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Multi-dimensional BSDEs with mean reflection*

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Abstract

In this paper, we consider multi-dimensional mean reflected backward stochastic differential equations (BSDEs) with possibly non-convex reflection domains along inward normal direction, which were introduced by Briand, Elie and Hu [6] in the scalar case. We first apply a fixed-point argument to establish the uniqueness and existence result under an additional bounded condition on the driver. Then, with the help of a priori estimates, we develop a successive approximation procedure to remove the additional bounded condition for the general case.

Keywords: mean reflected BSDE; non-convex domain; inward normal direction; a priori estimates.
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1 Introduction

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space under which B is a d-dimensional standard Brownian motion. Suppose $(\mathscr{F}_t)_{0\le t\le T}$ is the natural filtration generated by B augmented by the $\mathbf{P}$-null sets and $\mathcal{P}$ is the sigma algebra of all progressive sets of $\Omega\times[0,T]$. The present paper is devoted to the study of the following multi-dimensional BSDE with mean reflection over the time interval $[0,T]$,

$$\begin{cases} Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s + \eta_T - \eta_t, & \forall t\in[0,T], \\ \mathbf{E}[Y_t]\in\overline{D}, & \forall t\in[0,T], \end{cases} \tag{1.1}$$

in which the terminal condition ξ is an $\mathbb{R}^m$-valued $\mathscr{F}_T$-measurable random vector, the driver $f:\Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\to\mathbb{R}^m$ is a measurable map with respect to $\mathcal{P}\times$

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$\mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{R}^{m\times d})$ and the mean of the solution Y_t is forced to stay within a possibly non-convex domain $D \subset \mathbb{R}^m$. Our aim is to investigate the solvability of the mean reflected BSDE (1.1) under a natural Skorokhod type condition (see Definition 3.1), where the solution η is a deterministic function with bounded variation.

When the constraint is not on the mean of the solution but on the paths of the solution, i.e., $Y_t \in \overline{D}$, the reflected BSDE (1.1) was first introduced by El Karoui et al. [21] in the scalar case. Since then, great progress has been made in this field, as it has rich connections with partial differential equations (PDEs) and mathematical finance. For instance, the authors applied the reflected BSDEs to provide a probabilistic interpretation for an obstacle problem of parabolic PDEs in [21] and El Karoui, Pardoux and Quenez [22] found that the price of an American option could be represented as the unique solution to the reflected BSDE under a Skorokhod type condition. Moreover, various types of scalar-valued constrained BSDEs have been formulated due to their financial motivations. For examples, Buckdahn and Hu [9] used the constrained BSDEs driven by both a Wiener process and a Poisson random measure to analyze the option pricing with constrained portfolios in an incomplete market; Cvitanić and Karatzas [17] considered a type of Dynkin games via the BSDEs with two reflecting barriers. For more research on this field, we refer the reader to [18, 25, 26, 27, 28, 29, 37, 39] and the references therein.

In the reflected BSDEs theory, the research of the multidimensional case is significantly more difficult than that of the scalar case. In [24], Gegout-Petit and Pardoux first obtained the existence and uniqueness results for the case of normal reflection in convex domains, which was generalized by Klimsiak, Rozkosz and Słomiński to the case of time-dependent random convex regions in [34]. On the other hand, motivated by the optimal switching problems, certain types of multi-dimensional reflected BSDEs with oblique direction of reflection were also investigated, see e.g., [1, 12, 13, 14, 30, 33]. Recently, Chassagneux and Richou [16] established the solvability of general obliquely reflected BSDEs in convex reflection domains.

The theory of multidimensional reflected stochastic differential equations (SDEs) with non-convex reflection domains has been studied systematically (cf. [20, 36, 40]). However, there are only few papers dealing with the multidimensional reflected BSDEs with non-convex reflection domains due to the complicated structure. We would like to mention that [8, 23] considered certain types of multidimensional reflected BSDEs in non-convex domains. Recently, Chassagneux, Nadtochiy and Richou [15] first established well-posedness results for general multidimensional reflected BSDEs with non-convex domains satisfying a weak star-shape property.

