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Activation of metrologically useful genuine multipartite entanglement

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Abstract

We consider quantum metrology with several copies of bipartite and multipartite quantum states. We characterize the metrological usefulness by determining how much the state outperforms separable states. We identify a large class of entangled states that become maximally useful for metrology in the limit of large number of copies, even if the state is weakly entangled and not even more useful than separable states. This way we activate metrologically useful genuine multipartite entanglement. Remarkably, not only that the maximally achievable metrological usefulness is attained exponentially fast in the number of copies, but it can be achieved by the measurement of few simple correlation observables. We also make general statements about the usefulness of a single copy of pure entangled states. We surprisingly find that the multiqubit states presented in Hyllus *et al* (2010 *Phys. Rev.* A **82** 012337), which are not useful, become useful if we embed the qubits locally in qutrits. We discuss the relation of our scheme to error correction, and its possible use for quantum metrology in a noisy environment.

1. Introduction

Quantum entanglement plays a central role in quantum physics, as well as in quantum information processing applications [1–3]. There have been numerous experiments creating entanglement with photons, trapped cold ions, cold atoms and superconducting circuits [4–33]. In the multiparticle scenario, a state can be not only separable or entangled, but it can possess various levels of k-particle entanglement [9–15, 34–39]. The highest form of entanglement for N particles in this case is the *genuine multipartite entanglement* (GME), which is just N-particle entanglement. The main goal of quantum experiments is often proving that the quantum state created has a high level of multipartite entanglement or even showing that it has GME, which involves all the parties [16–33, 40]. In the latter case, something qualitatively new has been created compared to experiments with fewer particles.

In quantum metrology, it is known that quantum entanglement is needed to surpass the classical limit in the precision of parameter estimation [41]. Even weakly entangled bound entangled states can still be better than separable states [42–44]. It is also known that in order to reach a higher and higher metrological precision, higher and higher levels of multipartite entanglement is needed [45, 46]. This way, we can define metrologically useful k-particle entanglement, characterizing quantum states that are more useful metrologically than any quantum state with at most $(k-1)$-particle entanglement. States that do not possess metrologically useful k-particle entanglement form a convex set [45–47], similarly to other important sets of quantum states in entanglement theory, such as for instance separable states, which makes it possible to detect such entanglement with methods similar to the ones used in other areas of entanglement theory [1–3]. Then, a quantum state of N-particles possesses metrologically useful GME, if it has metrologically useful N-particle entanglement. From these we can see that GME is needed to reach the maximal precision in parameter estimation. On the other hand, surprisingly, there exist pure states containing GME that are not more useful for metrology than separable states [48]. Therefore, verifying the presence of metrologically useful entangled states is even more desirable than detecting entanglement without proving metrological usefulness [49–51].

The question naturally arises: how could one activate those entangled states that are not useful for metrology, using a scheme that is relatively simple to implement in the lab, thus we will avoid applying a distillation step or local operations and classical communication (LOCC). We will consider multicopy metrology such that only copies of the same party interact with each other, which is typical in activation schemes in quantum information science [52–59]. Looking for an optimal setup is challenging since the modeling of large quantum systems is needed.

We find that for N qudits of dimension d, there is a class of entangled states of at most rank-d that become maximally useful compared to separable states in the limit of large number of copies. Surprisingly, the maximal usefulness is attained *exponentially fast* in the number of copies. Unexpectedly, the operators to be measured in order to verify the presence of metrologically useful GME turn out to be *simple correlations*. For M copies, we need to measure a modest number of M correlation terms. It is remarkable that this class contains metrologically useless, weakly entangled states that can even have an arbitrarily large overlap with product states. Thus, such states attain metrologically useful GME in the multicopy scenario, which contributes to the recent intensive research on activating entanglement using many copies [57–59]. A similar approach has been used to study the activation of genuine multipartite nonlocality [60]. We will also consider how various relevant quantum states outside the subspace mentioned above perform in the multicopy scenario. Moreover, we will also show that embedding quantum states into higher dimensional spaces can also activate metrologically useful entanglement. We will support all our findings with powerful analytical and numerical methods that can describe multicopy metrology with large quantum systems.

Our method might offer an approach for quantum metrology in the noisy, intermediate-scale quantum (NISQ) era [61]. In particular, we will suggest a procedure to force states into the desired class where metrologically useful GME activation is guaranteed, if they left the subspace due to an imperfect preparation or noise during the dynamics.

The paper is structured as follows. In section 2, we describe quantum metrology and the basic quantities to characterize the metrological performance of quantum states. In section 3, we introduce the metrological scheme we use when considering multiple copies of quantum states. In section 4, we present the main result of the paper, that is, we identify a class of states for which metrologically useful GME activation is possible in the many copy limit. Moreover, we also provide the measurements that need to be performed in order to reach the maximal metrological precision. Then, we provide examples for quantum states inside and outside the above-mentioned subspace and evaluate their performance with our scheme. Furthermore, we also demonstrate the possibility of improving metrological performance simply by embedding quantum systems into larger dimensions. In section 5, we discuss a strategy that can be applied if the quantum states leave the subspace where metrologically useful GME activation is possible by making use of ideas from quantum error correction. Finally, in section 6 we conclude the paper.

2. Quantum metrology

Before discussing our main results, we review some of the fundamental relations of quantum metrology [62–72]. A basic metrological task in a *linear* interferometer is estimating the small angle θ for a unitary dynamics

$$\varrho_\theta = \exp\left(-i\mathcal{H}\theta\right)\varrho\exp\left(+i\mathcal{H}\theta\right), \tag{1}$$

where the Hamiltonian is the sum of *local* terms. In particular, for N-partite systems it is

$$\mathcal{H} = h_1 + h_2 + \cdots + h_N, \tag{2}$$

where h_n for $n = 1, 2, \ldots, N$ are single-subsystem operators[10]. The precision is limited by the Cramér–Rao bound as [62–76]

$$(\Delta\theta)^2 \geqslant \frac{1}{\nu \mathcal{F}_Q[\varrho, \mathcal{H}]}, \tag{3}$$

where ν is the number of independent repetitions, and the quantum Fisher information, a central quantity in quantum metrology is defined by the formula [62, 73–76]

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2. \tag{4}$$

Here, λ_k and $|k\rangle$ are the eigenvalues and eigenvectors, respectively, of the density matrix ϱ, which is used as a probe state for estimating θ. From equation (3) it can be seen that the larger the quantum Fisher information, the better precision we can achieve in parameter estimation. An efficient calculation method of the quantum Fisher information for large systems appears in supplement A.

We are interested in the ratio of the quantum Fisher information of a state and the maximum of the quantum Fisher information for the same Hamiltonian for separable states, which we call the metrological gain for that particular unitary dynamics [47]

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}. \tag{5}$$

The maximum for separable states is given as [43, 47, 77]

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\lambda_{\max}(h_n) - \lambda_{\min}(h_n)]^2, \tag{6}$$

where $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ denote the maximum and minimum eigenvalues of X, respectively. Note that for qubits, if

$$h_n = \sum_{l=x,y,z} c_{l,n}\sigma_l, \tag{7}$$

where $c_{l,n}$ are real numbers, and $|\vec{c}_n| = 1$, then $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = 4N$ [48]. We also define the metrological gain optimized over all local Hamiltonians as

$$g(\varrho) = \max_{\mathcal{H}} g_{\mathcal{H}}(\varrho). \tag{8}$$

If $g(\varrho) > 1$ then the state is entangled and we call it also metrologically useful. If h_n all have identical lowest and highest eigenvalues, then $g(\varrho) > k$ implies metrologically useful $(k+1)$-partite entanglement. If $g(\varrho) > N - 1$ then the state has metrologically useful GME, as discussed in appendix A. In general, for quantum states $g(\varrho) \leqslant N$ holds [47]. Note also that the metrological gain $g(\varrho)$ is convex in the quantum state [47].

Finally, we mention that the variance and the Wigner–Yanase skew information defined as [78]

$$I_\varrho(\mathcal{H}) = \mathrm{Tr}\left(\varrho\mathcal{H}^2\right) - \mathrm{Tr}\left(\sqrt{\varrho}\mathcal{H}\sqrt{\varrho}\mathcal{H}\right) \tag{9}$$

provide upper and lower bounds, respectively, on the quantum Fisher information as

$$4(\Delta H)^2_\varrho \geqslant \mathcal{F}_Q[\varrho, \mathcal{H}] \geqslant 4I_\varrho(\mathcal{H}). \tag{10}$$

It is often easier to calculate $I_\varrho(\mathcal{H})$ than $\mathcal{F}_Q[\varrho, \mathcal{H}]$, which will be used in our derivations.

[10] For simplicity, we will denote by h_n the operator acting on a single party as well as the operator acting on the entire system. The actual meaning can be inferred from the context.

