bounds in the theorem then follow by Markov's inequality.

The lower bound in Theorem 1.4(b) is established via certain submartingale conditions and lower bounds on the tails of return times to bounded sets in Lemma 3.7. As in the proof of Theorem 1.5 discussed above, Lemma 3.7 is again critical here (as the infinite moment criterion of [28] cannot be applied to obtain the lower bounds on the tail). Note that Lemma 3.7 is applicable in the entire recurrent regime. This is key in the proof of Theorem 1.4(b), in contrast to the proof of Theorem 1.5, where Lemma 3.7 is applied in the positive-recurrent case only.

As our modelling assumptions (C2), (D2), (V2) are asymptotic (i.e., they only specify the limiting behaviour both of the coefficients of SDE (2), the domain $\mathcal{D}$ and the reflection vector field as $x \to \infty$, see Section 2 below for details), it is natural for Theorem 1.4(b) to assert only the tail behaviour of return times, without information about the constants. However, we could provide some explicit constants, if we concentrated on the return times of the sufficiently large levels r only (see Propositions 5.1 and 5.2). Note also that, by Assumption (D2), the function b is sublinear as $x \to \infty$, implying that there exist positive constants $c < C$, such that $cx \leq \|z\|_{d+1} \leq Cx$ for all $z = (x, y) \in \mathcal{D} \cap (1, \infty) \times \mathbb{R}^d$. Theorem 1.4 may thus be

restated for return times of $Z = (X, Y)$, given in terms of its norm $\|Z\|_{d+1}$, instead of the scalar process X.

Recall that for the modulus of scalar Brownian motion, the critical exponent for return times equals $1/2$, with moments of order less (resp. greater) than $1/2$ being finite (resp. infinite). By Theorem 1.4(b), for domains with asymptotically increasing (resp. decreasing) boundary function b, i.e. $\beta > 0$ (resp. $\beta < 0$), the critical exponent m_c for the return times of the reflected process satisfies $0 < m_c < 1/2$ (resp. $1/2 < m_c$). Differently put, the reflected process in an asymptotically expanding (resp. narrowing) domain is, due to the asymptotically positive (resp. negative) projection of the reflection vector field in the x-direction, "less" (resp. "more") recurrent than the modulus of the scalar Brownian motion.

In the case $\beta = 0$, the boundary function b may (but need not, see Lemma B.1 below) be asymptotically constant, see the discussion in Remark 1.2 above. In this case, the projection of the reflection vector field converges to zero sufficiently fast that the critical moment exponent of the return time equals $m_c = 1/2$, regardless of other model parameters.

1.1.3. *The recurrence/transience dichotomy.* Recurrence/transience dichotomy for the reflected process defined by the SDE in (2) above is characterised by Theorem 1.3. Its proof relies on a generalisation to continuous time, given in Section 3.2 below, of the classical Foster–Lyapunov criteria for transience and recurrence. We stress that our approach is purely probabilistic: our criteria are phrased in terms of continuous-time supermartingale conditions. In particular, neither the Markov property nor any explicit knowledge of the infinitesimal characteristics of the process are required (cf. discussion about the approach in [32] in Section 1.3 below). Both of these features are crucial in the proof of Theorem 1.3.

The critical asymptotic growth rate β_c is always positive (see Assumptions (C2) and (V2)). By Theorem 1.3, if $\beta_c \geq 1$, then Z is recurrent for all boundary functions b satisfying (D2). This is for example the case for two-dimensional Brownian motion ($\sigma_1^2 = \sigma_2^2 = 1$ and $d = 1$) with normal reflection ($c_0 = s_0$) in $\mathcal{D} \subset \mathbb{R}^2$, as in this case we have $\beta_c = 1$. Note that, in any dimension $d \in \mathbb{N}$, the critical growth rate β_c may be greater than one, implying recurrence for all boundary functions satisfying (D2).

Theorem 1.4(a) strengthens transience of Z, stated in Theorem 1.3(b). Theorem 1.4(a) shows that for any $r > 0$ and $z \in \mathcal{D} \cap (r, \infty) \times \mathbb{R}^d$, the process Z does not visit the set $\mathcal{D} \cap [0, r] \times \mathbb{R}^d$ with positive probability (under $\mathbb{P}_z$), even though $x - r > 0$ can be arbitrarily small (where $z = (x, y)$). The proof of Theorem 1.3(b) establishes only that, with positive probability, the process Z does not return to $\mathcal{D} \cap [0, r] \times \mathbb{R}^d$ after reaching the set $\mathcal{D} \cap (r_1, \infty) \times \mathbb{R}^d$ for a sufficiently large $r_1 \in (r, \infty)$. As our assumptions are asymptotic, this extension requires proving, using basic analytical techniques, that, with positive probability, the reflected process reaches an arbitrarily high level before visiting a neighbourhood of the origin.

1.2. *A heuristic.* Recall that X denotes the x-component of the reflected process Z in (2). An informative heuristic argument in [32, pp. 679–680], based on the renewal theorem, es-

timates that the average local time accumulates as

$$\mathrm{d}L_t \approx \sigma_2^2/(2c_0 b(x))\mathrm{d}t, \qquad \text{when } X_t \text{ is at level } x. \tag{7}$$

By SDE (2) and the definition of β in (3), the total instantaneous drift of X (when $X_t = x$) is thus approximately equal to $s_0\sigma_2^2 b'(x)/(2c_0 b(x)) \approx s_0\sigma_2^2\beta/(2c_0 x)$ for large $x \in \mathbb{R}_+$. Consequently, the large-scale behaviour of the horizontal coordinate X of Z resembles that of the solution of the SDE $\mathrm{d}\widetilde{X}_t = s_0\sigma_2^2\beta/(2c_0\widetilde{X}_t)\mathrm{d}t + \sigma_1\mathrm{d}\widetilde{W}_t$ for large times t and values $\widetilde{X}_t$, where $\beta \in (-\infty, 1)$ and $\widetilde{W}$ is a one-dimensional Brownian motion. After time-changing the SDE for $\widetilde{X}$ by $t \mapsto t\sigma_1^2$, we obtain a Bessel-type process whose drift coefficient is determined by the ratio β/β_c. For $\beta > 0$, we have a Bessel process of dimension $2\beta/\beta_c$ and the transition between recurrence and transience in the heuristic matches that of the result in Theorem 1.3. In the case $\beta < 0$, the standard literature on Langevin diffusions [10] implies that the invariant distribution exists if $\beta < -\beta_c$, the tail of invariant distribution decays at the rate x^{β/β_c+1} (as $x \to \infty$) and the convergence to the invariant distribution is polynomial with the critical exponent $(1 + \beta/\beta_c)/2$. Again the heuristic coincides with our results in Theorem 1.5.

The definition of the asymptotically normal reflection requires the projection of the vector field ϕ in the x-direction to decay as $s_0 b'(x) \to 0$, when $x \to \infty$. The heuristic in (7) further motivates this definition: allowing a rate of decay of the projection of ϕ in the x-direction different from that of $b'(x)$ would miss phase transitions and other criticality phenomena. Moreover, heuristic (7) and the definition of β in (3) suggest that, in the super-linear case $\beta > 1$, the drift of X produced by the reflection at the boundary is asymptotically decays as $x^{-\beta}$ for large $x \in \mathbb{R}_+$. Such a drift is too weak to influence the long-term behaviour of the reflected process, because the process does not interact with the boundary sufficiently often.

Finally we note that the heuristic in (7) illustrates the difference between our domain $\mathcal{D}$, satisfying (D2), and the smooth domains studied in [38]. Recall from [29, Rem. 2.7(f)] that the domain $\mathcal{D}$ with asymptotically narrowing boundary does not satisfy the assumptions of [38]. This difference is more than just a technical caveat: one of the crucial features of the domains considered in [38] is the availability of bounds, uniform in the starting point, on the increments of expected local time. In contrast, for a narrowing domain $\mathcal{D}$ (i.e., $b(x) \to 0$ when $x \to \infty$), by (7), an increment of local time over a short time period cannot be bounded uniformly in the starting point because its growth is proportional to $1/b(x)$. We circumvent this issue via localisation, resulting in more involved proofs of fundamental properties such as the Feller continuity (see Theorem 6.5). Moreover, in the case of asymptotically oblique reflection, a narrowing domain $\mathcal{D}$ may result in an explosive reflected Brownian motion and explosive local time [29, Thm 2.2(i)], suggesting further that local time in $\mathcal{D}$ can grow arbitrarily fast. The long-term growth of local time is discussed in Section 7.3 below.

1.3. *Related literature.* The theory of reflecting diffusions began with [38]. A large literature is dedicated to reflecting diffusions in bounded domains (see e.g. [5, 15, 25]). In the unbounded case, the classical domains are cones (see e.g. [11, 17, 28, 39, 40]) and half-planes [6, 7]. Reflecting diffusions have been extensively studied due to their vast applications, including queueing theory [16, 31, 33] and mathematical finance [1, 19].

Our domain $\mathcal{D}$ in the case of (strictly) normal reflection has been studied in [32]. Theorem 1.3 can be viewed as a generalisation of a result in [32], which considers the case where $\Sigma(z)$ equals the identity matrix and the reflection vector field is given by the unit normal on the entire boundary $\partial\mathcal{D}$. In the context of our model, this setting is within the subclass $\sigma_1^2 = 1$, $\sigma_2^2 = d$ and $c_0 = s_0 = 1$ (recall that these constants specify only the limiting values of $\Sigma(z)$ and the reflection field ϕ on $\partial\mathcal{D}$ as $x \to \infty$).

By [32], the $(d+1)$-dimensional Brownian motion with normal reflection has phase transitions between transience and recurrence at $\beta_c = 1/d$, cf. (4). In particular, when dimension d is large, recurrence occurs only when the boundary function grows at rate slower than $1/d$, i.e., very slowly. In addition, Theorem 1.3(a) reveals that a $(d+1)$-dimensional reflected Brownian motion may be recurrent for a boundary function b with growth close to linear if the projection of the reflection vector field ϕ in the x-direction decays as $s_0 b'(x) \to 0$, when $x \to \infty$, for a sufficiently small constant $s_0 > 0$ (note that the projection in the x-direction of the normal reflection decays precisely at the rate $b'(x) \to 0$). Moreover, Theorem 1.3(a) and the formula in (4) imply that Z with normal reflection (i.e. $s_0 = c_0$) may be recurrent for the growth rate β (of the boundary function b) arbitrarily close to one, if the instantaneous variance of Z in the x-direction is greater than the sum of the variances in all other directions.

A general result in [32], for domains satisfying existence and uniqueness conditions of [38], states that a $(d+1)$-dimensional Brownian motion with normal reflection is positive recurrent if and only if $\mathcal{D}$ has finite volume. As in the case of Theorem 1.3 above, Theorem 1.4 and formula (6) contrast with the result of [32]. It is easy to see that the right choice of parameters results in positive (resp. null) recurrence in the domains with infinite (resp. finite) volume, e.g. $\beta < -1/(s_0 d)$ (resp. $-1/(s_0 d) < \beta < 1/(s_0 d)$) in the example in the previous paragraph.

As discussed in Section 1.2 above, some of the technical difficulties in our paper arise due to the fact that our domain $\mathcal{D}$ with the function b decreasing to zero (e.g. $\beta < 0$) violates the smoothness condition of domains in [38]. A general framework, via submartingale problems, for a large family of domains that fail to satisfy the assumptions [38] was developed in [22]. Similar ideas were employed in [21] to characterise stationary distributions for a large family of reflecting diffusions with piecewise smooth boundaries.

The question of convergence in general domains, beyond assumptions in [38], was studied in [4]. The main result of [4] gives the criteria for the uniform ergodicity of normally reflected planar Brownian motion. In particular, [4, Prop. 2.11] yields that the process in the domain $\mathcal{D}$ (with a boundary function b of polynomial decay defined in (1)) is not uniformly ergodic, a fact also implied by the lower bounds in Theorem 1.5 above. Moreover, the proofs in [4] use analytical tools such as potential theory and conformal mappings (the latter available in $\mathbb{R}^2$ only). In a domain $\mathcal{D} \subset \mathbb{R}^2$ in (1), uniform ergodicity arises if the decay of the boundary function b is superexponential [4, Prop. 2.11]. It is feasible that the probabilistic methods developed in our paper could generalise the results of [4] to domains $\mathcal{D}$ of arbitrary dimensions.

Finally we note that the invariant distribution and the rate of convergence of the reflecting Brownian motion with drift have been studied in [35, 36]. The motivation in these papers comes from applications in particle systems (see Section 1.1.1 above for a brief discussion of the relation between our result and those in [35, 36]).

2. Preliminary results: Itô's formula for the reflected process and Lyapunov functions. We start by noting that under Assumptions (C1), (D1) and (V1), by [29, Thm. A.1], SDE (2) has a unique strong solution $(Z, L, \tau_{\mathcal{E}})$ for any starting point in a generalised parabolic domain $\mathcal{D}$ defined in (1). In Section 4 we prove Theorem 4.1 (which shows that $\tau_{\mathcal{E}} = \infty$, a.s.) and Theorem 1.3. Sections 5 and 7 prove Theorems 1.4 and 1.5, respectively.

A key step in each of these proofs consists of the application of Itô's formula to an appropriate Lyapunov function. More precisely, let $f : \mathcal{D} \to \mathbb{R}$ denote a C^2-function on the open domain $\mathcal{D} \setminus \partial\mathcal{D}$, such that its gradient ∇f (i.e., the vector of the partial derivatives of f) has a continuous extension to the closed domain $\mathcal{D}$ (e.g. if f has a C^2-extension to an open set in $\mathbb{R}^{d+1}$ containing $\mathcal{D}$, which is typically the case in applications below). By Itô's formula [34, Thm. 3.3] we obtain

$$f(Z_t) = f(Z_0) + M_t + \frac{1}{2}\int_0^t \Delta_\Sigma f(Z_s)\mathrm{d}s + \int_0^t \langle \nabla f(Z_s), \phi(Z_s)\rangle \mathrm{d}L_s, \quad 0 \le t < \tau_{\mathcal{E}}, \tag{8}$$

where $\Delta_\Sigma f := \mathrm{Tr}\left(\Sigma^{1/2} H(f) \Sigma^{1/2}\right) = \mathrm{Tr}\left(\Sigma H(f)\right)$ is the Σ-Laplacian of f (recall that the Hessian matrix $H(f) : \mathcal{D} \to \mathcal{M}^+_{d+1}$ consists of the second partial derivatives of f). The (scalar) process M and its quadratic variation $[M]$ on the stochastic interval $[0, \tau_{\mathcal{E}})$ are given by

$$\begin{aligned} M_t &:= \int_0^t \langle \nabla f(Z_s), \Sigma^{1/2}(Z_s)\mathrm{d}W_s\rangle, \\ [M]_t &= \int_0^t \|\Sigma^{1/2}(Z_s)\nabla f(Z_s)\|^2_{d+1}\mathrm{d}s, \qquad 0 \le t < \tau_{\mathcal{E}}. \end{aligned} \tag{9}$$

The strategy of the proofs that follow consists of applying the continuous semimartingale results of Section 3 below to the process $\kappa = f(Z)$ for suitable C^2-Lyapunov functions f. In particular, we will use the representation of the quadratic variation $[M]$ in (9) to conclude that the appropriately stopped process M is a true martingale.

Pick $w \in \mathbb{R} \setminus \{0\}$. Define $k_w := 1 + \sup_{x \in \mathbb{R}_+} \left(|w(1-w)|^{1/2} b(x) - x\right)$. Note that, under (D2), the function b has sublinear growth at infinity (see Remark 1.2 above), implying $1 \le k_w < \infty$. For any $z = (x, y) \in \mathcal{D}$ and parameter $\gamma \in \mathbb{R}$ define

$$f_{w,1}(z) := (x + k_w)\left(1 + w(1-w)\frac{\|y\|^2_d}{2(x+k_w)^2}\right)^{1/w} \quad \& \quad f_{w,\gamma}(z) := f_{w,1}(z)^\gamma. \tag{10}$$

Since, for $(x, y) \in \mathcal{D}$, we have $\|y\|_d \le b(x)$, the definition of k_w implies

$$1/2 \le 1 + w(1-w)\|y\|^2_d/(2(x+k_w)^2) \le 3/2 \quad \text{for all } w \in \mathbb{R} \setminus \{0\}. \tag{11}$$

Thus $(x + k_w)2^{-1/|w|} \le f_{w,1}(x, y) \le (x + k_w)2^{1/|w|}$ for all parameter values $w \in \mathbb{R} \setminus \{0\}$. Moreover, for any $(x, y) \in \mathcal{D}$ we have

$$(x + k_w)^\gamma 2^{-|\gamma|/|w|} \le f_{w,\gamma}(x, y) \le (x + k_w)^\gamma 2^{|\gamma|/|w|} \qquad \text{for all } \gamma \in \mathbb{R}. \tag{12}$$

Note that, for $\gamma > 0$ (resp. $\gamma < 0$), the function $f_{w,\gamma}$ tends to infinity (resp. zero) as $x \to \infty$, making it suitable for the application of the results in Section 3. Moreover, it is clear that (for all choices of parameters γ and w) $f_{w,\gamma}$ is a C^2-function on the open domain $\mathcal{D} \setminus \partial\mathcal{D}$ and its gradient $\nabla f_{w,\gamma}$ has a continuous extension to the closed domain $\mathcal{D}$.

The Lyapunov function $f_{w,\gamma}$ is inspired by a generalisation of a polynomial approximation of the 2-dimensional harmonic function $h_w(z) = r^w \cos(w\theta)$ (given in polar coordinates $z = (r, \theta)$ of the plane), previously used in the analysis of the reflected processes in wedges [26, 28, 39].

The following lemma provides asymptotic properties of the relevant derivatives of $f_{w,\gamma}$.

LEMMA 2.1. *Let assumptions* (V2)*,* (D2) *and* (C2) *hold and fix* $\gamma \in \mathbb{R}$ *and* $w \in \mathbb{R} \setminus \{0\}$. *Then*

$$\Delta_\Sigma f_{w,\gamma}(z) = \gamma f_{w,1}(z)^{\gamma-2}(\sigma_1^2(\gamma-1) + \sigma_2^2(1-w) + o_{\mathcal{D}}(1)) \quad \text{as } x \to \infty. \tag{13}$$

There exists a constant $C > 0$*, such that*

$$\|\nabla f_{w,\gamma}(z)\|_{d+1}^2 \le C(x + k_w)^{2(\gamma-1)} \quad \text{for all } z = (x, y) \in \mathcal{D}. \tag{14}$$

Moreover, if $\gamma(s_0\beta/c_0 - 1 + w) < 0$ *(resp.* > 0*), then there exists a positive* x_0 *such that for all* $z = (x, y) \in \partial\mathcal{D} \cap [x_0, \infty) \times \mathbb{R}^d$ *we have* $\langle \nabla f_{w,\gamma}(z), \phi(z)\rangle < 0$ *(resp.* > 0*).*

REMARK 2.2. Note that the constants in $o_{\mathcal{D}}(1)$, as well as C and x_0, in Lemma 2.1 depend on the values of the parameters $\gamma \in \mathbb{R}$ and $w \in \mathbb{R} \setminus \{0\}$.

PROOF OF LEMMA 2.1. Denote $v(x, y) := y/(x + k_w) \in \mathbb{R}^d$ and define the scalar function $h(z) := 1 + w(1-w)\|v(z)\|_d^2/2$. For any $z = (x, y) \in \mathcal{D}$ we have $\|y\|_d \le b(x)$ and, by (D2) (see also Remark 1.2 above), $b(x) = o(x^{\beta'})$ as $x \to \infty$ for any $\beta' \in (\beta, 1)$. Hence $\|v(z)\|_d = o_{\mathcal{D}}(1)$ and, for any $r \in \mathbb{R}$, $h(z)^r = 1 + o_{\mathcal{D}}(1)$ as $x \to \infty$. Since $h(z) > 0$ (by (11)) and $f_{w,1}(z) = (x + k_w)h(z)^{1/w}$ for any $z = (x, y) \in \mathcal{D}$, we obtain

$$\begin{aligned}\nabla f_{w,1}(z) &= h(z)^{1/w}\left(e_x\left(1 - (1-w)\frac{\|v(z)\|_d^2}{h(z)}\right) + e_{\hat{y}}(1-w)\frac{\|v(z)\|_d}{h(z)}\right)\\ &= (1 + o_{\mathcal{D}}(1))\,(e_x + e_{\hat{y}}(1-w)\|v(z)\|_d) \quad \text{as } x \to \infty. \end{aligned} \tag{15}$$

(See the third paragraph of Section 1 for the definition of $\hat{y}$, e_x and $e_{\hat{y}}$.) Moreover, since $\|v(z)\|_d = o_{\mathcal{D}}(1)$ as $x \to \infty$ and $\nabla f_{w,\gamma}(z) = \gamma f_{w,\gamma-1}(z)\nabla f_{w,1}(z)$, by (12) there exists a positive constant C satisfying

$$\|\nabla f_{w,\gamma}(z)\|_{d+1}^2 = \gamma^2 f_{w,\gamma-1}(z)^2\,(1 + o_{\mathcal{D}}(1)) \le C(x + k_w)^{2(\gamma-1)} \quad \text{for all } z = (x, y) \in \mathcal{D}.$$

Recall that $f_{w,\gamma-1}(z) > 0$ by (12) for all $z \in \mathcal{D}$. Thus, for $z \in \partial\mathcal{D}$, the signs of $\langle \nabla f_{w,\gamma}(z), \phi(z)\rangle$ and $\gamma\langle \nabla f_{w,1}(z), \phi(z)\rangle$ are equal. Assumption (V2) implies $\langle e_x, \phi(z)\rangle = s_0 b'(x)(1 + o_{\partial\mathcal{D}}(1))$ and $\langle e_{\hat{y}}, \phi(z)\rangle = -c_0(1 + o_{\partial\mathcal{D}}(1))$ as $x \to \infty$. Note that for $z = (x, y) \in \partial\mathcal{D}$ we have $\|v(z)\|_d = b(x)/(x + k_w)$. By (15) we thus obtain

$$\gamma\langle \nabla f_{w,1}(z), \phi(z)\rangle = \gamma(s_0 b'(x) - c_0(1-w)\|v(z)\|_d)\,(1 + o_{\partial\mathcal{D}}(1))$$

$$
\begin{aligned}
&= \gamma b(x)(x+k_w)^{-1}(s_0(x+k_w)b'(x)/b(x) - c_0(1-w))(1+o_{\partial\mathcal{D}}(1)) \\
&= \gamma b(x)(x+k_w)^{-1}(s_0\beta - c_0(1-w) + o_{\partial\mathcal{D}}(1))(1+o_{\partial\mathcal{D}}(1)) \\
&= c_0 b(x)(x+k_w)^{-1}\left[\gamma\left(s_0\beta/c_0 - 1 + w\right) + o_{\partial\mathcal{D}}(1)\right],
\end{aligned}
$$

where the third equality in the display follows from the definition of β in (D2). By (V2), the model parameter c_0 is positive. Thus for $z=(x,y)\in\partial\mathcal{D}$ with x sufficiently large, the sign of $\gamma\langle\nabla f_{w,1}(z),\phi(z)\rangle$ equals that of $\gamma\left(s_0\beta/c_0 - 1 + w\right)$ as claimed in the lemma.

By definition we have $f_{w,\gamma}(z) = f_{w,1}(z)^\gamma$. Hence, for any $z\in\mathcal{D}$, the Hessian takes the form

$$
H(f_{w,\gamma})(z) = \gamma(\gamma-1)f_{w,1}(z)^{\gamma-2}\nabla f_{w,1}(z)(\nabla f_{w,1}(z))^\top + \gamma f_{w,1}(z)^{\gamma-1}H(f_{w,1})(z),
$$

where $(\nabla f_{w,1}(z))^\top$ denotes the $(d+1)$-dimensional row vector with coordinates given by the first partial derivatives of $f_{w,1}(z)$.

Since $\Delta_\Sigma f_{w,\gamma} = \mathrm{Tr}\left(\Sigma^{1/2}H(f_{w,\gamma})\Sigma^{1/2}\right) = \mathrm{Tr}\left(\Sigma H(f_{w,\gamma})\right)$, for $z=(x,y)\in\mathcal{D}$ we have

$$
\Delta_\Sigma f_{w,\gamma}(z) = \gamma f_{w,1}(z)^{\gamma-2}((\gamma-1)\langle\Sigma(z)\nabla f_{w,1}(z),\nabla f_{w,1}(z)\rangle + f_{w,1}(z)\Delta_\Sigma f_{w,1}(z)). \tag{16}
$$

Moreover, $\langle\Sigma(z)\nabla f_{w,1}(z),\nabla f_{w,1}(z)\rangle = \langle\Sigma(z)e_x,e_x\rangle + o_{\mathcal{D}}(1) = \sigma_1^2(1+o_{\mathcal{D}}(1))$ as $x\to\infty$ by (C2) and (15).

Note that $\partial_x h(z) = -w(1-w)\|y\|_d^2(x+k_w)^{-3}$ and $\partial_{y_i}h(z) = w(1-w)y_i(x+k_w)^{-2}$ for any $z=(x,y)\in\mathcal{D}$ (y_i is the i-th coordinate of $y\in\mathbb{R}^d$). An elementary (but tedious) calculation, based on the representation $f_{w,1}(z) = (x+k_w)h(z)^{1/w}$, yields

$$
\begin{aligned}
\partial_{y_i}^2 f_{w,1}(z) &= (1-w)(x+k_w)^{-1}h(z)^{1/w-1} + o_{\mathcal{D}}((x+k_w)^{-1}) \\
&= (1-w)(x+k_w)^{-1}(1+o_{\mathcal{D}}(1)) \qquad \text{as } x\to\infty
\end{aligned}
$$

for every $i\in\{1,\dots,d\}$ (recall that $h(z)^{1/w-1} = 1+o_{\mathcal{D}}(1)$). Moreover, all other elements of the Hessian $H(f_{w,1})(z)$ are of order $o_{\mathcal{D}}((x+k_w)^{-1})$ as $x\to\infty$. Thus, by definition $\Delta_\Sigma f_{w,1}(z) = \mathrm{Tr}(\Sigma(z)H(f_{w,1})(z))$ and the fact that Σ is bounded by (C1) (contained in (C2)) we get

$$
\Delta_\Sigma f_{w,1}(z) = \left(\mathrm{Tr}(\Sigma(z)) - \langle\Sigma(z)e_x,e_x\rangle\right)(1-w)(x+k_w)^{-1}(1+o_{\mathcal{D}}(1)) \quad \text{as } x\to\infty.
$$

By Assumption (C2), it thus follows that $\Delta_\Sigma f_{w,1}(z) = \sigma_2^2(1-w)(x+k_w)^{-1}(1+o_{\mathcal{D}}(1))$ as $x\to\infty$. The expression in (13) is now a direct consequence of (16). □

By Lemma 2.1, the function $f_{w,\gamma}$ controls the sign of the inner product $\langle\nabla f_{w,\gamma}(z),\phi(z)\rangle$ for $z=(x,y)\in\partial\mathcal{D}$ with sufficiently large $x\in\mathbb{R}_+$. Controlling the sign of $\langle\nabla f_{w,\gamma}(z),\phi(z)\rangle$ for *all* $z=(x,y)\in\partial\mathcal{D}$ is crucial for analysing the moments of Z_t at a fixed time t (see the proof of Theorem 1.3) as well as establishing drift conditions in the case of positive recurrence (see Lemma 5.4 in Section 5 below). This requires a slight modification of the function $f_{w,\gamma}$, which we now describe.

Fix arbitrary $x_0,x_1\in(0,\infty)$, satisfying $x_0<x_1$, and define the function $m:\mathbb{R}_+\times\mathbb{R}^d\to[0,1]$ as follows: for $z=(x,y)\in\mathbb{R}_+\times\mathbb{R}^d$ let

$$
m(z) := \exp\left((x_1-x_0)^{-2} - ((x_1-x_0)^2 - (x_1-x)^2)^{-1}\right)\mathbb{1}\{x_0<x<x_1\} + \mathbb{1}\{x_1\le x\}.
$$

The function m is smooth, $\partial_x m(z) \geq 0$ for all $z \in \mathbb{R}_+ \times \mathbb{R}^d$, and the following holds: for any $z_i = (x_i, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, $i \in \{0,1\}$, we have $m(z_0) = \partial_x m(z_0) = \partial_x^2 m(z_0) = 0$ and $m(z_1) = 1$, $\partial_x m(z_1) = \partial_x^2 m(z_1) = 0$. For any constant $k \in (0,\infty)$, define

$$F_{w,\gamma}(z) := f_{w,\gamma}(z)m(z) + k(1 - m(z)), \quad z \in \mathcal{D}. \tag{17}$$

The function $F_{w,\gamma}$ is clearly a C^2-function on the open domain $\mathcal{D} \setminus \partial\mathcal{D}$ and its gradient $\nabla F_{w,\gamma}$ has a continuous extension to the closed domain $\mathcal{D}$ (for all parameters $\gamma \in \mathbb{R}$ and $w \in \mathbb{R} \setminus \{0\}$).

LEMMA 2.3. *Let Assumptions* (V2)*,* (D2) *and* (C2) *hold and fix* $\gamma \in \mathbb{R}$ *and* $w \in \mathbb{R} \setminus \{0\}$. *Then, if* $\gamma(\beta s_0/c_0 - 1 + w) < 0$*, there exist* $k \in (0,\infty)$ *and* $0 < x_0 < x_1$ *such that*

$$\langle \nabla F_{w,\gamma}(z), \phi(z) \rangle \leq 0 \quad \textit{for all } z \in \partial\mathcal{D}.$$

PROOF. Note that for any constants $0 < k < \infty$ and $0 < x_0 < x_1$, for all $z \in \mathcal{D}$ we have

$$\langle \nabla F_{w,\gamma}(z), \phi(z) \rangle = (f_{w,\gamma}(z) - k)\partial_x m(z)\langle e_x, \phi(z)\rangle + m(z)\langle \nabla f_{w,\gamma}(z), \phi(z)\rangle. \tag{18}$$

Since $\gamma(\beta s_0/c_0 - 1 + w) < 0$, by Lemma 2.1, there exists $x_0' > 0$ such that

$$\langle \nabla f_{w,\gamma}(z), \phi(z)\rangle < 0 \quad \text{for all } z \in \partial\mathcal{D} \cap [x_0', \infty) \times \mathbb{R}^d.$$

By Assumption (V2), $\langle \phi(z), e_x\rangle = s_0 b'(x)(1 + f(z))$ for all $z \in \partial\mathcal{D}$, where the function $f : \partial\mathcal{D} \to \mathbb{R}$ satisfies $\sup_{y:(x,y)\in\partial\mathcal{D}} |f(x,y)| \to 0$ as $x \to \infty$. Pick $x_0'' \in \mathbb{R}_+$ such that $\sup_{y:(x,y)\in\partial\mathcal{D}} |f(x,y)| < 1/2$ for all $x \in [x_0'', \infty)$. There are two possibilities.