In contrast with the aforementioned paths constraints, the BSDEs with mean constraints were formulated recently to analyze partial hedging of financial derivatives in mathematical finance. In order to deal with quantile hedging problems, Bouchard, Elie and Réveillac [2] considered a new type of BSDEs, where the terminal value satisfies a type of mean constraint. Inspired by this, Briand, Elie and Hu [6] introduced the scalar-valued BSDEs with mean reflection to investigate the super-hedging problem under running risk management constraint. In this framework, the solution Y is required to satisfy the following type of mean constraint:

$$\mathbf{E}\big[\ell(t, Y_t)\big] \geq 0, \quad \forall t \in [0, T],$$

where $\ell(t, \cdot)$ is a collection of (possibly random) non-decreasing real-valued map.

Subsequently, Hibon et al. [31] considered quadratic BSDEs with mean reflection and Hu, Moreau and Wang [32] dealt with generalized mean reflected BSDEs, whose drivers also depend on the law of the solution Y. With the help of interacting particles systems, [5, 7, 8] studied the approximation of mean reflected SDEs and BSDEs. We

also refer to Djehiche, Elie and Hamadène [19] which formulated a mean-filed type of reflected BSDEs motivated by applications in pricing life insurance contracts with surrender options.

Recently, Briand et al. [4] introduced multi-dimensional mean reflected BSDEs with normal reflection, in which the marginal probability distribution $\mathbf{P}_{Y_t}$ of the solution Y is required to stay within a subset of $\mathcal{P}_2(\mathbb{R}^m)$[1], i.e.,

$$\mathbf{P}_{Y_t} \in \big\{\mu \in \mathcal{P}_2\big(\mathbb{R}^m\big),\ H(\mu) \geq 0\big\},\ \ \forall t \in [0,T].$$

Here the function $H : \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}$ is concave. Based on this, the authors also studied the associated propagation of chaos and established a probabilistic interpretation for an obstacle problem of PDEs stated on the Wasserstein space. Note that the framework of [4] coincides with the one of [6] when the constraint acts only on the mean of the solution, i.e.,

$$\ell(t,x) = x \text{ and } H(\mu) = \int x\mu(dx).$$

We refer to [4, Remark 4] for more details on their connections. Motivated by the results of [4, 6, 36], we want to investigate the multidimensional BSDE (1.1) with mean reflection in a possibly non-convex domain D.

In view of the arguments of [36, 40], we assume that the constraint domain D satisfies uniform exterior sphere and uniform interior cone conditions. Note that the well-posedness of deterministic multi-dimensional Skorokhod Problem is crucial for our main result as in [6]. Indeed, when the driver f does not depend on the unknowns y and z, it follows from (1.1) and the non-randomness of the solution η that

$$\mathbf{E}[Y_t] = \mathbf{E}\bigg[\xi + \int_t^T f_s ds\bigg] + \eta_T - \eta_t,\ \mathbf{E}[Y_t] \in \overline{D},\ \forall t \in [0,T],$$

which can be regarded as a deterministic Skorokhod problem. In this case, we can first define the component η and then solve a standard BSDE to find the components Y and Z. Based on this observation, we can construct an iteration map for the general case. The key point is to prove that the map is a contraction.

To this end, we firstly establish a priori estimates for solutions to the mean reflected BSDE (1.1) through Itô's formula and the associated Skorokhod condition. Compared with the one dimensional case, we cannot obtain the explicit form of the component η as in [6]. Fortunately, the solution η is deterministic and then we adapt the discretization technique introduced by [40] to obtain

$$d|\eta|_s^0 \leq C(r_0,\delta,\alpha)\mathbf{E}\big[|f(s,Y_s,Z_s)|\big]ds,\ ds\text{-a.e.}$$

which gives a priori estimate for the total variation $|\eta|_T^0$.

Note that the uniform estimate for $d|\eta^n|_t^0$ is crucial to construct a contraction map in the case of non-convex reflection domains, see Remark 4.2 for details. Then, we apply a fixed-point argument to establish the existence result under the following additional condition on the driver,

$\mathbf{E}[|f(t,Y_t^n,Z_t^n)|]$ is uniformly bounded by some integrable function $g_t \in L^1([0,T])$.