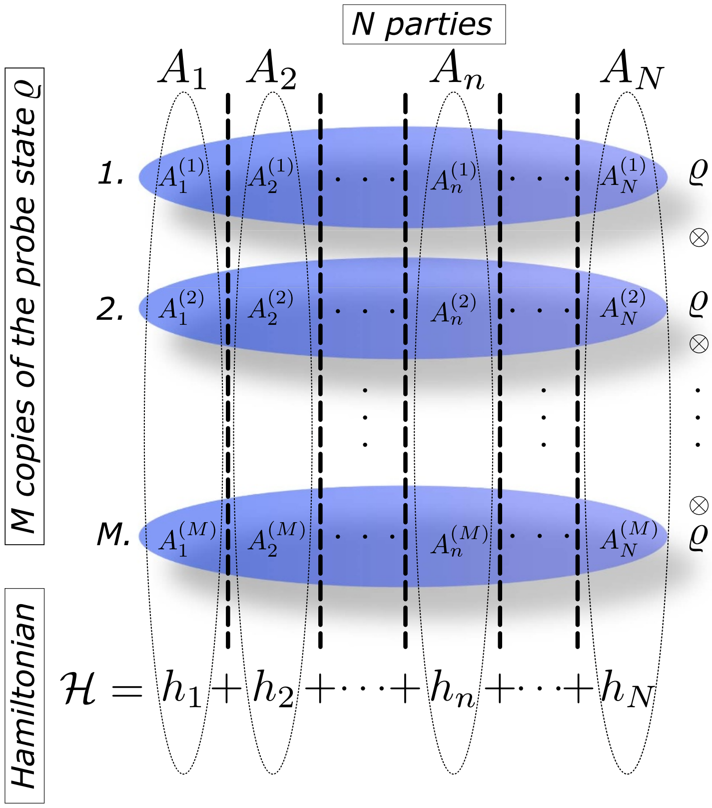

Figure 1. Metrology with M copies of an N-partite quantum state ϱ. Each horizontal ellipse represents a copy of the quantum state ϱ. The particles in the same vertical ellipse correspond to different copies of the same party A_n and they can interact with each other during the evolution generated by the Hamiltonian h_n for $n = 1, 2, \ldots, N$. However, there is no interaction between particles corresponding to different parties, which is stressed by vertical dashed lines separating the parties.

3. Multicopy scheme for activation

In this section, we will consider metrology with several copies of the quantum state. First, we show that without interaction during the evolution, one cannot obtain an improvement in gain. Thus, it is not at all trivial that by adding new copies, the metrological gain will increase. Then, we study the setup based on an interaction between the copies of the same party. In this case, even the maximal gain can be achieved with very weakly entangled copies.

Let us consider M copies of a quantum state, all undergoing a dynamics governed by the same Hamiltonian $\mathcal{H}$. Then, for the quantum Fisher information we obtain

$$\mathcal{F}_Q\left[\varrho^{\otimes M}, \mathcal{H}^{\otimes M}\right] = M\mathcal{F}_Q\left[\varrho, \mathcal{H}\right], \tag{11}$$

while the maximum for separable states also increases

$$\mathcal{F}_Q^{(\mathrm{sep})}\left(\mathcal{H}^{\otimes M}\right) = M\mathcal{F}_Q^{(\mathrm{sep})}\left(\mathcal{H}\right). \tag{12}$$

Thus, the metrological gain does not change

$$g_{\mathcal{H}^{\otimes M}}\left(\varrho^{\otimes M}\right) = g_{\mathcal{H}}\left(\varrho\right). \tag{13}$$

Here, this statement is true for all schemes realizing an unbiased estimator in which there is no interaction during the quantum dynamics, however, after the dynamics the final states of the copies can be processed with any quantum circuit, even with collective measurements acting on several copies. Note that the gain remained the same since the quantum Fisher information given in equation (11) increased M-fold, however, the performance of separable states also increased M-fold.

Hence, in order to increase the gain with the number of copies, we need to allow for interaction between the copies corresponding to the same party during the quantum dynamics. We consider M copies of the N-partite state ϱ acting on parties A_n, as shown in figure 1. The system consists of the subsystems $A_n^{(m)}$ for $m = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$. We calculate $g_{\mathcal{H}}[\varrho^{\otimes M}, \mathcal{H}]$ with local Hamiltonians of the form

$$h_n = \otimes_{m=1}^{M} h_{A_n^{(m)}} \tag{14}$$

for $n = 1, 2, \ldots, N$, as well as more general local Hamiltonians. Extensive numerics show that h_n of the form (14) can often reach the maximal metrological performance, which makes the implementation easier [47]. In supplement B, we consider a different type of h_n with only two-body correlations.

We already know that any entangled bipartite pure state is maximally useful metrologically, in the limit of large number of copies [47]. However, what is the situation in the multiqubit case, relevant for quantum metrology with particle ensembles? What can we tell about the usefulness of mixed states?

4. A metrologically useful GME activatable subspace

Here, we present the main result of the paper, that is, we identify a subspace in which all entangled multi-qudit quantum states can be made maximally useful. We also identify the measurements that have to be performed in order to achieve the maximal precision in metrology.

Result 1. Entangled states of $N \geqslant 2$ qudits of dimension d are maximally useful in the limit of large number of copies if they live in the

$$\{|000..00\rangle, |111..11\rangle, \ldots, |d-1, d-1, .., d-1\rangle\} \tag{15}$$

subspace. The maximally achievable metrological usefulness is attained *exponentially fast in the number of copies*. States that can be transformed to this form with local unitaries have also the same property.

Proof. Let us consider

$$\varrho = \sum_{k,l=0}^{d-1} c_{kl} \left(|k\rangle\langle l|\right)^{\otimes N}, \quad \mathcal{H} = \sum_{n=1}^{N} \left(D^{\otimes M}\right)_{A_n}, \tag{16}$$

with c_{kl} being the matrix elements of ϱ in the basis from equation (15) and $D = \mathrm{diag}(+1, -1, +1, -1, \ldots)$. To simplify the calculation, we use the mapping

$$\varrho \to \tilde{\varrho} = \sum_{k,l=0}^{d-1} c_{kl} |k\rangle\langle l|, \quad \mathcal{H} \to \tilde{\mathcal{H}} = N D^{\otimes M}, \tag{17}$$

for which $\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = \mathcal{F}_Q[\tilde{\varrho}^{\otimes M}, \tilde{\mathcal{H}}]$ holds. We can bound the quantum Fisher information as

$$\mathcal{F}_Q\left[\tilde{\varrho}^{\otimes M}, \tilde{\mathcal{H}}\right] \geqslant 4 I_{\tilde{\varrho}^{\otimes M}}\left(\tilde{\mathcal{H}}\right), \tag{18}$$

where the Wigner–Yanase skew information is

$$I_{\tilde{\varrho}^{\otimes M}}\left(\tilde{\mathcal{H}}\right) = N^2 \left[1 - \mathrm{Tr}\left(\sqrt{\tilde{\varrho}} D \sqrt{\tilde{\varrho}} D\right)^M\right]. \tag{19}$$

In the limit of large number of copies, if $[\sqrt{\tilde{\varrho}}, D] \neq 0$ then the skew information given in equation (19) converges to the maximum. In this case, the state overcomes $4N$, the separable limit of the quantum Fisher information given in equation (6), hence all such states are entangled. For $d \geqslant 3$, apart from the Hamiltonian D, we should try other Hamiltonians that are obtained from D by permuting its diagonal elements. If $\sqrt{\tilde{\varrho}}$ does not commute with one of these Hamiltonians, then, in the limit of large number of copies, the skew information with that Hamiltonian converges to the maximum, thus the state is entangled. If $\sqrt{\tilde{\varrho}}$ commutes with D and with all the Hamiltonians obtained after permuting the diagonal elements, then $c_{kl} = 0$ must hold for all $k \neq l$. Such a state is a mixture of product states.

The Wigner-Yanase skew information given in equation (19) can be written out as follows for $d = 2$

$$\frac{I}{N^2} = 1 - \left[\frac{8|c_{01}|^2 \sqrt{-c_{00}^2 + c_{00} - |c_{01}|^2} + 4\left(c_{00} - 1\right) c_{00} + 1}{\left(1 - 2c_{00}\right)^2 + 4|c_{01}|^2}\right]^M \tag{20}$$

if $c_{01} \neq 0$, otherwise $I = 0$. Moreover, if $c_{00} = c_{11} = 1/2$ then equation (20) can be simplified to

$$I(c_{01}, N) = N^2 \left[1 - \left(1 - 4|c_{01}|^2\right)^{M/2}\right]. \tag{21}$$

□

In result 1, we computed lower bounds on the quantum Fisher information. At this point, it could happen that the necessary measurements might be highly nonlocal operators. We will now show that, surprisingly, with simple operators it is possible to reach the maximal precision in parameter estimation. The operators to be measured are sums of *simple multiparty correlations*.