(I) There exists $x_0 \in [\max\{x_0', x_0''\}, \infty)$ such that $|b'(x_0)| > 0$. If $b'(x_0) > 0$ (resp. $b'(x_0) < 0$), by the continuity of b', there exists $x_1 \in (x_0, \infty)$, such that $b'(x) > 0$ (resp. $b'(x) < 0$) for all $x \in [x_0, x_1]$. Since $s_0 > 0$ by Assumption (V2), for any $z = (x,y) \in \partial\mathcal{D}$ with $x \in [x_0, x_1]$, we have $\langle \phi(z), e_x\rangle = s_0 b'(x)(1 + f(z)) > s_0 b'(x)/2 > 0$ (resp. $\langle \phi(z), e_x\rangle = s_0 b'(x)(1 + f(z)) < s_0 b'(x)/2 < 0$). By (12), $f_{w,\gamma}$ is a positive bounded function on the set $\mathcal{D} \cap [x_0, x_1] \times \mathbb{R}^d$. Thus we may pick k in the interval $(\sup_{(x,y)\in\partial\mathcal{D}: x_0 \leq x \leq x_1} f_{w,\gamma}(x,y), \infty)$ (resp. $(0, \inf_{(x,y)\in\partial\mathcal{D}: x_0 \leq x \leq x_1} f_{w,\gamma}(x,y))$). Since the function m, defined above, satisfies $\min\{\partial_x m(z), m(z)\} \geq 0$ for all $z \in \mathcal{D}$, $\partial_x m(z) = 0$ for all $z = (x,y)$ with $x \in [x_1, \infty)$ and $\partial_x m(z) = m(z) = 0$ for all $z = (x,y)$ with $x \in [0, x_0]$, by (18) it follows $\langle \nabla F_{w,\gamma}(z), \phi(z)\rangle \leq 0$ for all $z \in \partial\mathcal{D}$.

(II) $|b'(x)| = 0$ for all $x \in [\max\{x_0', x_0''\}, \infty)$. Thus, for any $x_0, x_1 \in [\max\{x_0', x_0''\}, \infty)$ with $x_0 < x_1$, we have $\partial_x m(z)\langle e_x, \phi(z)\rangle = \partial_x m(z) s_0 b'(x)(1 + f(z)) = 0$ for all $z \in \mathcal{D}$. Hence, for any $k \in (0,\infty)$, by (18) we have $\langle \nabla F_{w,\gamma}(z), \phi(z)\rangle = m(z)\langle \nabla f_{w,\gamma}(z), \phi(z)\rangle \leq 0$ for all $z \in \partial\mathcal{D}$. □

The function $f_{w,\gamma}$ suffices to establish Theorem 1.3(a)–(b) when the asymptotic exponent β is away from the critical value β_c defined in (4). In the critical case $\beta = \beta_c$, logarithmic (rather than polynomial) growth of the Lyapunov function is required. The function we now define for this purpose is inspired by the analysis of the reflecting Brownian motion and

random walk in a wedge in [26, 28]. Pick an arbitrary constant $\delta \in (0,\infty)$ and let $g_\delta : \mathcal{D} \to (1,\infty)$ be a continuous function, twice differentiable in the interior of $\mathcal{D}$, satisfying

$$g_\delta(z) = \log(x) - x^{-\delta} + \frac{\sigma_1^2}{\sigma_2^2}\frac{\|y\|_d^2}{2x^2}(1+\delta x^{-\delta}) + 1 \quad \text{for } z=(x,y)\in\mathcal{D} \text{ with } x \in (\mathrm{e},\infty), \tag{19}$$

with σ_1^2 and σ_2^2 given in (C2+). Since for any $(x,y)\in\mathcal{D}$ we have $\|y\|_d \le b(x)$ and b is sublinear by (D2+) (cf. Remark 1.2 above), there exists a positive constant $C_\delta \in (0,\infty)$ such that

$$\log x \le g_\delta(z) \le C_\delta + \log x, \qquad \text{for } z=(x,y)\in\mathcal{D} \text{ with } x > \mathrm{e}. \tag{20}$$

The relevant asymptotic properties of the derivatives of g_δ are in the next lemma.

LEMMA 2.4. *Assume $\beta = \beta_c$, where β (resp. β_c) is defined in* (3) *(resp.* (4)*). Let Assumptions* (V2+)*,* (D2+) *and* (C2+) *hold and choose $\delta \in (0, \min\{\varepsilon, 1-\beta\})$, where $\varepsilon > 0$ is the rate of decay in Assumption* (C2+)*. Then there exists an $x_0 > 0$ such that*

$$\Delta_\Sigma g_\delta < 0 \text{ on } \mathcal{D}\cap[x_0,\infty)\times\mathbb{R}^d \quad \text{and} \quad \langle \nabla g_\delta, \phi\rangle < 0 \text{ on } \partial\mathcal{D}\cap[x_0,\infty)\times\mathbb{R}^d. \tag{21}$$

PROOF. Recall that $\beta = \beta_c = c_0\sigma_1^2/(s_0\sigma_2^2)$. By (19), for $z=(x,y)\in\mathcal{D}$ with $x > \mathrm{e}$, we have

$$\begin{aligned}\nabla g_\delta(z) = e_x\left(x^{-1}(1+\delta x^{-\delta}) - \frac{\sigma_1^2}{\sigma_2^2}\frac{\|y\|_d^2}{x^3}(1-\delta x^{-\delta}(1+\delta/2))\right)\\ + e_{\hat{y}}\left(\frac{\sigma_1^2}{\sigma_2^2}\frac{\|y\|_d}{x^2}(1+\delta x^{-\delta})\right);\end{aligned} \tag{22}$$

see the first paragraph of Section 2 for the definition of $\hat{y}$, e_x and $e_{\hat{y}}$.

By (V2+), $\langle e_x, \phi(z)\rangle = s_0 b'(x)(1+o_{\partial\mathcal{D}}(b(x)^2x^{-2})) = s_0\beta b(x)x^{-1}(1+o_{\partial\mathcal{D}}(b(x)^2x^{-2}))$ holds as $x\to\infty$, where the second equality follows from (D2+), and $\langle e_{\hat{y}}, \phi(z)\rangle = -c_0(1+o_{\partial\mathcal{D}}(b(x)^2x^{-2}))$ as $x\to\infty$. Since $z=(x,y)\in\partial\mathcal{D}$ satisfies $\|y\|_d = b(x)$ as $x\to\infty$, by (22) we obtain

$$\begin{aligned}\langle\nabla g_\delta(z),\phi(z)\rangle &= -s_0^2\beta^2c_0^{-1}b(x)^3x^{-4}(1+b(x)^{-2}x^2o_{\partial\mathcal{D}}(b(x)^2x^{-2}) + o_{\partial\mathcal{D}}(x^{-\delta}))\\ &= -s_0^2c_0^{-1}\beta^2b(x)^3x^{-4} + o_{\partial\mathcal{D}}(b(x)^3x^{-4}) \quad \text{as } x\to\infty.\end{aligned}$$

Thus there exists $x_0 > 0$, such that $\langle\nabla g_\delta(z),\phi(z)\rangle < 0$ for all $z=(x,y)\in\partial\mathcal{D}\cap[x_0,\infty)\times\mathbb{R}^d$.

By Remark 1.2 above, for any $\beta' \in (\beta,1)$ it holds $b(x) = o(x^{\beta'})$ as $x\to\infty$, implying $\|y\|_d/x = o_{\mathcal{D}}(1)$ as $x\to\infty$. By assumption $\delta\in(0,1-\beta)$. Thus $b(x) = o(x^{\beta+\delta})$, implying $b(x)^2x^{-4} = o(x^{-2-\delta})$ and $b(x)x^{-3} = o(x^{-2-\delta})$ as $x\to\infty$. Recall from (22) that, for any $i\in\{1,\dots,d\}$, the i-th partial derivative $\partial_{y_i}g_\delta$ along the coordinate y_i of y equals $\partial_{y_i}g_\delta(z) = \beta s_0c_0^{-1}y_ix^{-2}(1+\delta x^{-\delta})$ for $x > \mathrm{e}$, implying $\partial^2_{y_i}g_\delta(z) = \beta s_0c_0^{-1}x^{-2}(1+\delta x^{-\delta})$. By the representation of the gradient (22) and the fact $\|y\|_d \le b(x)$ for every $z=(x,y)\in\mathcal{D}$, we have $\partial_x^2 g_\delta(z) = -x^{-2}(1+\delta(1+\delta)x^{-\delta}) + o_{\mathcal{D}}(x^{-2-\delta})$, while all mixed derivatives in the Hessian $H(g_\delta)(z)$ are of order $b(x)x^{-3} = o_{\mathcal{D}}(x^{-2-\delta})$ as $x\to\infty$. Thus, by the definition $\Delta_\Sigma g_\delta(z) = \mathrm{Tr}(\Sigma(z)H(g_\delta)(z))$ and the fact that $\partial^2_{y_i}g_\delta$ does not depend on the index

$i \in \{1, \dots, d\}$ and is bounded for $x > \mathrm{e}$, we get

$$\begin{aligned}\Delta_\Sigma g_\delta(z) =& \langle \Sigma(z)e_x, e_x\rangle \partial_x^2 g_\delta(z) + (\mathrm{Tr}(\Sigma(z)) - \langle \Sigma(z)e_x, e_x\rangle)\, \partial_{y_1}^2 g_\delta(z) + o_{\mathcal{D}}(x^{-2-\delta}) \\ =& -x^{-2}(1 + o_{\mathcal{D}}(x^{-\delta}))((\sigma_1^2 + o_{\mathcal{D}}(x^{-\varepsilon})) \\ &+ \beta s_0 c_0^{-1} x^{-2}(1 + \delta x^{-\delta})(\sigma_2^2 + o_{\mathcal{D}}(x^{-\varepsilon}))) + o_{\mathcal{D}}(x^{-2-\delta}) \\ =& -\sigma_1^2 \delta^2 x^{-2-\delta} + o_{\mathcal{D}}(x^{-2-\delta}) \quad \text{as } x \to \infty,\end{aligned}$$

where the last equality follows form the identity $s_0 \beta \sigma_2^2 / c_0 = \sigma_1^2$ and the fact $\delta < \varepsilon$. Hence, the sign of $\Delta_\Sigma g_\delta(z)$ is negative for $z \in (x, y) \in \mathcal{D}$ with x sufficiently large as claimed in lemma. □

3. Non-explosion, recurrence/transience criteria, and return times of continuous semimartingales. This section develops certain semimartingale tools for classifying asymptotic behaviour via Foster–Lyapunov criteria. The general theory developed in this section is expected to have broad applicability. In the present paper, it will be applied to study the reflected process Z, given by SDE (2), via the Lyapunov functions constructed and analysed in Section 2 above.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfying the usual conditions. Consider an $(\mathcal{F}_t)$-adapted continuous process $\kappa = (\kappa_t)_{t \in \mathbb{R}_+}$, taking values in $[0, \infty]$. Let $\mathcal{T}$ denote the set of all $[0, \infty]$-valued stopping times with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. For any $\ell, r \in \mathbb{R}_+$ and stopping time $T \in \mathcal{T}$, define the first entry times (after T) by

$$\lambda_{\ell,T} := T + \inf\{s \in \mathbb{R}_+ : T < \infty,\ \kappa_{T+s} \le \ell\}, \tag{23}$$

$$\rho_{r,T} := T + \inf\{s \in \mathbb{R}_+ : T < \infty,\ \kappa_{T+s} \ge r\}, \tag{24}$$

where we adopt the convention $\inf \emptyset := +\infty$. If $T = 0$, we write $\lambda_\ell := \lambda_{\ell,0}$ and $\rho_r := \rho_{r,0}$. Almost sure limits $\rho_\infty := \lim_{r \to \infty} \rho_r$ and $\rho_{\infty,T} = \lim_{r \to \infty} \rho_{r,T}$ exist by monotonicity. *Explosion* of the process κ occurs if the event $\{\rho_\infty < \infty\}$ has positive probability. Since $\rho_{r,T} = \rho_r$ on the event $\{T < \rho_r\}$, we have $\rho_\infty = \rho_{\infty,T}$ on the event $\{T < \rho_\infty\}$. For $r_0 \le r$, we define the first exit time from the interval $[r_0, r]$ after some stopping time $T \in \mathcal{T}$ by

$$S_{r,T} := \lambda_{r_0,T} \wedge \rho_{r,T}. \tag{25}$$

Here and throughout we denote $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$ for any $x, y \in [0, \infty]$.

3.1. *Non-explosion.* We first establish criteria for κ not to explode. The main application of this result in the present paper is to prove that the reflected process with asymptotically normal reflection cannot explode.

This should be contrasted with the case of the asymptotically oblique reflection, where explosion may occur, see the characterisation in [29, Thm 2.2]. The non-explosion criteria in [29, Thm 3.4] are more delicate than the ones in Lemma 3.1, but require transience of the underlying semimartingale, making them inapplicable to the entire class of processes considered here. The following result is more robust (i.e. with simpler assumptions), has an elementary proof and covers all the models analysed in this paper.

LEMMA 3.1. *Let $\kappa = (\kappa_t)_{t\in\mathbb{R}_+}$ be an $[0,\infty]$-valued $(\mathcal{F}_t)$-adapted continuous process and $V : \mathbb{R}_+ \to (0,\infty)$ a continuous function with $\lim_{x\to\infty} V(x) = \infty$. Suppose there exist $r_0, \eta \in \mathbb{R}_+$, such that for all $r \in (r_0,\infty)$ and any $T \in \mathcal{T}$, such that $\mathbb{E}[V(\kappa_T)\mathbb{1}\{T < \rho_\infty\}] < \infty$, the process $\zeta^{T,r} = (\zeta_t^{T,r})_{t\in\mathbb{R}_+}$, defined by*

$$\zeta_t^{T,r} := (V(\kappa_{(t+T)\wedge S_{r,T}}) - \eta(t \wedge (S_{r,T} - T)))\mathbb{1}\{T < \rho_\infty\}, \tag{26}$$

is an $(\mathcal{F}_{t+T})$-supermartingale, i.e., for $0 \le s \le t < \infty$,

$$\mathbb{E}[\zeta_t^{T,r}|\mathcal{F}_{s+T}] \le \zeta_s^{T,r}, \text{ a.s.} \tag{27}$$

Then $\mathbb{P}(\rho_\infty = \infty) = 1$.

PROOF. Choose $r_1 \in (r_0,\infty)$ and define recursively the upcrossing and downcrossing times of the process κ over the interval $[r_0, r_1]$ as follows: $\overline{\theta}_1 := 0$, and if $\overline{\theta}_k$ has been defined for some $k \in \mathbb{N}$, then $\underline{\theta}_k := \lambda_{r_0,\overline{\theta}_k}$ and $\overline{\theta}_{k+1} := \rho_{r_1,\underline{\theta}_k}$. Thus we have $0 = \overline{\theta}_1 \le \underline{\theta}_1 \le \cdots \le \overline{\theta}_k \le \underline{\theta}_k \le \overline{\theta}_{k+1} \le \cdots$. Moreover, for any $t \in \mathbb{R}_+$, we denote

$$D(t, r_0, r_1) := \sup\{k \in \mathbb{N};\ \underline{\theta}_k \le t\},$$

the number of downcrossings of the interval $[r_0, r_1]$ up to time t for the process κ. Since any continuous function on the compact interval $[0,t]$ crosses an interval of positive length at most finitely many times, we have $D(t, r_0, r_1) < \infty$ a.s.

Assume now $\mathbb{P}(\rho_\infty < \infty) > 0$. Then there exists $t_0 \in \mathbb{R}_+$ such that $\mathbb{P}(\rho_\infty < t_0) > 0$. We will prove by induction that $\{\overline{\theta}_k < \rho_\infty\} \cap \{\rho_\infty < t_0\} = \{\rho_\infty < t_0\}$ a.s. holds for all $k \in \mathbb{N}$. The induction hypothesis holds for $k = 1$ since $\overline{\theta}_1 = 0 < \rho_\infty$ a.s. Assume the almost-sure equality of events holds for some $k \in \mathbb{N}$. Set $T := \overline{\theta}_k$ and note $V(\kappa_T)\mathbb{1}\{T < \rho_\infty\} = V(r_1)\mathbb{1}\{T < \rho_\infty\}$, since the paths of κ are continuous and, on the event $\{T < \rho_\infty\}$, we have $T < \infty$. In particular, $V(\kappa_T)\mathbb{1}\{T < \rho_\infty\}$ is bounded and hence integrable. Pick any $r \in (r_1,\infty)$ and consider the supermartingale $\zeta^{T,r}$ defined in (26). Note that $(T \vee t_0) - T$ is a bounded $(\mathcal{F}_{T+t})$-stopping time since, for any $s \in \mathbb{R}_+$, we have $\{T \vee t_0 - T \le s\} = \{t_0 \vee T \le T + s\} \in \mathcal{F}_{T+s}$ as both $t_0 \vee T$ and $s + T$ are $(\mathcal{F}_t)$-stopping times. Applying the optional sampling theorem to $\zeta^{T,r}$ at $(T \vee t_0) - T$ yields:

$$\begin{aligned}\mathbb{P}(\rho_{r,T} < \lambda_{r_0,T} \wedge (T \vee t_0), T < \rho_\infty|\mathcal{F}_T)V(r) - \eta t_0 &\le \mathbb{E}[\zeta^{T,r}_{(T\vee t_0)-T}|\mathcal{F}_T] \\ &\le \zeta_0^{T,r} = V(r_1)\mathbb{1}\{T < \rho_\infty\}.\end{aligned}$$

Taking expectations on both sides, we obtain the following inequality for every $r \in (r_1,\infty)$:

$$\mathbb{P}(\rho_{r,T} < \lambda_{r_0,T} \wedge (T \vee t_0), T < \rho_\infty) \le (V(r_1) + \eta t_0)/V(r).$$

Thus, by the monotone convergence theorem, we obtain

$$\begin{aligned}0 \le \mathbb{P}(\rho_\infty \le \lambda_{r_0,T} \wedge (T \vee t_0), T < \rho_\infty) &= \lim_{r\to\infty} \mathbb{P}(\rho_{r,T} < \lambda_{r_0,T} \wedge (T \vee t_0), T < \rho_\infty) \\ &\le \limsup_{r\to\infty} (V(r_1) + t_0\eta)/V(r) = 0,\end{aligned} \tag{28}$$

implying $\{T < \rho_\infty\} = \{\rho_\infty > \lambda_{r_0,T} \wedge (T \vee t_0)\} \cap \{T < \rho_\infty\}$ a.s. We hence obtain

$$\begin{aligned}\{\rho_\infty < t_0\} &= \{T < \rho_\infty\} \cap \{\rho_\infty < t_0\} = \{\rho_\infty > \lambda_{r_0,T} \wedge (T \vee t_0)\} \cap \{T < \rho_\infty\} \cap \{\rho_\infty < t_0\} \\ &= \{\lambda_{r_0,T} < \rho_\infty\} \cap \{\rho_\infty < t_0\} \text{ a.s.}\end{aligned}$$

The first equality in this display holds by the induction hypothesis, the second holds by (28), and the third equality follows from the fact that $T \leq \lambda_{r_0,T}$ by definition (23). Since the equality $\{\underline{\theta}_k = \lambda_{r_0,T} < \rho_\infty\} = \{\rho_{r_1,\underline{\theta}_k} = \overline{\theta}_{k+1} < \rho_\infty\}$ holds almost surely, we proved that $\{\overline{\theta}_{k+1} < \rho_\infty\} \cap \{\rho_\infty < t_0\} = \{\rho_\infty < t_0\}$ almost surely, thus verifying the induction step.

We conclude that

$$\{\overline{\theta}_1 < \ldots < \overline{\theta}_k < \rho_\infty < t_0\} = \bigcap_{i=1}^{k} \{\overline{\theta}_i < \rho_\infty < t_0\} = \{\rho_\infty < t_0\} \quad \text{a.s. for all } k \in \mathbb{N}.$$

Since $\{\overline{\theta}_i < \overline{\theta}_{i+1}\} = \{\overline{\theta}_i < \underline{\theta}_i < \overline{\theta}_{i+1}\}$ for every $i \in \mathbb{N}$ and $D(t_0, r_0, r_1) < \infty$ a.s., it follows that $\mathbb{P}(\rho_\infty < t_0) = 0$, contradicting our assumption $\mathbb{P}(\rho_\infty < \infty) > 0$. □

3.2. *Transience and recurrence criteria for continuous semimartingales.* Lemmas 3.2 and 3.4 of the present subsection provide sufficient conditions for recurrence and transience, respectively, for a continuous semimartingale κ. They are continuous-time analogues to the Foster-Lyapunov criteria for discrete-time processes discussed in e.g. [27].

LEMMA 3.2. *Let $V : \mathbb{R}_+ \to (0, \infty)$ be a continuous function, such that $\lim_{x\to\infty} V(x) = \infty$, and let $\kappa = (\kappa_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted continuous process satisfying $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. If there exists such $r_0 > 0$, such that for all $t_0 \in \mathbb{R}_+$ and $r \in (r_0, \infty)$, the process $(V(\kappa_{(t+t_0)\wedge S_{r,t_0}}))_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_{t+t_0})$-supermartingale (recall that r_0 features in $S_{r,t_0} = \lambda_{r_0,t_0} \wedge \rho_{r,t_0}$ by definition (25)), i.e., $\mathbb{E}[V(\kappa_{t_0})] < \infty$ and for $0 \leq s \leq t < \infty$,*

$$\mathbb{E}[V(\kappa_{(t+t_0)\wedge S_{r,t_0}})|\mathcal{F}_{s+t_0}] \leq V(\kappa_{(s+t_0)\wedge S_{r,t_0}}),$$

then $\mathbb{P}(\liminf_{t\to\infty} \kappa_t \leq r_0) = 1$.

REMARK 3.3. Up to requiring verification over a smaller class of stopping times, the hypotheses of Lemma 3.2 essentially imply those of Lemma 3.1 with $\eta = 0$. Thus, a Lyapunov function that implies recurrence will also often yield non-explosion. In the case of a transient process, however, $\eta > 0$ is typically needed for Lemma 3.1 to be applicable.

PROOF OF LEMMA 3.2. Pick $t_0 \in \mathbb{R}_+$, $r \in (r_0, \infty)$. We will consider the $(\mathcal{F}_{t+t_0})$-supermartingale $(V(\kappa_{(t+t_0)\wedge S_{r,t_0}}))_{t\in\mathbb{R}_+}$. Note that the assumption $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. implies $S_{r,t_0} < \infty$ a.s. and hence it holds that $\lim_{t\to\infty} V(\kappa_{(t+t_0)\wedge S_{r,t_0}}) = V(\kappa_{S_{r,t_0}})$ a.s. The supermartingale property, Fatou's lemma, definition (25) and the continuity of κ imply

$$\begin{aligned}\infty > \mathbb{E}[V(\kappa_{t_0})] &\geq \liminf_{t\to\infty} \mathbb{E}[V(\kappa_{(t+t_0)\wedge S_{r,t_0}})] \geq \mathbb{E}[\liminf_{t\to\infty} V(\kappa_{(t+t_0)\wedge S_{r,t_0}})] \\ &= \mathbb{E}[V(\kappa_{S_{r,t_0}})] \geq \mathbb{P}(\rho_{r,t_0} < \lambda_{r_0,t_0}) V(r).\end{aligned}$$

Thus, for all $r \in (r_0, \infty)$, we obtain

$$\mathbb{P}\Big(\inf_{t\geq t_0} \kappa_t \leq r_0\Big) \geq \mathbb{P}(\lambda_{r_0,t_0} < \infty) \geq \mathbb{P}(\lambda_{r_0,t_0} < \rho_{r,t_0}) \geq 1 - \mathbb{E}[V(\kappa_{t_0})]/V(r).$$

Since $\lim_{r\to\infty} V(r) = \infty$ by assumption, it follows that $\mathbb{P}(\inf_{t\geq t_0} \kappa_t \leq r_0) = 1$ for any fixed $t_0 \in \mathbb{R}_+$, implying $\mathbb{P}(\liminf_{t\to\infty} \kappa_t \leq r_0) = \mathbb{P}(\cap_{t_0\in\mathbb{N}}\{\inf_{t\geq t_0} \kappa_t \leq r_0\}) = 1$. □

LEMMA 3.4. *Let $V : \mathbb{R}_+ \to (0,\infty)$ be a continuous function with $\lim_{x\to\infty} V(x) = 0$ and $\kappa = (\kappa_t)_{t\in\mathbb{R}_+}$ an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted continuous process satisfying $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. If there exists $r_0 > 0$, such that for all $T \in \mathcal{T}$, satisfying $T < \infty$ a.s., and $r \in (r_0,\infty)$, the process $(V(\kappa_{(t+T)\wedge S_{r,T}}))_{t\in\mathbb{R}_+}$, is an $(\mathcal{F}_{t+T})$-supermartingale, i.e., for $0 \leq s \leq t < \infty$,*

$$\mathbb{E}[V(\kappa_{(t+T)\wedge S_{r,T}})|\mathcal{F}_{s+T}] \leq V(\kappa_{(s+T)\wedge S_{r,T}}), \tag{29}$$

then $\mathbb{P}(\lim_{t\to\infty} \kappa_t = \infty) = 1$.

REMARK 3.5. Note that the function V in Lemma 3.4 is assumed to be continuous and have limit zero at infinity, making it bounded. Thus, for any $T \in \mathcal{T}$, satisfying $T < \infty$ a.s., we have $\mathbb{E}[V(\kappa_T)] < \infty$. Once we have such a candidate function V, in order to apply Lemma 3.4, we only need to check the $(\mathcal{F}_{t+T})$-supermartingale property in (29).

PROOF. Pick an arbitrary $\ell \in (r_0,\infty)$. For any $r, r_1 \in \mathbb{R}_+$, satisfying $\ell < r_1 < r$, define $T := \rho_{r_1}$. Since $\limsup_{t\to\infty} \kappa_t = \infty$ a.s., we have $T = \rho_{r_1} < \infty$ a.s. and $\rho_{r,T} < \infty$ a.s., implying further by definition (25) that $S_{r,T} = \lambda_{r_0,T} \wedge \rho_{r,T} \leq \rho_{r,T} < \infty$ a.s.

Define the process $\xi = (\xi_t)_{t\in\mathbb{R}_+}$ by $\xi_t := V(\kappa_{(t+T)\wedge\lambda_{\ell,T}\wedge\rho_{r,T}})$, $t \in \mathbb{R}_+$. Since $\lambda_{\ell,T} \wedge \rho_{r,T} \leq S_{r,T}$ a.s., the process ξ equals the $(\mathcal{F}_{t+T})$-supermartingale $(V(\kappa_{(t+T)\wedge S_{r,T}}))_{t\in\mathbb{R}_+}$ stopped at the $(\mathcal{F}_{t+T})$-stopping time $\lambda_{\ell,T} \wedge \rho_{r,T} - T$. Thus, by Remark 3.5 and [34, Ch. II, Thm 3.3], the process ξ is a positive $(\mathcal{F}_{t+T})$-supermartingale. Hence, for all $t \in \mathbb{R}_+$, we have

$$V(\kappa_T) = \xi_0 \geq \mathbb{E}[\xi_t|\mathcal{F}_T] \geq \mathbb{E}[\xi_t \mathbb{1}\{\lambda_{\ell,T} \leq \rho_{r,T}\}|\mathcal{F}_T].$$

Since κ is continuous, it holds that $\lim_{t\to\infty} \xi_t \mathbb{1}\{\lambda_{\ell,T} \leq \rho_{r,T}\} = V(\ell)\mathbb{1}\{\lambda_{\ell,T} \leq \rho_{r,T}\}$ and $V(\kappa_T) = V(r_1)$ a.s. The (conditional) Fatou lemma yields

$$\begin{aligned}\mathbb{P}(\lambda_{\ell,T} \leq \rho_{r,T}|\mathcal{F}_T)V(\ell) &= \mathbb{E}[\liminf_{t\to\infty} \xi_t \mathbb{1}\{\lambda_{\ell,T} \leq \rho_{r,T}\}|\mathcal{F}_T] \\ &\leq \liminf_{t\to\infty} \mathbb{E}[\xi_t \mathbb{1}\{\lambda_{\ell,T} \leq \rho_{r,T}\}|\mathcal{F}_T] \leq \xi_0 = V(\kappa_T) = V(r_1).\end{aligned}$$

Since, by assumption, $\kappa_t \in \mathbb{R}_+$ a.s. for all $t \in \mathbb{R}_+$, κ does not explode. Thus $\lim_{r\to\infty} \rho_{r,T} = \infty$ a.s. and, since $\rho_{r,T} < \infty$ a.s. for all $r \in (r_1,\infty)$, we get (recall $V(\ell) > 0$)

$$\mathbb{P}(\lambda_{\ell,T} < \infty) = \mathbb{P}(\cup_{r\in\mathbb{N}\cap(r_1,\infty)}\{\lambda_{\ell,T} \leq \rho_{r,T}\}) = \lim_{r\to\infty} \mathbb{P}(\lambda_{\ell,T} \leq \rho_{r,T}) \leq V(r_1)/V(\ell). \tag{30}$$

Recall $T = \rho_{r_1} < \infty$ a.s. and note $\{\liminf_{t\to\infty} \kappa_t \leq \ell\} \subset \{\lambda_{\ell,\rho_{r_1}} < \infty\}$ for all $r_1 \in (\ell,\infty)$. The inequality $\mathbb{P}(\liminf_{t\to\infty} \kappa_t \leq \ell) \leq \mathbb{P}(\lambda_{\ell,\rho_{r_1}} < \infty)$ for all $r_1 \in (\ell,\infty)$, the upper bound in (30) and the hypothesis $V(r_1) \to 0$ as $r_1 \to \infty$ imply

$$\mathbb{P}\Big(\liminf_{t\to\infty} \kappa_t \leq \ell\Big) \leq \limsup_{r_1\to\infty} \mathbb{P}(\lambda_{\ell,\rho_{r_1}} < \infty) = 0.$$

Thus $\liminf_{t\to\infty} \kappa_t > \ell$ a.s. Since $\ell \in (r_0,\infty)$ was arbitrary, transience follows. □

3.3. *Lower bounds on the tails of return times and associated additive functionals.* In this subsection we establish *lower* bounds for the tails of the return times of continuous semimartingales and associated additive functionals. In the Markovian setting, there exists a rich theory providing upper bounds for additive functionals considered here, in the context of establishing related upper bounds on the tails of the invariant distribution (see [9] and the references therein). Moreover, in the continuous semimartingale setting, [28] establishes upper bounds on the return-time moments. Comparatively, the literature dedicated to lower bounds on the tails of the return times is scarce. Some results in this direction can be found in [28], however the assumptions are too restrictive to be used in our model. Our approach, based on the maximal inequality in Proposition 3.6 below, is inspired by the discrete-time results in [18].

