Integrated Correlators in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory beyond Localization

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We study integrated correlators of four superconformal primaries $\mathcal{O}_p$ with arbitrary charges p in $\mathcal{N}=4$ super Yang-Mills theory. The $\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle$ integrated correlators can be computed by supersymmetric localization, whereas correlators with more general charges are currently not accessible from this method and, in general, contain complicated multiple zeta values. Nevertheless, we observe that, if one sums over the contributions from all different channels in a given correlator, then all the multiple zeta values (and products of ζ's) cancel, leaving only $\zeta(2\ell+1)$ at ℓ loops. We then propose an exact expression of such integrated correlators in the planar limit, valid for arbitrary 't Hooft coupling. The expression matches with the known exact localization-based results for specific charges, as well as with all existing perturbative and strong-coupling results in the literature for more general charges. As an application, our result is used to determine certain seven-loop Feynman integral periods and fix previously unknown coefficients in the correlators at strong coupling.

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Introduction.—Correlation functions of four superconformal primary operators, $\langle p_1p_2p_3p_4\rangle \coloneqq \langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$, in $\mathcal{N}=4$ super Yang-Mills (SYM) theory are quantities of great interest, which have been computed perturbatively in the planar theory to ten loops in the 't Hooft coupling λ [1–3], as well as at strong coupling to $O(\lambda^{-3})$ [4–12] (see the review [13] containing further references). Here $\mathcal{O}_p$ is a superconformal primary single-particle 1/2-Bogomol'nyi-Prasad-Sommerfield (BPS) operator with charge p,

$$\mathcal{O}_p(x,y)=\mathrm{Tr}[\phi(x,y)^p]+\cdots, \tag{1}$$

where the dots denote multitrace terms and $\phi(x,y)$ are the six fundamental scalars, with x the spacetime variable and y an internal equivalent variable packaging together the six scalars into a single object. There are not many quantities in quantum field theories with more than two dimensions that are known exactly. However, Ref. [7] showed that a special class of such four-point functions, $\langle 22pp\rangle$, may be computed exactly in both the Yang-Mills coupling τ and number of colors N when they are integrated over their spacetime dependence with a certain measure (equivalent to integrating over all four spacetime points modulo the conformal group [14]). This remarkable fact arises by relating the integrated correlators to the partition function of $\mathcal{N}=2^*$ SYM theory on the four-sphere that can be computed using localization [15–17]. Exact finite-N and finite-τ expressions are indeed obtained by exploring the localization formula, see Refs. [20,21] for $\langle 2222\rangle$ and [22–24] for more general $\langle 22pp\rangle$ correlators. The localization techniques, however, do not extend to general correlators beyond the $\langle 22pp\rangle$ case [7,25,26], which we will study in this Letter.

We consider integrated correlators for the most general four-point functions $\langle p_1p_2p_3p_4\rangle$. We observe that a great simplification occurs at five loops if we sum over all contributions from different $SU(4)_R$ channels. Inspired by this as well as explicit perturbative results, we will present a remarkably simple and exact expression (valid for arbitrary λ) for any integrated correlator, summed over $SU(4)_R$ channels, in the planar limit. In addition to reproducing the $\langle 22pp\rangle$ case in [7], the expression agrees with the existing results in the literature for more general correlators at both weak coupling [1,27] and strong coupling [10,11]. Importantly, the strong-coupling results had no input in obtaining the formula and thus provide very strong support. The strong-coupling results will also be used to constrain unfixed parameters in the dual anti–de Sitter (AdS) amplitudes [10,11].

Perturbative integrated correlators and periods.—We begin by considering integrated correlators perturbatively in the planar limit where all correlators are given by a single object due to a ten-dimensional (10D) symmetry discovered in [3] (see also [28,29]). In order to manifest this symmetry, it is useful to introduce a single operator that generates all the single-particle operators,

$$\mathcal{O}(x,y)=\sum_{p=2}^{\infty}\frac{1}{p}\left(\frac{16\pi^4}{c}\right)^{p/4}\mathcal{O}_p(x,y), \tag{2}$$

with $c=(N^2-1)/4$. Then (in the planar perturbation theory) the claim is that four-point functions of *all* single-particle 1/2-BPS operators $\langle p_1p_2p_3p_4\rangle$ combine together into the following "master correlator,"