Finally, with the help of a priori estimates, we develop an approximation procedure for the general case through an appropriate truncation argument. Consequently, we are able to prove the existence and uniqueness of solution to the BSDE (1.1) with mean reflection in a non-convex domain, which extends the relevant results in [4]. In addition,

[1] The collection of all probability measures over $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ with finite second moment.

when the constraint domain is convex, we show that the solution can be constructed by a penalization approach as in [24].

Since the constraint is only on the mean of the solution, the structure of the multi-dimensional mean re ected BSDE (1.1) is simpler than that of [4]. Therefore, we can employ the arguments of [36, 40] to tackle the non-convex re ection domains case. On the other hand, compared with the case of multidimensional re ected BSDEs with non-convex domains in [15], the non-randomness of the solution plays an important role in our arguments, which makes it easier to establish a priori estimates and construct a contraction map.

The paper is organized as follows. In Section 2, we recall some basic results of deterministic multi-dimensional Skorokhod Problem. In Section 3, we state the main result involving the existence and uniqueness for solution of the mean re ected BSDE (1.1), and some a priori estimates. Section 4 is devoted to the proof of the main result.

Let us nish this introduction by giving some notations which will be used frequently in this paper.

Notation

For each Euclidian space E, we denote by $h\,;\,i$ and $j\ j$ its scalar product and the associated norm, respectively. For each $p\ \ 1$, consider the following collections:

- $L^p(E)$ is the space of E-valued F_T-measurable random vectors satisfying $E[j\ j^p] < 1$;
- $H^p(E)$ is the space of E-valued F-progressively measurable processes $(z_t)_{0\ t\ T}$ satisfying
$$E\left[\int_0^T jz_tj^2dt\right]^{\frac{p}{2}} < 1 ;$$
- $H^{1;p}(E)$ is the space of E-valued F-progressively measurable processes $(z_t)_{0\ t\ T}$ satisfying
$$E\left[\int_0^T jz_tjdt\right]^{p} < 1 ;$$
- $S^p(E)$ is the space of E-valued F-adapted continuous processes $(y_t)_{0\ t\ T}$ satisfying
$$E\left[\sup_{0\ t\ T} jy_tj^p\right] < 1 ;$$
- $C(E)$ is the space of E-valued continuous functions on $[0;T]$;
- $V(E)$ is the space of E-valued continuous functions $(\ _t)_{0\ t\ T}$ satisfying $\ _0 = 0$ and $j\ j_T^0 < 1$, where $j\ j_t^s$ is the total variation on $[s;t]$ for each $0\ \ s\ \ t\ \ T$.

In what follows, for a given set of parameters , $C(\ )$ will denote a positive constant only depending on these parameters and may change from line to line.

2 Multi-dimensional Skorokhod problem

In this section, we will review some basic notions and results about multi-dimensional Skorokhod problem, which will be used in subsequent discussions.

Let D be a non-empty open connected subset of R^m with the boundary $@D$. Denote by $\overline{D}$ the closure of D. For each $x \in @D$, we de ne K_x as the set of all inward normal unit vectors at x, i.e.,

$$K_x := \bigcap_{r>0} K(x;r) \text{ and } K(x;r) := \{n \in R^m : |n| = 1, B(x - rn;r) \cap D = \emptyset\}, \tag{2.1}$$

where $B(x;r) := \{y \in R^m : |y - x| < r\}$.

De nition 2.1 (Skorokhod Problem). Given a function $\in C(R^m)$ satisfying $_T \in \overline{D}$. Then $(\ ;\ ) \in C(\overline{D}) \times V(R^m)$ is said to be a solution of Skorokhod problem for $(D;\ )$ (for short, SP $(D;\ )$) if for each $t \in [0;T]$

(i) $_t = _t + _t$;

(ii) $|_t| = \int_0^t 1_{\{ (s) \in @D\}} d|_s|$ and $_t = \int_0^t {}_s d|_s|$ for some Borel measurable function satisfying ${}_s \in K_{ s}$ $d|_s|$-a.e.

Next, we introduce the following two assumptions on the domain D, which have been used in Lions and Sznitman [36] and Saisho [40] to guarantee the well-posedness of SP$(D;\ )$.