The link between between additive functionals studied in the present subsection and the invariant distributions in the Markovian setting, described in [30], enables the application of Lemma 3.7 below in the proofs of lower bounds on the tails of the invariant distribution and the sub-exponential convergence rate in total variation (stated in Theorem 1.5). Lemma 3.7 is also crucial for establishing the lower bounds on the tails of return times in Theorem 1.4.

PROPOSITION 3.6 (Maximal inequality). *Let $\xi = (\xi_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted continuous process and $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ a measurable function. For some $r > 0$, let $\tau_r := \inf\{t \in \mathbb{R}_+ : \xi_t \geq r\}$ (with $\inf\emptyset = \infty$) and assume $(\xi_{t\wedge\tau_r} - \int_0^{t\wedge\tau_r} f(u, \xi_u)\mathrm{d}u)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)$-supermartingale. Then, for any $s \in (0,\infty)$, we have $\mathbb{P}(\sup_{0\leq t\leq s} \xi_t \geq r|\mathcal{F}_0) \leq r^{-1}(\xi_0 + \mathbb{E}[\int_0^{s\wedge\tau_r} f(u, \xi_u)\mathrm{d}u|\mathcal{F}_0])$.*

PROOF. Pick $s \in (0,\infty)$ and consider the stopping time $\tau_r \wedge s$, bounded above by s. By assumption we have $\mathbb{E}[\xi_{\tau_r\wedge s} - \int_0^{\tau_r\wedge s} f(u, \xi_u)\mathrm{d}u|\mathcal{F}_0] \leq \xi_0$, which implies that

$$\mathbb{E}[\xi_{\tau_r\wedge s}|\mathcal{F}_0] \leq \xi_0 + \mathbb{E}\left[\int_0^{\tau_r\wedge s} f(u, \xi_u)\mathrm{d}u\Big|\mathcal{F}_0\right]. \tag{31}$$

Moreover, by the definition of τ_r in the proposition we have $\{\tau_r \leq s\} = \{\sup_{u\in[0,s]} \xi_u \geq r\}$ a.s. Since the equality $\xi_{\tau_r\wedge s} = r$ holds on this event, by (31) we have

$$\mathbb{P}(\sup_{0\leq t\leq s} \xi_t \geq r|\mathcal{F}_0) = r^{-1}\,\mathbb{E}[\xi_{\tau_r\wedge s}\mathbb{1}\{\tau_r \leq s\}|\mathcal{F}_0] \leq r^{-1}\left(\xi_0 + \mathbb{E}\left[\int_0^{\tau_r\wedge s} f(u, \xi_u)\mathrm{d}u\Big|\mathcal{F}_0\right]\right),$$

implying the proposition. □

Note that, in the case $\mathbb{E}[\int_0^{s\wedge\tau_r} f(u, \xi_u)\mathrm{d}u] = \infty$, both the statement and the proof of Proposition 3.6 are formally correct, but not informative. In particular, when applying Proposition 3.6 we need $\mathbb{E}[\int_0^{s\wedge\tau_r} f(u, \xi_u)\mathrm{d}u] < \infty$, which follows easily if, for example, the function f is continuous (and hence bounded on $[0, s] \times [0, r]$).

LEMMA 3.7. *Let $\kappa = (\kappa_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted continuous process satisfying $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. Suppose that there exist $p \in (0,\infty)$, $\ell \in (0,\infty)$, and $C \in (0,\infty)$, such that the following hold for all $r \in (\ell,\infty)$:*

(a) *the process* $(\kappa^p_{t\wedge\lambda_\ell\wedge\rho_r})_{t\in\mathbb{R}_+}$ *is an* $(\mathcal{F}_t)$*-submartingale;*
(b) *for any* $q\in(0,1)$ *and* $r_q:=(1-q)^{-1}r$*, the process*

$$\left(\kappa^{-2}_{(\rho_{r_q}+t)\wedge\lambda_{r,\rho_{r_q}}}-C\int_{\rho_{r_q}}^{(\rho_{r_q}+t)\wedge\lambda_{r,\rho_{r_q}}}\kappa_u^{-4}\mathrm{d}u\right)_{t\in\mathbb{R}_+},$$

is an $(\mathcal{F}_{\rho_{r_q}+t})$*-supermartingale.*

Let $h:\mathbb{R}_+\to\mathbb{R}_+$ *be a non-decreasing measurable function. Then for all* $r\in(\ell,\infty)$*,* $q\in(0,1)$ *and* $\varepsilon\in(0,C^{-1}q(1-q)]$*, we have*

$$\mathbb{P}\left(\int_0^{\lambda_\ell}h(\kappa_s)\mathrm{d}s\geq\varepsilon h(r)r^2\Big|\mathcal{F}_0\right)\geq q\min\{(\kappa_0^p-\ell^p)(1-q)^pr^{-p},1\},\ \textit{on the event}\ \{\kappa_0>\ell\}.$$

In particular, for all $t\in(\ell,\infty)$ *(with* $h\equiv1$ *and* $\varepsilon=C^{-1}q(1-q)$*), we have*

$$\mathbb{P}(\lambda_\ell\geq t|\mathcal{F}_0)\geq q\min\{(\kappa_0^p-\ell^p)((1-q)^3q/C)^{p/2}t^{-p/2},1\},\quad\textit{on the event}\ \{\kappa_0>\ell\}.$$

PROOF. Pick $q\in(0,1)$, $r\in(\ell,\infty)$ and note that it suffices to prove the lemma for $\varepsilon=C^{-1}q(1-q)$. We start by establishing the following inequality:

$$\mathbb{P}(\lambda_{r,\rho_{r_q}}>\rho_{r_q}+\varepsilon r^2|\mathcal{F}_{\rho_{r_q}})\geq q\quad\text{a.s.}\tag{32}$$

Define $(\xi_t)_{t\in\mathbb{R}_+}$ by $\xi_t:=\kappa^{-2}_{\rho_{r_q}+t}$ and note that (23) yields $\tau_{r^{-2}}:=\inf\{t>0:\xi_t\geq r^{-2}\}=\lambda_{r,\rho_{r_q}}-\rho_{r_q}$. By Assumption (b), the process $(\xi_{t\wedge\tau_{r^{-2}}}-\int_0^{t\wedge\tau_{r^{-2}}}C\xi_u^2\mathrm{d}u)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_{\rho_{r_q}+t})$-supermartingale. By Proposition 3.6 applied to ξ and the stopping time $\tau_{r^{-2}}$, we obtain

$$\begin{aligned}\mathbb{P}(\lambda_{r,\rho_{r_q}}\leq\rho_{r_q}+t|\mathcal{F}_{\rho_{r_q}})&=\mathbb{P}(\tau_{r^{-2}}\leq t|\mathcal{F}_{\rho_{r_q}})=\mathbb{P}(\sup_{0\leq u\leq t}\xi_u\geq r^{-2}|\mathcal{F}_{\rho_{r_q}})\\&\leq r^2\left(\xi_0+\mathbb{E}\left[\int_0^{t\wedge\tau_{r^{-2}}}C\xi_u^2\mathrm{d}u\Big|\mathcal{F}_{\rho_{r_q}}\right]\right)\\&=r^2\left(\kappa^{-2}_{\rho_{r_q}}+\mathbb{E}\left[C\int_{\rho_{r_q}}^{(\rho_{r_q}+t)\wedge\lambda_{r,\rho_{r_q}}}\kappa_u^{-4}\mathrm{d}u\Big|\mathcal{F}_{\rho_{r_q}}\right]\right)\\&\leq r^2(r_q^{-2}+Ctr^{-4})\leq(1-q)^2+Cr^{-2}t\quad\text{for any } t\in(0,\infty).\end{aligned}$$

It follows that $\mathbb{P}(\lambda_{r,\rho_{r_q}}>\rho_{r_q}+t)\geq1-((1-q)^2+Cr^{-2}t)$. Taking $t=\varepsilon r^2$, we obtain (32).

Note that on the event $\{\lambda_{r,\rho_{r_q}}>\rho_{r_q}+\varepsilon r^2\}$, we have $h(\kappa_{\rho_{r_q}+t})\geq h(r)$ for all $t\in[0,\varepsilon r^2]$ and measurable non-decreasing functions $h:\mathbb{R}_+\to\mathbb{R}_+$. Since $r>\ell$, the inclusion

$$\left\{\int_0^{\lambda_\ell}h(\kappa_t)\mathrm{d}t\geq\varepsilon h(r)r^2\right\}\supset\{\rho_{r_q}<\lambda_\ell\}\cap\{\lambda_{r,\rho_{r_q}}>\rho_{r_q}+\varepsilon r^2\}$$

holds and, by the inequality in (32), we obtain

$$\begin{aligned}\mathbb{P}\left(\int_0^{\lambda_\ell}h(\kappa_t)\mathrm{d}t\geq\varepsilon h(r)r^2\Big|\mathcal{F}_0\right)&\geq\mathbb{E}\left[\mathbb{1}\{\rho_{r_q}<\lambda_\ell\}\mathbb{P}\left(\lambda_{r,\rho_{r_q}}>\rho_{r_q}+\varepsilon r^2|\mathcal{F}_{\rho_r}\right)\big|\mathcal{F}_0\right]\\&\geq q\mathbb{P}(\rho_{r_q}<\lambda_\ell|\mathcal{F}_0).\end{aligned}\tag{33}$$

By assumption $\limsup_{t\to\infty}\kappa_t=\infty$ a.s. and thus $\rho_{r_q}\wedge\lambda_\ell<\infty$ a.s. Since $(\kappa^p_{t\wedge\rho_{r_q}\wedge\lambda_\ell})_{t\in\mathbb{R}_+}$ is a continuous $(\mathcal{F}_t)$-submartingale by Assumption (a), dominated convergence implies

$$\kappa_0^p\le\lim_{t\to\infty}\mathbb{E}[\kappa^p_{t\wedge\lambda_\ell\wedge\rho_{r_q}}|\mathcal{F}_0]=\mathbb{E}[\kappa^p_{\lambda_\ell\wedge\rho_{r_q}}|\mathcal{F}_0]\le\ell^p+\mathbb{P}(\rho_{r_q}<\lambda_\ell|\mathcal{F}_0)r_q^p.$$

On the event $\{\kappa_0>\ell\}$ we obtain $\mathbb{P}(\rho_{r_q}<\lambda_\ell|\mathcal{F}_0)\ge((\kappa_0^p-\ell^p)r_q^{-p})\wedge 1$. Combining this result with (33) implies the general case of the lemma. The special case follows by setting $h\equiv 1$. □

4. Non-explosion and recurrence/transience dichotomy for the reflected process.

4.1. *Diffusivity and non-explosion under asymptotically normal reflection.* The non-explosion of Z is essentially due to Assumption (V2), which stipulates that the horizontal projection of the reflection vanishes sufficiently fast when Z_t is far from the origin. In contrast, in the asymptotically oblique case [29], the horizontal projection of the reflection vector field has a strictly positive limit as $\|Z_t\|_{d+1}\to\infty$, which in a domain $\mathcal{D}$ that narrows sufficiently fast, may lead to explosive behaviour $\tau_{\mathcal{E}}<\infty$ a.s.

THEOREM 4.1. *Suppose that Assumptions* (D2), (C2), (V2) *hold and let* (Z,L) *satisfy SDE* (2) *in the domain* $\mathcal{D}$ *on the stochastic interval* $[0,\tau_{\mathcal{E}})$. *For any starting point* $z\in\mathcal{D}$, *the process* Z *does not explode, i.e.* $\mathbb{P}_z(\tau_{\mathcal{E}}=\infty)=1$, *and the second moment of* Z_t *is finite and diffusive, i.e.,* $\sup_{t\in\mathbb{R}_+}\mathbb{E}_z\|Z_t\|^2_{d+1}/(1+t)<\infty$.

The finiteness of the first moment of $\|Z_t\|_{d+1}$, implied by Theorem 4.1, is crucial in the proof of recurrence in Theorem 1.3(a) and Theorem 1.3(c) (see Section 4.2 below for details). A minor modification of the final step in the proof of Theorem 4.1 would imply that the p-th moment of Z_t is also diffusive for any $p\in(0,\infty)$, i.e., $\sup_{t\in\mathbb{R}_+}\mathbb{E}_z\|Z_t\|^p_{d+1}/(1+t)^{p/2}<\infty$ (see Remark 4.2, after the proof of Theorem 4.1 below).

PROOF OF THEOREM 4.1. Assume (V2), (D2) and (C2) are satisfied. Consider the process $f_{w,\gamma}(Z)$ for any $\gamma\in(-\infty,1]$ and $w\in\mathbb{R}\setminus\{0\}$, where the function $f_{w,\gamma}$ is given in (10). Itô's formula applied to $f_{w,\gamma}(Z)$ on the stochastic interval $[0,\tau_{\mathcal{E}})$ is given in (8). The quadratic variation of the local martingale M in (8) grows at most linearly in time. Indeed, the representation in (9) and the bound in (14) of Lemma 2.1 (recall that $\gamma\le 1$) imply

$$[M]_t-[M]_s\le\int_s^t\|\Sigma(Zu)\|_{\mathrm{op}}\|\nabla f_{w,\gamma}(Z_u)\|^2_{d+1}\mathrm{d}u\le C_\gamma(t-s),\quad\text{for } 0\le s\le t<\tau_{\mathcal{E}},\tag{34}$$

for a constant $C_\gamma\in\mathbb{R}_+$. The inequality in (34) relies on the norm $\|\Sigma\|_{\mathrm{op}}=\|\Sigma^{1/2}\|^2_{\mathrm{op}}$ being bounded via Assumption (C1).

Our first task is to prove $\mathbb{P}_z(\tau_{\mathcal{E}}=\infty)=1$ for any $z\in\mathcal{D}$. Pick $w\in(-\infty,1-\beta s_0/c_0)\setminus\{0\}$, ensuring by Lemma 2.1 that $\langle\nabla f_{w,1}(z),\phi(z)\rangle<0$ for all $z=(x,y)\in\mathcal{D}$ with sufficiently large x, and define the process $\kappa:=f_{w,1}(Z)$. Recall the definition of the stopping times ρ_r (for any $r\in\mathbb{R}_+$ and $T=0$) in (24) and $\rho_\infty=\lim_{r\to\infty}\rho_r$, both given in Section 3 for the process κ. By (12), the function $f_{w,1}$ has linear growth at infinity, implying the equality of the events $\{\tau_{\mathcal{E}}<\infty\}=\{\rho_\infty<\infty\}$.

We will apply Lemma 3.1 with the identity function $V(r) = r$ for all $r \in (2^{-1/|w|-1}, \infty)$ to conclude $\mathbb{P}(\rho_\infty < \infty) = 0$. Since $\gamma = 1$, by (13) in Lemma 2.1, we have

$$\Delta_\Sigma f_{w,1}(z) = f_{w,1}(z)^{-1}\sigma_2^2(1-w) + o_{\mathcal{D}}(1) \quad \text{as } x \to \infty.$$

Thus, by (12), there exists $\eta > 0$ satisfying $|\Delta_\Sigma f_{w,1}(z)| < \eta$ for all $z \in \mathcal{D}$. Moreover, by Lemma 2.1, there exists $r_0 > 0$ such that $\langle \nabla f_{w,1}(z), \phi(z)\rangle < 0$ for all $z = (x,y) \in \mathcal{D}$ satisfying $x \geq r_0 2^{-1/|w|} - k_w$. Note that the upper bound on $f_{w,1}$ in (12) implies that any $z \in \mathcal{D}$ with $f_{w,1}(z) \geq r_0$ must satisfy $x \geq r_0 2^{-1/|w|} - k_w$ (and hence $\langle \nabla f_{w,1}(z), \phi(z)\rangle < 0$).

For any stopping time $T \in \mathcal{T}$, recall the definition in (25) of the exit time $S_{r,T}$ of the process κ from the interval (r_0, r) after time T. In order to apply Lemma 3.1, assume that the stopping time T is such that $\mathbb{E}[\kappa_T \mathbb{1}\{T < \rho_\infty\}] < \infty$. Then, by Itô's formula in (8) and the choice of the constants η and r_0, the process $\zeta^{T,r} = (\zeta_t^{T,r})_{t\in\mathbb{R}_+}$, defined for any $t \in \mathbb{R}_+$ by $\zeta_t^{T,r} := (\kappa_{(t+T)\wedge S_{r,T}} - \eta(t \wedge (S_{r,T} - T)))\mathbb{1}\{T < \rho_\infty\}$, satisfies

$$\zeta_t^{T,r} - \zeta_s^{T,r} \leq \left(M_{(t+T)\wedge S_{r,T}} - M_{(s+T)\wedge S_{r,T}}\right)\mathbb{1}\{T < \rho_\infty\} \quad \text{a.s. for any } 0 \leq s \leq t,$$

where M is the local martingale arising in (8) (for the function $f_{w,1}$). Since κ is continuous, on the event $\{T < \rho_\infty\}$ it holds that $S_{r,T} < \rho_\infty$. Thus, by (34), we get

$$[M]_{(t+T)\wedge S_{r,T}} - [M]_T \leq C_1((t+T) \wedge S_{r,T} - T) \leq C_1 t \quad \text{for all } t \in \mathbb{R}_+,$$

ensuring that $(M_{(t+T)\wedge S_{r,T}})_{t\in\mathbb{R}_+}$ is a true $(\mathcal{F}_{t+T})$-martingale, hence $\mathbb{E}[\zeta_t^{T,r} - \zeta_s^{T,r}|\mathcal{F}_{s+T}] \leq 0$ for all $0 \leq s \leq t$. Since $\zeta^{T,r}$ is an $(\mathcal{F}_{t+T})$-supermartingale for all $r \in (r_0, \infty)$, Lemma 3.1 yields $\mathbb{P}(\rho_\infty = \infty) = 1$. Thus, $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$ for all $z \in \mathcal{D}$. In the remainder of the section we assume that Z satisfies SDE (2) on the entire time interval $\mathbb{R}_+$.

We now prove there exists a constant $C_2 > 0$ such that $\mathbb{E}_z \|Z_t\|_{d+1}^2 \leq C_2(t + \|z\|_{d+1}^2 + 1)$ holds for any $z \in \mathcal{D}$ and $t \in \mathbb{R}_+$. Pick $w \in (-\infty, 1 - \beta s_0/c_0) \setminus \{0\}$, note $\beta s_0/c_0 - 1 + w < 0$ and apply Lemma 2.3 to find the constants $0 < x_0 < x_1$ and $k \in (0,\infty)$ such that the corresponding function $F_{w,2}$, defined in (17), satisfies $\langle \nabla F_{w,2}(z), \phi(z)\rangle \leq 0$ for all $z \in \partial\mathcal{D}$. Moreover, by (17), we have $\Delta_\Sigma F_{w,2}(z) = \Delta_\Sigma f_{w,2}(z)$ for all $z = (x,y) \in \mathcal{D} \cap [x_1, \infty) \times \mathbb{R}^d$. Since $F_{w,2}$ is smooth on a neighbourhood of $\mathcal{D}$ and, by (13) in Lemma 2.1, the function $z \mapsto |\Delta_\Sigma f_{w,2}(z)|$ is bounded on $\mathcal{D}$, there exists a constant $C_0' \in (0,\infty)$ satisfying $|\Delta_\Sigma F_{w,2}(z)| < C_0'$ for all $z \in \mathcal{D}$. By Assumption (D2), the boundary function b is sublinear (cf. Remark 1.2 above): there exist constants $C_1', C_2' \in (0,\infty)$ such that $b(x)^2 < C_1' x^2 + C_2'$ for all $x \in \mathbb{R}_+$. Thus, by the definition of $F_{w,2}$ in (17) and the lower bound on $f_{w,2}$ in (12), there exist positive constants $C_i' \in (0,\infty)$, $i \in \{3,4,5,6\}$, satisfying

$$\|z\|_{d+1}^2 \leq x^2 + b(x)^2 \leq (C_1'+1)x^2 + C_2' \leq C_3' F_{w,2}(z) + C_4' \leq C_5'\|z\|_{d+1}^2 + C_6', \ z \in \mathcal{D}. \tag{35}$$

Recall that the coordinates of $Z = (X, Y)$, taking values in $\mathcal{D}$, satisfy $X_t \in \mathbb{R}_+$ and $Y_t \in \mathbb{R}^d$ for all $t \in \mathbb{R}_+$. Define the passage time of the level $r \in \mathbb{R}_+$ for the process X by

$$\varrho_r := \inf\{t \in \mathbb{R}_+ : X_t \geq r\}, \tag{36}$$

(with $\inf\emptyset = \infty$). Fix r and assume that the starting point $z \in \mathcal{D}$ of the process Z lies in $[0, r) \times \mathbb{R}^d$ or, equivalently, $\mathbb{P}_z(\varrho_r > 0) = 1$. The definition of $F_{w,2}$ in the previous paragraph and Itô's formula in (8) applied to the process $(F_{w,2}(Z_{t\wedge\varrho_r}))_{t\in\mathbb{R}_+}$ yield the following

inequalities for all $t \in \mathbb{R}_+$:

$$F_{w,2}(Z_{t\wedge\varrho_r}) \le F_{w,2}(z) + \frac{1}{2}\int_0^{t\wedge\varrho_r} \Delta_\Sigma F_{w,2}(Z_s)\mathrm{d}s + M_{\varrho_r\wedge t} + \le F_{w,2}(z) + tC_0' + M_{\varrho_r\wedge t}.$$

Moreover, the quadratic variation of $M_{\varrho_r\wedge\cdot}$ in (9) is almost surely bounded and hence integrable, since the gradient $\nabla F_{w,2}$ is bounded on compact sets. Thus $\mathbb{E}_z M_{t\wedge\varrho_r} = 0$ for all $t, r \in \mathbb{R}_+$, implying the inequality $\mathbb{E}_z F_{w,2}(Z_{t\wedge\varrho_r}) \le tC_0' + F_{w,2}(z)$ for all $t, r \in \mathbb{R}_+$ and $z \in \mathcal{D} \cap [0, r) \times \mathbb{R}^d$. Since Z does not explode in finite time and has continuous paths, we have $\varrho_r \to \infty$ a.s. as $r \to \infty$ and hence $Z_{t\wedge\varrho_r} \to Z_t$ a.s. as $r \to \infty$. By Fatou's lemma and the inequalities in (35), for all $t \in \mathbb{R}_+$ and $z \in \mathcal{D}$, we obtain

$$\mathbb{E}_z \|Z_t\|_{d+1}^2 = \mathbb{E}_z \liminf_{r\to\infty} \|Z_{t\wedge\varrho_r}\|_{d+1}^2 \le \liminf_{r\to\infty} \mathbb{E}_z \|Z_{t\wedge\varrho_r}\|_{d+1}^2 \le tC_0'C_3' + C_5'\|z\|_{d+1}^2 + C_6',$$

concluding the proof of Theorem 4.1. □

REMARK 4.2. The proof of Theorem 4.1 can be modified, for any $p \in (0, \infty)$, to obtain $\sup_{t\in\mathbb{R}_+} \mathbb{E}_z \|Z_t\|_{d+1}^{2p}/(1+t)^p < \infty$. For $p \in \mathbb{N}$, we note that $F_{w,2p}$ can be constructed by Lemma 2.3 as above, so that $\langle \nabla F_{w,2p}(z), \phi(z)\rangle \le 0$ for all $z \in \partial\mathcal{D}$. By (13) in Lemma 2.1 there exists $C_0' \in (0, \infty)$ satisfying $|\Delta_\Sigma F_{w,2p}(z)| < C_0' F_{w,2(p-1)}(z)$ for all $z \in \mathcal{D}$. This inequality, combined with the modification of (35) for $\|z\|_{d+1}^{2p}$ and $F_{w,2p}(z)$, and induction on $p \in \mathbb{N}$ yields the diffusive property of moments for all even powers. The statement for all positive real powers p can be deduced from the even powers via Lyapunov's inequality. As the generalisation of the case $p = 1$ is not essential for the development in the present paper, the details are omitted for brevity.

4.2. *Proof of recurrence/transience classification.* By Theorem 4.1, in the remainder of the section we may assume without loss of generality that Z satisfies SDE (2) on the entire time interval $\mathbb{R}_+$. The proof of Theorem 1.3 starts with a lemma about non-confinement, essential for the applications of Lemmas 3.2 and 3.4 in the proof of recurrence/transience dichotomy.

LEMMA 4.3. *Under* (V1)*,* (D1) *and* (C1)*, the process $Z = (X, Y)$ defined by SDE* (2)*, started at any $z \in \mathcal{D}$, satisfies*

$$\limsup_{t\to\infty} X_t = \limsup_{t\to\infty} f_{w,1}(Z_t) = \limsup_{t\to\infty} F_{w,1}(Z_t) = \limsup_{t\to\infty} g_\delta(Z_t) = \infty \quad \mathbb{P}_z\text{-a.s.}$$

for any $w \in \mathbb{R} \setminus \{0\}$ and $\delta \in (0, \infty)$, where the functions $f_{w,1}$, $F_{w,1}$ and g_δ are given in (10)*,* (17) *and* (19)*, respectively.*

PROOF. Under (V1), (D1), (C1), [29, Thm 4.1] implies that the process $Z = (X, Y)$ cannot be confined to a compact set: $\limsup_{t\to\infty} X_t = \infty$ $\mathbb{P}_z$-a.s. for all $z \in \mathcal{D}$. In particular, by the inequality in (12) (resp. (20)), for any parameter value $w \in \mathbb{R} \setminus \{0\}$ (resp. $\delta \in (0, \infty)$) the process $f_{w,1}(Z)$ (resp. $g_\delta(Z)$) is also not confined: $\limsup_{t\to\infty} f_{w,1}(Z_t) = \infty$ (resp. $\limsup_{t\to\infty} g_\delta(Z_t) = \infty$) a.s. Since, by (17), the functions $f_{w,1}$ and $F_{w,1}$ coincide on the complement of a neighbourhood of the origin, the lemma follows. □

PROOF OF THEOREM 1.3. Since Assumptions (V1), (D1), (C1) hold if (V2), (D2), (C2) are satisfied, we may apply Lemma 4.3 in the proofs of this section.

(a) Recurrence for $\beta < \beta_c$. The definition of β_c in (4) and the assumption $\beta < \beta_c$ imply that $\beta s_0/c_0 < \sigma_1^2/\sigma_2^2$ (recall the the definition of σ_1^2, σ_2^2 and s_0, c_0 in Assumptions (C2) and (V2), respectively). Pick $w \in (1 - \sigma_1^2/\sigma_2^2, 1 - \beta s_0/c_0) \setminus \{0\}$ and note that $0 < 1 - (1-w)\sigma_2^2/\sigma_1^2$. Choose $\gamma \in (0, \min\{1, 1 - (1-w)\sigma_2^2/\sigma_1^2\})$ and observe the inequalities:

$$\sigma_1^2(\gamma - 1) + \sigma_2^2(1 - w) < 0 \quad \text{and} \quad \gamma(s_0\beta/c_0 - 1 + w) < 0. \tag{37}$$

Lemma 2.1 and the inequalities in (37) and (12) imply that there exists $r_0 \in (0, \infty)$ such that for all $z = (x, y) \in \mathcal{D}$ with $x \geq 2^{-1/|w|} r_0 - k_w$ we have

$$\Delta_\Sigma f_{w,\gamma}(z) < 0 \quad \text{and} \quad \langle \nabla f_{w,\gamma}(z), \phi(z) \rangle < 0. \tag{38}$$

Consider the process $\kappa = f_{w,1}(Z)$ and a continuous function $V : \mathbb{R}_+ \to (0, \infty)$, satisfying $V(r) = r^\gamma$ for all $r \in (2^{-1/|w|-1}, \infty)$. Since, by (12), $f_{w,1}(z) \geq 2^{-1/|w|}$ for all $z \in \mathcal{D}$, we have $V(\kappa) = f_{w,\gamma}(Z)$. Moreover, by (12) and the fact that $0 < \gamma < 1$, there exist constants $D_1, D_2 \in (0, \infty)$ such that the inequality $f_{w,\gamma}(z) \leq D_1 \|z\|_{d+1} + D_2$ holds for all $z \in \mathcal{D}$. Thus, by Theorem 4.1, we get $0 < \mathbb{E}[V(\kappa_t)] = \mathbb{E}_z[f_{w,\gamma}(Z_t)] \leq D_1 \, \mathbb{E}_z \, \|Z_t\|_{d+1} + D_2 < \infty$ for any $t \in \mathbb{R}_+$ and $z \in \mathcal{D}$.

For any fixed $t_0 \in \mathbb{R}_+$ and any $r \in (r_0, \infty)$, recall the definition in (25) of the exit time S_{r,t_0} of the process κ from the interval (r_0, r) after time t_0. The choice of r_0, the inequalities in (38), and Itô's formula in (8), applied to the process $V(\kappa_{(\cdot + t_0) \wedge S_{r,t_0}}) = f_{w,\gamma}(Z_{(\cdot+t_0)\wedge S_{r,t_0}})$ imply

$$f_{w,\gamma}(Z_{(t+t_0)\wedge S_{r,t_0}}) - f_{w,\gamma}(Z_{t_0}) - (M_{(t+t_0)\wedge S_{r,t_0}} - M_{t_0}) \leq 0 \quad \text{a.s.}$$

The local martingale $(M_t)_{t \in \mathbb{R}_+}$ has integrable quadratic variation by (34), making it a true martingale and implying $\mathbb{E}[V(\kappa_{(t+t_0)\wedge S_{r,t_0}}) - V(\kappa_{(s+t_0)\wedge S_{r,t_0}}) | \mathcal{F}_{s+t_0}] \leq 0$ for all $0 \leq s \leq t$. Since $V(\kappa_{t_0 \wedge S_{r,t_0}}) = V(\kappa_{t_0})$ is integrable, the process $(V(\kappa_{(t+t_0)\wedge S_{r,t_0}}))_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_{t+t_0})$-supermartingale for all $r \in (r_0, \infty)$. By Lemma 4.3, we have $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. Since $\lim_{r\to\infty} V(r) = \infty$, we may apply Lemma 3.2 to conclude that $\kappa = f_{w,1}(Z)$ is recurrent. By (12) and (35), the recurrence of Z follows.