Result 2. To achieve the maximal usefulness for the states appearing in result 1, the following operator has to be measured

$$\mathcal{M} = \sum_{m=1}^{M} Z^{\otimes(m-1)} \otimes Y \otimes Z^{\otimes(M-m)}, \tag{22}$$

where we define the multi-qubit operators acting on a single copy

$$Y = \begin{cases} \sigma_y^{\otimes N} & \text{for odd } N, \\ \sigma_x \otimes \sigma_y^{\otimes(N-1)} & \text{for even } N, \end{cases}$$
$$Z = \sigma_z \otimes \mathbb{1}^{\otimes(N-1)}, \tag{23}$$

where σ_x, σ_y, and σ_z are the Pauli spin matrices. By taking sufficiently many copies, the precision corresponding to the metrologically useful GME ($g = N$) can be approached fast and the required number of copies depends on how noisy the state is. The proof is given in appendix B.

Next, we will look at some consequences and applications of result 1, and we will also test the performance of our scheme for states living outside the subspace of result 1. Moreover, we will also provide some results concerning the single-copy case.

4.1. Noisy GHZ states

The method given in result 1 can be used to calculate the precision of the multicopy metrology with the state

$$\varrho_p = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\,\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2}, \tag{24}$$

where the Greenberger–Horne–Zeilinger (GHZ) state, playing a central role in quantum information science and quantum metrology, is [79]

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right). \tag{25}$$

The state ϱ_p is the one obtained after the particles of a GHZ state pass through a phase-noise channel, which is a relevant type of noise in many physical systems [80].

Let us see a concrete example with $M = 2$ copies of the three-qubit case of the state ϱ_p given in equation (24) with $p = 0.8$. Then, we have

$$\mathcal{F}_Q[\varrho, \mathcal{H}_{M=2}] = 28.0976, \tag{26}$$

while in the single-copy case we have

$$\mathcal{F}_Q[\varrho, \mathcal{H}_{M=1}] = 23.0400. \tag{27}$$

Here, $\mathcal{H}_{M=2}$ and $\mathcal{H}_{M=1}$ are the Hamiltonians for the two-copy and single-copy cases, respectively, as defined in equation (16). For these two cases, the upper bounds for the quantum Fisher information for separable states are

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}_{M=1}) = \mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}_{M=2}) = 12. \tag{28}$$

Hence, for the corresponding metrological gains

$$g_{M=1} = 1.92 < g_{M=2} = 2.34 \tag{29}$$

holds.

The state ϱ_p turns out to be maximally useful in the limit of very many copies if $p > 0$ as it is an entangled element of the set in result 1. For ϱ_p, a lower bound on the quantum Fisher information can be obtained using the relation with the quantum Fisher information and the Wigner-Yanase skew information given in equation (18) and the inequality given in equation (21) with $c_{01} = p/2$. We plot the lower bound on the metrological gain in figure 2. It can be seen that the lower bound approaches the theoretical maximum rapidly, as the number of copies is increasing.

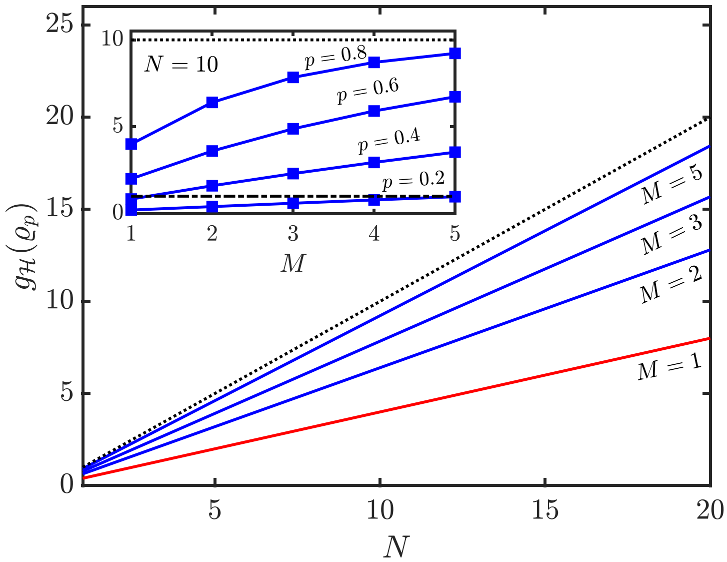

Figure 2. Multicopy metrology with the noisy GHZ state given in equation (24) for $p = 0.8$. (solid) Lower bound on the metrological gain as a function of N. M denotes the number of copies. The state and the Hamiltonian are given in equation (16). We used the lower bound on the quantum Fisher information given in equations (18) and (21), where, for the case of ϱ_p we have to set $c_{01} = p/2$. Moreover, the separable limit is $\mathcal{F}_Q^{(\mathrm{sep})} = 4N$. (dotted) The maximal gain, $g_{\max} = N$. (inset) The increase of the gain with the number of copies for $N = 10$ for various values for p. (dotted) The maximal gain $g_{\max} = 10$. (dashed dotted) Curve corresponding to $g = 1$.

4.2. GHZ-like states with qudits of dimension $d > 2$

Result 1 also implies that all entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k |k\rangle^{\otimes N} \tag{30}$$

with $\sum_k |\sigma_k|^2 = 1$ are maximally useful in the limit of large number of copies. This can be seen as follows. Pure states given in equation (30) form a subset of the states considered in result 1. Among them, only states of the type $|k\rangle^{\otimes N}$are separable, therefore this is the only case where result 1 does not apply.

Surprisingly, in the single-copy case not all entangled states of the type given in equation (30) are useful. In particular, a single copy of the state given in equation (30) for $d = 2$ and $N \geqslant 3$ is useful metrologically if and only if [48]

$$1/N < E := 4|\sigma_0\sigma_1|^2. \tag{31}$$

For a short proof of this fact, which is more general than the one presented in [48], see appendix C. In contrast, in the bipartite ($N = 2$) case all entangled pure states are metrologically useful [47] and to some extent the following result can be considered as a generalization of this fact.

Result 3. All entangled pure states of the form given in equation (30) with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geqslant 3$ and $N \geqslant 3$.

Proof. Let us see first the case of odd $d \geqslant 3$ and the block diagonal matrix

$$h_n^{(\mathrm{odd})} = \mathrm{diag}(1, X_{d-1}), \tag{32}$$

where X_{d-1} is a $(d-1) \times (d-1)$ matrix with 1's in the antidiagonal and all other elements being 0. Hence,

$$\mathcal{F}_Q\left[\varrho, \mathcal{H}^{(\mathrm{odd})}\right] = 4N + 4N|\sigma_1|^2 \left[N\left(1 - |\sigma_1|^2\right) - 1\right], \tag{33}$$

where the separable limit is $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}^{(\mathrm{odd})}) = 4N$. With an appropriate permutation of the local basis states, from $\mathcal{H}^{(\mathrm{odd})}$ we can obtain a Hamiltonian for which σ_k appears in the place of σ_1 in equation (33). Hence, if for any σ_k

$$0 < |\sigma_k|^2 < (N-1)/N \tag{34}$$

holds then the given state is useful. If the state is entangled, then at least two of the σ_k are nonzero, and one of them fulfills equation (34).

Let us now consider the case of even $d \geqslant 4$, with the block diagonal matrix

$$h_n^{(\mathrm{even})} = \mathrm{diag}(1, 1, X_{d-2}), \tag{35}$$

for which the quantum Fisher information is obtained as

$$\mathcal{F}_Q\left[\varrho, \mathcal{H}^{(\mathrm{even})}\right] = 4N + 4N\left(|\sigma_1|^2 + |\sigma_2|^2\right)\left[N\left(1 - |\sigma_1|^2 - |\sigma_2|^2\right) - 1\right] \tag{36}$$

holds. Similarly to the case of odd d, with an appropriate permutation of the local basis states, from $\mathcal{H}^{(\mathrm{even})}$ we can obtain a Hamiltonian for which σ_k and σ_l with $l \neq k$ appear in the place of σ_1 and σ_2, respectively, in equation (36). Hence, if

$$0 < |\sigma_k|^2 + |\sigma_l|^2 < (N-1)/N \tag{37}$$

then the state is useful. If the state is entangled, then at least two of the σ_k are nonzero. We have to examine the following cases. If the number of nonzero σ_k is three or more, then two of the σ_k will fulfill equation (37). If only two of them are nonzero then we can consider a problem of odd d with $d = 3$ with one σ_k set to zero, and the state is useful. □

4.3. Activation of metrologically useful entanglement by embedding

A surprising consequence of result 3 is that all entangled states of the form given in equation (30) are useful for $d = 2$, if we embed the qubits locally in qutrits, and consider a state as in equation (30) for $d = 3$ by setting $\sigma_3 = 0$. Thus, just by increasing the local dimension of the system, the states that have been found useless in [48] can be activated and made useful. It is also clear from the proof of result 3 that in this case, if we take $1/N = 4|\sigma_0\sigma_1|^2$ (cf equation (31)) then for the asymptotic case of $N \to \infty$, and for a single copy we obtain

$$\mathcal{F}_Q = 5N, \qquad g = 5/4, \tag{38}$$

by embedding, while $\mathcal{F}_Q^{(\mathrm{sep})} = 4N$. Here, in the spirit of the proof of result 3, we consider $\mathcal{H}^{(\mathrm{odd})}$ and also other Hamiltonians obtained from it after the local basis states are permuted. Thus, we improve metrological performance in the single-copy scenario, just by embedding the quantum states locally into a higher dimensional system. This can happen since after the embedding, new types of dynamics become possible that lead out from the original space. If such dynamics is allowed then the quantum state can outperform all separable states. Activation by embedding is related to activation by ancillas studied in [47]. The effect of ancillas have also been studied in [81].