$$\langle\mathcal{O}(x_1,y_1)\mathcal{O}(x_2,y_2)\mathcal{O}(x_3,y_3)\mathcal{O}(x_4,y_4)\rangle=\text{free part}\\+\frac{\mathcal{I}_4(x_i,y_j)}{2c}\sum_{\ell=1}^{\infty}\left(\frac{\lambda}{4\pi^2}\right)^{\ell}\frac{1}{\ell!}\int\frac{d^4x_5}{(-4\pi^2)}\cdots\frac{d^4x_{4+\ell}}{(-4\pi^2)}f^{(\ell)}(\mathbf{x}_{ij}^2), \tag{3}$$

where $\lambda=Ng_{\mathrm{YM}}^2$ is the 't Hooft coupling, and $\mathcal{I}_4(x_i,y_j)$ is a known prefactor arising from superconformal symmetry. The bold spacetime invariants $\mathbf{x}_{ij}^2$ combine the external and internal variables into a natural 10D object,

$$\mathbf{x}_{ij}^2:=x_{ij}^2-y_{ij}^2=x_{ij}^2(1-g_{ij}), \tag{4}$$

with $g_{ij}:=y_{ij}^2/x_{ij}^2$, and $y_i=0$ if $i>4$ (thus, $g_{ij}=0$ if i or $j>4$). The function $f^{(\ell)}(\mathbf{x}_{ij}^2)$ is given as a sum over planar f graphs [1,27] (see also [2,30]),

$$f^{(\ell)}(\mathbf{x}_{ij}^2)=\sum_{\alpha}c_{\alpha}^{(\ell)}f_{\alpha}^{(\ell)}(\mathbf{x}_{ij}^2), \tag{5}$$

$$f_{\alpha}^{(\ell)}(\mathbf{x}_{ij}^2)=\frac{1}{|\mathrm{aut}(\alpha)|}\sum_{\sigma\in S_{\ell+4}}\prod_{i,j=1}^{4+\ell}\frac{1}{(\mathbf{x}_{\sigma_i\sigma_j}^2)^{e_{ij}^{\alpha}}}, \tag{6}$$

where α are $(4+\ell)$-point graphs with net weight 4 at each vertex, and e_{ij}^{α} is the number of edges between points i and j (we choose a particular labeling of the graph to define e_{ij}^{α}; numerator edges between i and j are negative). Here we sum over full permutation symmetry $S_{\ell+4}$ [1,27], and $|\mathrm{aut}(\alpha)|$ is the symmetry factor of the graph.

We now consider the integrated correlators. These have been defined (in the $\langle 22pp\rangle$ cases) by integrating the correlators [omitting the "free part" and divided by the prefactor $\mathcal{I}_4(x_i,y_j)$ as well as the factor g_{34}^{p-2}] over a certain measure. As shown in [14], this measure is equivalent to simply integrating over the four external variables (modulo the conformal group),

$$\int d\mu\cdots=-\frac{1}{\pi^2}\int\frac{d^4x_1,\ldots,d^4x_4}{\mathrm{vol}[\mathrm{SO}(2,4)]}\cdots. \tag{7}$$

Note that it only makes sense to integrate in this way conformally covariant objects with conformal weight 4 at each of the four points x_i against this measure. Dividing by g_{34}^{p-2} gives the correct weight to be integrated.

This also shows the natural way to define integrated correlators of arbitrary charges $\langle p_1p_2p_3p_4\rangle$. They are more general polynomials of g_{ij}, so we cannot simply divide by g_{ij} factors; we instead integrate separately each coefficient of the g_{ij} polynomial. Equivalently, we write the correlator as a function of x_{ij}^2, g_{ij} rather than x_{ij}^2, y_{ij}^2 and treat g_{ij} as constants.

Combined with the 10D symmetry (3) then all such integrated correlators are obtained from the integrated master correlator

$$\mathcal{C}(\lambda;g_{ij}):=-\sum_{\ell=1}^{\infty}\left(\frac{\lambda}{4\pi^2}\right)^{\ell}\\\times\int\frac{d^4x_1,\ldots,d^4x_{4+\ell}}{\mathrm{vol}[\mathrm{SO}(2,4)]}\frac{f^{(\ell)}\left(x_{ij}^2(1-g_{ij})\right)}{\pi^2\ell!(-4\pi^2)^{\ell}}. \tag{8}$$