(b) Transience for $\beta > \beta_c$. The definition of β_c in (4) and the assumption $\beta > \beta_c$ imply that $\sigma_1^2/\sigma_2^2 < \beta s_0/c_0$. Pick $w \in (1 - \beta s_0/c_0, 1 - \sigma_1^2/\sigma_2^2) \setminus \{0\}$ and note that $1 - \sigma_2^2/\sigma_1^2(1-w) < 0$. Choose $\gamma \in (1 - \sigma_2^2/\sigma_1^2(1-w), 0)$ and observe the inequalities:

$$\gamma(\sigma_1^2(\gamma - 1) + \sigma_2^2(1 - w)) < 0 \quad \text{and} \quad \gamma(s_0\beta/c_0 - 1 + w) < 0. \tag{39}$$

Lemma 2.1, along with the inequalities in (39) and (12) imply that there exists $r_0 \in (0, \infty)$ such that for all $z = (x, y) \in \mathcal{D}$ with $x \geq 2^{-1/|w|} r_0 - k_w$ we have

$$\Delta_\Sigma f_{w,\gamma}(z) < 0 \quad \text{and} \quad \langle \nabla f_{w,\gamma}(z), \phi(z) \rangle < 0.$$

Consider the process $\kappa = f_{w,1}(Z)$ and a continuous function $V : \mathbb{R}_+ \to (0, \infty)$, satisfying $V(r) = r^\gamma$ for all $r \in (2^{-1/|w|-1}, \infty)$. Since, by (12), $f_{w,1}(z) \geq 2^{-1/|w|}$ for all $z \in \mathcal{D}$, we have $V(\kappa) = f_{w,\gamma}(Z)$. For any stopping time $T \in \mathcal{T}$, satisfying $T < \infty$ a.s., we have

$\mathbb{E}_z V(\kappa_T) < \infty$ for all $z \in \mathcal{D}$ since the function V is bounded. Pick $r \in (r_0, \infty)$ and recall the definition in (25) of the exit time $S_{r,T}$ of the process κ from the interval (r_0, r) after time T. The choice of r_0 and Itô's formula in (8), applied to the process $V(\kappa_{(\cdot+T)\wedge S_{r,T}}) = f_{w,\gamma}(Z_{(\cdot+T)\wedge S_{r,T}})$, imply

$$f_{w,\gamma}(Z_{(t+T)\wedge S_{r,T}}) - f_{w,\gamma}(Z_T) - (M_{(t+T)\wedge S_{r,T}} - M_T) \le 0 \quad \text{a.s.}$$

Moreover, local martingale $(M_t)_{t\in\mathbb{R}_+}$ has integrable quadratic variation by (34), making it a true martingale, and for any $t \in \mathbb{R}_+$, $\mathbb{E}[V(\kappa_t)] < \infty$ by the fact that V is bounded. This implies $\mathbb{E}[V(\kappa_{(t+T)\wedge S_{r,T}}) - V(\kappa_{(s+T)\wedge S_{r,T}})|\mathcal{F}_{s+T}] \le 0$ for all $0 \le s \le t$. Thus, $(V(\kappa_{(t+T)\wedge S_{r,T}})_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_{t+T})$-supermartingale for all $r \in (r_0, \infty)$. Since $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. and $\lim_{r\to\infty} V(r) = 0$, Lemma 3.4 yields transience of $\kappa = f_{w,1}(Z)$. By (12), the transience of Z follows.

(c) The critical case $\beta = \beta_c$. Assume $\beta = \beta_c$ and (D2+), (V2+), (C2+). Consider the process $\kappa := g_\delta(Z)$ with g_δ defined in (19) and the parameter δ chosen to satisfy the assumption in Lemma 2.4. Then, by Lemma 2.4, there exists $x_0 > 0$ such that the inequalities in (21) hold. Define $r_0 := C_\delta + \log x_0$ and note that, by (20), the inequality $r_0 \ge g_\delta(z)$ (where $z = (x, y)$) implies $x \in [x_0, \infty)$. Set $V(r) = r$ for all $r \in (1/2, \infty)$ and note $V(\kappa_t) = \kappa_t = g_\delta(Z_t)$ (recall $g_\delta > 1$ on $\mathcal{D}$). Pick $t_0 \in \mathbb{R}_+$ and $r \in (r_0, \infty)$ and recall the definition in (25) of the exit time S_{r,t_0} of the process κ from the interval (r_0, r) after time t_0. The choice of r_0, the inequalities in (21) and Itô's formula in (8), applied to the process $\kappa_{(\cdot+t_0)\wedge S_{r,t_0}} = g_\delta(Z_{(\cdot+t_0)\wedge S_{r,t_0}})$, imply

$$g_\delta(Z_{(t+t_0)\wedge S_{r,t_0}}) - g_\delta(Z_{t_0}) - (M_{(t+t_0)\wedge S_{r,t_0}} - M_{t_0}) \le 0 \quad \text{a.s.}$$

By continuity, the gradient $\|\nabla g_\delta(z)\|_{d+1}^2$ is bounded on compact sets and $\|\Sigma\|_{\text{op}}$ is bounded by Assumption (C1). Thus, by the representation in (9), we can bound the quadratic variation

$$[M]_{(t+t_0)\wedge S_{r,t_0}} - [M]_{t_0} \le \int_{t_0}^{(t+t_0)\wedge S_{r,t_0}} \|\Sigma(Z_s)\|_{\text{op}} \|\nabla g_\delta(Z_s)\|_{d+1}^2 ds \le \widetilde{C}_1 t \quad \text{a.s.},$$

where $\widetilde{C}_1 \in (0, \infty)$ is a positive constant. Thus the process $(M_{(t+t_0)\wedge S_{r,t_0}} - M_{t_0})_{t\in\mathbb{R}_+}$ is a true martingale. Moreover, since $g_\delta(z) \le \widetilde{C}_2 \|z\|_{d+1} + \widetilde{C}_3$ holds for all $z \in \mathcal{D}$ for some positive constants $\widetilde{C}_2, \widetilde{C}_3$, Theorem 4.1 implies $\mathbb{E}_z[\kappa_t] = \mathbb{E}_z[g_\delta(Z_t)] \le \widetilde{C}_2 \mathbb{E}_z \|Z_t\|_{d+1} + \widetilde{C}_3 < \infty$ for any $t \in \mathbb{R}_+$ and $z \in \mathcal{D}$. Thus, $\mathbb{E}[\kappa_{(t+t_0)\wedge S_{r,t_0}} - \kappa_{(s+t_0)\wedge S_{r,t_0}}|\mathcal{F}_{s+t_0}] \le 0$ for all $0 \le s \le t$, and hence $(\kappa_{(t+t_0)\wedge S_{r,t_0}})_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_{t+t_0})$-supermartingale for all $r \in (r_0, \infty)$ and any $t_0 \in \mathbb{R}_+$. Moreover, by Lemma 4.3, we have $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. Since $\lim_{r\to\infty} V(r) = \infty$, we may apply Lemma 3.2 to conclude that $\kappa = g_\delta(Z)$ is recurrent. By (20) and (35), the recurrence of Z follows. □

5. Return times and drift conditions. The tails of return times are controlled by Propositions 5.1 and 5.2, established in this section. The two propositions are crucial in the proof of Theorem 1.4(b); see Remark 5.3 below for more details. Moreover, Proposition 5.2 is key in obtaining the lower bounds on the tails of the invariant distribution of Z in the positive-recurrent regime: see the proof of Theorem 1.5 in Section 7 below. The drift conditions in Lemma 5.4, proved in the present section, are used for establishing finite moments

(and hence upper bounds on the tails) of the invariant distribution and the rate of convergence to stationarity of Z in the positive-recurrent regime (see the proof of Proposition 7.1 in Section 7 below). The common theme of the proofs of the results in this section is that they are all based on the supermartingale property of certain processes.

Recall the definitions of the return time ς_r (for $r \in (0,\infty)$) in (5) and of the critical exponent $m_c = (1-\beta/\beta_c)/2$ in (6).

PROPOSITION 5.1. *Suppose that* (D2), (C2), (V2) *hold and* $\beta < \beta_c$. *Then, for every* $p \in (0, m_c)$, *there exists* $x_0 > 0$ *such that for all* $x_1 \in [x_0, \infty)$ *and* $z = (x,y) \in \mathcal{D}$ *there exists a constant* $C \in \mathbb{R}_+$ *(depending only on* x *and* p*) for which* $\mathbb{E}_z[\varsigma_{x_1}^p] \le C$.

PROOF. Pick $\gamma \in (0, (1-\beta/\beta_c)/2)$. Then choose $w \in (-\infty, 1-\beta s_0/c_0) \setminus \{0\}$, such that $\sigma_1^2(2\gamma-1) + \sigma_2^2(1-w) < 0$, and pick any $\varepsilon \in (0, -(\sigma_1^2(2\gamma-1)+\sigma_2^2(1-w))/(2\gamma))$. Lemma 2.1 implies the existence of $\ell_0 \in (0,\infty)$, such that for all $z=(x,y) \in \mathcal{D}$ with $x > 2^{-1/|w|}\ell_0 - k_w$ (the constant k_w is given above display (10)), the function $f_{w,2\gamma}$ defined in (10) satisfies

$$\Delta_\Sigma f_{w,2\gamma}(z) + \varepsilon f_{w,2\gamma-2}(z) < 0 \quad \text{and} \quad \langle \nabla f_{w,2\gamma}(z), \phi(z)\rangle < 0. \tag{40}$$

Having chosen the parameters γ, w, and ε, consider the process $\kappa = f_{w,1}(Z)$. The key step in the proof of the proposition consists of the application of [28, Thm 2.1] to deduce that, for any $z \in \mathcal{D}$ with $\kappa_0 = f_{w,1}(z) > \ell_0$, the return time λ_{ℓ_0} of κ below the level ℓ_0, defined in (23), has finite γ-moment, i.e., $\mathbb{E}_z[\lambda_{\ell_0}^\gamma] < \infty$. This will hold by [28, Thm 2.1] if we establish that the process $(\xi_t)_{t\in\mathbb{R}_+}$, given by $\xi_t := \kappa_{t\wedge\lambda_{\ell_0}}^{2\gamma} + \varepsilon \int_0^{t\wedge\lambda_{\ell_0}} \kappa_u^{2\gamma-2}\mathrm{d}u$, is a supermartingale.

With this in mind, take an arbitrary $r \in (\ell_0,\infty)$ and consider the stopped process $(\xi_{t\wedge\rho_r})_{t\in\mathbb{R}_+}$, where the stopping time ρ_r, defined in (24), is the first time the process κ reaches level r. Since $0 \le \xi_{t\wedge\rho_r} \le \max\{r^{2\gamma}, f_{w,2\gamma}(z)\} + \varepsilon t \max\{r^{2\gamma-2}, {\ell_0}^{2\gamma-2}\}$ for all $t \in \mathbb{R}_+$, we have $\mathbb{E}_z[\xi_{t\wedge\rho_r}] < \infty$. Moreover, by Itô's formula in (8), the inequalities in (40) and the fact that $\kappa_t \ge \ell_0$ for all $t \in \mathbb{R}_+$, for any two times $0 \le s \le t < \infty$ we have $\xi_{t\wedge\rho_r} - \xi_{s\wedge\rho_r} - M_{t\wedge\lambda_{\ell_0}\wedge\rho_r} + M_{s\wedge\lambda_{\ell_0}\wedge\rho_r} \le 0$ a.s. Since the local martingale M has, by (9) and (C2), bounded quadratic variation for each $t \in \mathbb{R}_+$ with probability one, the stopped process $(\xi_{t\wedge\rho_r})_{t\in\mathbb{R}_+}$ is a supermartingale for any $r \in (\ell_0,\infty)$. By Theorem 4.1 we have $\lim_{r\to\infty}\rho_r = \infty$. Thus $\xi_t = \liminf_{r\to\infty}\xi_{t\wedge\rho_r}$ for all $t \in \mathbb{R}_+$. Since the process $(\xi_t)_{t\in\mathbb{R}_+}$ is non-negative, the conditional Fatou lemma implies that it is a supermartingale. Thus, we may apply [28, Thm 2.1] to deduce that for any $p \in (0,\gamma)$ and $z \in \mathcal{D}$ there exist $C_1, C_2 \in (0,\infty)$, such that $\mathbb{E}_z[\lambda_{\ell_0}^p] \le C_1 f_{w,2\gamma}(z) + C_2 \le C_1 2^{2\gamma/|w|}(x+k_w)^{2\gamma} + C_2$, where the second inequality follows from (12).

Recall that $f_{w,1}(Z) = \kappa$. Hence the second inequality in (12) implies that, for $x_0 := 2^{1/|w|}\ell_0 - k_w$, we have $\varsigma_{x_0} \le \lambda_{\ell_0}$ $\mathbb{P}_z$-a.s. for every $z \in \mathcal{D}$. For every $p \in (0, m_c)$ and $x \in (0,\infty)$, define $C := C_1 2^{2\gamma/|w|}(x+k_w)^{2\gamma} + C_2$, where $\gamma \in (p, m_c)$. Thus, we have $\mathbb{E}_z[\varsigma_{x_0}^p] \le \mathbb{E}_z[\lambda_{\ell_0}^p] \le C$ for any $z=(x,y) \in \mathcal{D}$. Moreover, for any $x_1 \in (x_0,\infty)$, we have $\mathbb{P}_z(\varsigma_{x_0} \ge \varsigma_{x_1}) = 1$ for every $z \in \mathcal{D}$, implying $\mathbb{E}_z[\varsigma_{x_1}^p] \le C$. □

The next proposition provides *lower bounds* on the tails of return times and related path functionals in the recurrent case. This result is a key ingredient in the proof of Theorem 1.4(b), as well as in the proof of lower bounds in Theorem 1.5. The proof of Proposition 5.2 is based on an application of Lemma 3.7 of Section 3.3 above. Recall $m_c = (1-\beta/\beta_c)/2$ defined in (6).

PROPOSITION 5.2. *Suppose that* (D2), (C2), (V2) *hold with $\beta < \beta_c$. Then, for every $p \in (2m_c, \infty)$, there exist $x_0 \in (0,\infty)$ and constants $c_1, c_2 \in (1,\infty)$ such that, for every non-decreasing measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$, $q \in (0,1)$, $x_1 \in (x_0,\infty)$ and $z = (x,y) \in \mathcal{D} \cap (c_1 x_1 + c_2, \infty) \times \mathbb{R}^d$ we have*

$$\mathbb{P}_z\left(\int_0^{\varsigma_{x_1}} h\left(c_1(X_s + c_2)\right) \mathrm{d}s \geq \varepsilon r^2 h(r)\right) \geq q \min\{(c_1^{-p}(x-c_2)^p - x_1^p)(1-q)^p r^{-p}, 1\},$$

for all $r \in (c_1 x_1 + c_2, \infty)$ and all sufficiently small $\varepsilon > 0$. In particular,

$$\mathbb{P}_z(\varsigma_{x_1} \geq t) \geq q \min\{(c_1^{-p}(x-c_2)^p - x_1^p)\varepsilon^{p/2}(1-q)^p t^{-p/2}, 1\},$$

for every $t \in (c_1 x_1 + c_2, \infty)$ and all sufficiently small $\varepsilon > 0$.

REMARK 5.3. Propositions 5.1 and 5.2 provide crucial estimates in the proof of Theorem 1.4(b) in Section 6.2 below. The only assertion of Theorem 1.4(b) not contained in Propositions 5.1 and 5.2 is that the bounds in Propositions 5.1 and 5.2 actually holds for all $x_0 \in (0,\infty)$ and $z \in \mathcal{D} \cap (x_0,\infty) \times \mathbb{R}^d$ and not only for large enough x_0 and the starting points z sufficiently far (in the x-direction) from x_0. This generalisation requires uniform ellipticity and will be established in Section 6.2.

PROOF OF PROPOSITION 5.2. Pick $p \in (1-\beta/\beta_c, \infty)$ and note $p > 0$. Then there exists $w \in (1 - \beta s_0/c_0, \infty) \setminus \{0\}$, such that $p > 1 - \sigma_2^2/\sigma_1^2(1-w)$. Lemma 2.1 implies that there exist $\ell_0 > 0$ and a constant $C \in \mathbb{R}_+$ such that, for all $z = (x,y) \in \mathcal{D}$ with $x > 2^{-1/|w|}\ell_0 - k_w$ (the constant k_w is defined above display (10)), we have

$$\Delta_\Sigma f_{w,p}(z) > 0 \quad \text{and} \quad \langle \nabla f_{w,p}(z), \phi(z)\rangle > 0, \tag{41}$$

$$\Delta_\Sigma f_{w,-2}(z) \leq C f_{w,-4}(z) \quad \text{and} \quad \langle \nabla f_{w_1,-2}(z), \phi(z)\rangle < 0. \tag{42}$$

Define $\kappa := f_{w,1}(Z)$ and recall the return time λ_{ℓ_0} of κ below the level ℓ_0, defined in (23). By (12), on the event $\{\kappa_t = f(Z_t) \geq \ell_0\}$, the first coordinate X_t of Z_t satisfies $X_t > 2^{-1/|w|}\ell_0 - k_w$. Itô's formula in (8) applied to $\kappa = f_{w,1}(Z)$ and the inequalities in (41) imply that, for all $r > \ell \geq \ell_0$ and $0 \leq s \leq t < \infty$, we have $\kappa^p_{t\wedge\lambda_\ell\wedge\rho_r} - M_{t\wedge\lambda_\ell\wedge\rho_r} \geq \kappa^p_{s\wedge\lambda_\ell\wedge\rho_r} - M_{s\wedge\lambda_\ell\wedge\rho_r}$. The process M is a true martingale, since its quadratic variation is bounded by (9) and (C2). Thus, since $\mathbb{E}_z[\kappa^p_{t\wedge\lambda_\ell\wedge\rho_r}] \leq \max\{r^p, f_{w,p}(z)\}$, the process $(\kappa^p_{t\wedge\lambda_\ell\wedge\rho_r})_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)$-submartingale.

Pick $r \in (\ell_0,\infty)$, $q \in (0,1)$ and set $r_q := r/(1-q)$. Recall that $f_{w,1}(Z_t) = \kappa$ and define the process $\xi = (\xi_t)_{t\in\mathbb{R}_+}$ by $\xi_t := \kappa^{-2}_{(\rho_{r_q}+t)\wedge\lambda_{r,\rho r_q}} - C\int_{\rho_{r_q}}^{t\wedge\lambda_{r,\rho r_q}} \kappa_u^{-4}\mathrm{d}u$, where the constant $C > 0$ is as in (42). By Itô's formula in (8) and the inequalities in (42), for every $0 \leq s \leq t < \infty$, we

have

$$\xi_t - \xi_s - (M_{(\rho_{r_q}+t)\wedge\lambda_{r,\rho_{r_q}}} - M_{(\rho_{r_q}+s)\wedge\lambda_{r,\rho_{r_q}}}) \leq 0 \quad \text{a.s.}$$

Since the process ξ is bounded, an analogous argument to the one in the previous paragraph implies that ξ is a supermartingale.

We have now proved that Assumptions (a) and (b) of Lemma 3.7 are satisfied. Moreover, by Lemma 4.3, we have $\limsup_{t\to\infty} \kappa_t = \infty$ a.s. Hence, by Lemma 3.7, for any $z \in \mathcal{D}$ and $\ell > \ell_0$, satisfying $f_{w,1}(z) > \ell$, any non-decreasing measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ and all sufficiently small $\varepsilon > 0$, we obtain

$$\mathbb{P}_z\left(\int_0^{\lambda_\ell} h(f_{w,1}(Z_s))\mathrm{d}s \geq \varepsilon r^2 h(r)\right) \geq q\min\{(f_{w,1}(z)^p - \ell^p)(1-q)^p r^{-p}, 1\}, \qquad r \in (\ell, \infty). \tag{43}$$

Define $x_0 := 2^{-1/|w|}\ell_0 - k_w$, $c_1 := 2^{1/|w|}$ and $c_2 := k_w$. Then, for any $x_1 \in (x_0, \infty)$ there exists $\ell > \ell_0$, such that $x_1 = 2^{-1/|w|}\ell - k_w$. The second inequality in (12) implies $\varsigma_{x_1} \geq \lambda_\ell$ and $h(f_{w,1}(Z_s)) \leq h\left(2^{1/|w|}(X_s + k_w)\right)$ for all non-decreasing measurable functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ and $s \in [0, \lambda_\ell]$. Thus the inequality $\int_0^{\lambda_\ell} h(f_{w,1}(Z_s))\mathrm{d}s \leq \int_0^{\varsigma_{x_0}} h\left(2^{1/|w|}(X_s + k_w)\right)\mathrm{d}s$ holds, implying by (43) the inequality in the proposition for all $r \in (2^{1/|w|}x_0 + k_w, \infty)$. The special case follows by choosing $h \equiv 1$. □

The next result establishes a drift condition (in the positive-recurrent case), used in the proofs of the upper bounds of Theorem 1.5 concerning the finite moments of the invariant distribution π of Z and the total variation distance between $\mathbb{P}_z(Z_t \in \cdot)$ and π. The proof of Lemma 5.4 is analogous to the proof of Proposition 5.1. For any $r \in \mathbb{R}_+$, denote

$$\mathcal{D}^{(r)} := \mathcal{D} \cap [0, r] \times \mathbb{R}^d. \tag{44}$$

LEMMA 5.4. *Suppose that* (C2), (V2) *and* (D2) *hold with* $\beta < -\beta_c$. *Then, for any* $\gamma \in (0, 1 - \beta/\beta_c)$, *there exist parameters* $w \in (-\infty, 1 - \beta s_0/c_0) \setminus \{0\}$ *and* $x_0, x_1, k \in \mathbb{R}_+$, *defining the function* $F_{w,\gamma}$ *in* (17), *and* $x_2 \in \mathbb{R}_+$, $C_1, C_2 \in \mathbb{R}_+$, *such that the process* $\xi = (\xi_t)_{t\in\mathbb{R}_+}$,

$$\xi_t := F_{w,\gamma}(Z_t) + C_1\int_0^t F_{w,\gamma-2}(Z_u)\mathrm{d}u - C_2\int_0^t \mathbb{1}_{\mathcal{D}^{(x_2)}}(Z_u)\mathrm{d}u, \tag{45}$$

is an $(\mathcal{F}_t)$*-supermartingale.*

Note that the process $F_{w,\gamma}(Z)$ in Lemma 5.4 gets a non-positive push (by Lemma 2.3) when Z hits the boundary $\partial\mathcal{D}$. The constant C_1 (resp. C_2) needs to be sufficiently small (resp. large) for the process ξ to have non-positive drift in the entire interior of $\mathcal{D}$.

PROOF OF LEMMA 5.4. Pick $\gamma \in (0, 1 - \beta/\beta_c)$ and note that $1 + \sigma_1^2/\sigma_2^2(\gamma - 1) < 1 - \beta s_0/c_0$, since $\beta_c = c_0\sigma_1^2/(s_0\sigma_2^2)$ by definition (4). Pick $w \in (1 + \sigma_1^2/\sigma_2^2(\gamma-1), 1 - \beta s_0/c_0) \setminus \{0\}$ and note that $\gamma(\beta s_0/c_0 - 1 + w) < 0$. Lemma 2.3 implies that there exist $x_0, x_1, k \in$

$(0,\infty)$ such that the function $F_{w,\gamma}$ defined in (17) satisfies $\langle \nabla F_{w,\gamma}(z), \phi(z)\rangle < 0$ for all $z \in \partial\mathcal{D}$.

By (17) we have $F_{w,\gamma}(z) = f_{w,\gamma}(z)$ on $z \in \mathcal{D} \cap (x_1,\infty)\times\mathbb{R}^d$ and (13) yields

$$\Delta_\Sigma F_{w,\gamma}(z) = \gamma F_{w,1}(z)^{\gamma-2}(\sigma_1^2(\gamma-1)+\sigma_2^2(1-w)+o_{\mathcal{D}}(1)) \quad \text{as } x\to\infty,$$

where $o_{\mathcal{D}}(1)$ is defined after Assumption (V2). Since $\sigma_1^2(\gamma-1)+\sigma_2^2(1-w)<0$, there exists $x_2 \in (x_1,\infty)$, such that for $C_1 := -(\sigma_1^2(\gamma-1)+\sigma_2^2(1-w))\gamma/4$ we have

$$\frac{1}{2}\Delta_\Sigma F_{w,\gamma}(z) + C_1 F_{w,\gamma-2}(z) \le 0 \quad \text{on } z \in \mathcal{D}\cap(x_2,\infty)\times\mathbb{R}^d.$$

Thus, since the functions $\Delta_\Sigma F_{w,\gamma}$ and Σ are bounded on the compact set $\mathcal{D}_{x_2}$, there exists $C_2 \in \mathbb{R}_+$ such that

$$\frac{1}{2}\Delta_\Sigma F_{w,\gamma}(z) + C_1 F_{w,\gamma-2}(z) \le C_2 \mathbb{1}_{\mathcal{D}^{(x_2)}}(z) \quad \text{for all } z\in\mathcal{D}. \tag{46}$$

Recall the definition of ξ in (45) and set $\kappa := F_{w,1}(Z)$. Note that by definition (17), there exist $\delta_0 > 0$ such that $\inf_{z\in\mathcal{D}} F_{w,1}(z) > \delta_0$. For any $r > \max\{1,\delta_0\}$, the stopped process $(\xi_{t\wedge\rho_r})_{t\in\mathbb{R}_+}$, where the stopping time ρ_r, defined in (24) as the first time the process κ crosses level r, satisfies $-C_2 t \le \mathbb{E}_z[\xi_{t\wedge\rho_r}] \le \max\{F_{w,\gamma}(z), r^\gamma\} + C_1 t \max\{r^{\gamma-2}, \delta_0^{\gamma-2}\}$ for all $t \in \mathbb{R}_+$. Thus $\mathbb{E}_z|\xi_{t\wedge\rho_r}| < \infty$ for all $t\in\mathbb{R}_+$ and $z\in\mathcal{D}$. Moreover, the inequality (46) and Itô's formula (8) applied to $F_{w,\gamma}(Z)$ imply that, for any $0\le s\le t<\infty$, we have $\xi_{t\wedge\rho_r} - \xi_{s\wedge\rho_r} - (M_{t\wedge\rho_r} - M_{s\wedge\rho_r}) \le 0$ a.s. Since, by (9) and (C2), $[M]_{t\wedge\rho_r} \le C_0 t$ a.s. for all $t\in\mathbb{R}_+$ and some constant $C_0>0$, $(\xi_{t\wedge\rho_r})_{t\in\mathbb{R}_+}$ is a supermartingale for any $r\in\mathbb{R}_+$. By Theorem 4.1 we have $\lim_{r\to\infty}\rho_r = \infty$. Thus, $\xi_s = \liminf_{r\to\infty}\xi_{s\wedge\rho_r}$ for all $s\in\mathbb{R}_+$. Since $\xi_s \ge -C_2 t$ for all $s\in[0,t]$, the conditional Fatou lemma implies that for any $0\le s\le t<\infty$, we have

$$\mathbb{E}_z[\xi_t|\mathcal{F}_s] = \mathbb{E}_z[\liminf_{r\to\infty}\xi_{t\wedge\rho_r}|\mathcal{F}_s] \le \liminf_{r\to\infty}\mathbb{E}_z[\xi_{t\wedge\rho_r}|\mathcal{F}_s] \le \xi_s,$$

in addition we deduce the integrability of ξ_t by choosing $s=0$ and noting that $\mathbb{E}_z[\xi_0] = F_{w,\gamma}(z)$, hence ξ is an $(\mathcal{F}_t)$-supermartingale. □

6. Feller continuity and irreducibility of the reflected process and applications. The existence of the invariant distribution of Z requires positive recurrence (see definition preceding Theorem 1.5 above). The study of its moments requires certain technical results established in the present section. Section 6.1 is dedicated to the proofs of Feller continuity and irreducibility of the process Z. In Section 6.3 we apply these two properties to prove that the reflected process Z is Harris recurrent with an irreducible skeleton chain and that the set $\mathcal{D}\cap[0,r]\times\mathbb{R}^d$ is petite for any $r\in(0,\infty)$. Moreover, in Section 6.3 we will also show that every petite set for Z is bounded.

6.1. *Feller continuity and irreducibility of the reflected process.* In this section we prove that the reflected process Z is Feller-continuous (see Theorem 6.5 below) and that the $(d+1)$-dimensional Lebesgue measure $\mathfrak{m}_{d+1}$ on the Borel σ-algebra $\mathcal{B}(\mathcal{D})$ on $\mathcal{D}$ is absolutely continuous with respect to its marginals of the reflected process at positive times (Proposition 6.1 below). We start with the latter.

PROPOSITION 6.1. *Let* (D2), (C2), (V2) *hold. Then, for any* $z \in \mathcal{D}$ *and* $t \in (0,\infty)$ *and any* $A \in \mathcal{B}(\mathcal{D})$*, such that* $\mathfrak{m}_{d+1}(A) > 0$*, we have* $\mathbb{P}_z(Z_t \in A) > 0$.

The proofs of Proposition 6.1 and Theorem 6.5 below require the following lemma. For small $h > 0$, define a "thin" neighborhood of $\partial\mathcal{D}$ in $\mathcal{D}$ by $\mathcal{D}_h := \{z \in \mathcal{D} : \exists z' \in \partial\mathcal{D}$ such that $\|z - z'\|_{d+1} < h\}$.

LEMMA 6.2. *Let* (D1), (C1), (V1) *hold. Then there exists a function* $G : \mathcal{D} \to \mathbb{R}_+$*, twice-differentiable on a neighbourhood of* $\mathcal{D} \subset \mathbb{R}^{d+1}$ *and strictly positive on the open set* $\mathcal{D} \setminus \partial\mathcal{D}$*. Moreover, the function* $g(z) := G(z)^2$ *satisfies* $\nabla g(z) = 0$ *for all* $z \in \partial\mathcal{D}$ *and for any* $r > 0$ *there exist* $h_r > 0$*,* $\delta > 0$*, such that the following hold*

(a) $\|\Sigma^{1/2}(z)\nabla G(z)\|^2_{d+1} > \delta$ *for all* $z \in \mathcal{D}_{h_r} \cap [0,r] \times \mathbb{R}^d$*;*
(b) $\langle \phi(z), \nabla G(z)\rangle > \delta$ *for all* $z \in \partial\mathcal{D} \cap [0,r] \times \mathbb{R}^d$.

REMARK 6.3. Recall that Assumptions (D2), (C2), (V2) imply Assumptions (D1), (C1), (V1).