4.4. Activation of metrologically useful GME from a non-useful state

Let us take an entangled state of the form given in equation (30) for $d = 2$ and $N \geqslant 3$ such that equation (31) does not hold, which means that the state is non-useful ($g \leqslant 1$). We note that such a state can even be arbitrarily close to the fully polarized state. Despite being non-useful, according to result 1, just by having several copies, we can reach metrologically useful GME with the above state ($g = N$). For further details see supplement C. Thus, metrologically useful GME can be detected in these states [45, 46].

The related problem of activating GME is at the center of attention in entanglement theory [57–59]. Note, however, that in our examples all states have been GME even in the single-copy case. It remains an important open problem whether metrologically useful GME can be activated using several copies of a quantum state without GME.

4.5. Mixtures of W and $\overline{W}$ states

Now, in order to demonstrate that states living outside of the subspace described in result 1 can also be improved with our multicopy scheme let us consider a mixture of W state defined as

$$|W\rangle = \frac{1}{\sqrt{N}}\left(|100\ldots00\rangle + |010\ldots00\rangle + \cdots + |000\ldots01\rangle\right), \tag{39}$$

and the $\overline{W}$ state given as

$$|\overline{W}\rangle = \frac{1}{\sqrt{N}}\left(|011\ldots11\rangle + |101\ldots11\rangle + \cdots + |111\ldots10\rangle\right). \tag{40}$$

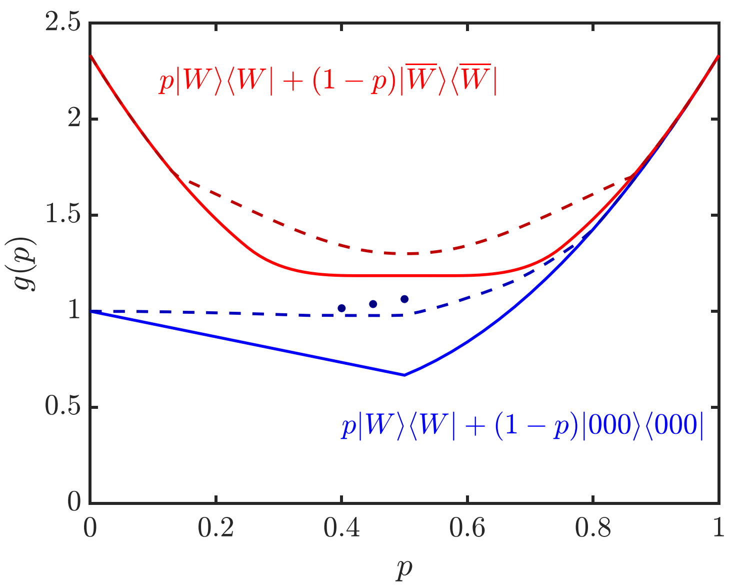

Figure 3. Dependence of the metrological gain on p for three-qubit mixtures of the W state and the $\overline{W}$ state and the three-qubit product state $|000\rangle$ given in equation (42). (solid) Single copy and (dashed) two copies. (top two curves) States with $r=0$, that is, mixtures of the three-qubit W and $\overline{W}$ states. (bottom two curves) States with $q=0$, mixtures of the three-qubit W state and $|000\rangle$. (Individual dots) Three-copy case for $q=0$ and $p=0.4, 0.45$, and 0.5. For all of them, $g>1$ holds.

It is known that for three-qubit pure states, genuine multipartite entangled states can be equivalent to GHZ states or W states under stochastic LOCC (SLOCC) [34]. Thus, W states represent a type of entanglement very different from that of GHZ states.

Let us examine first the $N=3$ case. Using the numerical methods of [47] that maximize the gain over Hamiltonians, we find that the optimal Hamiltonian for the $|W\rangle$ and the $|\overline{W}\rangle$ state is

$$\mathcal{H}=\sum_{n=1}^{N}\sigma_x^{(n)}. \tag{41}$$

Now, let us now consider a broader family of states given by the mixture

$$\varrho_{p,q}=p|W\rangle\langle W|+q|\overline{W}\rangle\langle\overline{W}|+r|000\rangle\langle 000|, \tag{42}$$

where $p,q,r\geqslant 0$ and $p+q+r=1$. First, let us examine the $r=0$ case. Such states have been studied for odd N, since they are genuine multipartite entangled, still possess no multipartite correlations for $p=1/2$ [82–84]. We find that the metrological gain for states given in equation (42) for $N=3$ is minimal for $p=1/2$ and the optimal Hamiltonian is

$$\mathcal{H}=\sigma_z^{(1)}+\sigma_z^{(2)}-\sigma_z^{(3)}, \tag{43}$$

which is translationally not invariant. For $p\approx 1$ and for $p\approx 0$ the maximal gain for two copies is the same as for a single copy, while for intermediate p values the gain for two copies is larger, as can be seen in figure 3.

Next, let us consider states given in equation (42) in the $q=0$ case. The maximal gain for two copies is always larger than for a single copy for $0<p<1$ as can be seen in figure 3. We also tested the three-copy case for some p values for which $g\leqslant 1$. The metrological gain increases and states around $p=0.5$ are activated. Note that the state corresponding to $p=0.5$ is the state obtained from a four-qubit W-state, after a particle is lost. Thus, we can make such a state useful, if several copies are available.

For $N=4$ qubits and for a single copy, we find that the optimal Hamiltonian is of the type given in equation (41), for states $|W\rangle$, $|\overline{W}\rangle$, and $\varrho_{1/2,1/2}$. In such cases, the calculation of the quantum Fisher information can be simplified as described in supplement D.

Note that W states have been created experimentally, for instance, in trapped cold ions and photons [20, 85], while quantum metrology has also been considered with the W and $\overline{\text{W}}$ states [86]. The type of noise considered can also be realized experimentally. We will discuss later the experiments creating GHZ states.

4.6. Cluster states

In this section, we study cluster states. They are highly entangled and can be used as a resource in measurement-based quantum computing [87, 88]. Certain type of cluster states are known to be useless in the single-copy case in linear interferometers. Here, we show that surprisingly they remain useless even for multicopy metrology.

Result 4. The ring cluster states for $N \geqslant 5$ are not useful even in the limit of large number of copies.

Proof. Since the state is pure, the quantum Fisher information equals the variance times four. Let us consider an N-qubit ring cluster state $|R_N\rangle$, which is defined by the equations

$$\sigma_z^{(n-1)}\sigma_x^{(n)}\sigma_z^{(n+1)}|R_N\rangle = |R_N\rangle \tag{44}$$

for $n = 1, 2, \ldots, N$, where $\sigma_z^{(0)} \equiv \sigma_z^{(N)}$ and $\sigma_z^{(N+1)} \equiv \sigma_z^{(1)}$. For ring cluster states for $N \geqslant 5$ all two-qubit reduced density matrices are the completely mixed state [48]. Hence, for the reduced two-qubit states $\langle \sigma_k \otimes \sigma_l \rangle = 0$ holds for $k, l = x, y, z$, while for all reduced single-qubit states $\langle \sigma_l \rangle = 0$ holds for $l = x, y, z$. It is easy to see that this statement is also true for the multicopy case. Due to that, the variance equals the variance of the completely mixed state and can be obtained as

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 4(\Delta\mathcal{H})^2_{\varrho_{\text{cm}}} = 4\sum_{n=1}^{N} \text{Tr}\left(h_n^2\right)/2^M, \tag{45}$$

where we assumed $\text{Tr}(h_n) = 0$, and h_n are now $2^M \times 2^M$ matrices representing operators acting on A_n. The quantum Fisher information given in equation (45) is never larger than the separable limit in equation (6) based on the well-known relation [89, 90]

$$\sum_{k=1}^{2^M} \lambda_k^2(h_n)/2^M \leqslant [\lambda_{\max}(h_n) - \lambda_{\min}(h_n)]^2/4, \tag{46}$$

where $\lambda_k(h_n)$ denote the eigenvalues of h_n. □

4.7. Two-qubit isotropic state

Here, we consider M copies of the two-qubit isotropic state, which is defined as [91, 92]

$$\varrho_{\text{iso}}(p) = p|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{\mathbb{1}}{4}, \tag{47}$$

where p is a noise parameter, and the maximally entangled state is

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{48}$$

For the Hamiltonian, let us choose $h_n = \sigma_z^{\otimes M}$ for $n = 1, 2$.