(H1) (uniform exterior sphere condition) There exists a constant $r_0 > 0$ such that

$$K_x = K(x;r_0) \neq \emptyset, \text{ for all } x \in @D.$$

(H2) There exist two constants > 0 and $\in (0;1]$ satisfying the following property: for any $x \in @D$, there exists a unit vector l_x such that

$$\langle l_x, n\rangle \geq \text{ for any } n \in \bigcap_{y \in B(x;\ ) \cap @D} K_y.$$

Remark 2.2. (i) It is easy to check that $n \in K(x;r)$ if and only if $\langle x - y, n\rangle \geq \frac{1}{2r}|x - y|^2$ for all $y \in \overline{D}$ (see Remark 1.2 in [36] or Remark 1.1 in [40]).

(ii) If D is a convex set, it is easy to see that D satis es (H1) for any $r_0 > 0$.

Remark 2.3. (i) The following uniform interior cone condition is very useful, which is slightly stronger than (H2):

(H2$'$) (uniform interior cone condition) There exist two constants > 0 and $\in (0;1)$ satisfying the following property: for any $x \in @D$, there exists a unit vector l_x such that

$$C(y;l_x;\ ) \cap B(x;\ ) \subset D \text{ for any } y \in B(x;\ ) \cap @D;$$

where $C(y;l_x;\ )$ is the convex cone with vertex y, i.e.,

$$C(y;l_x;\ ) := \{z \in R^m : \langle z - y, l_x\rangle > |z - y|\ \}.$$

(ii) It was shown in Bramson, Burdzy and Kendall (see [3, Section 2]) that the uniform interior cone condition is equivalent to the following Lipschitz boundary condition.

Lipschitz domain: A domain $D \subset R^m$ is said to be Lipschitz if there exists > 0 such that for all $x \in @D$, there exists an orthonormal basis $e_1, e_2, \ldots, e_m$ and a Lipschitz function $f : R^{m-1} \to R$ such that

$$B(x;\ ) \cap D = \{y \in B(x;\ ) : f(y_1, \ldots, y_{m-1}) < y_m\ \},$$

where $y_i := \langle y, e_i\rangle$, $i = 1, \ldots, m$.

According to Remark 2.3, we give some interesting examples of D which satisfy (H1) and (H2) .

Example 2.4. (1) Any bounded convex domain with piecewise smooth boundary satis es (H1) and (H2) . Such as balls, polyhedrons and cylinders.

(2) (Unbounded) half planes, polyhedral cones and circular cones satisfy (H1) and (H2) , e.g., see Figure 1 (a).

(3) If the domain D satis es (H1) and (H2) , and $\{B_i\}_{i=1}^n$ is a family of balls such that $\{\partial B_i\}_{i=1}^n$; ∂D are non-tangential, then $D \cap (\bigcap_{i=1}^n B_i)$ satis es (H1) and (H2) , e.g., see Figure 1 (b).

(a) Cone. (b) Non-convex domain.

Figure 1: Domains satisfying (H1) and (H2)

In this paper, we need to consider the following multi-dimensional Backward Skorokhod Problem.

De nition 2.5 (Backward Skorokhod Problem) . Given a function $\in C(\mathbb{R}^m)$ with $_T \in \overline{D}$. Then $(\ ;\ ) \in C(\overline{D}) \times V(\mathbb{R}^m)$ is said to be a solution of Backward Skorokhod problem for $(D;\ )$ (for short, BSP $(D;\ )$) if for each $t \in [0,T]$

(i) $_t = \ + \int_t^T \ - \int_t^T \ ;$

(ii) $|\ |_t^0 = \int_0^t 1_{\{ (s)\in\partial D\}} d|\ |_s^0$ and $_t = \int_0^t \ _s d|\ |_s^0$ for some Borel measurable function satisfying $_s \in K_{\ _s} \ d|\ |_s^0$-a.e.

Remark 2.6. Note that $(\ ;\ )$ is a solution to BSP $(D;\ )$ if and only if $(\ ,\ )$ is a solution to SP $(D;\ )$, where $(\ _t;\ _t;\ _t) = (\ _T - \ _t;\ _T - \ _t;\ _T - \ _T - \ _t)$ for any $t \in [0,T]$.