PROOF OF LEMMA 6.2. Extend the function $G : \mathcal{D} \to \mathbb{R}_+$, defined by $G(z) := b(x)^2 - \|y\|_d^2$ for any $z = (x,y) \in \mathcal{D}$, to a C^2-function on a neighbourhood of $\mathcal{D}$ in $\mathbb{R}^{d+1}$. Hence $\|\nabla G(z)\|^2_{d+1} = 4(b(x)^2 b'(x)^2 + \|y\|_d^2)$ for all $z = (x,y) \in \mathcal{D}$ and $\nabla g(z) = 2G(z)\nabla G(z) = 0$ for all $z \in \partial\mathcal{D}$, since $G(z) = 0$ on $\partial\mathcal{D}$. Recall that (D1) yields $\liminf_{x\to 0} b(x)b'(x) > 0$. Thus, for any $r > 0$, there exists a sufficiently small $h_r > 0$, such that $0 < \delta_{h_r} := \inf_{z\in\mathcal{D}_{h_r}\cap[0,r]\times\mathbb{R}^d} \|\nabla G(z)\|^2_{d+1}$ (we restrict to $x \in [0,r]$ because of functions b with $\beta \le 0$, e.g. example in Lemma B.1). Moreover, by (C1), there exists $\delta_\Sigma > 0$ such that

$$\|\Sigma^{1/2}(z)\nabla G(z)\|^2_{d+1} = \langle \Sigma(z)\nabla G(z), \nabla G(z)\rangle \ge \delta_\Sigma \|\nabla G(z)\|^2_{d+1} \ge \delta_\Sigma \cdot \delta_{h_r} > 0$$

for all $z \in \mathcal{D}_{h_r} \cap [0,r] \times \mathbb{R}^d$, implying (a).

Note that gradient $\nabla G(z)$ for any $z \in \partial\mathcal{D}$ equals $n(z)\|\nabla G(z)\|_{d+1}$, where $n(z)$ is the inwards-pointing unit normal vector to $\partial\mathcal{D}$ at z. Hence, by (V1), there exists a $\delta_\phi > 0$ such that $\langle \phi(z), \nabla G(z)\rangle \ge \delta_{h_r}^{1/2} \langle \phi(z), n(z)\rangle > \delta_\phi \cdot \delta_{h_r}^{1/2} > 0$ for all $z \in \partial\mathcal{D} \cap [0,r] \times \mathbb{R}^d$. □

PROOF OF PROPOSITION 6.1. We start by proving that Z spends no time at the boundary.
Claim 0. The equality $\int_0^\infty \mathbb{1}\{Z_t \in \partial\mathcal{D}\}\mathrm{d}t = 0$ holds $\mathbb{P}_z$-a.s. for any starting point $z \in \mathcal{D}$.
Proof of Claim 0. Let G be a function whose existence is guaranteed by Lemma 6.2. Define the non-negative continuous semimartingale $\xi = (\xi_t)_{t\in\mathbb{R}_+}$, $\xi_t := G(Z_t)$, denote its local time field by $(L_t^u(\xi))_{t,u\in\mathbb{R}_+}$ (see [34, Ch. VI] for definition and properties) and note that $\xi_t = 0$ if and only if $Z_t \in \partial\mathcal{D}$, since $G > 0$ on $\mathcal{D} \setminus \partial\mathcal{D}$ by Lemma 6.2. Thus the local-time process L in SDE (2) satisfies $L^0(\xi) = L$. Itô's formula in (8), applied to $G(Z)$, yields that the quadratic variation $[\xi]$ equals $[M]$ given in (9). Pick any $t, r \in (0,\infty)$. The occupation times formula in [34, Cor VI.1.6] applied to the indicator $u \mapsto \mathbb{1}_{\{0\}}(u)$ of zero, the representation of the quadratic variation $[M]$ in (9), the property in Lemma 6.2(a) and the fact that $\xi_s = 0$ is

equivalent to $Z_s \in \partial\mathcal{D}$ for all $s \in \mathbb{R}_+$ yield

$$0 = \int_{\mathbb{R}_+} \mathbb{1}_{\{0\}}(u) L^u_{t\wedge\varrho_r}(\xi)\mathrm{d}u = \int_0^{t\wedge\varrho_r} \mathbb{1}_{\{0\}}(\xi_s)\mathrm{d}[M]_s \geq \delta \int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}s \geq 0,$$

implying $0 = \int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}s$. Since ϱ_r, given in (36), satisfies $\lim_{r\to\infty} \varrho_r = \infty$ a.s. (by Theorem 4.1) and $t > 0$ is arbitrary, our claim follows.

Fubini's theorem and Claim 0 yield $\int_0^\infty \mathbb{P}_z(Z_t \in \partial\mathcal{D})\mathrm{d}t = \mathbb{E}_z \int_0^\infty \mathbb{1}\{Z_t \in \partial\mathcal{D}\}\mathrm{d}t = 0$ for any $z \in \mathcal{D}$ (since Z is continuous, it is progressively measurable, implying the various integrals are well defined and measurable). In particular, denoting $\mathrm{Int}(\mathcal{D}) := \mathcal{D} \setminus \partial\mathcal{D}$, for any $z \in \mathcal{D}$ and $t > 0$, it holds that $\int_0^t \mathbb{P}_z(Z_s \in \mathrm{Int}(\mathcal{D}))\mathrm{d}s = t$.

Claim 1. For every $z \in \mathcal{D}$, $t > 0$ and $A \in \mathcal{B}(\mathrm{Int}(\mathcal{D}))$ the following holds: if $\mathfrak{m}_{d+1}(A) > 0$ then $\int_0^t \mathbb{P}_z(Z_s \in A)\mathrm{d}s > 0$.

In order to prove Claim 1, we need Claims 2 and 3 below. For $z \in \mathcal{D}$ and $h > 0$, define the open ball in $\mathcal{D}$ by $B(z,h) := \{z' \in \mathcal{D} : \|z - z'\|_{d+1} < h\}$.

Claim 2. Pick any $z \in \mathcal{D}$, $s \in \mathbb{R}_+$ and any ball $B(z',h) \subset \mathrm{Int}(\mathcal{D})$ and $A \in \mathcal{B}(\mathrm{Int}(\mathcal{D}))$. The inequalities $\mathbb{P}_z(Z_s \in B(z',h)) > 0$ and $\mathfrak{m}_{d+1}(B(z',h) \cap A) > 0$ imply $\mathbb{P}_z(Z_v \in B(z',h) \cap A) > 0$ for all $v \in (s, s+h^2)$.

Proof of Claim 2. Since $B(z',h) \subset \mathrm{Int}(\mathcal{D})$, the stopping time $\tau_{\partial B(z',h)} := \inf\{t \in \mathbb{R}_+ : Z_t \notin B(z',h)\}$ is strictly positive $\mathbb{P}_{z''}$-a.s. for all $z'' \in B(z',h)$. Moreover, the process Z on the stochastic interval $[0, \tau_{\partial B(z',h)})$, started at any $z'' \in B(z',h)$, coincides with a uniformly elliptic diffusion on $\mathbb{R}^{d+1}$, stopped upon exiting the ball $B(z',h)$. Thus, [37, Thm II.1.3] is applicable and, together with the strong Markov property of Z [29, Thm A.1], yields the claim.

Claim 3. For any $z \in \mathcal{D}$, $t > 0$ and $A_0 \in \mathcal{B}(\mathrm{Int}(\mathcal{D}))$, such that $\mathfrak{m}_{d+1}(A_0) > 0$, there exist $z_0 \in \mathrm{Int}(\mathcal{D})$, $h_0 > 0$ and $s \in (0,t)$ satisfying $B(z_0,h_0) \subset \mathrm{Int}(\mathcal{D})$, $\mathfrak{m}_{d+1}(A_0 \cap B(z_0,h_0)) > 0$ and $\mathbb{P}_z(Z_s \in B(z_0,h_0)) > 0$.

Proof of Claim 3. Since $\int_0^t \mathbb{P}_z(Z_s \in \mathrm{Int}(\mathcal{D}))\mathrm{d}s = t$, there exist $s < t$, $z' \in \mathrm{Int}(\mathcal{D})$ and $h > 0$, such that $B(z',h) \subset \mathrm{Int}(\mathcal{D})$ and $\mathbb{P}_z(Z_s \in B(z',h)) > 0$. Moreover, the assumption $\mathfrak{m}_{d+1}(A_0) > 0$ implies that there exists a ball $B(z_0,h_0) \subset \mathrm{Int}(\mathcal{D})$ such that $\mathfrak{m}_{d+1}(B(z_0,h_0) \cap A_0) > 0$.

It remains to prove that for some $s \in (0,t)$ we have $\mathbb{P}_z(Z_s \in B(z_0,h_0)) > 0$. If $B(z_0,h_0) \cap B(z',h) \neq \emptyset$, then, since $\mathfrak{m}_{d+1}(B(z_0,h_0) \cap B(z',h)) > 0$, by Claim 2 applied with $A := B(z_0,h_0)$ and $\mathbb{P}_z(Z_s \in B(z',h)) > 0$, there exist $v \in (s,t)$ with $\mathbb{P}_z(Z_v \in B(z_0,h_0)) > 0$. If $B(z_0,h_0) \cap B(z',h) = \emptyset$, then there exists a sequence of $n \in \mathbb{N}$ balls $B(z_i,h_i) \subset \mathrm{Int}(\mathcal{D})$, where $i \in \{1,\ldots,n\}$, such that $z_n = z_0$, $h_n = h_0$ and $z_1 = z'$, $h_1 = h$ and $B(z_i,h_i) \cap B(z_{i+1},h_{i+1}) \neq \emptyset$ for all $i \in \{1,\ldots,n-1\}$. Since $\mathbb{P}_z(Z_s \in B(z_1,h_1)) > 0$, by Claim 2 (applied with $A := B(z_2,h_2)$), there exists time $v_1 \in (s,t)$, such that $\mathbb{P}_z(Z_{v_1} \in B(z_2,h_2)) > 0$. The Markov property at v_1 and Claim 2 imply the existence of $v_2 \in (v_1,t)$ such that $\mathbb{P}_z(Z_{v_2} \in B(z_3,h_3)) > 0$. Construct inductively the increasing sequence $v_1, v_2, \ldots, v_{n-1} \in (0,t)$, set $s := v_{n-1}$ and note $\mathbb{P}_z(Z_{v_{n-1}} \in B(z_n,h_n)) > 0$, implying Claim 3.

Proof of Claim 1. Assume that Claim 1 does not hold. More precisely, there exist $t > 0$, $z \in \mathcal{D}$ and $A_0 \in \mathcal{B}(\mathrm{Int}(\mathcal{D}))$, such that $\mathfrak{m}_{d+1}(A_0) > 0$ and $\int_0^t \mathbb{P}_z(Z_v \in A_0)\mathrm{d}v = 0$. By Claim 3

there exist a ball $B(z_0, h_0)$ in $\mathrm{Int}(\mathcal{D})$ and $s \in (0,t)$ such that $\mathfrak{m}_{d+1}(A_0 \cap B(z_0,h_0)) > 0$ and $\mathbb{P}_z(Z_s \in B(z_0,h_0)) > 0$. Claim 2 (applied with $z' := z_0$, $h := h_0$ and $A := A_0$) yields the contradiction: $0 = \int_0^t \mathbb{P}_z(Z_v \in A_0)\mathrm{d}v \geq \int_s^{t\wedge(s+h_0^2)} \mathbb{P}_z(Z_v \in B(z_0,h_0)\cap A_0)\mathrm{d}v > 0$.

To conclude the proof of the proposition, we strengthen Claim 1. Suppose there exist $z \in \mathcal{D}$, $t > 0$ and $A \in \mathcal{B}(\mathrm{Int}(\mathcal{D}))$ with $\mathfrak{m}_{d+1}(A) > 0$, such that $\mathbb{P}_z(Z_t \in A) = 0$. Since there exists a ball $B(z',h) \in \mathcal{B}(\mathrm{Int}(\mathcal{D}))$, such that $\mathfrak{m}_{d+1}(A \cap B(z',h)) > 0$, by Claim 1 applied to $A \cap B(z',h)$ we have $\int_0^{h^2} \mathbb{P}_z(Z_v \in A \cap B(z',h))\mathrm{d}v > 0$ for all $z \in \mathcal{D}$. We may assume $h^2 \in (0,t)$. Since Z is Markov, we have $\int_{t-h^2}^t \mathbb{P}_z(Z_s \in A \cap B(z',h))\mathrm{d}s > 0$ and hence $\mathbb{P}_z(Z_s \in A \cap B(z',h)) > 0$ for some $s \in (t-h^2, t)$. Thus $t \in (s, s+h^2)$. By Claim 2 we get $\mathbb{P}_z(Z_t \in A) \geq \mathbb{P}_z(Z_t \in A \cap B(z',h)) > 0$, completing the proof of the proposition. □

REMARK 6.4. By Claim 0 in the proof of Proposition 6.1 above, for any $z \in \mathcal{D}$, the equality $\mathbb{P}_z(Z_t \in \partial\mathcal{D}) = 0$ holds for Lebesgue almost every $t \in \mathbb{R}_+$. Note also that the proof of Claim 0 uses only the occupation times formula for continuous semimartingales and basic properties of the solution of SDE (2).

Domain $\mathcal{D}$, defined in (1), with increasing boundary (e.g. $\beta > 0$, see (3) for definition) satisfies the conditions of [38], which establishes Feller continuity for reflecting processes Z. However, as explained in [29, Rem. 2.3(f)], the assumptions of [38] are not satisfied if the boundary function b decreases to zero (e.g. $\beta < 0$). In the case $\beta = 0$, the domain $\mathcal{D}$, may but need not, satisfy the assumptions of [38], see example in Lemma B.1 below. Since the case $\beta < 0$ is when positive recurrence occurs, we develop a new approach to Feller continuity of Z, relying on the localisation of the process. This is more involved than the standard approach in the literature (see e.g. [8]) due to the difficulty of obtaining a global bound on the growth of the local time in the case $\beta < 0$, which requires localisation.

THEOREM 6.5. *Let* (D2), (C2), (V2) *hold. For a continuous bounded function $f : \mathcal{D} \to \mathbb{R}_+$, $t \in \mathbb{R}_+$, and a convergent sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ with limit $\lim_{n\to\infty} z_n = z_\infty \in \mathcal{D}$, we have*

$$\mathbb{E}_{z_n}[f(Z_t)] \to \mathbb{E}_{z_\infty}[f(Z_t)] \quad \text{as } n \to \infty.$$

The proof of Theorem 6.5 requires Lemmas 6.6 and 6.7. Lemma 6.6 provides growth estimates required for the proof of tightness via the Aldus's criterion [20, VI. Thm 4.5].

LEMMA 6.6. *Let* (D2), (C2), (V2) *hold. Fix any $r > 0$, $T > 0$ and $\theta \in (0,T]$. Then there exist positive constants C_i, for $i \in \{1,2,3,4\}$, such that for any $z \in \mathcal{D}$ and $(\mathcal{F}_t)$-stopping times S_1, S_2 ($(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the Brownian filtration in* (2)*), satisfying $S_1 \leq S_2 \leq S_1 + \theta \leq T$, for any $\varepsilon > 0$ the following hold:*

(a) $\mathbb{P}_z(L_{S_2\wedge\varrho_r} - L_{S_1\wedge\varrho_r} \geq \varepsilon) \leq (C_1\theta + C_2\theta^{1/2})/\varepsilon$;
(b) $\mathbb{P}_z(\|Z_{S_2\wedge\varrho_r} - Z_{S_1\wedge\varrho_r}\|_{d+1}^2 \geq \varepsilon) \leq (C_3\theta + C_4\theta^{1/2})/\varepsilon$.

Moreover, $\mathbb{E}_z[L_{T\wedge\varrho_r}] \leq C_1T + C_2T^{1/2} < \infty$ holds.

PROOF. Let G be the function whose existence is guaranteed by Lemma 6.2 and let $g(z) := G(z)^2$, $z \in \mathcal{D}$. Since the first and second derivatives of functions G and g are continuous on $\mathcal{D}$, and Σ is bounded by (C2), it follows that $\|\Sigma^{1/2}\nabla G\|_{d+1}^2$, $\|\Sigma^{1/2}\nabla g\|_{d+1}^2$, $|\Delta_\Sigma G|$ and $|\Delta_\Sigma g|$ are bounded on $\mathcal{D}^{(r)} = \mathcal{D} \cap [0,r] \times \mathbb{R}^d$ for any $r > 0$.

Pick $r > 0$, $T > 0$ and $\theta \in (0,T]$. Recall from Lemma 6.2 that $\nabla g(z) = 0$ and $\langle \phi(z), \nabla G(z)\rangle > \delta_r$ for all $z \in \partial\mathcal{D} \cap [0,r] \times \mathbb{R}^d$ and some positive constant δ_r. Let the bounded stopping times S_1, S_2 be as in the statement of the lemma. Itô's formula (8) applied to the processes $g(Z)$ and $G(Z)$ on the stochastic interval $[S_1 \wedge \varrho_r, S_2 \wedge \varrho_r]$ (recall the definition of ϱ_r in (36)) yields

$$g(Z_{S_2\wedge\varrho_r}) = g(Z_{S_1\wedge\varrho_r}) + M^g_{S_2\wedge\varrho_r} - M^g_{S_1\wedge\varrho_r} + \frac{1}{2}\int_{S_1\wedge\varrho_r}^{S_2\wedge\varrho_r} \Delta_\Sigma g(Z_u)\mathrm{d}u,$$

$\mathbb{P}_z$-a.s., and

$$\delta_r(L_{S_2\wedge\varrho_r} - L_{S_1\wedge\varrho_r}) \le G(Z_{S_2\wedge\varrho_r}) - G(Z_{S_1\wedge\varrho_r}) - M^G_{S_2\wedge\varrho_r} + M^G_{S_1\wedge\varrho_r} - \frac{1}{2}\int_{S_1\wedge\varrho_r}^{S_2\wedge\varrho_r} \Delta_\Sigma G(Z_u)\mathrm{d}u,$$

$\mathbb{P}_z$-a.s., for any $z \in \mathcal{D}$. Note that $(M^g_{t\wedge\varrho_r})_{t\in\mathbb{R}_+}$ and $(M^G_{t\wedge\varrho_r})_{t\in\mathbb{R}_+}$ are true martingales by (9). The optional sampling theorem at the bounded stopping time $S_2 \wedge \varrho_r$ yields $\mathbb{E}_z[M^g_{S_2\wedge\varrho_r} - M^g_{S_1\wedge\varrho_r}|\mathcal{F}_{S_1\wedge\varrho_r}] = 0$, $\mathbb{P}_z$-a.s. and $\mathbb{E}_z[M^G_{S_2\wedge\varrho_r} - M^G_{S_1\wedge\varrho_r}|\mathcal{F}_{S_1\wedge\varrho_r}] = 0$, $\mathbb{P}_z$-a.s. There exists a constant $C_1 > 0$ such that $|\Delta_\Sigma g(z)| \le 2C_1$ for all $z \in \mathcal{D}^{(r)}$, which implies that

$$\mathbb{E}_z[g(Z_{S_2\wedge\varrho_r})|\mathcal{F}_{S_1\wedge\varrho_r}] \le g(Z_{S_1\wedge\varrho_r}) + C_1\theta, \quad \mathbb{P}_z\text{-a.s.} \tag{47}$$

holds for any $z \in \mathcal{D}$. Moreover, there exists a constant C_2 such that $|\Delta_\Sigma G(z)| \le 2C_2$ for $z \in \mathcal{D}^{(r)}$. Using the fact that $g(z) = G(z)^2$, and applying the optional sampling theorem at the bounded stopping time $S_2 \wedge \varrho_r$ implies

$$\begin{aligned}
\delta_r\,\mathbb{E}_z[L_{S_2\wedge\varrho_r} - L_{S_1\wedge\varrho_r}] &\le \mathbb{E}_z\left[\mathbb{E}_z[G(Z_{S_2\wedge\varrho_r})|\mathcal{F}_{S_1\wedge\varrho_r}] - G(Z_{S_1\wedge\varrho_r}) + C_2(S_2 - S_1)\right] \\
&\le \mathbb{E}_z\left[\mathbb{E}_z[g(Z_{S_2\wedge\varrho_r})|\mathcal{F}_{S_1\wedge\varrho_r}]^{1/2} - G(Z_{S_1\wedge\varrho_r}) + C_2\theta\right] \\
&\le \mathbb{E}_z\left[(C_1\theta + g(Z_{S_1\wedge\varrho_r}))^{1/2} - G(Z_{S_1\wedge\varrho_r}) + C_2\theta\right] \\
&\le (C_1\theta)^{1/2} + C_2\theta,
\end{aligned} \tag{48}$$

where we used the Cauchy-Schwarz inequality, the inequality in (47) and the triangle inequality in the second, third and fourth inequalities in the display above, respectively. Application of Markov inequality to (48) implies (a). Moreover, since $L_0 = 0$, setting $S_1 = 0, S_2 = T, \theta = T$ in (48), we get $\mathbb{E}_z[L_{T\wedge\varrho_r}] \le (C_1T)^{1/2} + C_2T < \infty$, as claimed.

To prove (b), define the function $V_{z_0} : \mathcal{D} \to \mathbb{R}_+$ by $V_{z_0}(z) := \|z - z_0\|_{d+1}^2$ for a parameter $z_0 \in \mathcal{D}$. Pick $r > 0, T > 0$ and $\theta \in (0,T]$. Since first and second derivatives of $V_{z_0}(z)$ are continuous on $\mathcal{D}$ in both variable z and parameter z_0, it follows by (V2) and (C2) that there exist constants $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3$ such that $\sup_{z_0\in\mathcal{D}^{(r)}} \langle \nabla V_{z_0}(z), \phi(z)\rangle \le \widetilde{C}_1$, $\sup_{z_0\in\mathcal{D}^{(r)}} |\Delta_\Sigma V_{z_0}(z)| \le 2\widetilde{C}_2$, and $\sup_{z_0\in\mathcal{D}^{(r)}} \|\Sigma(z)^{1/2}\nabla V_{z_0}(z)\|_{d+1}^2 \le \widetilde{C}_3$ hold for every $z \in \mathcal{D}^{(r)}$. Itô's formula (8) applied to $V_{z_0}(Z)$ yields the following equality $\mathbb{P}_{z_0}$-a.s.,

$$V_{z_0}(Z_{S\wedge\varrho_r}) = M^{V_{z_0}}_{S\wedge\varrho_r} + \int_0^{S\wedge\varrho_r} \langle \nabla V_{z_0}(Z_u), \phi(Z_u)\rangle \mathrm{d}L_u + \frac{1}{2}\int_0^{S\wedge\varrho_r} \Delta_\Sigma V_{z_0}(Z_u)\mathrm{d}u,$$

for any stopping time S and any $z_0 \in \mathcal{D}$. Moreover, by (9) the process $(M^{V_{z_0}}_{t\wedge\varrho_r})_{t\in\mathbb{R}_+}$ is a true martingale. By (48), for S satisfying $S \le \theta$, we have $\sup_{z_0\in\mathcal{D}^{(r)}} \mathbb{E}_{z_0}[L_{S\wedge\varrho_r}] \le (C_1\theta)^{1/2} + C_2\theta$. Hence, the optional sampling theorem at the bounded stopping time $S \wedge \varrho_r$ yields

$$\mathbb{E}_{z_0}[\|Z_{S\wedge\varrho_r} - Z_0\|^2_{d+1}] \le \widetilde{C}_1\,\mathbb{E}_{z_0}[L_{S\wedge\varrho_r}] + \widetilde{C}_2\theta \le \widetilde{C}_1((C_1\theta)^{1/2} + C_2\theta) + \widetilde{C}_2\theta,$$

for any $S \le \theta$ and $z_0 \in \mathcal{D}^{(r)}$, where C_1, C_2 come from part (a) of this lemma. It follows from the strong Markov property that for any stopping times S_1, S_2 satisfying assumptions of the lemma, and any $z_0 \in \mathcal{D}$ we have

$$\begin{aligned}\mathbb{E}_{z_0}[\|Z_{S_2\wedge\varrho_r} - Z_{S_1\wedge\varrho_r}\|^2_{d+1}] &= \mathbb{E}_{z_0}[\mathbb{E}_{z_0}[\|Z_{S_2\wedge\varrho_r} - Z_{S_1\wedge\varrho_r}\|^2_{d+1}|\mathcal{F}_{S_1\wedge\varrho_r}]]\\ &\le \widetilde{C}_1((C_1\theta)^{1/2} + C_2\theta) + \widetilde{C}_2\theta.\end{aligned}$$

Application of Markov inequality gives part (b) of the lemma. □

LEMMA 6.7. *Let* (D2)*,* (C2)*,* (V2) *hold. Suppose that for any $t > 0$, any continuous bounded function $f : \mathcal{D} \to \mathbb{R}$, and a convergent sequence $z_n \to z_\infty \in \mathcal{D}$ (as $n \to \infty$), there exists a sequence $(r_k)_{k\in\mathbb{N}}$ satisfying $\lim_{k\to\infty} r_k = \infty$, such that for every $k \in \mathbb{N}$ the following holds:*

$$\mathbb{E}_{z_n}[f(Z_{t\wedge\varrho_{r_k}})] \to \mathbb{E}_{z_\infty}[f(Z_{t\wedge\varrho_{r_k}})] \qquad \text{as } n \to \infty. \tag{49}$$

Then the process Z is Feller continuous, that is, $\mathbb{E}_{z_n}[f(Z_t)] \to \mathbb{E}_{z_\infty}[f(Z_t)]$ as $n \to \infty$.

PROOF. Fix $t > 0$. For any $r > 0$, define the events $\mathcal{A}_r := \{\varrho_r \le t\}$ (and their complements $\mathcal{A}_r^c$).

Claim. For any convergent sequence $z_n \to z_\infty \in \mathcal{D}$ and $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r \in [r_0, \infty)$ we have $\sup_{n\in\mathbb{N}} \mathbb{P}_{z_n}(\mathcal{A}_r) \le \varepsilon$ and $\mathbb{P}_{z_\infty}(\mathcal{A}_r) \le \varepsilon$.

Proof of claim. Pick $w \in (-\infty, 1 - \beta s_0/c_0)$. Lemma 2.3 guarantees the existence of constants $k, x_0, x_1 \in \mathbb{R}_+$ in the definition of the function $F_{w,1}$ such that $\langle \nabla F_{w,1}(z), \phi(z)\rangle \le 0$ for all $z \in \partial\mathcal{D}$. By (17), we have $F_{w,1}(z) = f_{w,1}(z)$ for all $z = (x, y) \in \mathcal{D} \cap [x_1, \infty) \times \mathbb{R}^d$. Since $F_{w,1}$ is smooth on a neighbourhood of $\mathcal{D}$, by (13) and (14) in Lemma 2.1, the functions $z \mapsto |\Delta_\Sigma F_{w,1}(z)|$ and $z \mapsto \|\Sigma(z)^{1/2}\nabla F_{w,1}(z)\|^2_{d+1}$ are bounded on $\mathcal{D}$. Hence, there exist constants $\widetilde{C}_1, \widetilde{C}_2 \in (0, \infty)$ satisfying $|\Delta_\Sigma F_{w,1}(z)| \le 2\widetilde{C}_1$ and $\|\Sigma(z)^{1/2}\nabla F_{w,1}(z)\|^2_{d+1} \le \widetilde{C}_2$ for all $z \in \mathcal{D}$. Define the process $\kappa := 2^{1/|w|}F_{w,1}(Z)$, and recall the definition of ρ_r in (24). Since $F_{w,1}(z) = f_{w,1}(z)$ for $z \in \mathcal{D} \cap [x_1, \infty) \times \mathbb{R}^d$, (12) implies that for $r \in [x_1, \infty)$ we have $\rho_r \le \varrho_r$. Moreover, application of Itô's formula (8) implies that the process $\kappa_{t\wedge\rho_r} - C(t \wedge \rho_r)$ is a supermartingale for $C = 2^{1/|w|}\widetilde{C}_1$, and any $r > 0$, where the local martingale appearing in Itô's formula is a true martingale by (9). Thus, applying Proposition 3.6 (with $\xi = \kappa$ and a constant function $f \equiv C$) we infer that for any $r \ge x_1$ and any $z \in \mathcal{D}$,

$$\mathbb{P}_z(\mathcal{A}_r) \le \mathbb{P}_z(\rho_r \le t) \le r^{-1}(2^{1/|w|}F_{w,1}(z) + Ct).$$

Since $(2^{1/|w|}\sup_{n\in\mathbb{N}} F_{w,1}(z_n) + Ct)$ is finite by the continuity of $F_{w,1}$, there exists r_0 as claimed.

To prove the lemma, pick an arbitrary continuous bounded function $f : \mathcal{D} \to \mathbb{R}$, $\varepsilon > 0$, and a convergent sequence $z_n \to z_\infty$. Let the sequence $(r_k)_{k\in\mathbb{N}}$ tend to infinity and satisfy (49)

for every $k \in \mathbb{N}$. Since f is bounded, the Claim above implies that there exists $k \in \mathbb{N}$ such that $\sup_{z\in\mathcal{D}}|f(z)|(\mathbb{P}_{z_n}(\mathcal{A}_{r_k}) + \mathbb{P}_{z_\infty}(\mathcal{A}_{r_k})) < \varepsilon/3$ holds for every $n \in \mathbb{N}$. Thus, as $f(Z_t) = f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}^c_{r_k}) + f(Z_t)\mathbb{1}(\mathcal{A}_{r_k})$, for every $n \in \mathbb{N}$ we get

$$|\,\mathbb{E}_{z_n}[f(Z_t)] - \mathbb{E}_{z_\infty}[f(Z_t)]| \le |\,\mathbb{E}_{z_n}[f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}^c_{r_k})] - \mathbb{E}_{z_\infty}[f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}^c_{r_k})]| + \varepsilon/3. \tag{50}$$

Next, by (49), for all large $n \in \mathbb{N}$ we have $|\mathbb{E}_{z_n}[f(Z_{t\wedge\varrho_{r_k}})] - \mathbb{E}_{z_\infty}[f(Z_{t\wedge\varrho_{r_k}})]| < \varepsilon/3$. Since $f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}^c_{r_k}) = f(Z_{t\wedge\varrho_{r_k}}) - f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}_{r_k})$, the triangle inequality yields

$$\begin{aligned}&|\mathbb{E}_{z_n}[f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}^c_{r_k})] - \mathbb{E}_{z_\infty}[f(Z_{t\wedge\varrho_{r_k}})\mathbb{1}(\mathcal{A}^c_{r_k})]|\\ &\quad\le |\mathbb{E}_{z_n}[f(Z_{t\wedge\varrho_{r_k}})] - \mathbb{E}_{z_\infty}[f(Z_{t\wedge\varrho_{r_k}})]| + \sup_{z\in\mathcal{D}}|f(z)|(\mathbb{P}_{z_n}(\mathcal{A}_{r_k}) + \mathbb{P}_{z_\infty}(\mathcal{A}_{r_k})) \le 2\varepsilon/3,\end{aligned}$$

which, together with (50), implies $|\,\mathbb{E}_{z_n}[f(Z_t)] - \mathbb{E}_{z_\infty}[f(Z_t)]| < \varepsilon$ for all large $n \in \mathbb{N}$. □

PROOF OF THEOREM 6.5. By Lemma 6.7, it suffices to prove that for any $f : \mathcal{D} \to \mathbb{R}_+$, $T \in \mathbb{R}_+$ and a convergent sequence $z_n \to z_\infty \in \mathcal{D}$ (as $n \to \infty$), there exists a sequence $(r_k)_{k\in\mathbb{N}}$ tending to infinity, such that for any $k \in \mathbb{N}$ we have

$$\mathbb{E}_{z_n}[f(Z_{T\wedge\varrho_{r_k}})] \to \mathbb{E}_{z_\infty}[f(Z_{T\wedge\varrho_{r_k}})] \quad \text{as } n \to \infty.$$

The first step is to prove that, for a fixed $r > 0$, the laws of $(Z_{\cdot\wedge\varrho_r}, L_{\cdot\wedge\varrho_r}, W_\cdot)$ under $\mathbb{P}_{z_n}$ are tight. The second step consists of proving that every subsequence converges to the law of $(Z_{\cdot\wedge\varrho_r}, L_{\cdot\wedge\varrho_r}, W_\cdot)$ under $\mathbb{P}_{z_\infty}$.