The results are illustrated in figure 4 for two different noise parameters. From this, we can see that there is an optimal number of copies above which the metrological performance does not improve. In the example with lower noise, the state is useful and the performance improves for $M = 2, 3$ copies, but the gain starts to decrease from $M = 4$ copies. In the example with higher noise, the quantum Fisher information is increasing with M but still does not overcome the separable limit.

To support the above observations from figure 4, we can obtain a general upper bound on the quantum Fisher information as

$$\mathcal{F}_Q \leqslant 4\langle\mathcal{H}^2\rangle = 4\sum_n w_n^2 + 4\sum_{n\neq n'}\langle h_n \otimes h_{n'}\rangle \leqslant 4\left(\sum_n w_n\right)^2, \tag{49}$$

where we used that

$$h_n^2 = w_n^2\mathbb{1}, \quad \mathcal{F}_Q^{(\text{sep})}(\mathcal{H}) = 4\sum_n w_n^2 \tag{50}$$

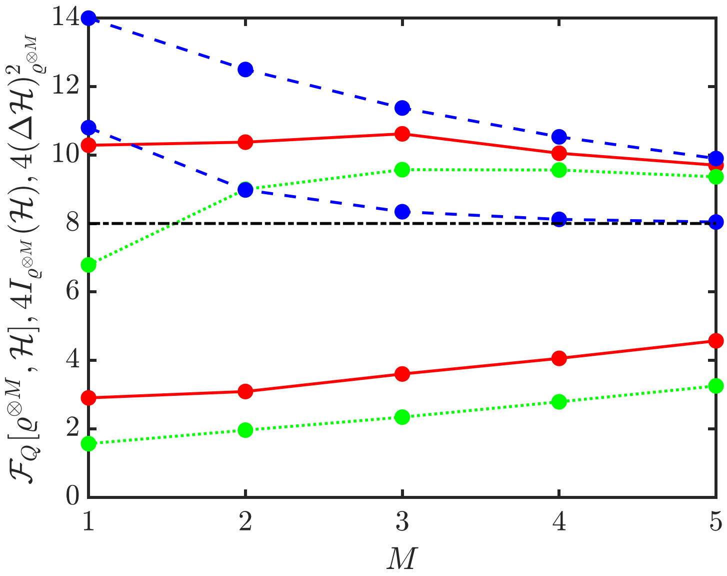

Figure 4. The quantum Fisher information, and the upper and lower bounds given in equation (10) as a function of the number of copies M for the isotropic state of two qubits with $h_n=\sigma_z^{\otimes M}$. (solid) $\mathcal{F}_Q[\varrho_{\rm iso}^{\otimes M},\mathcal{H}]$, (dashed) the variance $4(\Delta\mathcal{H})^2_{\varrho_{\rm iso}^{\otimes M}}$, (dotted) $4I_{\varrho_{\rm iso}^{\otimes M}}(\mathcal{H})$ are plotted, as well as (dashed dotted) the maximal quantum Fisher information for separable states, $\mathcal{F}_Q^{(\rm sep)}(\mathcal{H})$. The noise parameter of the isotropic state is (top three curves) $p=0.75$ and (bottom three curves) $p=0.35$.

hold, where $w_n>0$ is some constant[11]. Then, we have

$$\langle h_n\otimes h_{n'}\rangle=\left(\langle\sigma_z^{(n)}\otimes\sigma_z^{(n')}\rangle_\varrho\right)^M. \tag{51}$$

Thus, if $\langle\sigma_z^{(n)}\otimes\sigma_z^{(n')}\rangle_\varrho<1$ then the upper bound on the quantum Fisher information given in equation (49) is going to the separable limit for large M.

5. Relation to the bitflip code

Even if the initial state is ideally within the desired subspace of result 1, in practice, due to imperfect preparation or noise during the dynamics it can eventually be outside, where metrologically useful GME activation is no longer guaranteed. Here, we suggest a method to transform states not living in the subspace of result 1 into the subspace of result 1 based on ideas stemming from quantum error correction. This is to make sure that they can achieve maximal metrological performance with our scheme.

Error correction has been considered for quantum metrology [93–95]. Using a bitflip error correcting code, a single qubit is stored in many qubits in the subspace mentioned in result 1 [80, 96–98]. The syndrome measurements and restoring steps of the bitflip code can be used to move the state into the desired subspace even in our case. Note that we do not need to restore or protect a given quantum state, which is usually the case in error correction. We need only to obtain a quantum state that has a large metrological gain. The error correcting step mentioned above can be carried out initially, and also throughout the dynamics. We can even avoid applying the correcting step. It is sufficient to employ a different Hamiltonian to states with different syndrome measurement results. We also analyze further relations to error correction in supplement E.

GHZ states have been realized in trapped cold ions [16, 17, 25, 31], as well as error correction has also been carried out [99–102]. The GHZ states created have errors both in the diagonal and the off-diagonal elements of the density matrix when given in the computational basis. During the metrology, the main type of error is the decay of the off-diagonal elements [25]. Errors in the off-diagonal elements can be mitigated by multicopy metrology using $h_n=\sigma_z^{\otimes M}$, while errors in the diagonal elements can be overcome by the error correction scheme above. Even if the error correcting steps are not applied, the multicopy scheme can

[11] It is sufficient to consider the case when $h_n^2\propto\mathbb{1}$ for $n=1,2,..,N$. If an operator h_n is not of the above form, then we can consider the local Hamiltonians h_n' with eigenvalues $\pm\lambda_{\max}(h_n')$ and eigenvectors identical to those of h_n, which fulfill $(h_n')^2\propto\mathbb{1}$. It has been shown that the metrological gain of at least one of the Hamiltonians $\mathcal{H}$ constructed from the local Hamiltonians above is larger or equal to that of the original Hamiltonian [47].

increase the metrological gain. Superconducting circuits have also been used to create GHZ states [24, 30] and implement error correction [103–108]. Error correction has recently been carried out in reconfigurable atom arrays [109]. Our proposal might offer a viable information processing scenario in the NISQ era [61], in which simple error mitigation technics are needed [110–121].

6. Conclusion

In summary, we have presented an approach to activate metrologically useful multipartite entanglement. If the state is in a certain subspace, then, even if it was weakly entangled, it becomes maximally useful compared to separable states in the limit of large number of copies, hence it will possess metrologically useful GME. Operations similar to the ones applied in error correction can be used to force the state into the desirable subspace. Our method involves simple measurements and can immediately be tested in present day quantum devices requiring moderate resources.

We have also shown that our scheme can improve the metrological performance of states living outside the above-mentioned subspace, like, for example for the two-qubit isotropic state. Moreover, we have also demonstrated the possibility of improving metrological performance by embedding quantum states locally into higher dimensions.

Deciding whether a quantum state is entangled or not is a hard problem, apart from small quantum systems. Deciding whether a quantum state possess metrologically useful entanglement for a given Hamiltonian is an easy task [1]. However, deciding whether a state is metrologically useful in general needs an optimization over all local Hamiltonians [47]. It would be interesting to clarify whether this task is also computationally hard.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Metrological gain and multipartite entanglement

We generalize the findings of [45, 46] from qubits to higher dimensional systems. Let us assume that $\lambda_{\max}(h_n)$ and $\lambda_{\min}(h_n)$ are identical for all parties $n = 1, 2, \ldots, N$. Then, based on equation (6),

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = N\Lambda^2 \tag{A.1}$$

holds, where we define the difference between the largest and the smallest eigenvalue as

$$\Lambda = \lambda_{\max}(h_1) - \lambda_{\min}(h_1). \tag{A.2}$$

Let us consider a pure state of the form

$$\otimes_l |\Psi_{k_l}\rangle, \tag{A.3}$$

where $|\Psi_{k_l}\rangle$ denotes a pure state of k_l qudits and let us define the entanglement depth as $k = \max_l k_l$. Then, for the quantum Fisher information we obtain that a series of inequalities

$$\mathcal{F}_Q[\varrho, \mathcal{H}] \leqslant \Lambda^2 \sum_l k_l^2 \leqslant \Lambda^2 \left[sk^2 + (N - sk)^2\right] \leqslant \Lambda^2 Nk, \tag{A.4}$$

where $s = \lfloor N/k \rfloor$. The first inequality is based on [45, 46]. For the second one, we need to prove that (page 68, [122])

$$sk^2 + (N - sk)^2 \leqslant Nk. \tag{A.5}$$

By substracting sk^2 from both sides of equation (A.5), we arrive at

$$(N - sk)^2 \leqslant k(N - sk). \tag{A.6}$$

Equation (A.6) is evidently true, knowing that $0 \leqslant N - sk < k$.