Theorem 2.7. Suppose that assumptions (H1) and (H2) are satis ed. Then there is a unique solution to BSP $(D;\ )$. Moreover, $(t;\ )$ and $(t;\ )$ are continuous in $(t;\ )$.

Proof. The proof is immediate from Saisho [40, Theorem 4.1] and Remark 2.6. □

The following lemmas are crucial for our main results.

Lemma 2.8. Assume that assumptions (H1) and (H2) are ful lled. Suppose that $(\ ^n;\ ^n)$ is the unique solution of BSP $(D;\ _n)$ for each $n \geq 1$ and there exist three functions $\ ;\ ;$ such that

$$\sup_{n\geq 1} |\ ^n|_T^0 < \infty \quad \text{and} \quad \lim_{n\to\infty} \sup_{0\leq t\leq T} |\ ^n_t - \ _t| + |\ ^n_t - \ _t| + |\ ^n_t - \ _t| = 0.$$

Then $(\ ;\ )$ is the unique solution of BSP $(D;\ )$.

Proof. The proof is immediate from the proof of [36, Theorem 1.1] or that of [40, Theorem 4.1]. □

Lemma 2.9. Let assumptions (H1) and (H2) be satis ed and (;) be the unique solution of BSP (D;). Assume that has nite total variation. Then, there exists a constant C(r0; ;) such that for any 0 s t T,

$$j\ j_t^s \quad C(r_0; ;)j\ j_t^s: \tag{2.2}$$

Proof. According to Theorem 2.7 and Remark 2.6, (e; e) solves SP (D; e) with

$$e_t = \ _T \ _t; \ e_t = \ _T \ _t; \ e_t = \ _T \ \ _T \ _t; \ t \ 2 \ [0;T]:$$

Note that $j\ j_t^s = je j_T^T \ _s^t$ and $j\ j_t^s = j\ e j_T^T \ _s^t$, it suf ces to show that (2.2) holds for e and e. The proof is from [40, Theorem 4.1] by some appropriate modi cations.

For each n 1, we denote n by

$$e_t^n = e_{k2^{\ n}}; \ k2^{\ n} \quad t < (k+1)2^{\ n}; \ k \quad 0:$$

In view of Remark 1.4 in [40], there exists a unique solution (e^n; e^n) to SP (D; e^n), i.e.,

$$e_t^n = e_t^n + e_t^n; \ t \ 2 \ [0;T]:$$

It follows from the proof of [40, p. 467] that e_n converges to e uniformly on [0;T]. De ne

$$\begin{aligned} T^{n;0} &= \inf \ t \quad 0 : e_t^n \ 2 \ @D \ ^ \ T; \\ t^{n;l} &= \inf \ t > T_{n;l \ 1} : j\ e_t^n \quad e^n_{T_{n;l \ 1}} j \quad =2 \ ^ \ T; \\ T^{n;l} &= \inf \ t \quad t_{n;l} : e_t^n \ 2 \ @D \ ^ \ T: \end{aligned}$$

Recalling the proof of [40, pp. 465–466] and using the fact that $_{s;t}(e) \quad _{0;T;jt \ sj}(e)$, we can nd an integer n_0 1 and a constant h > 0 such that for all n n_0,

$$T^{n;l} \quad T^{n;l \ 1} \quad h; \ \text{if } T_{n;l} < T; \tag{2.3}$$

and for each l 1, we have for all s;t 2 $[T^{n;l \ 1}; T^{n;l}]$

$$je^n j_t^s \quad C(r_0; ;) \ 1 + \exp \ C(r_0; ;) \ 1 + \ _{0;T;jt \ sj} \ e^n \ _{s;t} \ e^n \ ; \tag{2.4}$$

where for any 0 s t T and > 0

$$\begin{aligned} _{s;t}(e) &:= \sup fj\ e_{t_1} \quad e_{t_2} j : s \quad t_1; t_2 \quad tg; \\ _{0;T;}(e) &:= \sup \ j e(t_1) \quad e(t_2)j : 0 \quad t_1; t_2 \quad T; \ jt_1 \quad t_2 j \quad : \end{aligned}$$