Let $E^{(1)} := E_1 \times E_2 \times E_3$, where $E_1 := \mathcal{D}$, $E_2 := \mathbb{R}_+$ and $E_3 := \mathbb{R}^{d+1}$, and denote by $D_{E^{(1)}}([0,\infty))$ the space of càdlàg functions (i.e. right-continuous functions with left limits), mapping the interval $[0,\infty)$ into the metric space $E^{(1)}$. Denote by $C_{E^{(1)}}([0,\infty)) \subset D_{E^{(1)}}([0,\infty))$ the subspace of continuous functions. Endow $D_{E^{(1)}}([0,\infty))$ with the Skorohod topology and its Borel σ-algebra $\mathcal{M}^1$, see e.g. [2, Ch. 16] for details. Any function $\omega \in D_{E^{(1)}}([0,\infty))$ can be expressed as $\omega = (a^{(1)}, a^{(2)}, a^{(3)})$, for some "coordinate" càdlàg functions $a^{(i)} : [0,\infty) \to E_i$ for $i = 1,2,3$. For any $t \in [0,\infty)$, define maps $A^{(i)}_t : D_{E^{(1)}}([0,\infty)) \to E_i$ by $A^{(i)}_t(\omega) := a^{(i)}(t)$, $i \in \{1,2,3\}$. Define a σ-algebra $\mathcal{M}^1_t := \sigma\{A^{(i)}_s : 0 \le s \le t, i \in \{1,2,3\}\} \subset \mathcal{M}^1$ generated by continuous maps $A^{(i)}_s$. By [2, Thm 16.6] we have $\mathcal{M}^1_\infty = \mathcal{M}^1$.

For any $z \in \mathcal{D}$, it follows from [29, Thm A.1], that there exists a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathcal{F}, \mathbb{P}_z)$, supporting the processes (Z, L, W), taking values in $E^{(1)}$ such that, under $\mathbb{P}_z$, the SDE in (2) holds and W is a standard $(\mathcal{F}_t)$-Brownian motion. We may assume that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is complete (i.e. $\mathcal{F}_0$ contains all $\mathbb{P}_z$-null sets of $\mathcal{F}$) and right continuous (i.e. $\mathcal{F}_t = \cap_{t<s}\mathcal{F}_s$). Pick arbitrary $r_0 \in \mathbb{R}_+$ and $T > 0$, and recall the stopping time ϱ_{r_0}, defined in (36), is the first time the coordinate X of $Z = (X, Y)$ reaches level r_0. The stopped process $(Z_{t\wedge\varrho_{r_0}\wedge T}, L_{t\wedge\varrho_{r_0}\wedge T}, W_{t\wedge T})_{t\in\mathbb{R}_+}$ produces a measurable map $(Z_{\cdot\wedge\varrho_{r_0}\wedge T}, L_{\cdot\wedge\varrho_{r_0}\wedge T}, W_{\cdot\wedge T}) : (\Omega, \mathcal{F}) \to (D_{E^{(1)}}([0,\infty)), \mathcal{M}^1)$ and induces the probability measure $Q^1_z(\cdot) := \mathbb{P}_z((Z, L, W) \in \cdot)$ on $\mathcal{M}^1$.

Denote by $\mathcal{P}(D_{E^{(1)}}([0,\infty)),\mathcal{M}^1)$ the space of probability measures on the measurable space $(D_{E^{(1)}}([0,\infty)),\mathcal{M}^1)$ and endow it with the topology of weak convergence (see [20, VI. 3.] for details). We say that the sequence of measures $(Q_n)_{n\in\mathbb{N}}\in\mathcal{P}(D_{E^{(1)}}[0,\infty),\mathcal{M}^1)$ is *C-tight*, if every subsequence of $(Q_n)_{n\in\mathbb{N}}$ has a convergent subsequence and the limiting probability measure Q_* charges only the set $C_{E^{(1)}}([0,\infty))$, i.e. $Q_*(C_{E^{(1)}}[0,\infty))=1$.

Recall that the measures Q^1_z, for $z\in\mathcal{D}$, are defined by the laws of the stopped processes $(Z_{\cdot\wedge\varrho_r\wedge T},L_{\cdot\wedge\varrho_r\wedge T},W_{\cdot\wedge T})$ under the measure $\mathbb{P}_z$. Lemma 6.6, and the fact that W is a Brownian motion, imply that for any $\theta>0$ there exist positive constants C_1,C_2 such that for any stopping times $S_1,S_2\in\mathcal{F}_T$ satisfying $S_1\le S_2\le S_1+\theta\le T$, for any $n\in\mathbb{N}$ we have

$$\mathbb{P}_{z_n}(\|(Z_{S_2\wedge\varrho_{r_0}},L_{S_2\wedge\varrho_{r_0}},W_{S_2})-(Z_{S_1\wedge\varrho_{r_0}},L_{S_1\wedge\varrho_{r_0}},W_{S_1})\|^2_{2d+3}\ge\varepsilon)\le(C_1\theta+C_2\theta^{1/2})/\varepsilon.$$

There exists a positive constant $C>0$, such that $\sup_{t\in\mathbb{R}_+}\|Z_{t\wedge\varrho_{r_0}}\|_{d+1}\le C$ $\mathbb{P}_{z_n}$-a.s. for all $n\in\mathbb{N}$. Since Lemma 6.6 bounds the expected growth of the local time L, for any $\varepsilon>0$, there exists $K>0$, such that for all $n\in\mathbb{N}$ we have $\mathbb{P}_{z_n}(\sup_{0\le s\le T}\|(Z_{s\wedge\varrho_{r_0}},L_{s\wedge\varrho_{r_0}},W_s)\|_{2d+3}\ge K)\le\varepsilon$. Thus we may apply Aldous's tightness criterion [20, VI. Thm 4.5] to deduce that the sequence of measures $(Q^1_{z_n})_{n\in\mathbb{N}}$ is tight. Moreover, since $Q^1_{z_n}(C[0,\infty))=1$ for every $n\in\mathbb{N}$, [20, VI. Prop. 3.26] implies that the sequence $(Q^1_{z_n})_{n\in\mathbb{N}}$ is C-tight. It follows that there exists a subsequence $(Q^1_{z_{n_k}})_{k\in\mathbb{N}}$ that converges weakly to a probability measure Q^1_*, satisfying $Q^1_*(C[0,\infty))=1$. For notational convenience we assume that the sequence $(Q^1_{z_n})_{n\in\mathbb{N}}$ itself converges to Q^1_*.

Our aim is to prove that $(A^{(1)},A^{(2)},A^{(3)})$, under the measure Q^1_*, solves SDE (2). Consider the process $\mathcal{Z}:=(Z_{\cdot\wedge\varrho_{r_0}\wedge T},L_{\cdot\wedge\varrho_{r_0}\wedge T},W_{\cdot\wedge T},\bar{\mathcal{Z}}_{\cdot\wedge T})$, where

$$\bar{\mathcal{Z}}:=\left(\int_0^{\cdot}\Sigma^{1/2}(Z_u)\mathrm{d}W_u,\int_0^{\cdot\wedge\varrho_{r_0}}\phi(Z_u)\mathrm{d}L_u,\int_0^{\cdot\wedge\varrho_{r_0}}(b(X_u)^2-\|Y_u\|_d^2)\mathrm{d}L_u\right).$$

The state space of $\mathcal{Z}$ is $E^{(2)}:=E^{(1)}\times E_4\times E_5\times E_6$, where $E_4=\mathbb{R}^{d+1},E_5=\mathbb{R}^{d+1}$ and $E_6=\mathbb{R}$. Let $(D_{E^{(2)}}([0,\infty)),(\mathcal{M}^2_t)_{t\in[0,\infty)},\mathcal{M}^2)$ be the filtered measurable space with coordinate projections $A^{(i)}:D_{E^{(2)}}([0,\infty))\to D_{E_i}([0,\infty))$ for $i\in\{1,\dots,6\}$ (note that $A^{(i)}$, for $i\in\{1,2,3\}$, agrees with the definition in the beginning of the second paragraph of this proof). Moreover, a measurable map $\mathcal{Z}:(\Omega,\mathcal{F})\to(D_{E^{(2)}}([0,\infty)),\mathcal{M}^2)$ induces the probability measure $Q^2_z(\cdot):=\mathbb{P}_z(\mathcal{Z}\in\cdot)$ on $\mathcal{M}^2$. Note that $Q^2_z((A^{(1)},A^{(2)},A^{(3)})^{-1}(\cdot))=Q^1_z(\cdot)$ for all $z\in\mathcal{D}$. Hence the first three coordinates converge weakly under $(Q^2_{z_n})_{n\in\mathbb{N}}$.

Denote $A^{(1)}=(A^{(1,X)},A^{(1,Y)})$, where $A^{(1,X)}:D_{E^{(2)}}([0,\infty))\to D_{\mathbb{R}_+}([0,\infty))$ and $A^{(1,Y)}:D_{E^{(2)}}([0,\infty))\to D_{\mathbb{R}^d}([0,\infty))$. Note that for each $z\in\mathcal{D}$, L and W under the measures $\mathbb{P}_z$ are adapted to the complete, right-continuous filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and have continuous sample paths, which are thus in $D_{E_2}([0,\infty))$ and $D_{E_3}([0,\infty))$, respectively.[1] Moreover, the processes W,L are semimartingales, which satisfy $\mathbb{E}_{z_n}[L_{t\wedge\varrho_{r_0}\wedge T}]<\infty$ (by Lemma 6.6) and $\mathbb{E}_{z_n}[[W]_{t\wedge T}]\le T$ for any $t\in\mathbb{R}_+$ and $n\in\mathbb{N}$. Thus, Assumption [24, C2.2(i)] is satisfied with the deterministic time $\tau^\alpha_n:=\alpha+1$, where $\alpha\in(0,\infty)$, $n\in\mathbb{N}$ are arbitrary parameters and τ^α_n

[1] We work with càdlàg paths because we apply [24, Thm 2.2] to conclude the stability of the stochastic integrals.

is a sequence of stopping times in [24, C2.2(i)]. Let

$$\bar{A} := \left(\int_0^\cdot \Sigma^{1/2}(A_u^{(1)}) \mathrm{d}A_u^{(3)}, \int_0^\cdot \phi(A_u^{(1)}) \mathrm{d}A_u^{(2)}, \int_0^\cdot (b(A_u^{(1,X)})^2 - \|A_u^{(1,Y)}\|_d^2) \mathrm{d}A_u^{(2)} \right),$$

and define the process $\widetilde{A} := (A_\cdot^{(1)}, A_\cdot^{(2)}, A_\cdot^{(3)}, \bar{A})$. Since the functions $z \mapsto \Sigma(z)$, $z \mapsto \phi(z)$, $z \mapsto b(x)^2 - \|y\|_{d+1}^2$ are continuous in $z = (x, y) \in \mathcal{D}$ and $\mathbb{P}_z(\mathcal{Z} \in \cdot) = Q_z^2(\cdot)$, by [24, Thm 2.2] there exists a probability measure Q_*^2 on $(D_{E^{(2)}}([0,\infty)), \mathcal{M}^2)$, such that $\widetilde{A}$ under the measure $Q_{z_n}^2$ converges weakly (as $n \to \infty$) to $\widetilde{A}$ under Q_*^2. In particular, [24, Thm 2.2] ensures that $A^{(2)}, A^{(3)}$ are semimartingales under Q_*^2. Note that all limiting processes have continuous paths Q_*^2-a.s., which follows from the fact that $A^{(1)}, A^{(2)}$ and $A^{(3)}$ have continuous paths Q_*^2-a.s. We now extend (using [13, Lem 4.3]) the weak convergence to stopping times ϱ_r and the corresponding stopped processes.

For any $a^{(1,X)} \in D_{\mathbb{R}_+}([0,\infty))$ and $r > 0$, denote the first hitting time of the level r by $\tau_r(a^{(1,X)}) := \inf\{t > 0 : a_t^{(1,X)} \geq r \text{ or } a_{t-}^{(1,X)} \geq r\}$, where $a_{t-}^{(1,X)}$ denotes the left limit of $a^{(1,X)}$ at time t. Since, for any $z \in \mathcal{D}$, the process X has $\mathbb{P}_z$-a.s. continuous paths, we have $\tau_r(X) = \varrho_r$, $\mathbb{P}_z$-a.s. We now make the final extension of the state space, which will capture the convergence of the stopping times. Denote $E^{(3)} := E^{(2)} \times \mathbb{R}_+$, and let $(D_{E^{(3)}}([0,\infty)), (\mathcal{M}_t^3)_{t\in[0,\infty)}, \mathcal{M}^3)$ be the filtered measurable space with coordinate projections $A^{(i)}$, for $i \in \{1, \ldots, 7\}$. For any $r \in (0,\infty)$, a measurable map $(\mathcal{Z}, \varrho_r) : (\Omega, \mathcal{F}) \to (D_{E^{(3)}}([0,\infty)), \mathcal{M}^3)$ induces the probability measure $Q_z^{3,r}(\cdot) := \mathbb{P}_z((\mathcal{Z}, \varrho_r) \in \cdot)$ on $\mathcal{M}^3$. Furthermore, for any $r \in (0,\infty)$, it holds that $Q_z^{3,r}(\widetilde{A}^{-1}(\cdot)) = Q_z^2(\cdot)$.

As before, $\widetilde{A}$ under $Q_{z_n}^{3,r}$ converge to $\widetilde{A}$ under some probability measure $Q_*^{3,r}$. Moreover, $A^{(1,X)}$ has $Q_*^{3,r}$-a.s. continuous paths. Thus, [13, Rem 4.8 and Lem 4.3] implies that for all but at most countably many $r \in (0, r_0)$, the map $D_{\mathbb{R}_+}([0,\infty)) \to \mathbb{R}_+$, given by $a^{(1,X)} \mapsto \tau_r(a^{(1,X)})$, is continuous at $A^{(1,X)}$, $Q_*^{3,r}$-a.s. (recall that $r_0 > 0$ is an arbitrary number, fixed at the beginning of the proof). Pick $r_1 \in (r_0/2, r_0)$ such that $a^{(1,X)} \mapsto \tau_{r_1}(a^{(1,X)})$ is continuous at $a^{(1,X)}$, $Q_*^{r_1}$-a.s. The Continuous mapping theorem [2, Thm 2.7] implies that $(\widetilde{A}, \tau_{r_1}(A^{(1,X)}))$, under $Q_{z_n}^{3,r_1}$, converges weakly to $(\widetilde{A}, \tau_{r_1}(A^{(1,X)}))$ under Q_*^{3,r_1}. The convergence of the stopped processes follows by applying [24, Theorem 2.2] to the stochastic integral $\int_0^t \mathbb{1}\{s \leq \varrho_{r_1}\} \mathrm{d}\mathcal{Z}_s$ (condition [24, C2.2(i)] is satisfied with $\tau_n^\alpha := \alpha + 1$ as above). Since $\mathbb{P}_z((\mathcal{Z}, \varrho_{r_1}) \in \cdot) = Q_z^{3,r_1}(\cdot)$, it follows that $(\widetilde{A}_{\cdot\wedge\tau_{r_1}(A^{(1,X)})})$, under $Q_{z_n}^{3,r_1}$, converges weakly as $n \to \infty$ to $(\widetilde{A}_{\cdot\wedge\tau_{r_1}(A^{(1,X)})})$ under the probability measure Q_*^{3,r_1}.

To conclude the proof, we have to show that the process $(A^{(1)}, A^{(2)}, A^{(3)})$ under Q_*^{3,r_1} solves SDE (2). Denote by $\mathbf{0} \in D_{E_1}([0,\infty))$ the function mapping every $t \in [0,\infty)$ into the origin of $\mathbb{R}^{d+1}$. For every $n \in \mathbb{N}$, we have

$$A_{\cdot\wedge\tau_{r_1}(A^{(1,X)})}^{(1)} - z_n - \int_0^{\cdot\wedge\tau_{r_1}(A^{(1,X)})} \phi(A_u^{(1)}) \mathrm{d}A_u^{(2)} - \int_0^{\cdot\wedge\tau_{r_1}(A^{(1,X)})} \Sigma^{1/2}(A_u^{(1)}) \mathrm{d}A_u^{(3)} \equiv \mathbf{0},$$

$Q_{z_n}^{3,r_1}$-a.s. Since the set $\{\mathbf{0}\}$ is closed in $D_{E_1}([0,\infty))$, [2, Thm 2.1(iii)] implies that

$$A_{\cdot\wedge\tau_{r_1}(A^{(1,X)})}^{(1)} - z_\infty - \int_0^{\cdot\wedge\tau_{r_1}(A^{(1,X)})} \Sigma^{1/2}(A_u^{(1)}) \mathrm{d}A_u^{(3)} - \int_0^{\cdot\wedge\tau_{r_1}(A^{(1,X)})} \phi(A_u^{(1)}) \mathrm{d}A_u^{(2)} \equiv \mathbf{0},$$

Q_*^{3,r_1}-a.s. Note that since $A^{(3)}$ is a Brownian motion under $Q_{z_n}^{3,r_1}$ for every $n \in \mathbb{N}$, it is also a Brownian motion under the weak limit Q_*^{3,r_1}.

It remains to prove that, under Q^{3,r_1}_*, $A^{(2)}$ is a local time of $A^{(1)}$ at the boundary $\partial\mathcal{D}$. For any $z \in \mathcal{D}$, the local time has to satisfy $L_{t\wedge\varrho_r} = \int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}L_s$, $\mathbb{P}_z$-a.s. This requirement is equivalent to

$$\int_0^{t\wedge\varrho_r} (b(X_s)^2 - \|Y_s\|_d^2)\mathrm{d}L_s = 0, \quad \mathbb{P}_z\text{-a.s.} \tag{51}$$

Indeed, since $b(x)^2 - \|y\|_d^2 = 0$ for all $(x,y) \in \partial\mathcal{D}$, we have $\int_0^{t\wedge\varrho_r}(b(X_s)^2 - \|Y_s\|_d^2)\mathrm{d}L_s = \int_0^{t\wedge\varrho_r}(b(X_s)^2 - \|Y_s\|_d^2)\mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}L_s = 0$, $\mathbb{P}_z$-a.s., by the definition of local time. Conversely, since $b(x)^2 - \|y\|_d^2 > 0$ for all $(x,y) \in \mathcal{D} \setminus \partial\mathcal{D}$, the sets $\mathcal{D}_k := \{(x,y) \in \mathcal{D} : b(x)^2 - \|y\|_d^2 \in [1/(k+1), 1/k)\}$ are pairwise disjoint and satisfy $\cup_{k\in\mathbb{N}}\mathcal{D}_k = \mathcal{D} \setminus \partial\mathcal{D}$. Moreover, we have

$$0 \le \int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \mathcal{D}_k\}\mathrm{d}L_s \le (k+1)\int_0^{t\wedge\varrho_r} (b(X_s)^2 - \|Y_s\|_d^2)\mathbb{1}\{Z_s \in \mathcal{D}_k\}\mathrm{d}L_s = 0,$$

implying (51) via

$$L_{t\wedge\varrho_r} = \int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}L_s + \sum_{k\in\mathbb{N}}\int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \mathcal{D}_k\}\mathrm{d}L_s = \int_0^{t\wedge\varrho_r} \mathbb{1}\{Z_s \in \partial\mathcal{D}\}\mathrm{d}L_s.$$

Note that $\int_0^{\cdot\wedge\tau_{r_1}(A^{(1,X)})}(b(A^{(1,X)})^2 - \|A^{(1,Y)}\|_d^2)\mathrm{d}A^{(2)}_u \equiv \mathbf{0}$, $Q^{3,r_1}_{z_n}$-a.s., for every $n \in \mathbb{N}$, where $\mathbf{0}$ now denotes the zero function in $D_{E_6}([0,\infty))$ (recall that $E_6 = \mathbb{R}$). Thus, by [2, Thm. 2.1(iii)], we get $\int_0^{\cdot\wedge\tau_{r_1}(A^{(1,X)})}(b(A^{(1,X)})^2 - \|A^{(1,Y)}\|_d^2)\mathrm{d}A^{(2)}_u \equiv 0$, Q^{3,r_1}_*-a.s. Hence, the condition in (51) implies that $A^{(2)}$ is indeed the local time of $A^{(1)}$ at the boundary $\partial\mathcal{D}$.

We have thus proved that $(A^{(1)}_{\cdot\wedge\tau_{r_1}(A^{(1,X)})}, A^{(2)}_{\cdot\wedge\tau_{r_1}(A^{(1,X)})}, A^{(3)}_{\cdot\wedge\tau_{r_1}(A^{(1,X)})})$, under Q^{3,r_1}_* (and hence under Q^1_*), solves SDE (2) on the stochastic interval $[0, \varrho_{r_1}]$. The pathwise uniqueness [29, Thm A.1.] of solutions of SDE (2) implies that every sub-sequential limit Q^1_* of the sequence of $(Q^1_{z_n})_{n\in\mathbb{N}}$ equals $Q^1_{z_\infty}$, implying the Feller continuity for the process stopped at ϱ_{r_1}, i.e. the limit in (49) holds for r_1. Since r_0 was chosen arbitrarily and $r_1 \in (r_0/2, r_0)$, we can inductively construct a sequence $(r_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} r_k = \infty$, such that (49) holds for every $k \in \mathbb{N}$. An application of Lemma 6.7 concludes the proof of Theorem 6.5. □

6.2. *Application to return times.* An easy consequence of Feller continuity and irreducibility is the non-confinement of the reflected process Z in any compact set. We will use this property to extend the asymptotic results about return times in Section 5 to the entire domain $\mathcal{D}$.

Recall the definition of the first passage time ϱ_r (over r) in (36), the return time ς_r (below r) in (5) and the neighbourhood $\mathcal{D}_h = \{z \in \mathcal{D} : \exists z' \in \partial\mathcal{D}$ such that $\|z - z'\|_{d+1} < h\}$ of the boundary $\partial\mathcal{D}$ for $h \in (0,\infty)$. We start with the following proposition.

PROPOSITION 6.8. *For any $0 < r_0 < r_1 < \infty$, the following statements hold.*

(a) *For all sufficiently small $h > 0$ and $z \in (\mathcal{D} \setminus \mathcal{D}_h) \cap (r_0, r_1) \times \mathbb{R}^d$, we have $\mathbb{P}_z(\varrho_{r_1} < \varsigma_{r_0}) > 0$.*

(b) *For any $\delta \in (0, r_1 - r_0)$ and sufficiently small $h > 0$, there exists $\varepsilon > 0$, such that $\mathbb{P}_z(\varsigma_{r_0} < \varrho_{r_1}) > \varepsilon$ for any $z \in (\mathcal{D} \setminus \mathcal{D}_h) \cap [r_0, r_1 - \delta] \times \mathbb{R}^d$.*

(c) *All moments of the first exit time of the interval* (r_0, r_1) *are finite:* $\mathbb{E}_z[(\varsigma_{r_0} \wedge \varrho_{r_1})^k] < \infty$ *for all* $k \in \mathbb{N}$ *and* $z \in \mathcal{D}$.

PROOF. Since proofs of both (a) and (b) follow from the same PDE argument, we will prove them together. Let $i \in \{0,1\}$. Fix $0 < h' < h < \infty$. Let $\mathcal{D}_h$ be as above and set $\mathcal{D}_{r_0,r_1,h} := (\mathcal{D} \setminus \mathcal{D}_h) \cap (r_0, r_1) \times \mathbb{R}^d$. Note that $\mathcal{D}_{r_0,r_1,h} \subset \mathcal{D}_{r_0,r_1,h'}$. For any $h'' \in (h', h)$ there exists a closed domain $\hat{\mathcal{D}}_{r_0,r_1,h'}$ with C^2 boundary, satisfying

$$\mathcal{D}_{r_0,r_1,h''} \subset \hat{\mathcal{D}}_{r_0,r_1,h'} \subset \mathcal{D}_{r_0,r_1,h'} \quad \text{and} \quad \mathcal{D}_{r_0,r_1,h''} \cap \{r_i\} \times \mathbb{R}^d = \hat{\mathcal{D}}_{r_0,r_1,h'} \cap \{r_i\} \times \mathbb{R}^d$$

for $i \in \{0,1\}$. Closed domain $\hat{\mathcal{D}}_{r_0,r_1,h'}$ can be obtained from $\mathcal{D}_{r_0,r_1,h'}$ by smoothing its corners appropriately.

Choose continuous functions $f_i : \partial\hat{\mathcal{D}}_{r_0,r_1,h'} \to \mathbb{R}$, such that $f_i \equiv 1$ on $\mathcal{D}_{r_0,r_1,h} \cap \{r_i\} \times \mathbb{R}^d = \partial\mathcal{D}_{r_0,r_1,h} \cap \{r_i\} \times \mathbb{R}^d$ and $f_i \equiv 0$ on $\partial\hat{\mathcal{D}}_{r_0,r_1,h'} \setminus (\hat{\mathcal{D}}_{r_0,r_1,h'} \cap \{r_i\} \times \mathbb{R}^d)$ for $i \in \{0,1\}$. Dirichlet problems on $\hat{\mathcal{D}}_{r_0,r_1,h'}$ with boundary conditions $f_i : \partial\hat{\mathcal{D}}_{r_0,r_1,h'} \to \mathbb{R}$ are given by

$$\frac{1}{2}\Delta_\Sigma u_i = 0 \text{ on } \hat{\mathcal{D}}_{r_0,r_1,h'} \setminus \partial\hat{\mathcal{D}}_{r_0,r_1,h'}; \tag{52}$$

$$u_i = f_i, \text{ on } \partial\hat{\mathcal{D}}_{r_0,r_1,h'}. \tag{53}$$

Then, by [23, pp. 364–366], the functions

$$u_i(z) := \mathbb{E}_z[f_i(Z_\tau)], \quad \text{where } \tau := \inf\{t \in \mathbb{R}_+ : Z_t \in \partial\hat{\mathcal{D}}_{r_0,r_1,h'}\},$$

solve the respective Dirichlet problem in (52)–(53) on $\hat{\mathcal{D}}_{r_0,r_1,h'}$ for $i \in \{0,1\}$. Moreover, $f_0(Z_\tau) \le \mathbb{1}\{\varsigma_{r_0} < \varrho_{r_1}\}$, $\mathbb{P}_z$-a.s., and $f_1(Z_\tau) \le \mathbb{1}\{\varrho_{r_1} < \varsigma_{r_0}\}$, $\mathbb{P}_z$-a.s., on $\hat{\mathcal{D}}_{r_0,r_1,h'}$, implying $u_0(z) \le \mathbb{P}_z(\varsigma_{r_0} < \varrho_{r_1})$ and $u_1(z) \le \mathbb{P}_z(\varrho_{r_1} < \varsigma_{r_0})$. Since f_i are continuous, $\hat{\mathcal{D}}_{r_0,r_1,h'}$ has a C^2 boundary and thus satisfies the inside the sphere property (see [12, p. 59] for definition), and the coefficients in (52)–(53) are continuous and uniformly elliptic by assumption (C2), the maximum principle [12, Thm. 21, p. 55], applied to $-u_i$, yields $u_i(z) > 0$ for all $z \in \hat{\mathcal{D}}_{r_0,r_1,h'}$. This directly implies part (a). Moreover, since $\mathcal{D}_{r_0,r_1,h} \cap [r_0, r_1 - \delta] \times \mathbb{R}^d$ is compact and u_0 is continuous with $u_0(z) > 0$ for all $z \in \mathcal{D}_{r_0,r_1,h} \cap [r_0, r_1 - \delta] \times \mathbb{R}^d$, part (b) follows.

Recall the notation $Z_t = (X_t, Y_t) \in \mathcal{D}$, where $X_t \in \mathbb{R}_+$, for all $t \in \mathbb{R}_+$. To prove part (c), consider a function $p(z) := \mathbb{P}_z(X_1 > r_1)$. Proposition 6.1 implies that $p(z) > 0$ for all $z \in \mathcal{D}$. Moreover, by Theorem 6.5 and [2, Thm 2.1], the function p is lower semi-continuous. Since the set $\mathcal{D} \cap [0, r_1] \times \mathbb{R}^d$ is compact, there exists $\varepsilon > 0$ such that $p(z) > \varepsilon$ for all $z \in \mathcal{D} \cap [0, r_1] \times \mathbb{R}^d$. Thus, $\mathbb{P}_z(X_1 \le r_1) \le 1 - \varepsilon$ for all $z \in \mathcal{D} \cap [0, r_1] \times \mathbb{R}^d$. This, together with the Markov property, implies that for every $n \in \mathbb{N}$ and $z \in \mathcal{D}$ we have $\mathbb{P}_z(\varsigma_{r_0} \wedge \varrho_{r_1} > n) \le \mathbb{P}_z(\cap_{k=1}^n \{X_k \le r_1\}) \le (1 - \varepsilon)^n$. Thus, $\mathbb{E}_z[(\varsigma_{r_0} \wedge \varrho_{r_1})^k] < \infty$ holds for all $k \in \mathbb{N}$ and every $z \in \mathcal{D}$. □

PROOF OF THEOREM 1.4. Transience and the lower bounds in the recurrent case both require the following claim.

Claim 1. For any $0 < r_0 < r_1 < \infty$ and $z \in \mathcal{D} \cap (r_0, r_1) \times \mathbb{R}^d$ we have $\mathbb{P}_z(\varrho_{r_1} < \varsigma_{r_0}) > 0$.