Due to the convexity of the quantum Fisher information, the same bound holds also for quantum states that are mixtures of pure states with an entanglement depth at most k. Hence, we obtain that

$$g > k \tag{A.7}$$

implies $(k+1)$-partite entanglement. From equation (A.4), a more complicated, but somewhat stronger relation also follows

$$g > \left[sk^2 + (N - sk)^2\right]/N. \tag{A.8}$$

Appendix B. Optimal measurements for the GME activatable subspace in result 1

First, we will present the measurement operators explicitly for the N-qubit state

$$\varrho(p, q, r) = p|\mathrm{GHZ}_q\rangle\langle \mathrm{GHZ}_q| + (1 - p)\left[r(|0\rangle\langle 0|)^{\otimes N} + (1 - r)(|1\rangle\langle 1|)^{\otimes N}\right], \tag{B.1}$$

where we choose a parametrization convenient for our derivation. Here, $0 < p \leqslant 1$, $0 \leqslant r \leqslant 1$, and the generalized GHZ state is [79]

$$|\mathrm{GHZ}_q\rangle = \sqrt{q}|000..00\rangle + \sqrt{1 - q}|111..11\rangle, \tag{B.2}$$

where $0 < q < 1$ is real. Thus, the state $\varrho(p, q, r)$ turns out to be maximally useful in the limit of very many copies if $p > 0$ and $0 < q < 1$, otherwise the state is a separable state.

The error propagation formula, essentially, characterizes the uncertainty of the parameter estimation if we measure $\mathcal{M}$ as

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M}, \mathcal{H}]\rangle^2}. \tag{B.3}$$

The minimum is taken when $\mathcal{M}$ equals the symmetric logarithmic derivative, $\mathcal{M}_{\mathrm{opt}}$ which can be obtained from an explicit formula for a given ϱ and $\mathcal{H}$ [64–68]. For $M = 1$, it is well known that [16, 17, 25]

$$\mathcal{M}_{\mathrm{opt}} = \sigma_x^{\otimes N}. \tag{B.4}$$

On the other hand, for $M = 2$ and $N = 3$, for $q = r = 1/2$ the optimal operator fulfills the following relations

$$\begin{aligned} \langle\overline{00}|\mathcal{M}_{\mathrm{opt}}|\overline{00}\rangle &= 0, & \langle\overline{00}|\mathcal{M}_{\mathrm{opt}}|\overline{11}\rangle &= 0, \\ \langle\overline{01}|\mathcal{M}_{\mathrm{opt}}|\overline{11}\rangle &= -i, & \langle\overline{00}|\mathcal{M}_{\mathrm{opt}}|\overline{01}\rangle &= +i, \end{aligned} \tag{B.5}$$

where $|\overline{0}\rangle = |000\rangle$ and $|\overline{1}\rangle = |111\rangle$. Based on these, one can obtain the optimal operator as

$$\mathcal{M}_{\mathrm{opt}} = \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} + \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y. \tag{B.6}$$

Each correlation term is inverting the qubits at one of the copies and adds an additional phase conditioned on the state of the other copy.

Based on these observations, with straightforward algebra, it is easy to see that the operator, given by equation (22) leads to the following error propagation formula

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1/[4q(1-q)] + (M-1)p^2}{4MN^2p^2}. \tag{B.7}$$

If the condition

$$1/[4q(1-q)] \ll (M-1)p^2 \tag{B.8}$$

is fulfilled and $M \gg 1$ holds, we have

$$(\Delta\theta)^2_{\mathcal{M}} \approx \frac{1}{4N^2}, \tag{B.9}$$

which corresponds to the Heisenberg limit, which is the best achievable precision. Thus, measuring the operators, we can reach the Heisenberg limit for all the entangled states of the subspace defined in result 1 in the multiqubit case.

Thus, for a given noise level p and for a given value of the parameter q, we need

$$M \approx \frac{1}{4q(1-q)p^2} \tag{B.10}$$

copies, which already leads to an almost optimal precision. Much more copies will increase the precision somewhat, but will not lead to a much better performance. Moreover, the number of copies needed, M, does not increase even if N is increasing. Note that the operator given by equation (22) is not the optimal one for $M \geqslant 3$. The optimal operator contains more correlation terms.

The c matrix used in the proof of result 1 can be given for the multiqubit state given in equation (B.1) as

$$c = \begin{pmatrix} (1-p)r + pq & p\sqrt{(1-q)q} \\ p\sqrt{(1-q)q} & (1-p)(1-r) + p(1-q) \end{pmatrix}. \tag{B.11}$$

This shows that we considered the most general c with the only exception that c_{01} is real and positive. The variable c_{01} can always be made real and positive with an appropriate basis transformation

$$|1\rangle \to \exp(-i\phi)|1\rangle, \tag{B.12}$$

where ϕ is an angle. Consequently, the $\mathcal{M}$ operator corresponding to a state with a complex or a negative c_{01} can be obtained by carrying out the inverse of such a basis transformation on σ_x and σ_y in the definition given in equation (22).

Let us now consider the case of higher even dimensions d. Let us consider a concrete example, a state of higher dimensional qudits that is analogous to a GHZ state given as

$$|\mathrm{A}_d\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k\rangle^{\otimes N}, \tag{B.13}$$

where d is the dimensional of the qudits. We can obtain $\mathcal{M}$ and all the h_n operators for metrology with the state given in equation (B.13) from the operators used in the qubit case using the substitution

$$\sigma_l \to \oplus_{k=1}^{d/2}\sigma_l \tag{B.14}$$

for $l = x, y, z$. After the transformation, the operator to be measured is of the form given in equation (22), where X, Y, and Z are defined as

$$Y = \begin{cases} \left(\oplus_{k=1}^{d/2}\sigma_y\right)^{\otimes N} & \text{for odd } N, \\ \oplus_{k=1}^{d/2}\sigma_x \otimes \left(\oplus_{k=1}^{d/2}\sigma_y\right)^{\otimes(N-1)} & \text{for even } N, \end{cases}$$
$$Z = \left(\oplus_{k=1}^{d/2}\sigma_z\right) \otimes \mathbb{1}_d^{\otimes(N-1)}, \tag{B.15}$$

where $\mathbb{1}_d$ is a $d \times d$ identity matrix. Note that

$$\oplus_{k=1}^{d/2}\sigma_z = D, \tag{B.16}$$

where D appears in the definition of the Hamiltonian given in equation (16). Thus, we really map the Hamiltonian given in result 1 for qubit systems to the Hamiltonian given in result 1 for qudit systems. The qubit states can also be mapped to qudit states as

$$|0\rangle \to |\tilde{0}\rangle = \frac{1}{\sqrt{d/2}}\left(|0\rangle + |2\rangle + |4\rangle + \ldots\right), \tag{B.17}$$

and

$$|1\rangle \to |\tilde{1}\rangle = \frac{1}{\sqrt{d/2}}\left(|1\rangle + |3\rangle + |5\rangle + \ldots\right). \tag{B.18}$$

Based on the transformations above, the multiqubit state given in equation (24) corresponds to the multiqudit state

$$p|A_d\rangle\langle A_d| + (1-p)\,\frac{\left(|\tilde{0}\rangle\langle\tilde{0}|\right)^{\otimes N} + \left(|\tilde{1}\rangle\langle\tilde{1}|\right)^{\otimes N}}{2}. \tag{B.19}$$

For the $\mathcal{M}$ and $\mathcal{H}$ obtained via equation (B.14) from the operators used for the qubit case, for the error propagation formula we obtain the same as in the qubit case we get for the state given in equation (24), i.e. equation (B.7) for $q = 1/2$. Thus, the setup reaches the Heisenberg limit given in equation (B.9) in the case of sufficiently large M, if $p > 0$.

Finally, let us consider the noisy state

$$p|A_d\rangle\langle A_d| + (1-p)\,\frac{1}{d}\sum_{k=0}^{d-1}|k\rangle\langle k|^{\otimes N}. \tag{B.20}$$

The error propagation formula with the transformed $\mathcal{M}$ and $\mathcal{H}$ give the same result for the state given in equation (B.20) as for the state given in equation (B.19), since the operators appearing in the error propagation formula cannot distinguish the two states from each other. In particular, they cannot distinguish the superposition of $|n\rangle$ and $|n+2\rangle$ from their mixture. Thus, even for the state given in equation (B.20), the error propagation formula is given by equation (B.7) for $q = 1/2$ and the setup reaches the Heisenberg limit given in equation (B.9) in the case of sufficiently large M, if $p > 0$.