It follows from (2.3) that for any s;t 2 [0;T], there exist integers 1 i_n j_n such that s 2 $[T^{n;i_n \ 1}; T^{n;i_n}]$ and t 2 $[T^{n;j_n \ 1}; T^{n;j_n}]$. Then for any > 0, there is a partition $s = t_0 \quad t_1 \quad \quad t_n = t$ such that

$$[t_{k \ 1}; t_k] \quad T^{n;l \ 1}; T^{n;l} \ \text{for some } i_n \quad l \quad j_n; \ \text{and } jt_k \quad t_{k \ 1} j \quad ; \ \text{for all } k = 1; 2; : : : ; n :$$

Thus applying (2.4) yields that for any > 0,

$$\begin{aligned} je^n j_t^s &= \sum_{k=1}^{n} je^n j_{t_{k \ 1}}^{t_k} \\ &\quad C(r_0; ;) \ 1 + \exp \ C(r_0; ;) \ 1 + \ _{0;T;} \ e^n \sum_{k=1}^{n} \ _{t_{k \ 1};t_k} \ e^n \\ &\quad C(r_0; ;) \ 1 + \exp \ C(r_0; ;) \ 1 + \ _{0;T; +2 \ ^{(n \ 1)}}(e) \quad j\ e^n j_t^s: \end{aligned} \tag{2.5}$$

Sending $\to 0$ in (2.5), we have

$$|e^n|_t^s \le C(r_0;\ ;\ )\ 1+\exp\ C(r_0;\ ;\ )\ 1+\ _{0;T;2^{(n\ 1)}}(e)\ \ |e^n|_t^s.$$

Note that e_n converges to e uniformly on $[0,T]$ and

$$\lim_{n\to\infty}\ _{0;T;2^{(n\ 1)}}(e)=0;\ |e^n|_t^s\ \ |e|_{(t+2^{\ n})\wedge T}^{(s\ 2^{\ n})\vee 0}.$$

Consequently, we obtain

$$|e|_t^s\ \ \liminf_{n\to\infty}|e^n|_t^s\ \ C(r_0;\ ;\ )|e|_t^s;$$

which ends the proof. □

3 Mean re ected BSDEs

The main purpose of this section is to study the solvability of the multidimensional mean re ected BSDE (1.1) . In what follows, we make use of the following conditions on the terminal value and the driver f .

(H3) The terminal condition $\in L^2(\mathbb{R}^m)$ satis es that $\in \overline{D}$ and the driver $f(t;0;0)$ is in the space of $H^{1;2}(\mathbb{R}^m)$.

(H4) There exists a constant >0 such that for any $t\in[0,T]$, $y_1,y_2\in\mathbb{R}^m$ and $z_1,z_2\in\mathbb{R}^{m\ d}$

$$|f(t;y_1;z_1)\ \ f(t;y_2;z_2)|\ \ (|y_1\ \ y_2|+|z_1\ \ z_2|).$$

De nition 3.1. A triplet $(Y,Z,\ )\in S^2(\mathbb{R}^m)\ H^{\ 2}(\mathbb{R}^{m\ d})\ V(\mathbb{R}^m)$ is said to be a solution to the BSDE (1.1) with mean re ection if it satis es equation (1.1) and the component $_t$ changes only when $E[Y_t]$ is on the boundary of D such that

(i) $|\ |_t^0=\int_0^t 1_{\{E[Y_s]\in\partial D\}}\,d|\ |_s^0$;

(ii) there exists a measurable function $:[0,T]\to\mathbb{R}^m$ such that $_s\in K_{E[Y_s]}\ d|\ |_s^0$-a.e. and

$$_t=\int_0^t\ _s d|\ |_s^0.$$

Remark 3.2. In [4], Briand et al. introduced the following multi-dimensional mean re ected BSDEs with normal re ection:

$$\begin{cases} Y_t=\ +\int_t^T f(s;Y_s;Z_s)ds\ \ \int_t^T Z_s dB_s+\int_t^T D\ H(P_{Y_s})(Y_s)dK_s;\quad \forall t\in[0,T];\\ P_{Y_t}\ 2\quad 2\ P_2\ \mathbb{R}^m\ ;\ H(\ )\ \ 0\ ;\quad \forall t\in[0,T];\ \int_0^T H(P_{Y_t})dK_t=0; \end{cases} \tag{3.1}$$