Proof of Claim 1. By Proposition 6.8(a), for $h > 0$ sufficiently small and $z \in (\mathcal{D} \setminus \mathcal{D}_h) \cap (r_0, r_1)$, we have $\mathbb{P}_z(\varrho_{r_1} < \varsigma_{r_0}) > 0$. Pick $z \in \partial\mathcal{D} \cap (r_0, r_1) \times \mathbb{R}^d$ and define the stopping

time $v_h := \inf\{t > 0 : Z_t \in \mathcal{D} \setminus \mathcal{D}_h\}$. The continuity of paths implies $\mathbb{E}_z[\varsigma_{r_0}] > 0$. Moreover, Assumptions (D2), (V2) and (C2) and [29, Lem. 4.5] imply that for some $\delta_\Sigma > 0$ and all sufficiently small $h > 0$, we have $\mathbb{E}_z[v_h \wedge \varrho_{r_1}] \le 2h^2/\delta_\Sigma$. Hence, if $h \in (0, (\delta_\Sigma \mathbb{E}_z[\varsigma_{r_0}]/2)^{1/2})$ then $\mathbb{E}_z[v_h \wedge \varrho_{r_1}] < \mathbb{E}_z[\varsigma_{r_0}]$. In particular, this implies $\mathbb{P}_z(v_h \wedge \varrho_{r_1} < \varsigma_{r_0}) > 0$. If $\mathbb{P}_z(\{v_h \wedge \varrho_{r_1} < \varsigma_{r_0}\} \cap \{\varrho_{r_1} < v_h\}) > 0$ the proof is complete. Otherwise, $\{v_h \wedge \varrho_{r_1} < \varsigma_{r_0}\} = \{v_h \wedge \varrho_{r_1} < \varsigma_{r_0}\} \cap \{\varrho_{r_1} > v_h\} = \{v_h < \varsigma_{r_0}\}$, $\mathbb{P}_z$-a.s. Thus $\mathbb{P}_z(v_h < \varsigma_{r_0}) = \mathbb{P}_z(v_h \wedge \varrho_{r_1} < \varsigma_{r_0}) > 0$. By the strong Markov property at v_h and Proposition 6.8(a), Claim 1 follows.

Proof of (a). In this case we have $\beta > \beta_c$. By Theorem 1.3 the process Z is transient. The Lyapunov function $f_{w,\gamma}(x, y)$ (with $\gamma < 0$) tends to zero by (12) as $x \to \infty$. Analogous to the proof of Theorem 1.3(b), an application of Lemma 3.4 (and in particular the bound in (30) with $\kappa = f_{w,1}(Z)$, $V(u) = u^\gamma$), implies that for every $x_0 \in \mathbb{R}_+$, there exist $c(x_0)$, such that for all $z \in \mathcal{D} \cap [x_0 + c(x_0), \infty) \times \mathbb{R}^d$, we have $\mathbb{P}_z(\varsigma_{x_0} = \infty) > 0$. Moreover, Claim 1 implies that for all $z \in \mathcal{D} \cap (x_0, \infty) \times \mathbb{R}^d$ we have $\mathbb{P}_z(\varrho_{x_0+c(x_0)} < \varsigma_{x_0}) > 0$. The strong Markov property at the stopping time $\varrho_{x_0+c(x_0)}$ concludes the proof of Theorem 1.4(a).

Proof of the lower bound in (b). In this case we have $\beta < \beta_c$. Recall the definition of m_c in (6). Moreover, Proposition 5.2 implies that, for any $p \in (m_c, \infty)$, there exist constants $x_0 \in (0, \infty)$ and $c_1, c_2 \in (1, \infty)$, such that for all $x_1 \in [x_0, \infty)$ and $z \in \mathcal{D} \cap (c_1(x_1 + 1) + c_2, \infty) \times \mathbb{R}^d$ we have $\mathbb{P}_z(\varsigma_{x_1} > t) \ge Ct^{-p}$, for some constant C and all $t \ge 1$, say. Pick $x_1 \in [x_0, \infty)$. Note that by Claim 1, for any $x_2 \in (0, x_1]$ and any $z \in \mathcal{D} \cap (x_2, c_1(x_1+1)+c_2) \times \mathbb{R}^d$ we have $\mathbb{P}_z(\varrho_{c_1(x_1+1)+c_2} < \varsigma_{x_2}) > 0$. Moreover, $\varsigma_{x_2} \ge \varsigma_{x_1}$, $\mathbb{P}_z$-a.s. Thus, the strong Markov property applied at the stopping time $\varrho_{c_1(x_1+1)+c_2}$ implies the lower bound in Theorem 1.4(b).

Claim 2. For any $0 < r_0 < r_1 < \infty$ and any $\delta \in (0, \min\{(r_1 - r_0)/2, 1\})$, there exists an $\varepsilon > 0$ such that $\mathbb{P}_z(\varsigma_{r_0} < \varrho_{r_1}) > \varepsilon$ for any $z \in \mathcal{D} \cap [r_0, r_1 - 2\delta] \times \mathbb{R}^d$.

Proof of Claim 2. For any $z = (x, y) \in \mathcal{D} \cap [r_0, r_1 - 2\delta] \times \mathbb{R}^d$, Itô's formula (8) implies that $X_{t\wedge\varrho_{r_1-\delta}} = x + M_{t\wedge\varrho_{r_1-\delta}} + \int_0^{t\wedge\varrho_{r_1-\delta}} \langle e_x, \phi(Z_s)\rangle \mathrm{d}L_s$. Moreover, by (9) and Assumption (C1), M is a local martingale. By (V1) and Lemma 6.6 there exist constants $C_1, C_2 \in (1, \infty)$, such that

$$\mathbb{E}_z[X_{t\wedge\varrho_{r_1-\delta}}] \le x + C_1 t + C_2 t^{1/2} \le x + (C_1 + C_2)t^{1/2} \le r_1 - 3\delta/2 \quad \text{for all } t \in [0, t_0], \tag{54}$$

where $t_0 := \delta^2/(4(C_1 + C_2)^2)$, since $x \in [r_0, r_1 - 2\delta]$. Thus, $\varepsilon_1 := \inf\{\mathbb{P}_z(\varrho_{r_1-\delta} > t_0) : z \in \mathcal{D} \cap [r_0, r_1 - 2\delta] \times \mathbb{R}^d\} > 0$ and hence $\mathbb{E}_z[\varrho_{r_1-\delta}] > \varepsilon_1 t_0 > 0$. Otherwise, there would exist $\varepsilon > 0$ and $z \in \mathcal{D} \cap [r_0, r_1 - 2\delta] \times \mathbb{R}^d$ such that $\mathbb{E}_z[X_{t\wedge\varrho_{r_1-\delta}}] \ge (r_1 - \delta)(1 - \varepsilon) > r_1 - 3\delta/2$, contradicting the inequality in (54).

By (D2), (V2), (C2) and [29, Lem. 4.5], for some $\delta_\Sigma > 0$ and all sufficiently small $h > 0$, we have $\mathbb{E}_z[v_h \wedge \varrho_{r_1-\delta}] \le 2h^2/\delta_\Sigma$ for all $z \in \mathcal{D}$. Thus, for all $h \in (0, (\delta_\Sigma \varepsilon_1 t_0)^{1/2}/2)$ and $z \in \mathcal{D} \cap [0, r_1 - 2\delta] \times \mathbb{R}^d$, the strict inequality $\mathbb{E}_z[v_h \wedge \varrho_{r_1-\delta}] \le t_0\varepsilon_1/2 < t_0\varepsilon_1 \le \mathbb{E}_z[\varrho_{r_1-\delta}]$ implies $\mathbb{P}_z(v_h < \varrho_{r_1-\delta}) > \varepsilon_2$, for some $\varepsilon_2 > 0$. Moreover, by Proposition 6.8(b) there exists $h_0 \in (0, \infty)$ such that the following holds: for any $h \in (0, h_0)$ there exists $\varepsilon_3 > 0$, such that for all $z \in \mathcal{D} \setminus \mathcal{D}_h \cap [r_1, r_0 - \delta)$, we have $\mathbb{P}_z(\varsigma_{r_0} < \varrho_{r_1}) > \varepsilon_3$. Pick $h = \min\{h_0/2, (\delta_\Sigma \varepsilon_1)^{1/2}/2\}$. The strong Markov property at time v_h yields the claim:

$$\mathbb{P}_z(\varsigma_{r_0} < \varrho_{r_1}) \ge \mathbb{P}_z(\varsigma_{r_0} < \varrho_{r_1}, v_h < \varrho_{r_1-\delta}) \ge \mathbb{E}_z[\mathbb{1}\{v_h < \varrho_{r_1-\delta}\} \cdot \mathbb{P}_{Z_{v_h}}(\varsigma_{r_0} < \varrho_{r_1})]$$

$$\geq \mathbb{E}_z[\varepsilon_3 \mathbb{1}\{v_h < \varrho_{r_1-\delta}\}] \geq \varepsilon_2\varepsilon_3 \quad \text{for all } z \in \mathcal{D} \cap [0, r_1 - 2\delta] \times \mathbb{R}^d.$$

Proof of the upper bounds in (b). Pick $p \in (0, m_c)$. Proposition 5.1 implies the following: there exists $x_0 \in (0,\infty)$ such that for every $x_1 \in [x_0, \infty)$ and $z \in \mathcal{D}$ we have $\mathbb{E}_z[\varsigma_{x_1}^p] < \infty$. We now extend this result to all $x_1 \in (0,\infty)$. Thus for any $q \in (p, m_c)$, there exist $x_0, \widetilde{C} \in (0,\infty)$, such that $\mathbb{E}_z[\varsigma_{x_0}^q] \leq \widetilde{C}$ for all $z \in \mathcal{D} \cap \{x_0 + 1\} \times \mathbb{R}^d$. Pick $x_1 \in (0, x_0)$ and $z \in \mathcal{D}$. In order to prove $\mathbb{E}_z[\varsigma_{x_1}^p] < \infty$ for all $x_1 \in (0, x_0)$, we introduce the sequence of stopping times $T_0 := 0$,

$$S_k := \inf\{t > T_{k-1} : X_t \leq x_0\} \quad \text{and} \quad T_k := \{t > S_k : X_t \notin (x_1, x_0 + 1)\}, \quad \text{for } k \in \mathbb{N}.$$

There exist constants $\varepsilon, C_1, C_2 \in (0,\infty)$ such that, for every $k \in \mathbb{N}$, we have: $\mathbb{P}_z(Z_{T_k} \leq x_1) > \varepsilon$ (resp. $\mathbb{E}_z[(S_k - T_{k-1})^q] \leq C_1$, $\mathbb{E}_z[(T_k - S_k)^q] \leq C_2$) by Prop. 6.8(b) (resp. Prop. 5.1, Prop. 6.8(c)) and the strong Markov property at S_k (resp. T_{k-1}, S_k).

Since $T_k = \sum_{j=1}^k (S_j - T_{j-1} + T_j - S_j)$, we have $\mathbb{P}_z(T_k < \infty) = 1$, for all $k \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}$, we have $\{Z_{T_k} \leq x_1\} = \{T_k = \varsigma_{x_1}\}$, $\mathbb{P}_z$-a.s. (recall that on the event $\{Z_{T_k} \leq x_1\}$, we have $T_n = T_k = \varsigma_{x_1}$ for all $n \geq k$). Hence, by the strong Markov property at S_j, for $j \in \{1, \ldots, k\}$, it follows that $\mathbb{P}_z(\varsigma_{x_1} > T_k) = \mathbb{P}_z(\cap_{j=1}^k Z_{T_j} > x_1) \leq (1-\varepsilon)^k$. We thus conclude $\mathbb{P}_z(\varsigma_{x'} < \infty) = 1$ for all $x' \in (0, x_0)$.

Define the indicator $\mathbb{1}(A_k) := \mathbb{1}\{Z_{T_k} \leq x_1\} \prod_{j=1}^{k-1} \mathbb{1}\{Z_{T_j} > x_1\}$. Note that the events A_k, $k \in \mathbb{N}$, are pairwise disjoint, $\mathbb{P}_z(A_k) \leq (1-\varepsilon)^{k-1}$ and, since $\mathbb{P}_z(\varsigma_{x_1} < \infty) = 1$, the following equality holds

$$\varsigma_{x_1} = \sum_{k=1}^{\infty} \mathbb{1}(A_k) T_k \quad \mathbb{P}_z\text{-a.s.}$$

Thus, by Hölder's inequality with exponents p/q and $1 - p/q$, we obtain

$$\begin{aligned}
\mathbb{E}_z[\varsigma_{x_1}^p] &= \mathbb{E}_z\left[\sum_{k=1}^{\infty} \mathbb{1}(A_k) T_k^p\right] = \sum_{k=1}^{\infty} \mathbb{E}_z\left[\mathbb{1}(A_k)\left(\sum_{j=1}^{k}(T_j - S_j) + \sum_{j=1}^{k}(S_j - T_{j-1})\right)^p\right] \\
&\leq \sum_{k=1}^{\infty} (2k)^p\, \mathbb{E}_z\left[\mathbb{1}(A_k) \max_{j\in\{1,\ldots,k\}} \{(T_j - S_j)^p, (S_j - T_{j-1})^p\}\right] \\
&\leq \sum_{k=1}^{\infty} (2k)^p \mathbb{P}_z(A_k)^{1-p/q}\, \mathbb{E}\left[\max_{j\in\{1,\ldots,k\}} \{(T_j - S_j)^q, (S_j - T_{j-1})^q\}\right]^{p/q} \\
&\leq \sum_{k=1}^{\infty} (2k)^p (1-\varepsilon)^{(k-1)(1-p/q)}\, \mathbb{E}_z\left[\sum_{j=1}^{k} ((T_j - S_j)^q + (S_j - T_{j-1})^q)\right]^{p/q} \\
&\leq \sum_{k=1}^{\infty} (2k)^{p+1} (1-\varepsilon)^{(k-1)(1-p/q)} (C_1 + C_2)^{p/q} < \infty,
\end{aligned}$$

where the first equality in the display above holds since A_k are pairwise disjoint. We conclude that for every $x_1 \in (0,\infty)$, all $p \in (0, m_c)$ and every $z \in \mathcal{D}$ we have $\mathbb{E}_z[\varsigma_{x_1}^p] < \infty$. Since Markov's inequality implies $\mathbb{P}_z(\varsigma_{x_1} > t) \leq \mathbb{E}_z[\varsigma_{x_1}^p]/t^p < \infty$ for all $t \in (0,\infty)$, we have $\mathbb{P}_z(\varsigma_{x_1} > t) \leq Ct^{-p}$ for all $t \in [1,\infty)$ and $z \in \mathcal{D}$ with $C := \mathbb{E}_z[\varsigma_{x_1}^p]$. □

6.3. *Application to petite sets and Harris recurrence.* The family of probability measures $\mathbb{P}_z(Z_t \in \cdot)$ on $\mathcal{B}(\mathcal{D})$, indexed by $t \in \mathbb{R}_+$ and $z \in \mathcal{D}$ constitutes a Markov transition kernel by [29, Thm A.1] and Theorem 6.5. A non-empty set $B \in \mathcal{B}(\mathcal{D})$ is called *petite* if there exists a probability measure a on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, which does not charge zero (i.e. $a(\{0\}) = 0$), and a non-trivial measure φ_a on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ satisfying

$$K_a(z, \cdot) \geq \varphi_a(\cdot) \qquad \text{for all } z \in B, \tag{55}$$

where the Markov transition function $K_a : \mathcal{D} \times \mathcal{B}(\mathcal{D}) \to \mathbb{R}_+$ is given by

$$K_a(z, \cdot) := \int_{\mathbb{R}_+} \mathbb{P}_z(Z_t \in \cdot) a(\mathrm{d}t). \tag{56}$$

The measurability of $z \mapsto K_a(z, A)$ for any $A \in \mathcal{B}(\mathcal{D})$ follows from [34, Ch. III. Prop. 1.6].

Let φ be a σ-finite measure on the Borel σ-algebra $\mathcal{B}(\mathcal{D})$. The process Z is *Harris recurrent* if $\varphi(A) > 0$ implies $\int_0^\infty \mathbb{1}_A(Z_s)\mathrm{d}s = \infty$, $\mathbb{P}_z$-a.s., for all $A \in \mathcal{B}(\mathcal{D})$ and $z \in \mathcal{D}$.

PROPOSITION 6.9. *Suppose that* (C2), (V2) *and* (D2) *hold. Then all compact subsets of $\mathcal{D}$ are petite. Moreover, if $\beta < \beta_c$, the process Z is Harris recurrent.*

PROOF. Fix $z_0 \in \mathcal{D} \setminus \partial\mathcal{D} = \mathrm{Int}(\mathcal{D})$ and $h_0 > 0$, such that $B(z_0, 2h_0) = \{z \in \mathcal{D} : \|z - z_0\|_{d+1} < 2h_0\} \subset \mathrm{Int}(\mathcal{D})$. We now prove that the ball $B(z_0, h_0)$ is petite. Recall that the stopping time $\tau_{\partial B(z_0,2h_0)} := \inf\{t \in \mathbb{R}_+ : Z_t \notin B(z_0, 2h_0)\}$ is strictly positive $\mathbb{P}_z$-a.s. for all $z \in B(z_0, h_0)$. Moreover, the process Z on the stochastic interval $[0, \tau_{\partial B(z_0,2h_0)})$, started at any $z \in B(z_0, h_0)$, coincides with a uniformly elliptic diffusion on $\mathbb{R}^{d+1}$, stopped upon exiting the ball $B(z_0, 2h_0)$. Thus, by [37, Thm II.1.3], we have $\inf_{z \in B(z_0,h_0)} \mathbb{P}_z(Z_{h_0^2} \in \cdot) \geq \varphi(\cdot)$, where φ is the Lebesgue measure multiplied by a positive scalar and supported in $B(z_0, h_0)$. Hence, for any $z \in B(z_0, h_0)$, condition (55) holds with $a_1(\mathrm{d}t) = \delta_{h_0^2}(\mathrm{d}t)$, where $\delta_{h_0^2}$ is the Dirac delta concentrated at $h_0^2 > 0$, and taking the non-trivial measure $\varphi_{a_1} := \varphi$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$.

We now prove that an arbitrary compact set D in $\mathcal{D}$ is also petite. Since $B(z_0, h_0)$ is an open set, Theorem 6.5 and [2, Thm 2.1] imply that the function $z \mapsto \mathbb{P}_z(Z_1 \in B(z_0, h_0))$ is lower semi-continuous on $\mathcal{D}$. Moreover, by Proposition 6.1, we have $\mathbb{P}_z(Z_1 \in B(z_0, h_0)) > 0$ for all $z \in D$. Hence, by compactness of D and the lower semi-continuity of the function $z \mapsto \mathbb{P}_z(Z_1 \in B(z_0, h_0))$, we get $\inf_{z \in D} \mathbb{P}_z(Z_1 \in B(z_0, h_0)) > 0$. For any $z \in D$, the Markov property of Z implies

$$\begin{aligned}\mathbb{P}_z(Z_{1+h_0^2} \in \cdot) &\geq \mathbb{P}_z(Z_1 \in B(z_0, h_0)) \inf_{z' \in B(z_0,h_0)} \mathbb{P}_{z'}(Z_{h_0^2} \in \cdot) \\ &\geq \inf_{z'' \in D} \mathbb{P}_{z''}(Z_1 \in B(z_0, h_0)) \varphi_{a_1}(\cdot).\end{aligned}$$

Thus the set D satisfies condition (55) with the probability measure $a_2(\mathrm{d}t) = \delta_{1+h_0^2}(\mathrm{d}t)$ and non-trivial measure $\varphi_{a_2} := \inf_{z \in D} \mathbb{P}_z(Z_1 \in B(z_0, h_0)) \varphi_{a_1}$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$, making D petite.

In particular, the set $\mathcal{D} \cap [r_0, \infty) \times \mathbb{R}^d$ is a petite set for every $r_0 \in (0, \infty)$. Moreover, if $\beta < \beta_c$, Proposition 5.1 implies that, for all sufficiently large $r_0 \in (0, \infty)$, we have $\mathbb{P}_z(\varsigma_{r_0} < \infty) = 1$ for all $z \in \mathcal{D}$. We conclude that the process Z is Harris recurrent by [30, Thm. 1.1]. $\square$

The following proposition is crucial for establishing the lower bounds on the tail of the invariant distribution of the reflected process Z in Theorem 1.5 (the proof of Theorem 1.5 requires an estimate of the return times to an arbitrary petite set).

PROPOSITION 6.10. *Suppose that* (C2), (V2) *and* (D2) *hold with* $\beta < \beta_c$. *Then every petite set is bounded.*

PROOF. Let B be an arbitrary petite set and let the probability measure a on $\mathbb{R}_+$ and a non-zero measure φ_a on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ be such that (55) holds, with K_a as in (56). Denote $\mathcal{D}^{(r)} = \mathcal{D} \cap [0, r] \times \mathbb{R}^d$ for any $r > 0$. Since φ_a is a non-trivial measure on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$, there exists $r_0 \in (0, \infty)$, such that $c := \varphi_a(\mathcal{D}^{(r_0)}) > 0$.

Proposition 5.2 implies that for every $q \in (0, 1)$ and $p \in (1 - \beta/\beta_c, \infty)$, there exist $r_1 \in (0, \infty)$, $c_1, c_2 \in (1, \infty)$ and $\varepsilon \in (0, \infty)$, such that for every $r \in (r_1, \infty)$ and $z = (x, y) \in \mathcal{D}$ we have

$$\mathbb{P}_z(\varsigma_r \geq t_0) \geq q \min\{(c_1^{-p}(x - c_2)^p - r^p)(1 - q)^p \varepsilon^{-1/2} t_0^{-p/2}, 1\} \tag{57}$$

for all $t_0 \in (c_1 r_1 + c_2, \infty)$ (recall that the return time ς_{r_0} is defined in (5)). Since a is a probability measure, there exists $t_0 \in (c_1 r_1 + c_2, \infty)$ satisfying $a([t_0, \infty)) < c/2$.

We now show that there exists $x_0 \in (0, \infty)$, such that $\mathbb{P}_z(\varsigma_{r_0} < t_0) < c/2$ for all $z \in \mathcal{D} \setminus \mathcal{D}^{(x_0)}$. Indeed, fix $r \geq \max\{r_0, r_1\}$ (note $\varsigma_r \leq \varsigma_{r_0}$), $q \in (1 - c/2, 1)$ and $p \in (1 - \beta/\beta_c, \infty)$. Pick $x_0 > 0$, such that $(c_1^{-p}(x - c_2)^p - r^p)(1 - q)^p \varepsilon^{-1/2} t_0^{-p/2} \geq 1$ for all $x \in (x_0, \infty)$. Note that this choice of x_0 implies $x_0 > r$ and, in particular, $x_0 \in (r_0, \infty)$. For any $z = (x, y) \in \mathcal{D} \setminus \mathcal{D}^{(x_0)}$, the inequality in (57) implies $\mathbb{P}_z(\varsigma_{r_0} < t_0) \leq \mathbb{P}_z(\varsigma_r < t_0) < 1 - q < c/2$.

By (56), we have $K_a(z, \mathcal{D}^{(r_0)}) \leq \int_0^{t_0} \mathbb{P}_z(Z_t \in \mathcal{D}^{(r_0)}) a(\mathrm{d}t) + a([t_0, \infty))$. Since, for all $z \in \mathcal{D} \setminus \mathcal{D}^{(x_0)}$, we have $\mathbb{P}_z(Z_t \in \mathcal{D}^{(r_0)}) \leq \mathbb{P}_z(\varsigma_{r_0} < t_0) < c/2$ for all $t \in [0, t_0]$, the inequality $\varphi_a(\mathcal{D}^{(r_0)}) = c > K_a(z, \mathcal{D}^{(r_0)})$ holds for all $z \in \mathcal{D} \setminus \mathcal{D}^{(x_0)}$. Since the petite set B satisfies (55), we must have $B \subset \mathcal{D}^{(x_0)}$, making B bounded. □

7. Stability: the proof of Theorem 1.5. Existence and uniqueness of the invariant distribution of Z, the upper bounds on the tails of the invariant distribution and upper bounds on the rate of convergence of Z to stationarity will be established using the drift condition (i.e. supermartingale property) given in Lemma 5.4 (see Section 5) and the fact that every compact set in $\mathcal{D}$ is petite for the reflected process Z (see Proposition 6.9 in Section 6.3 above). The lower bounds on the tails of the invariant distribution and the rate of convergence to stationarity will follow from the fact that every petite set of Z is bounded (see Proposition 6.10 above) and the control we have established on the return time and length of excursions away from bounded sets (see Proposition 5.2). Theorem 1.5 follows easily from Propositions 7.1 and 7.4 proved in this section.

7.1. *Existence, uniqueness, and upper bounds.* The upper bounds on the tails of the invariant distribution are obtained by establishing finiteness of certain moments and applying the Markov inequality. In Section 7.2 below we show that these bounds cannot be improved.

PROPOSITION 7.1. *Suppose that* (C2), (V2) *and* (D2) *hold with* $\beta < -\beta_c$ *and recall* $M_c = -(1+\beta/\beta_c)/2 > 0$. *Then there exists the unique invariant distribution* π *on* $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ *for the process* Z. *Pick* $\varepsilon > 0$. *There exists a constant* $C_\pi \in (0,\infty)$, *such that*

$$\pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \leq C_\pi r^{-2M_c+\varepsilon} \qquad \textit{for all } r \in [1,\infty).$$

Furthermore, for every $z \in \mathcal{D}$ *there exists a constant* $C_{\mathrm{TV}} \in (0,\infty)$, *such that*

$$\|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{\mathrm{TV}} \leq C_{\mathrm{TV}} t^{-M_c+\varepsilon} \quad \textit{for all } t \in [1,\infty).$$

PROOF. By Proposition 6.1, the process Z admits an irreducible skeleton chain. Moreover, by Proposition 6.9 the sets $\mathcal{D}^{(r)} = \mathcal{D} \cap [0,r] \times \mathbb{R}^d$, defined in (44), are petite for every $r > 0$. Pick arbitrary $\varepsilon \in (0, 1 - \beta/\beta_c - 2)$ and note that $\gamma := 1 - \beta/\beta_c - \varepsilon > 2$. By Lemma 5.4, the process ξ, defined in (45), is a supermartingale. Note that by definition of $F_{w,\gamma}$ in (17), we have $F_{w,\gamma}(z)^{(\gamma-2)/\gamma} = F_{w,\gamma-2}(z)$ for $z \in \mathcal{D} \cap [x_1,\infty) \times \mathbb{R}^d$, where x_1 is the constant appearing in the definition of the function m in (17). We may thus apply [9, Prop. 3.1] (with $V = F_{w,\gamma}$, $\phi(u) = C_1 u^{(\gamma-2)/\gamma}$, $b = C_2$ and the petite set $\mathcal{D}^{(x_2)}$ from Lemma 5.4), to deduce the existence and uniqueness of the invariant distribution π of Z and $\int_{\mathcal{D}} F_{w,\gamma-2}(z)\pi(\mathrm{d}z) < \infty$. From the definition of $F_{w,\gamma}(z)$ in (17) and the lower bound in (12) it follows that $\widetilde{C}_\pi := \int_{z=(x,y)\in\mathcal{D}} x^{2M_c-\varepsilon}\pi(\mathrm{d}z) < \infty$. Moreover, Markov's inequality implies

$$\pi(\mathcal{D} \cap [r,\infty) \times \mathbb{R}^d) \leq \int_{\mathcal{D}} (x/r)^{2M_c-\varepsilon}\pi(\mathrm{d}z) \leq \widetilde{C}_\pi r^{-2M_c+\varepsilon} \qquad \text{for all } r \in [1,\infty).$$

Recall that for any $z = (x,y) \in \mathcal{D}$ we have $x \leq \|z\|_{d+1} \leq (x^2 + b(x)^2)^{1/2}$ and the boundary function b has sublinear growth (cf. Remark 1.2), implying $(x^2 + b(x)^2)^{1/2}/x \to 1$ as $x \to \infty$. Thus, the upper bound $\pi(\mathcal{D} \cap [r,\infty) \times \mathbb{R}^d) \leq \widetilde{C}_\pi r^{-2M_c+\varepsilon}$ implies the existence of the constant $C_\pi \in (0,\infty)$ such that the bound on the tail $\pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \leq C_\pi r^{-2M_c+\varepsilon}$ holds for all $r \in [1,\infty)$ as claimed in the proposition.

Recall that $\mathfrak{m}_{d+1}$ is a Lebesgue measure on $\mathbb{R}^{d+1}$. The process Z admits an $\mathfrak{m}_{d+1}$-*irreducible skeleton* chain, since for every $A \in \mathcal{B}(\mathcal{D})$, such that $\mathfrak{m}_{d+1}(A) > 0$, and $z \in \mathcal{D}$, by Proposition 6.1, we have $\mathbb{P}_z(Z_k \in A) > 0$ for every $k \in \mathbb{N}$. In particular, Assumption (i) in [9, Thm 3.2] is satisfied for Z. For $\gamma = 1 - \beta/\beta_c - \varepsilon$, Lemma 5.4 ensures that Assumption (ii) of [9, Thm 3.2] is satisfied. By [9, Thm 3.2, Eq. (3.5)], with the pair of functions $\Psi = (\mathrm{Id}, \mathbf{1})$, where $\mathrm{Id}, \mathbf{1} : \mathbb{R}_+ \to \mathbb{R}_+$ denote the identity and the constant functions, respectively, we obtain

$$r_*(t)\|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{\mathrm{TV}} \leq F_{w,\gamma}(z) \quad \text{for all } t \geq 0,$$

where $r_*(t) = \varphi \circ H_\varphi^{-1}(t)$. Here, the function φ is positive, satisfying $\varphi(u) = \widetilde{C}_1 u^{(\gamma-2)/\gamma}$ for $u \geq 1$, and H_φ^{-1} is the inverse of the increasing function H_φ, satisfying $H_\varphi(u) = \int_1^u \varphi(s)^{-1}\mathrm{d}s$ for $u \geq 1$. This implies $r_*(t) = \widetilde{C}_2(t+1)^{\gamma/2-1}$ for a positive constant $\widetilde{C}_2 \in (0,\infty)$ and $t \in (0,\infty)$. □

7.2. *Lower bounds.* The lower bounds on the tails of the invariant distribution π of the reflected process Z are closely related to the tail behaviour of certain additive functionals of

the paths of Z until the return time to a petite set. For a measurable subset $D \subset \mathcal{D}$ and $\delta > 0$, define the return time of the process Z to the set D after the time $\delta > 0$ by $\tau_\delta(D) := \inf\{t > \delta : Z_t \in D\}$ (with convention $\inf \emptyset = \infty$).