Appendix C. GHZ-like states

The condition for the usefulness of multipartite states given in equation (30) for $d = 2$ and for $N \geqslant 3$ has been presented already in [48], considering Hamiltonians given in equation (7), where $c_{l,n}$ are real numbers, and $|\vec{c}_n| = 1$. For completeness, we present a very short proof, which also includes the case $|\vec{c}_n| \neq 1$.

Result 5. A single copy of the state given in equation (30) for $d = 2$ is useful metrologically if and only if equation (31) holds.

Proof. Let us consider local Hamiltonians of the type given in equation (7), where $|\vec{c}_n| = L_n$. For this case, we obtain

$$\begin{aligned}\langle\mathcal{H}^2\rangle &= \sum_n L_n^2 + \left(\sum_n c_{z,n}\right)^2 - \left(\sum_n c_{z,n}^2\right),\\ \langle\mathcal{H}\rangle^2 &= \left(\sum_n c_{z,n}\right)^2(1-E).\end{aligned} \tag{C.1}$$

Let us first assume that $E > 1/N$. Then, we have the series of inequalities

$$\begin{aligned} g_{\mathcal{H}} &= 1 + \frac{E\left(\sum_n c_{z,n}\right)^2 - \sum_n c_{z,n}^2}{\sum_n L_n^2} \leqslant 1 + \frac{(E-1/N)\left(\sum_n c_{z,n}\right)^2}{\sum_n L_n^2}\\ &\leqslant 1 + \frac{(E-1/N)\left(\sum_n L_n\right)^2}{\sum_n L_n^2} \leqslant N\cdot E. \end{aligned} \tag{C.2}$$

In the first inequality, we used the inequality between the arithmetic and quadratic means for the $c_{z,n}$, $\left(\sum_n c_{z,n}\right)^2/N \leqslant \sum_n c_{z,n}^2$. In the third inequality, we used the same relation for L_n. All inequalities are saturated if $c_{z,n} = L_n$ for all n and they are all equal to each other. Thus, in this case we have $g_{\mathcal{H}} > 1$ for a certain choice of local $\mathcal{H}$, hence the state is useful.

Next, let us consider the $E \leqslant 1/N$ case. Now, the first inequality in equation (C.2) is still valid and it leads to $g_{\mathcal{H}} \leqslant 1$ for any choice of local $\mathcal{H}$, hence the state is not useful.

Thus, we obtain that the state is useful if and only if equation (31) is fulfilled. □

Note also that for odd N, states given in equation (30) for $d = 2$ do not violate Bell inequalities with full correlation terms and two two-outcome observables per party if $E < 1/2^{N-1}$ [123] (cf equation (31)).

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References

[1] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865–942
[2] Gühne O and Tóth G 2009 *Phys. Rep.* **474** 1–75
[3] Friis N, Vitagliano G, Malik M and Huber M 2019 *Nat. Rev. Phys.* **1** 72–87
[4] Julsgaard B, Kozhekin A and Polzik E S 2001 *Nature* **413** 400–3
[5] Esteve J, Gross C, Weller A, Giovanazzi S and Oberthaler M 2008 *Nature* **455** 1216–9
[6] Behbood N, Martin Ciurana F, Colangelo G, Napolitano M, Tóth G, Sewell R J and Mitchell M W 2014 *Phys. Rev. Lett.* **113** 093601
[7] Bohnet J G, Sawyer B C, Britton J W, Wall M L, Rey A M, Foss-Feig M and Bollinger J J 2016 *Science* **352** 1297–301
[8] Kong J, Jiménez-Martínez R, Troullinou C, Lucivero V G, Tóth G and Mitchell M W 2020 *Nat. Commun.* **11** 2415
[9] Riedel M F, Böhi P, Li Y, Hänsch T W, Sinatra A and Treutlein P 2010 *Nature* **464** 1170–3
[10] Gross C, Zibold T, Nicklas E, Esteve J and Oberthaler M K 2010 *Nature* **464** 1165–9
[11] Lücke B, Peise J, Vitagliano G, Arlt J, Santos L, Tóth G and Klempt C 2014 *Phys. Rev. Lett.* **112** 155304
[12] McConnell R, Zhang H, Hu J, Ćuk S and Vuletić V 2015 *Nature* **519** 439–42
[13] Hosten O, Engelsen N J, Krishnakumar R and Kasevich M A 2016 *Nature* **529** 505–8
[14] Zou Y-Q, Wu L-N, Liu Q, Luo X-Y, Guo S-F, Cao J-H, Tey M K and You L 2018 *Proc. Natl Acad. Sci. USA* **115** 6381–5
[15] Xin L, Barrios M, Cohen J T and Chapman M S 2023 *Phys. Rev. Lett.* **131** 133402
[16] Sackett C *et al* 2000 *Nature* **404** 256–9
[17] Leibfried D *et al* 2005 *Nature* **438** 639–42
[18] Kiesel N, Schmid C, Weber U, Tóth G, Gühne O, Ursin R and Weinfurter H 2005 *Phys. Rev. Lett.* **95** 210502
[19] Walther P, Resch K J, Rudolph T, Schenck E, Weinfurter H, Vedral V, Aspelmeyer M and Zeilinger A 2005 *Nature* **434** 169–76
[20] Häffner H *et al* 2005 *Nature* **438** 643–6
[21] Kiesel N, Schmid C, Tóth G, Solano E and Weinfurter H 2007 *Phys. Rev. Lett.* **98** 063604
[22] Wieczorek W, Krischek R, Kiesel N, Michelberger P, Tóth G and Weinfurter H 2009 *Phys. Rev. Lett.* **103** 020504
[23] Prevedel R, Cronenberg G, Tame M S, Paternostro M, Walther P, Kim M S and Zeilinger A 2009 *Phys. Rev. Lett.* **103** 020503
[24] Song C *et al* 2017 *Phys. Rev. Lett.* **119** 180511
[25] Monz T, Schindler P, Barreiro J T, Chwalla M, Nigg D, Coish W A, Harlander M, Hänsel W, Hennrich M and Blatt R 2011 *Phys. Rev. Lett.* **106** 130506
[26] Huang Y-F, Liu B-H, Peng L, Li Y-H, Li L, Li C-F and Guo G-C 2011 *Nat. Commun.* **2** 546
[27] Wang X-L *et al* 2018 *Phys. Rev. Lett.* **120** 260502
[28] Friis N *et al* 2018 *Phys. Rev.* X **8** 021012
[29] Gong M *et al* 2019 *Phys. Rev. Lett.* **122** 110501
[30] Song C *et al* 2019 *Science* **365** 574–7
[31] Pogorelov I *et al* 2021 *PRX Quantum* **2** 020343
[32] Thomas P, Ruscio L, Morin O and Rempe G 2022 *Nature* **608** 677–81
[33] Cao S *et al* 2023 *Nature* **619** 738–42
[34] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[35] Acín A, Bru D, Lewenstein M and Sanpera A 2001 *Phys. Rev. Lett.* **87** 040401
[36] Sørensen A S and Mølmer K 2001 *Phys. Rev. Lett.* **86** 4431–4
[37] Gühne O, Tóth G and Briegel H J 2005 *New J. Phys.* **7** 229
[38] Vitagliano G, Apellaniz I, Kleinmann M, Lücke B, Klempt C and Tóth G 2017 *New J. Phys.* **19** 013027
[39] Vitagliano G, Colangelo G, Martin Ciurana F, Mitchell M W, Sewell R J and Tóth G 2018 *Phys. Rev.* A **97** 020301
[40] Tóth G and Gühne O 2005 *Phys. Rev. Lett.* **94** 060501
[41] Pezzé L and Smerzi A 2009 *Phys. Rev. Lett.* **102** 100401
[42] Czekaj Ł, Przysieżna A, Horodecki M and Horodecki P 2015 *Phys. Rev.* A **92** 062303
[43] Tóth G and Vértesi T 2018 *Phys. Rev. Lett.* **120** 020506
[44] Pál K F, Tóth G, Bene E and Vértesi T 2021 *Phys. Rev. Res.* **3** 023101
[45] Tóth G 2012 *Phys. Rev.* A **85** 022322
[46] Hyllus P, Laskowski W, Krischek R, Schwemmer C, Wieczorek W, Weinfurter H, Pezzé L and Smerzi A 2012 *Phys. Rev.* A **85** 022321