in which $D\ H$ denotes the Lions' derivative (see [11, 35]). When the constraint acts only on the mean of the solution, i.e., $H(\ )=\int_{\mathbb{R}}x\ (dx)$, (3.1) reduces to

$$\begin{cases} Y_t=\ +\int_t^T f(s;Y_s;Z_s)ds\ \ \int_t^T Z_s dB_s+(K_T\ \ K_t);\forall t\in[0,T];\\ E[Y_t]\ \ 0;\forall t\in[0,T]; \qquad \int_0^T E[Y_t]dK_t=0; \end{cases}$$

which is the same as the equation (1.1) with the convex domain $D=\{x\in\mathbb{R}^m;\ x>0\}$. We refer to [4, Remark 4] for more details on this topic.

Now we are ready to state the main result of this paper.

Theorem 3.3. Suppose that assumptions (H1) –(H4) hold. Then the BSDE (1.1) with mean re ection admits a unique square integrable solution $(Y,Z,\ )\in S^2(\mathbb{R}^m)\ H^2(\mathbb{R}^{m\ d})\ V(\mathbb{R}^m)$.

Remark 3.4. In [15], Chassagneux, Nadtochiy and Richou studied the multidimensional re ected BSDEs with the following possibly non-convex domain D:

$$D = \{ y \in R^m : \ (y) < 0 \ ;$$

where $\in C^2(R^m)$ satis es compactness, smoothness and a weak star-shape property (see [15 , Assumption 1.1]). It is easy to check that the domain D satis es conditions (H1) and (H2) . Due to the adaptedness issues, the a priori estimate for the total variation $|dj|_t^0$ is much more complicated in this case, which has quadratic terms in z (see [15 , Lemma 2.1]).

In what follows, we are going to prove Theorem 3.3. Firstly, we state some useful a priori estimates for solutions to the BSDE (1.1) with mean re ection, which is much more delicate and involved compared with the scalar-valued case.

Lemma 3.5. Assume that assumptions (H1) –(H4) are satis ed. Let $(Y; Z;)$ be a square integrable solution to the mean re ected BSDE (1.1) . Then

$$E\Big[\sup_{0 \ s \ T} |Y_s|^2 + \int_0^T |Z_s|^2 ds + |\ |_T^0|^2\Big] \ C(r_0; \ ; \ ; \ ; T \ )E\Big[|\ |^2 + \int_0^T |f(s;0;0)| ds\Big]^2 : \tag{3.2}$$

Proof. Note that the Lipschitz continuity of f implies

$$2 \ y; f(t; y; z) \ \ 2|y||f(t;0;0)| + \ 2 \ + 3 \ ^2 \ |y|^2 + \frac{1}{3}|z|^2; \ 8(y; z) \in R^m \ R^{m \ d}: \tag{3.3}$$

Setting $a = (2 \ + 3 \ ^2) + b$ for some constant $b > 0$ which is to be determined later. Using the inequality (3.3) and applying Itô's formula to $e^{at}|Y_t|^2$ yields that

$$\begin{aligned}
&e^{at}|Y_t|^2 + b\int_t^T e^{as}|Y_s|^2 ds + \frac{2}{3}\int_t^T e^{as}|Z_s|^2 ds \\
&\quad e^{aT}|\ |^2 + 2\int_t^T e^{as}|Y_s||f(s;0;0)| ds + 2\int_t^T e^{as}\langle Y_s; d \ _s\rangle \quad 2\int_t^T e^{as}\langle Y_s; Z_s dB_s\rangle \\
&\quad e^{aT}|\ |^2 + 2\sup_{0 \ s \ T}|Y_s| \int_t^T e^{as}|f(s;0;0)| ds + 2\int_t^T e^{as}\langle Y_s; d \ _s\rangle \quad 2\int_t^T e^{as}\langle Y_s; Z_s dB_s\rangle :
\end{aligned} \tag{3.4}$$