Here we integrate over *all* $(4+\ell)$ (internal and external) points. For every individual term in the permutation sum in a particular f-graph contribution $f_{\alpha}^{(\ell)}$ (6), the x_{ij} and $(1-g_{ij})$ terms completely factorize and, after integration, the x_{ij} contributions from a given f graph contribute equally, all giving the period of the graph $f_{\alpha}^{(\ell)}$. Thus, all the integrated correlators package together as

$$\mathcal{C}(\lambda;g_{ij})=-\sum_{\ell=1}^{\infty}\left(\frac{\lambda}{4\pi^2}\right)^{\ell}\\\times\frac{1}{\ell!(-4)^{\ell+1}}\sum_{\alpha}c_{\alpha}^{(\ell)}\mathcal{P}_{f_{\alpha}^{(\ell)}}f_{\alpha}^{(\ell)}(1-g_{ij}), \tag{9}$$

where the periods are defined as

$$\mathcal{P}_{f_{\alpha}^{(\ell)}}=\frac{1}{(\pi^2)^{\ell+1}}\int\frac{d^4x_1,\ldots,d^4x_{4+\ell}}{\mathrm{vol}[\mathrm{SO}(2,4)]}f_{\alpha}^{(\ell)}(x_{ij}^2). \tag{10}$$

Periods of Feynman integrals of the form (10) have been studied quite extensively [31–38], which have been utilized for integrated correlator $\langle 2222\rangle$ [14].

Let us consider some examples. At one loop, we have

$$f^{(1)}(x_{ij}^2)=\frac{1}{\prod_{1\le i<j\le 5}x_{ij}^2}, \tag{11}$$

and so the integrated master correlator is simply (recall $y_5=0$ and $g_{i5}=0$)

$$-\frac{1}{1!(-4)^1}\frac{\mathcal{P}_{f^{(1)}}}{\prod_{1\le i<j\le 4}(1-g_{ij})}, \tag{12}$$

with $\mathcal{P}_{f^{(1)}}=6\zeta(3)$ [31,39]. The correlators $\langle 22pp\rangle$ are extracted from this by taking the coefficient of g_{34}^{p-2} and setting all g's to zero. In this case, they are all equal, in agreement with [7,22]. At two loops, we have

$$f^{(2)}(x_{ij}^2)=\frac{1}{48}\sum_{\sigma\in S_6}\frac{x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6}}{\prod_{1\le i<j\le 6}x^2_{ij}}, \tag{13}$$

and the integrated master correlator becomes

$$-\frac{\mathcal{P}_{f^{(2)}}}{2!(-4)^2}\frac{g_{12}g_{34}+g_{13}g_{24}+g_{14}g_{23}-3\sum_{1\le i<j\le 4}g_{ij}+15}{\prod_{1\le i<j\le 4}(1-g_{ij})},$$

where $\mathcal{P}_{f^{(2)}}=20\zeta(5)$ [31,39]. For $\langle 22pp\rangle$, after we set g_{ij} to zero except g_{34}, it gives

$$-\frac{\mathcal{P}_{f^{(2)}}}{2!(-4)^2}\left(\frac{12}{1-g_{34}}+3\right). \tag{14}$$

Expanding in g_{34} yields integrated correlators for $\langle 22pp\rangle$: $-75\zeta(5)/8$ for $\langle 2222\rangle$ and $-15\zeta(5)/2$ for all others.

Proceeding to higher loops in a similar way, we obtain integrated correlators of all single-trace 1/2-BPS operators in terms of the f-graph periods at higher loops. When specifying to the $\langle 22pp\rangle$ case, the resulting expressions all agree with the known results [7,22].

Simplification and all-order expression.—Starting from five loops, we find multiple zeta values and products of ζ's on integrating higher-charge correlators. For example, for $\langle 4444\rangle$ at five loops, we find

$$\begin{aligned}&\frac{g_{12}^2g_{34}^2}{40320}\Big[7560\zeta(5,3,3)+14685615\zeta(11)+56700\pi^2\zeta(9)\\&\quad+252\pi^4\zeta(7)+31500\zeta(5)^2+6300\zeta(3)^2\zeta(5)-20\pi^6\zeta(5)\Big]\\&-\frac{g_{12}g_{23}g_{34}g_{14}}{40320}\Big[7560\zeta(5,3,3)+569205\zeta(11)\\&\quad+56700\pi^2\zeta(9)+252\pi^4\zeta(7)+31500\zeta(5)^2\\&\quad+6300\zeta(3)^2\zeta(5)-20\pi^6\zeta(5)\Big],\end{aligned} \tag{15}$$

with additional terms from crossing. However, we note, intriguingly, in the sum of the two terms in (15) (despite different g_{ij} factors) everything but $\zeta(11)$ cancels.