[47] Tóth G, Vértesi T, Horodecki P and Horodecki R 2020 *Phys. Rev. Lett.* **125** 020402
[48] Hyllus P, Gühne O and Smerzi A 2010 *Phys. Rev.* A **82** 012337
[49] Lücke B *et al* 2011 *Science* **334** 773–6
[50] Krischek R, Schwemmer C, Wieczorek W, Weinfurter H, Hyllus P, Pezzé L and Smerzi A 2011 *Phys. Rev. Lett.* **107** 080504
[51] Strobel H, Muessel W, Linnemann D, Zibold T, Hume D B, Pezzé L, Smerzi A and Oberthaler M K 2014 *Science* **345** 424–7
[52] Horodecki P, Horodecki M and Horodecki R 1999 *Phys. Rev. Lett.* **82** 1056–9
[53] Navascués M and Vértesi T 2011 *Phys. Rev. Lett.* **106** 060403
[54] Palazuelos C 2012 *Phys. Rev. Lett.* **109** 190401
[55] Shor P W, Smolin J A and Thapliyal A V 2003 *Phys. Rev. Lett.* **90** 107901
[56] Nawareg M, Muhammad S, Horodecki P and Bourennane M 2017 *Sci. Adv.* **3** e1602485
[57] Yamasaki H, Morelli S, Miethlinger M, Bavaresco J, Friis N and Huber M 2022 *Quantum* **6** 695
[58] Palazuelos C and de Vicente J I 2022 *Quantum* **6** 735
[59] Morelli S, Sauerwein D, Skotiniotis M and Friis N 2022 *Quantum* **6** 722
[60] Contreras-Tejada P, Palazuelos C and de Vicente J I 2021 *Phys. Rev. Lett.* **126** 040501
[61] Preskill J 2018 *Quantum* **2** 79
[62] Petz D 2008 *Quantum Information Theory and Quantum Statistics* (Springer)
[63] Giovannetti V, Lloyd S and Maccone L 2004 *Science* **306** 1330–6
[64] Paris M G A 2009 *Int. J. Quantum Inf.* **07** 125–37
[65] Demkowicz-Dobrzanski R, Jarzyna M and Kolodynski J 2015 *Prog. Opt.* **60** 345–435
[66] Pezze L and Smerzi A 2014 Quantum theory of phase estimation *Atom Interferometry (Proc. Int. School of Physics 'Enrico Fermi', Course 188, Varenna)* ed G Tino and M Kasevich (IOS Press) pp 691–741
[67] Tóth G and Apellaniz I 2014 *J. Phys. A: Math. Theor.* **47** 424006
[68] Pezzè L, Smerzi A, Oberthaler M K, Schmied R and Treutlein P 2018 *Rev. Mod. Phys.* **90** 035005
[69] Braun D, Adesso G, Benatti F, Floreanini R, Marzolino U, Mitchell M W and Pirandola S 2018 *Rev. Mod. Phys.* **90** 035006
[70] Sidhu J S and Kok P 2020 *AVS Quantum Sci.* **2** 014701
[71] Polino E, Valeri M, Spagnolo N and Sciarrino F 2020 *AVS Quantum Sci.* **2** 024703
[72] Barbieri M 2022 *PRX Quantum* **3** 010202
[73] Helstrom C 1976 *Quantum Detection and Estimation Theory* (Academic)
[74] Holevo A 1982 *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland)
[75] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72** 3439–43
[76] Braunstein S L, Caves C M and Milburn G J 1996 *Ann. Phys., NY* **247** 135–73
[77] Ciampini M A, Spagnolo N, Vitelli C, Pezzè L, Smerzi A and Sciarrino F 2016 *Sci. Rep.* **6** 28881
[78] Wigner E P and Yanase M M 1963 *Proc. Natl Acad. Sci. USA* **49** 910–8
[79] Greenberger D M, Horne M A, Shimony A and Zeilinger A 1990 *Am. J. Phys.* **58** 1131–43
[80] Nielsen M A and Chuang I L 2011 *Quantum Computation and Quantum Information: 10th Anniversary Edition* 10th edn (Cambridge University Press)
[81] Morris B, Yadin B, Fadel M, Zibold T, Treutlein P and Adesso G 2020 *Phys. Rev.* X **10** 041012
[82] Kaszlikowski D, Sen(De) A, Sen U, Vedral V and Winter A 2008 *Phys. Rev. Lett.* **101** 070502
[83] Bennett C H, Grudka A, Horodecki M, Horodecki P and Horodecki R 2011 *Phys. Rev.* A **83** 012312
[84] Schwemmer C, Knips L, Tran M C, de Rosier A, Laskowski W, Paterek T and Weinfurter H 2015 *Phys. Rev. Lett.* **114** 180501
[85] Eibl M, Kiesel N, Bourennane M, Kurtsiefer C and Weinfurter H 2004 *Phys. Rev. Lett.* **92** 077901
[86] Li Y and Ren Z 2023 *Phys. Rev.* A **107** 012403
[87] Briegel H J and Raussendorf R 2001 *Phys. Rev. Lett.* **86** 910–3
[88] Raussendorf R and Briegel H J 2001 *Phys. Rev. Lett.* **86** 5188–91
[89] Popoviciu T 1935 *Mathematica* **9** 20
[90] Bhatia R and Davis C 2000 *Am. Math. Mon.* **107** 353–7
[91] Horodecki M and Horodecki P 1999 *Phys. Rev.* A **59** 4206–16
[92] Horodecki M, Horodecki P and Horodecki R 1999 *Phys. Rev.* A **60** 1888–98
[93] Dür W, Skotiniotis M, Fröwis F and Kraus B 2014 *Phys. Rev. Lett.* **112** 080801
[94] Kessler E M, Lovchinsky I, Sushkov A O and Lukin M D 2014 *Phys. Rev. Lett.* **112** 150802
[95] Zhou S, Zhang M, Preskill J and Jiang L 2018 *Nat. Commun.* **9** 78
[96] Shor P W 1995 *Phys. Rev.* A **52** R2493–6
[97] Steane A 1996 *Proc. R. Soc.* A **452** 2551–77
[98] Gottesman D 1996 *Phys. Rev.* A **54** 1862–8
[99] Chiaverini J *et al* 2004 *Nature* **432** 602–5
[100] Schindler P, Barreiro J T, Monz T, Nebendahl V, Nigg D, Chwalla M, Hennrich M and Blatt R 2011 *Science* **332** 1059–61
[101] Egan L *et al* 2021 *Nature* **598** 281–6
[102] Postler L *et al* 2022 *Nature* **605** 675–80
[103] Reed M D, DiCarlo L, Nigg S E, Sun L, Frunzio L, Girvin S M and Schoelkopf R J 2012 *Nature* **482** 382–5
[104] Córcoles A D, Magesan E, Srinivasan S J, Cross A W, Steffen M, Gambetta J M and Chow J M 2015 *Nat. Commun.* **6** 6979
[105] Ofek N *et al* 2016 *Nature* **536** 441–5
[106] Zhao Y *et al* 2022 *Phys. Rev. Lett.* **129** 030501
[107] Krinner S *et al* 2022 *Nature* **605** 669–74
[108] Acharya R *et al* 2023 *Nature* **614** 676–81
[109] Bluvstein D *et al* 2023 *Nature* (https://doi.org/10.1038/s41586-023-06927-3)
[110] Li Y and Benjamin S C 2017 *Phys. Rev.* X **7** 021050
[111] Temme K, Bravyi S and Gambetta J M 2017 *Phys. Rev. Lett.* **119** 180509
[112] Endo S, Benjamin S C and Li Y 2018 *Phys. Rev.* X **8** 031027
[113] Kandala A, Temme K, Córcoles A D, Mezzacapo A, Chow J M and Gambetta J M 2019 *Nature* **567** 491–5
[114] McArdle S, Yuan X and Benjamin S 2019 *Phys. Rev. Lett.* **122** 180501
[115] Koczor B 2021 *Phys. Rev.* X **11** 031057

[116] Yamamoto K, Endo S, Hakoshima H, Matsuzaki Y and Tokunaga Y 2022 *Phys. Rev. Lett.* **129** 250503
[117] Bharti K *et al* 2022 *Rev. Mod. Phys.* **94** 015004
[118] Rattew A G, Sun Y, Minssen P and Pistoia M 2021 *Quantum* **5** 609
[119] Czarnik P, Arrasmith A, Coles P J and Cincio L 2021 *Quantum* **5** 592
[120] Strikis A, Qin D, Chen Y, Benjamin S C and Li Y 2021 *PRX Quantum* **2** 040330
[121] Cerezo M *et al* 2021 *Nat. Rev. Phys.* **3** 625–44
[122] Tóth G 2021 Entanglement detection and quantum metrology in quantum optical systems *D Sc Thesis* Hungarian Academy of Sciences, Budapest (available at: http://real-d.mtak.hu/1230/7/dc_1593_18_doktori_mu.pdf)
[123] Żukowski M, Brukner Č, Laskowski W and Wieśniak M 2002 *Phys. Rev. Lett.* **88** 210402