PROPOSITION 7.2. *Suppose that* (D2), (C2), (V2) *hold with $\beta < \beta_c$ and pick $p \in (1 - \beta/\beta_c, \infty)$, a bounded measurable set $D \subset \mathcal{D}$ and $z \in \mathcal{D}$. Then there exist constants $\delta \in (0, \infty)$ and $C, r_0, c_1, c_2, \varepsilon \in (0, \infty)$, such that for every non-decreasing continuous function $H : \mathbb{R}_+ \to \mathbb{R}_+$ with $r_H := \inf\{r' \geq : H(r') > 0\} < \infty$, we have*

$$\mathbb{P}_z\left(\int_0^{\tau_\delta(D)} H\left(c_1(X_s + c_2)\right) \mathrm{d}s \geq r\right) \geq C/G(r/\varepsilon)^p \quad \textit{for all } r \in (r_0, \infty),$$

where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of the strictly increasing function defined on $[r_H, \infty)$ by the formula $v \mapsto v^2 H(v)$.

REMARK 7.3. (a) The assumption $\beta < \beta_c$ in Proposition 7.2 covers both the null-recurrent and positive-recurrent cases. However, the main application of Proposition 7.2 in the proof of Lemma 7.5 below requires only the positive-recurrent case. Since Lemma 7.5 is crucial in the proof of Proposition 7.4, the bound in Proposition 7.2 is key for the lower bounds in Theorem 1.5.
(b) The statement of Proposition 7.2 in fact holds for every $\delta > 0$. In the proof of Proposition 7.2 below, we apply the non-confinement property of Z, given in Lemma 4.3, to conclude that $\delta > 0$ exists. However, since by Proposition 6.1 Z is irreducible, the event $\{X_\delta > c_1(d_0 + 1) + c_2\}$ has positive probability for every $\delta > 0$. Since the existence of $\delta > 0$ is sufficient for our analysis of the lower bound on the tail of the invariant distribution and non-confinement is weaker (and easier to prove) than irreducibility, we use the formulation of Proposition 7.2 above.
(c) The assumed continuity of the function H in Proposition 7.2 is not necessary: measurability would be sufficient but it would complicate the formulation of the proposition.

PROOF OF PROPOSITION 7.2. Pick a non-decreasing continuous function $H : \mathbb{R}_+ \to \mathbb{R}_+$ with $r_H = \inf\{r' : H(r') > 0\} < \infty$. Proposition 5.2 implies that for every $p \in (1 - \beta/\beta_c, \infty)$ and $q \in (0, 1)$, there exist constants $x_0 \in (0, \infty)$, $c_1, c_2 \in (1, \infty)$ and $\varepsilon(q) \in (0, \infty)$, such that for every $x_1 \in (x_0, \infty)$ and $z = (x, y) \in \mathcal{D} \cap (c_1 x_1 + c_2, \infty) \times \mathbb{R}^d$ and function $\bar{H}(r) := H(c_1(r + c_2))$, $r \in \mathbb{R}_+$, we have

$$\mathbb{P}_z\left(\int_0^{\varsigma_{x_1}} \bar{H}\left(X_s\right) \mathrm{d}s \geq \varepsilon v^2 H(v)\right) \geq q \min\{(c_1^{-p}(x - c_2)^p - x_1^p)(1 - q)^p v^{-p}, 1\}, \tag{58}$$

for all $v \in (c_1 x_1 + c_2, \infty)$. Recall here that ς_{x_1}, defined in (5) above, is the return time of the first coordinate X (of Z) below the level x_1 and $X_0 = x$ $\mathbb{P}_z$-a.s. Note that the following elementary inequality holds since the function is monotonically increasing as $p > 0$:

$$c_1^{-p}(x - c_2)^p - x_1^p \geq (x_1 + 1)^p - x_1^p \quad \text{for all } x \in (c_1(x_1 + 1) + c_2, \infty).$$

Since $v \mapsto v^2 H(v)$ is strictly increasing on $[r_H, \infty)$ with range equal to $\mathbb{R}_+$, for any $r \in (c_1 x_1 + c_2, \infty)$ we can define $v := G(r/\varepsilon)$, implying $r = \varepsilon v^2 H(v)$. Thus, for every $z =$

$(x,y) \in \mathcal{D} \cap (c_1(x_1+1)+c_2, \infty) \times \mathbb{R}^d$ and any $C_{x_1} \in (0, q((x_1+1)^p - x_1^p)(1-q)^p)$ we have

$$\mathbb{P}_z\left(\int_0^{\varsigma_{x_1}} \bar{H}(X_s)\,\mathrm{d}s \geq r\right) \geq q\min\{((x_1+1)^p - x_1^p)(1-q)^p/G(r/\varepsilon)^p, 1\} \geq C_{x_1}/G(r/\varepsilon)^p \quad \text{for all } r \in (c_1x_1+c_2, \infty), \tag{59}$$

where the first inequality in (59) follows from (58) above.

Fix $z \in \mathcal{D}$ and a bounded set $D \subset \mathcal{D}$. Let $m_D := \sup\{x : (x,y) \in D\} \in \mathbb{R}_+$ satisfy $D \subset [0, m_D] \times \mathbb{R}^d \cap \mathcal{D}$ (note that $m_D < \infty$ by assumption on D) and denote $d_0 := \max\{m_D, x_1\}$. Since, by Lemma 4.3, X is not confined to any compact set, there exists $\delta > 0$, such that $\mathbb{P}_z(X_\delta > c_1(d_0+1)+c_2) > 0$. Note that, on the event $\{X_\delta > c_1(d_0+1)+c_2\}$, $\mathbb{P}_z$-a.s. we have $Z_\delta \in \mathcal{D} \cap (c_1(x_1+1)+c_2, \infty) \times \mathbb{R}^d$. The Markov property of Z and (59) thus imply that for every $p \in (1-\beta/\beta_c, \infty)$ and $r_0 := c_1d_0 + c_2$, there exists a constant $C_{d_0} \in (0, q((d_0+1)^p - d_0^p))$, such that

$$\begin{aligned}
\mathbb{P}_z\left(\int_0^{\tau_\delta(D)} \bar{H}(X_s)\,\mathrm{d}s \geq r\right) &\geq \mathbb{P}_z\left(\int_0^{\tau_\delta(D)} \bar{H}(X_s)\,\mathrm{d}s \geq r, X_\delta > r_0 + c_1\right) \\
&\geq \mathbb{E}_z\left[\mathbb{1}\{X_\delta > r_0+c_1\} \cdot \mathbb{P}_{Z_\delta}\left(\int_0^{\tau_\delta(D)} \bar{H}(X_s)\,\mathrm{d}s \geq r\right)\right] \\
&\geq \mathbb{E}_z\left[\mathbb{1}\{X_\delta > r_0+c_1\} \cdot \mathbb{P}_{Z_\delta}\left(\int_0^{\varsigma_{d_0}} \bar{H}(X_s)\,\mathrm{d}s \geq r\right)\right] \\
&\geq \mathbb{P}_z(X_\delta > r_0+c_1) C/G(r/\varepsilon)^p \text{ for all } r \in (r_0, \infty). \quad \square
\end{aligned}$$

We can now establish the lower bounds on the tail of the the invariant distribution of Z.

PROPOSITION 7.4. *Suppose that* (D2), (C2), (V2) *hold with $\beta < -\beta_c$ and recall $M_c = -(1+\beta/\beta_c)/2$. Let π be the invariant distribution of the process Z and pick $\varepsilon > 0$. There exists a constant $c_\pi \in (0,\infty)$ such that*

$$c_\pi r^{-2M_c-\varepsilon} \leq \pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \qquad \textit{for all } r \in [1,\infty).$$

Furthermore, for any $z \in \mathcal{D}$, there exists a constant $c_{\mathrm{TV}} \in (0,\infty)$, such that

$$c_{\mathrm{TV}} t^{-M_c-\varepsilon} \leq \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{\mathrm{TV}} \qquad \textit{for all } t \in [1,\infty).$$

The key step in the proof of Proposition 7.4 is the following lemma.

LEMMA 7.5. *Suppose that* (D2), (C2), (V2) *hold with $\beta < -\beta_c$. For any $\varepsilon > 0$ there exists a constant $c_\pi \in (0,\infty)$, such that $c_\pi r^{1+\beta/\beta_c-\varepsilon} \leq \pi(\mathcal{D} \cap [r,\infty) \times \mathbb{R}^d)$ for all $r \in [1,\infty)$.*

PROOF OF LEMMA 7.5. The reflected process Z is *positive Harris recurrent*, i.e., Z is Harris recurrent (by Proposition 6.9) and admits an invariant distribution (by Proposition 7.1). Thus, by [30, Theorem 1.2(b)], a measurable function $f : \mathcal{D} \to [1,\infty)$ satisfies $\int_{z\in\mathcal{D}} f(z)\pi(\mathrm{d}z) < \infty$ if and only if $\sup_{z\in D} \mathbb{E}_z[\int_0^{\tau_\delta(D)} f(Z_s)\mathrm{d}s] < \infty$ for some closed petite

set $D \subset \mathcal{D}$ and all $\delta > 0$, where $\tau_\delta(D) = \inf\{t > \delta : Z_t \in D\}$ is the return time to the set D after time δ (defined before the statement of Proposition 7.2 above). Since by Proposition 6.10 all petite sets are bounded, Proposition 7.2 implies that for any closed petite set D of Z there exists $\delta > 0$ such that for any non-decreasing continuous function $\widetilde{H} : \mathbb{R}_+ \to [1, \infty)$ we have $\mathbb{E}_z[\int_0^{\tau_\delta(D)} \widetilde{H}(c_1(X_s + c_2))\mathrm{d}s] \geq C \int_{r_0}^\infty G(r/\varepsilon)^{-p}\mathrm{d}r$ for all $z \in \mathcal{D}$ and $p \in (1 - \beta/\beta_c, \infty)$, where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of the function $v \mapsto v^2\widetilde{H}(v)$ and the positive constants $\varepsilon, r_0, C, c_1, c_2$ are as in Proposition 7.2. By the criterion in [30, Theorem 1.2(b)] stated above we thus obtain that

$$\int_{r_0}^\infty G(r/\varepsilon)^{-p}\mathrm{d}r = \infty \quad \text{implies} \quad \int_{z=(x,y)\in\mathcal{D}} \widetilde{H}(c_1(x + c_2))\pi(\mathrm{d}z) = \infty. \tag{60}$$

The proof of Lemma 7.5 proceeds by contradiction. Note that the statement in the lemma is equivalent to the following: for every $\varepsilon > 0$ there exists $r_0 \in (0, \infty)$ such that

$$r^{1+\beta/\beta_c-\varepsilon} \leq \pi(\mathcal{D} \cap [r, \infty) \times \mathbb{R}^d) \qquad \text{for all } r \in [r_0, \infty).$$

Assume to the contrary that there exists $\varepsilon > 0$, such that *for every* $r_0 > 0$ there exists $r_1 \in [r_0, \infty)$ satisfying $r_1^{1+\beta/\beta_c-\varepsilon} > \pi(\mathcal{D} \cap [r_1, \infty) \times \mathbb{R}^d)$. We may pick $r_0 > 1$ and $r_1 > 2r_0$. Using a recursive construction, we obtain a sequence $(r_n)_{n\in\mathbb{N}}$, such that $r_{n+1} > 2r_n$ and $r_n^{1+\beta/\beta_c-\varepsilon} > \pi(\mathcal{D} \cap [r_n, \infty) \times \mathbb{R}^d)$ for all $n \in \mathbb{N}$. Using this sequence, we now construct a function H satisfying $\int_{\mathcal{D}} H(x)\pi(\mathrm{d}z) < \infty$ but which violates the implication in (60).

Set $\alpha := -(1 + \beta/\beta_c) + \varepsilon$ and define the function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ by $\mu(x) := 1$ for $x \in [0, r_1)$ and $\mu(x) := r_n^{-\alpha}$ for $x \in [r_n, r_{n+1})$, $n \in \mathbb{N}$. Since the function $x \mapsto \pi(\mathcal{D} \cap [x, \infty) \times \mathbb{R}^d)$ is non-increasing, we have $\pi(\mathcal{D} \cap [x, \infty) \times \mathbb{R}^d) \leq \mu(x)$ for all $x \in \mathbb{R}_+$. Let $H : \mathbb{R}_+ \to [1, \infty)$ be a differentiable function such that $H(x) = 1$ for $x \in [0, r_1)$ and, for $x \in [r_n, r_{n+1})$, we have

$$H'(x) = \begin{cases} r_n^{\alpha-\varepsilon/2}, & x \in [r_n, r_n + 1); \\ 1/(r_{n+1} - r_n), & x \in [r_n + 1, r_{n+1}). \end{cases}$$

Since μ is non-increasing by definition, for $x \in [r_n, r_{n+1})$, we have $\mu(x) \leq r_n^{-\alpha}$ and

$$H'(x)\mu(x) = \begin{cases} r_n^{-\varepsilon/2}, & x \in [r_n, r_n + 1); \\ r_n^{-\alpha}/(r_{n+1} - r_n), & x \in [r_n + 1, r_{n+1}). \end{cases}$$

The inequality $\pi(\mathcal{D} \cap [x, \infty) \times \mathbb{R}^d) \leq \mu(x)$ and Tonelli's theorem (with $H(0) = 1$) imply

$$\begin{aligned} \int_{z=(x,y)\in\mathcal{D}} H(x)\pi(\mathrm{d}z) &\leq 1 + \int_{r_1}^\infty H'(x)\pi(\mathcal{D} \cap [x, \infty) \times \mathbb{R}^d)\mathrm{d}x \leq 1 + \int_{r_1}^\infty H'(x)\mu(x)\mathrm{d}x \\ &= 1 + \sum_{n=1}^\infty \left(\int_{r_n}^{1+r_n} H'(x)\mu(x)\mathrm{d}x + \int_{1+r_n}^{r_{n+1}} H'(x)\mu(x)\mathrm{d}x \right) \\ &\leq 1 + \sum_{n=1}^\infty r_n^{-\varepsilon/2} + \sum_{n=1}^\infty r_n^{-\alpha} < \infty, \end{aligned} \tag{61}$$

where the final inequality follows from $2^{n-1}r_1 \leq r_n$ for every $n \in \mathbb{N}$ and $\alpha > \varepsilon > 0$.

Let $\varepsilon > 0$ and $c_1, c_2 \in (1, \infty)$ whose existence is guaranteed by Proposition 7.2. Let $\widetilde{H} : \mathbb{R}_+ \to [1, \infty)$ be a non-decreasing differentiable function satisfying $\widetilde{H}(x) = H(x/c_1 - c_2)$ for all $x \in (c_1 c_2, \infty)$. Pick $p \in (1 - \beta/\beta_c, 1 - \beta/\beta_c + \varepsilon/2)$ and note that $p > 2$ and $2 - p + \alpha - \varepsilon/2 > 0$. Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be the inverse of the function $u \mapsto u^2 \widetilde{H}(u)$. Introduce the substitution $r = \varepsilon u^2 \widetilde{H}(u)$ into the first integral in (60) to obtain

$$\begin{aligned}
\int_1^\infty G(r/\varepsilon)^{-p} \mathrm{d}r &= \varepsilon \int_{G(1/\varepsilon)}^\infty u^{-p} (2u\widetilde{H}(u) + u^2 \widetilde{H}'(u)) \mathrm{d}u \\
&\geq \varepsilon \sum_{n=n_0}^\infty \int_{c_1(r_n + c_2)}^{c_1(2r_n + c_2)} u^{1-p} \widetilde{H}(u) \mathrm{d}u = \varepsilon \sum_{n=n_0}^\infty H(r_n) \int_{c_1(r_n+c_2)}^{c_1(2r_n+c_2)} u^{1-p} \mathrm{d}u \\
&= \varepsilon/(p-2) \sum_{n=n_0}^\infty r_n^{\alpha - \varepsilon/2} c_1^{2-p} ((r_n - c_2)^{2-p} - (2r_n - c_2)^{2-p}) \\
&\geq \varepsilon/(p-2) \sum_{n=n_0}^\infty r_n^{\alpha-\varepsilon/2} (r_n - c_2)^{2-p} (1 - ((2r_n - c_2)/(r_n - c_2))^{2-p}) \\
&\geq \varepsilon(1 - 2^{2-p})/(p-2) \sum_{n=n_0}^\infty (r_n - c_2)^{2-p+\alpha-\varepsilon/2} = \infty, \qquad (62)
\end{aligned}$$

where $n_0 \in \mathbb{N}$ is sufficiently large so that $c_1(r_n + c_2) > G(1/\varepsilon)$ and $r_n > c_2$ hold for all $n \geq n_0$. The first inequality in the previous display uses the fact that $\widetilde{H}$ is non-decreasing and positive, while the second follows from $c_1 > 1$ and $2 - p < 0$.

Note that $\widetilde{H}(c_1(x + c_2)) = H(x)$ for every $x \in \mathbb{R}_+$. Thus, the implication in (60) and the estimate in (62) yield

$$\infty = \int_{z=(x,y)\in\mathcal{D}} \widetilde{H}(c_1(x + c_2)) \pi(\mathrm{d}z) = \int_{z=(x,y)\in\mathcal{D}} H(x) \pi(\mathrm{d}z).$$

This contradicts (61) and concludes the proof of the lemma. □

PROOF OF PROPOSITION 7.4. Note that $\pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \geq \pi(\mathcal{D} \cap [r, \infty) \times \mathbb{R}^d)$ for every $r \in \mathbb{R}_+$. Thus, the lower bound on the invariant distribution π follows form Lemma 7.5.

Pick arbitrary $\varepsilon' \in (0, 1)$. Lemma 5.4 implies that for and $\gamma := 1 - \beta/\beta_c - \varepsilon'$, there exists $w \in (-\infty, 1 - \beta s_0/c_0)$, some constant $x_0, x_1, d \in \mathbb{R}_+$, defining the function $F_{w,\gamma}$, and a constant $C_3 \in (0, \infty)$ such that $\mathbb{E}_z[F_{w,\gamma}(Z_t)] \leq F_{w,\gamma}(z) + C_3 t$ for all $t \in \mathbb{R}_+$ and $z \in \mathcal{D}$. Since $F_{w,\gamma}(z) = f_{w,\gamma}(z)$ on $z = (x, y) \in \mathcal{D} \cap [x_1, \infty) \times \mathbb{R}^d$, the upper bound in (12) implies that

$$\pi(\{z \in \mathcal{D} : F_{w,\gamma}(z) \geq r\}) \geq \pi(\{z = (x, y) \in \mathcal{D} : x \geq 2^{\gamma/|w|} r^{1/\gamma} - k_w\}) \text{ for all } r \in (x_1, \infty).$$

This inequality and the lower bound on the invariant distribution in Lemma 7.5 imply that there exists a constant $C_4 \in (0, 1)$ such that

$$\pi(\{z \in \mathcal{D} : F_{w,\gamma}(z) \geq r\}) \geq C_4 r^{(1+\beta/\beta_c - \varepsilon')/\gamma} = C_4 r^{(1+\beta/\beta_c-\varepsilon')/(1-\beta/\beta_c-\varepsilon')}$$

for all $r \in (x_1, \infty)$. By further reducing $C_4 > 0$ if necessary, we may assume that the inequality in the last display holds for all $r \in [1, \infty)$.

Define the functions $f : [1, \infty) \to (0, 1]$, $f(a) := C_4 a^{(1+\beta/\beta_c-\varepsilon')/(1-\beta/\beta_c-\varepsilon')}$ and $F(a) := af(a) = C_4 a^{(2-2\varepsilon')/(1-\beta/\beta_c-\varepsilon')}$ and note that F is strictly increasing with $\lim_{a\to\infty} F(a) = \infty$. By Lemma A.1, applied with functions $G(z) = F_{w,\gamma}(z)$ (recall from the previous display that $\pi(\{z \in \mathcal{D} : G(z) \geq r\}) \geq f(r)$ for all $r \in [1, \infty)$) and $g(z, t) := F_{w,\gamma}(z) + C_3 t \geq \mathbb{E}_z[G(Z_t)]$, yields constants $C_2, C_5 \in (0, \infty)$, such that for all $t \in [1, \infty)$ we get

$$C_2 t^{(1+\beta/\beta_c-\varepsilon')/(2-2\varepsilon')} \leq \frac{C_5}{2}(2g(z, t))^{(1+\beta/\beta_c-\varepsilon')/(2-2\varepsilon')} \leq \|\mathbb{P}_z(Z_{t_n} \in \cdot) - \pi\|_{\mathrm{TV}}. \tag{63}$$

Pick arbitrary $\varepsilon \in (0, 1)$ and let $\varepsilon' \in (0, 1)$ be such that $0 > (1 + \beta/\beta_c - \varepsilon')/(2 - 2\varepsilon') > (1 + \beta/\beta_c)/2 - \varepsilon$. Then the bound in the proposition follows from (63). □

7.3. *Concluding remarks.* In the case of the asymptotically oblique reflection in the domain $\mathcal{D}$, the local time L_t either explodes in finite time or is proportional (as $t \to \infty$) to the integral of the boundary function b (which in this case tends to infinity) [29]. In the case of the asymptotically normal reflection considered in this paper, the long-term behaviour of the local time L is much harder to determine. As our assumptions in any compact region of $\mathcal{D}$, given by (D1), (C1), (V1), are non-quantitative (and, in fact, equal to the general existence and uniqueness assumptions in [25]), the limiting behaviour of L_t as $t \to \infty$ appears to be most tractable in the transient case, where the process spends all of its time (from some point on) in the region where the asymptotic assumptions in (D2), (C2), (V2) can be applied. The recurrent case appears to be much harder in this generality.

Heuristic (7) in Section 1.2 above suggest that the expected local time $\mathbb{E}_z[L_t]$ grows as $\int_0^t 1/(1 + b(X_s))\mathrm{d}s$ when $t \to \infty$, implying that $\mathbb{E}_z[L_t] \to \infty$ as $t \to \infty$ in all the cases. Theorem 4.1 suggests that X is diffusive. Thus, in the transient case (i.e. $0 < \beta_c < \beta < 1$ by Theorem 1.3), where X_t is expected approximately to equal to $t^{1/2}$ for large t, the expectation $\mathbb{E}_z[L_t]$ is approximately of the order $t^{1-\beta/2} \approx \int_0^t 1/(1 + s^{\beta/2})\mathrm{d}s$ as $t \to \infty$ (recall that β in (3) is the growth rate of b).

It is feasible that the methods developed in this paper could be applied to find deterministic (law-of-iterated-logarithm type) bounds for X_t of order $t^{1/2}$, which would reveal that the asymptotic behaviour of $\mathbb{E}_z[L_t]$ as $t \to \infty$ is of order $t^{1-\beta/2}$. The lack of quantitative assumptions on any compact set (discussed in the first paragraph of this section), where the process spends most of its time in the recurrent case, makes it difficult to quantify the growth of L_t. This is left as an open problem.

In our proofs, we circumvent the problem of having to quantify the long-term behaviour of local time L by localising the process and/or controlling the sign of the local time term via a suitable choice of the state space transformation. However, unlike in the asymptotically oblique case [29] (where the long-term behaviour of local time can be deduced from the results and the SDE itself), in the asymptotically normal case the results obtained in this paper do not provide directly any information about the growth of local time.

APPENDIX A: A LOWER BOUND ON THE CONVERGENCE TO STATIONARITY OF A MARKOV PROCESS

Fix $m \in \mathbb{N}$ and let $\kappa = (\kappa_t)_{t\in\mathbb{R}_+}$ be a Markov process on an unbounded domain $\mathcal{D}_\kappa$ in $\mathbb{R}^m$ with an invariant distribution π_κ satisfying $\pi_\kappa(\cdot) = \int_{\mathcal{D}_\kappa} \mathbb{P}_u(\kappa_t \in \cdot)\pi_\kappa(\mathrm{d}u)$ for every $t > 0$. Via a suitable Lyapunov function, the following lemma converts a lower bound estimate on the tail of invariant distribution π_κ into a lower bound on the rate of convergence in total variation of the law of κ_t to the invariant distribution π_κ. The elementary proof of Lemma A.1 below is adapted from [14, Thm 5.1 and Cor 5.2]. This lemma is key in the proof of the lower bound on the rate of convergence in total variation stated in Theorem 1.5(b).

LEMMA A.1. *Let $\mathcal{D}_\kappa$ be an an unbounded domain in $\mathbb{R}^m$ and $\kappa = (\kappa_t)_{t\in\mathbb{R}_+}$ a $\mathcal{D}_\kappa$-valued Markov process with invariant distribution π_κ. Assume the function $G : \mathcal{D}_\kappa \to [1,\infty)$ satisfies:*

(a) *there exists $f : [1,\infty) \to (0,1]$, such that the function $F : y \mapsto yf(y)$ is increasing, $\lim_{y\uparrow\infty} F(y) = \infty$ and $\pi_\kappa(\{v \in \mathcal{D}_\kappa : G(v) \geq y\}) \geq f(y)$ for all $y \in [1,\infty)$;*
(b) *there exists $g : \mathcal{D}_\kappa \times \mathbb{R}_+ \to [1,\infty)$, such that for every $u \in \mathcal{D}_\kappa$ the function $t \mapsto g(u,t)$ is continuous and increasing to infinity and $\mathbb{E}_u[G(\kappa_t)] \leq g(u,t)$ for all $t \in \mathbb{R}_+$.*

Then, for any starting point $u \in \mathcal{D}_\kappa$ we have

$$\|\pi_\kappa(\cdot) - \mathbb{P}_u(\kappa_t \in \cdot)\|_{\mathrm{TV}} \geq \frac{1}{2} f\left(F^{-1}(2g(u,t))\right) \qquad \text{for all } t \in \mathbb{R}_+.$$

REMARK A.2. A good choice for the function G in Lemma A.1 has the following properties: the expectation $\mathbb{E}_u[G(\kappa_t)]$ is bounded as a function of the starting point u and time t *and* the function $y \mapsto \pi_\kappa(G^{-1}([y,\infty)))$ satisfies $\lim_{y\to\infty} \pi_\kappa(G^{-1}([y,\infty))) = 0$ and $\lim_{y\to\infty} y\pi_\kappa(G^{-1}([y,\infty))) = \infty$. The proof of Lemma A.1 shows that if the assumption in (a) holds for y sufficiently large, then the conclusion of the lemma is valid for all t sufficiently large.

PROOF OF LEMMA A.1. The definition of the total variation distance (together with assumption (a)) and the Markov inequality (together with assumption (b)) imply that for every $u \in \mathcal{D}_\kappa$ and $t \in \mathbb{R}_+$ the following inequalities hold for all $y \in [1,\infty)$:

$$\|\pi_\kappa(\cdot) - \mathbb{P}_u(\kappa_t \in \cdot)\|_{\mathrm{TV}} \geq \pi_\kappa(\{v \in \mathbb{R}^d : G(v) \geq y\}) - \mathbb{P}_u(G(\kappa_t) \geq y) \geq f(y) - \frac{g(u,t)}{y}.$$

Since $F(y) \to \infty$ (as $y \to \infty$) and $t \mapsto g(u,t)$ is increasing and continuous, for all $y \in [F^{-1}(2g(u,0)),\infty)$ (where F^{-1} is the inverse of the increasing function F defined in (a)), there exists a unique $t \in \mathbb{R}_+$ satisfying $F(y) = 2g(u,t) \in [1,\infty)$. Differently put, for every $t \in \mathbb{R}_+$, there exists $y_t \in [F^{-1}(1),\infty) \subset [1,\infty)$ satisfying $y_t = F^{-1}(2g(u,t))$. Thus, for every $t \in \mathbb{R}_+$, we have $f(y_t) - g(u,t)/y_t = f(F^{-1}(2g(u,t)))/2$. □

APPENDIX B: ASYMPTOTICALLY OSCILLATING DOMAIN

LEMMA B.1. *Let $b : \mathbb{R}_+ \to (0,\infty)$ be a C^2 function with $b(0) = 0$, satisfying*

$$b(x) = \log\log x(1 + (\log\log x)^{-2} + \sin\log\log x) \quad \text{for } x > \exp(\mathrm{e}+1).$$

Then $\limsup_{x\to\infty} b(x) = \infty$ *and* $\liminf_{x\to\infty} b(x) = 0$. *Moreover the function* b *satisfies assumption* (D2) *with* $\beta = 0$, *i.e.* $\lim_{x\to\infty} xb'(x)/b(x) = \beta = 0$, *and* $\lim_{x\to\infty} b'(x) = \lim_{x\to\infty} b''(x) = 0$

PROOF. To show $\liminf_{x\to\infty} b(x) = 0$, consider $\ell_k := \exp(\exp(-\pi/2 + 2k\pi))$, for any $k \in \mathbb{N}$. It follows that

$$\lim_{k\to\infty} b(\ell_k) = \lim_{k\to\infty} (\log\log \ell_k)^{-1} = \lim_{k\to\infty} (-\pi/2 + 2k\pi)^{-1} = 0.$$

Similarly, to show $\limsup_{x\to\infty} b(x) = \infty$, consider $\widetilde{\ell}_k := \exp(\exp(2k\pi))$, for any $k \in \mathbb{N}$. We obtain

$$\lim_{k\to\infty} b(\widetilde{\ell}_k) = \lim_{k\to\infty} (\widetilde{\ell}_k + (\log\log \widetilde{\ell}_k)^{-1}) = \lim_{k\to\infty} 2k\pi + (2k\pi)^{-1} = \infty.$$

The first two derivatives of b on $x > \exp(\mathrm{e} + 1)$ take the form

$$b'(x) = (x\log x)^{-1}(1 - (\log\log x)^{-1} + \sin\log\log x + \cos\log\log x),$$

$$b''(x) = (x\log x)^{-2}((1 + \log x)(1 - (\log\log x)^{-2} + \sin\log\log x + \cos\log\log x) + o(x)),$$

implying that $\lim_{x\to\infty} b'(x) = \lim_{x\to\infty} b''(x) = 0$. For the result about β it is enough to show $\limsup_{x\to\infty} |xb'(x)|/b(x) \le 0$. We estimate

$$\begin{aligned}\limsup_{x\to\infty} \frac{xb'(x)}{b(x)} &= \limsup_{x\to\infty} \frac{(\log x)^{-1}|1 - (\log\log x)^{-1} + \sin\log\log x + \cos\log\log x)}{\log\log x(1 + (\log\log x)^{-2} + \sin\log\log x)} \\ &\le \limsup_{x\to\infty} \frac{\log\log x(3 + (\log\log x)^{-1})}{\log x} = 0,\end{aligned}$$

this concludes the proof. □

Acknowledgments. We thank Krzysztof Burdzy for drawing our attention to the literature on the uniform ergodicity for normally reflected Brownian motion and an anonymous referee for improving significantly the structure of the introduction.

Funding. MB is funded by EPSRC grant EP/V009478/1 and was supported by the CDT in Mathematics and Statistics at The University of Warwick. AM is supported by EPSRC grants EP/W006227/1 & EP/V009478/1. The work of AW is supported by EPSRC grant EP/W00657X/1.

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