Remarkably, this pattern continues at higher charges: for any correlator at five loops, if we sum over all contributions in this way, so that all $\mathrm{SU}(4)_R$ channels contribute equally, then one is always left with a single $\zeta(11)$. This sum over channels can be implemented automatically by formally replacing $g_{ij}\to\gamma_i\gamma_j$. With this replacement, the different g_{ij} factors in a given correlator contribute equally and we can see the above simplification directly at the level of the integrated master correlator. Note $g_{ij}\to\gamma_i\gamma_j$ is irrelevant for $\langle 22pp\rangle$, which has only a single $\mathrm{SU}(4)_R$ channel. The γ_i carry the information of the operator charges, so correlator $\langle p_1p_2p_3p_4\rangle$ arises as the $\gamma_1^{p_1-2}\gamma_2^{p_2-2}\gamma_3^{p_3-2}\gamma_4^{p_4-2}$ coefficient.

We therefore observe the striking feature in the integrated correlators with $g_{ij}\to\gamma_i\gamma_j$ that all multi-ζ's cancel. We then initiated a careful examination of perturbative results, which we knew up to five loops for all charges. By comparing with all-order results of $\langle 22pp\rangle$ [7], we were then able to guess all-order expressions for new correlators, such as $\langle 33pp\rangle$, $\langle 44pp\rangle$, $\langle 23pp+1\rangle$. Although these nontrivial results are obtained based on observation from lower-loop results, we are confident of their validity due to the strong-coupling matching as well as other checks, which we will discuss later.

Crucially, on lifting these results to the master correlator and rewriting the resulting symmetric polynomial in γ_1, γ_2, γ_3, γ_4 in terms of Schur polynomials, we found further dramatic simplification. The result of these investigations is then a proposal for the planar integrated master correlator (8) (summed over channels) via the following remarkably simple formula:

$$\mathcal{C}(\lambda;\gamma_i\gamma_j)=\sum_{\ell=1}^{\infty}\lambda^{\ell}\sum_{\nu=2}^{\infty}\frac{4(-1)^{\nu+\ell+1}\Gamma(\ell+\frac{3}{2})^2\zeta(2\ell+1)}{\pi^{2\ell+1}\Gamma(\ell+2-\nu)\Gamma(\ell+\nu+1)}F_{\nu}(\gamma_i), \tag{16}$$

where ℓ is the number of loops and we have set $g_{ij}\to\gamma_i\gamma_j$. The factor $F_\nu(\gamma_i)$, which contains the information of the operator charges, is given by

$$F_\nu(\gamma_i)=\frac{\mathcal{S}_{\nu-2,\nu-2,0,0}(\gamma_i)-\mathcal{S}_{\nu-2,\nu-2,1,1}(\gamma_i)}{\prod_{1\le i<j\le 4}(1-\gamma_i\gamma_j)}, \tag{17}$$

where $\mathcal{S}_{\nu_i}(\gamma_i)$ are the Schur polynomials,

$$\mathcal{S}_{\nu_1,\nu_2,\nu_3,\nu_4}(\gamma_i)=\frac{\det(\gamma_i^{4+\nu_j-j})_{i,j=1,2,3,4}}{\prod_{1\le i<j\le 4}(\gamma_i-\gamma_j)}. \tag{18}$$

Some comments are in order to illuminate the expression of $F_\nu(\gamma_i)$. The structure of the denominator is expected from (9) and (6). The fact that the numerator is given by some symmetric polynomial in γ_i is also not surprising because the integrated correlators are permutation symmetric due to the symmetric integration measure. What is striking is the appearance of the Schur polynomials with very special partitions and, more importantly, the remarkable simplicity of the overall formula, which is hidden unless written in terms of Schur polynomials.

The integrated $\langle p_1p_2p_3p_4\rangle$ correlator (summed over channels) is then extracted from (16) by simply taking the coefficient of $\gamma_1^{p_1-2}\gamma_2^{p_2-2}\gamma_3^{p_3-2}\gamma_4^{p_4-2}$,

$$\mathcal{C}_{p_1p_2p_3p_4}(\lambda)\coloneqq\mathcal{C}(\lambda;\gamma_i\gamma_j)\big|_{\gamma_1^{p_1-2}\gamma_2^{p_2-2}\gamma_3^{p_3-2}\gamma_4^{p_4-2}}. \tag{19}$$

We remark that, for correlators at a fixed loop order l or for a fixed correlator of charges p_i, the summation over ν

in (16) is naturally truncated: the $\Gamma(\ell+2-\nu)$ in the denominator means all terms with $\nu > l+2$ vanish; whereas Schur polynomials with $\nu > \max(p_i)$ will not contribute to the $\langle p_1 p_2 p_3 p_4 \rangle$ correlator [40].

We now comment on the 10D symmetry and its role in our construction. It is incredibly useful in obtaining results for arbitrary charge correlators, which we used to deduce (16). However, our formula (16) is not a simple consequence of the 10D symmetry, which is valid only at integrand level, destroyed upon integration. Furthermore, the 10D symmetry *is* broken at strong coupling, but as we will see the expression (16) correctly reproduces integrated correlators at strong coupling.

Resummed expression.—The proposed all-order expression (16) can be resummed through a modified Borel transform (see, e.g., [41,42]) by using the following integral identity:

$$\zeta(n)\Gamma(n+1) = 2^{n-1}\int_0^\infty dw \frac{w^n}{\sinh^2(w)}. \tag{20}$$

Replacing $\zeta(2\ell+1)$ in (16) by its integral representation using the above identity and performing the resummation, we obtain

$$\mathcal{C}(\lambda;\gamma_i\gamma_j) = \int_0^\infty \frac{wdw}{\sinh^2(w)} \sum_{\nu=2}^{\infty} [J_{\nu-1}(u)^2 - J_\nu(u)^2] F_\nu(\gamma_i), \tag{21}$$

where $u = (w\sqrt{\lambda}/\pi)$ and $J_\nu(u)$ are Bessel functions. For $\langle 22pp \rangle$, (21) reduces to the known result of [7]

$$\mathcal{C}_{2,2,p,p}(\lambda) = \int_0^\infty \frac{wdw}{\sinh^2(w)} (J_1(u)^2 - J_p(u)^2), \tag{22}$$

while for $\langle 33pp \rangle$, for instance, we find

$$\begin{aligned}\mathcal{C}_{3,3,p,p}(\lambda) = \int_0^\infty \frac{wdw}{\sinh^2(w)} \big(&3J_1(u)^2 + 4J_2(u)^2 \\ &+ J_3(u)^2 - 2J_{p-1}(u)^2 - 4J_p(u)^2 \\ &- 2J_{p+1}(u)^2\big).\end{aligned} \tag{23}$$

Similar expressions can be obtained for all other cases.

Importantly, the expression (21) is valid for arbitrary λ, allowing us to study the strong-coupling regime, which we will consider in the next section.

Strong coupling.—The strong-coupling expansion of the integrated correlators can be obtained straightforwardly from (21) by using the Mellin-Barnes representation of products of Bessel functions [7], and we find

$$\mathcal{C}(\lambda;\gamma_i\gamma_j)\Big|_{\text{strong}} = \sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu(\nu-1)} + \sum_{n=1}^{\infty} \frac{4n(-1)^n\Gamma(n+\frac{1}{2})\Gamma(\nu+n-\frac{1}{2})\zeta(2n+1)}{\lambda^{n+\frac{1}{2}}\sqrt{\pi}\Gamma(n)\Gamma(\nu-n+\frac{1}{2})}\right) F_\nu(\gamma_i). \tag{24}$$

This can then be compared with known results for four-point correlators of general charges at strong coupling, which can be determined using a simple effective action of a massless 10D scalar on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [10]. The correlators are completely fixed up to order $\lambda^{-5/2}$ and λ^{-3} [9–11] under a certain $\mathbb{Z}_2$-symmetry assumption [43]. The 10D effective action is very different from the 10D symmetry of the "perturbative integrands." The tree-level supergravity result does possess the 10D symmetry, but α' corrections break this. The key point for us is that the effective action generates correlators of all charges at strong coupling efficiently; one can then integrate their spacetime dependence with the measure (7) analytically using the techniques outlined in Appendix B of [44]. These give highly nontrivial functions of γ_i, which perfectly agree with our predictions (24) for all the correlators [45].

At order $\lambda^{-7/2}$ there remain unfixed coefficients, but our result (24) is both consistent with these and indeed constrains them further. There are 11 coefficients occurring as possible nonequivalent terms in the effective action, and comparison with our result (24) fixes five of them. If in addition we utilize the recently available $\langle 2222 \rangle$ correlator at this order [12], we further fix three coefficients. The $\mathbb{Z}_2$ symmetry fixes two more coefficients to zero, left with a single undetermined coefficient. Explicitly, in the notation of [10] we fix nine coefficients,

$$\begin{aligned}&A_4 = -\frac{1575\zeta(7)}{4}, \quad C_2 = \frac{641\zeta(7)}{16}, \quad D_1 = 0, \\ &E_1 = 0, \quad F_0 = \frac{\zeta(7)}{2}, \quad G_{1;0} = -\frac{11\zeta(7)}{64}, \\ &G_{2;0} = \frac{71\zeta(7)}{64}, \quad G_{3;0} = \frac{141\zeta(7)}{256}, \quad G_{5;0} = -\frac{51\zeta(7)}{64},\end{aligned} \tag{25}$$

together with $B_2 = 20G_{4;0} + [259\zeta(7)/4]$. Finally, in [11] the authors found one additional constraint arising from the operator product expansion, which goes beyond what the effective action predicts [46], which gives $G_{4;0} = [12\,619\zeta(7)/6656]$.

Unlike the small-λ expansion (16), the large-λ expansion (24) is asymptotic and not Borel summable. The series should be completed with exponentially decayed terms $\Delta\mathcal{C}(\lambda;\gamma_i\gamma_j)$, which can be obtained by the means of resurgence [47]. Following [20,48], we find

$$\Delta\mathcal{C}(\lambda;\gamma_i\gamma_j) = \pm\frac{i}{2}\sum_{\nu=2}^{\infty}(-1)^{\nu}(2\nu-1)^2\bigg(\frac{8\mathrm{Li}_0(z)}{(2\nu-1)^2}+\frac{2\mathrm{Li}_1(z)}{\lambda^{1/2}} + \frac{(4\nu^2-4\nu+5)\mathrm{Li}_2(z)}{4\lambda}+\cdots\bigg)F_{\nu}(\gamma_i), \quad (26)$$

where $z=e^{-2\sqrt{\lambda}}$. Holographically, these terms behave as $e^{-2L^2/\alpha'}$ (L is the AdS length scale), which indicates they may arise from world-sheet instantons.

Modular invariance.—The S duality of $\mathcal{N}=4$ SYM theory [49,50] implies that the correlators of the operators we consider here should be $\mathrm{SL}(2,\mathbb{Z})$ invariant. Therefore, the large-N expansion of the integrated correlators with *fixed* coupling τ should be expressed in terms of modular functions. More explicitly, as in the cases of $\langle 2222\rangle$ [20,51,52] (and, more generally, $\langle 22pp\rangle$ [22–24]), we expect the power series terms (24) are replaced by the nonholomorphic Eisenstein series $E(s;\tau,\bar\tau)$, whereas the exponentially decayed terms (26) should be expressed in terms of $D_N(s;\tau,\bar\tau)$ introduced in [52] (see also [53] in a different context). These modular functions are defined as

$$E(s;\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)}\frac{\tau_2^s}{\pi^s|m+n\tau|^{2s}},$$
$$D_N(s;\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} e^{-4\sqrt{N\pi}\frac{|m+n\tau|}{\sqrt{\tau_2}}}\frac{\tau_2^s}{\pi^s|m+n\tau|^{2s}}, \quad (27)$$

where $\tau \coloneqq \theta/(2\pi)+i4\pi/g_{\mathrm{YM}}^2=\tau_1+i\tau_2$. The results (24) and (26) allow us to determine the first few orders in the large-N (fixed-τ) expansion [54],

$$\mathcal{C}(\tau,\bar\tau;\gamma_i\gamma_j) = \sum_{\nu=2}^{\infty}\bigg[\frac{1}{2(\nu-1)\nu}-\frac{2\nu-1}{2^4N^{\frac{3}{2}}}E(3/2;\tau,\bar\tau) + \frac{3(2\nu-3)(4\nu^2-1)}{2^8N^{\frac{5}{2}}}E(5/2;\tau,\bar\tau)+\cdots \pm 2i(-1)^{\nu}D_N(0;\tau,\bar\tau)+\cdots\bigg]F_{\nu}(\gamma_i). \quad (28)$$

The expression agrees with known results in the literature when there is an overlap in charges and to the orders determined here. The fixed-τ result is beyond the ‘t Hooft limit; in particular, it contains instanton contributions.

New periods and the 10D lightlike limit.—Finally, we return to the perturbative regime where the all-order expression (16) can be compared directly with the results obtained from four-point functions integrated in terms of periods (9). At six loops, there are 26 periods and the integrands can be found in [1]. They are highly nontrivial seven-loop Feynman integral periods [55]. We have evaluated 16 of them explicitly using HYPERLOGPROCEDURES [36], and we find the proposed expression (16) is perfectly consistent with these Feynman integral results. Furthermore, (16) allows us to then determine the remaining unknown periods. We find (16) fixes all the periods except a single one, which can be further determined by exploiting a fascinating connection [3] between the master correlator and the “octagon” $\mathbb{O}$ introduced in [56,57], as we will discuss now.

As proposed in [3], in a 10D lightlike limit $\mathbf{x}_{i,i+1}^2 = x_{i,i+1}^2 - y_{i,i+1}^2 \to 0$, the master correlator (3) reduces to the octagon $\mathbb{O}$ (squared),

$$\lim_{\mathbf{x}_{i,i+1}^2\to 0}\frac{\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle}{\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_{\mathrm{free}}} = M^2, \quad (29)$$

where $M=\mathbb{O}/\mathbb{O}_{\mathrm{free}}$ [58]. The octagon is given in terms of products of known ladder integrals [59], whereas the correlator is expressed as four-point conformal integrals. This relation then predicts nontrivial relations between these conformal integrals, which has been previously confirmed (numerically) to four loops [3].

The 10D lightlike limit together with $g_{ij}\to\gamma_i\gamma_j$ implies $\gamma_1=\gamma_3=1/\gamma_2=1/\gamma_4 \coloneqq \gamma$. Making this substitution and integrating over both sides of (29) with the measure (7) allows an all-orders comparison. For the lhs we use our formula (16); the rhs, arising from the octagon [3], is also known to all orders in this limit [60]. We find both sides of (29) lead to exactly the same result,

$$-\sum_{\ell=1}^{\infty}\left(\frac{\lambda}{4\pi^2}\right)^{\ell}\frac{\ell+1}{2^{2\ell+1}}\binom{2\ell+2}{\ell+1}\frac{(\gamma^2-1)^{2\ell}}{\gamma^{2\ell}}\zeta(2\ell+1). \quad (30)$$

We have further verified the relation (29) up to six loops (integrating over external points) without imposing $g_{ij}\to\gamma_i\gamma_j$, using the results of Feynman integral periods determined from (16). All these simultaneously provide further consistency checks of both our proposal (16) and the relation (29). In the process, we also obtain many integral relations, which allow us to express the aforementioned unfixed period in terms of certain (integrated) triple products of ladders that then can be computed using HYPERLOGPROCEDURES. More detailed discussion of six-loop integrated correlators and the evaluation of periods at this order can be found in the Supplemental Material [61] as well as the attached *Mathematica* notebook.

Conclusion.—We have presented an exact formula for the integrated correlators of arbitrary charges in the planar limit. We should emphasize that the formula was obtained purely based on examining data to five loops in perturbation theory (but for all correlators). The fact that the formula is then consistent with the periods entering six loops, as well as the octagon to all orders, and furthermore agrees with all strong-coupling results to $\lambda^{-7/2}$ for all correlators provides very strong evidence for the proposal.

Our result reveals remarkable simplicities of these general integrated correlators at both weak and strong coupling. Given that they are currently inaccessible by supersymmetric localization, a natural question that arises then is what the origin of the simplicities is and, in particular, what is the meaning of, or reason for, the $g_{ij} \to \gamma_i \gamma_j$ replacement. A related question is if there are some other operations in g_{ij} space one could perform, which still yield simple and interesting results.

It has been shown that integrated correlators $\langle 22pp \rangle$ obey Laplace-difference equations that relate them with different N and charges [20,22,62,63]. It would be fascinating if these general integrated correlators studied in this Letter obey similar relations. Seeing any such structures requires results beyond the planar limit.

We would also like to explore integrated correlators with a different integration measure [44,64], which was originally introduced for $\langle 2222 \rangle$. In perturbation theory, they can again be understood as periods of Feynman integrals [14]. We leave this Letter as a future direction.

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