Now set $_t = E[Y_t]$ and $_t = E[\ + \int_t^T f(s; Y_s; Z_s) ds]$. It is easy to check that $(\ ; \ )$ is the unique solution to BSP $(D; \ )$. Since has nite total variation and

$$|\ |_t^s \ \int_s^t |E[\ f(r; Y_r; Z_r)\ ]| dr \quad \text{for all} \quad 0 \ s \ t \ T;$$

it follows from (2.2) in Lemma 2.9 that

$$|dj|_t^0 \ C(r_0; \ ; \ )|E[\ f(t; Y_t; Z_t)\ ]| dt: \tag{3.5}$$

Note that $_t = \int_0^t {}_s |dj|_s^0$ for some Borel measurable function satisfying $_s \in K_{E[Y_s]}$ $|dj|_s^0$-a.e. Thus, we obtain

$$\begin{aligned}
&2E\Big[\int_t^T e^{as}\langle Y_s; d \ _s\rangle\Big] \\
&\quad 2C(r_0; \ ; \ )\int_t^T e^{as}E[|Y_s|]|E[\ f(s; Y_s; Z_s)\ ]| ds \\
&\quad C(r_0; \ ; \ ) \ 2 \ + 3C(r_0; \ ; \ )^2 \int_t^T e^{as}E\big[|Y_s|^2\big] ds + \frac{1}{3}\int_t^T e^{as}E\big[|Z_s|^2\big] ds \\
&\quad + 2C(r_0; \ ; \ )E\Big[\sup_{0 \ s \ T}|Y_s|\Big]\int_t^T e^{as}E[|f(s;0;0)|] ds:
\end{aligned} \tag{3.6}$$

Setting $b = C(r_0; ;)(2 + 3C(r_0; ;)^2)$ and taking expectations on both sides of (3.4), we have

$$\begin{aligned}
&\sup_{0 \le s \le T} e^{as} E|Y_s|^2 + E\int_0^T e^{as}|Z_s|^2 ds \\
&\le 3E\left[e^{aT}|\xi|^2 + 6E\left[\sup_{0 \le s \le T}|Y_s| \int_t^T e^{as}|f(s;0;0)|ds\right]\right. \\
&\left. + 6C(r_0; ;)E\left[\sup_{0 \le s \le T}|Y_s| E\int_0^T e^{as}|f(s;0;0)|ds\right]\right.:
\end{aligned}$$

Then we deduce that for any $\varepsilon \in (0;1)$

$$\begin{aligned}
&\sup_{0 \le s \le T} E|Y_s|^2 + E\int_0^T |Z_s|^2 ds \\
&\le \frac{1}{\varepsilon} C(r_0; ; ; ;T)E|\xi|^2 + \left(\int_0^T |f(s;0;0)|ds\right)^2 + \varepsilon E\left[\sup_{0 \le s \le T}|Y_s|^2\right]:
\end{aligned} \tag{3.7}$$

According to (3.5) and (3.7), we derive that

$$|\xi|_T^0 \le C(r_0; ; ; ;T)\left(\frac{1}{\sqrt{\varepsilon}} E|\xi|^2 + \left(\int_0^T |f(s;0;0)|ds\right)^2\right)^{\frac{1}{2}} + \sqrt{\varepsilon}\, E\left[\sup_{0 \le s \le T}|Y_s|^2\right]^{\frac{1}{2}}: \tag{3.8}$$

On the other hand, it follows from the de nition of (1.1) that for any $s \in [0;T]$,

$$|Y_s| \le |\xi| + |\xi|_T^0 + \int_0^T |f(r;0;0)|dr + \int_0^T |Z_r|dr + \sup_{0 \le r \le T}\left|\int_r^T Z_u dB_u\right| + \int_s^T |Y_r|dr:$$

Applying Gronwall's inequality, we have

$$|Y_s| \le e^{(T-s)}\left(|\xi| + |\xi|_T^0 + \int_0^T |f(r;0;0)|dr + \int_0^T |Z_r|dr + \sup_{0 \le r \le T}\left|\int_r^T Z_u dB_u\right|\right);$$

which together with Burkholder-Davis-Gundy's (BDG's) inequality implies

$$E\left[\sup_{0 \le s \le T}|Y_s|^2\right] \le C(;T)\left(E|\xi|^2 + \left(|\xi|_T^0\right)^2 + \left(\int_0^T |f(s;0;0)|ds\right)^2 + \int_0^T |Z_s|^2 ds\right): \tag{3.9}$$

Putting (3.7), (3.8) and (3.9) together, we conclude that

$$E\left[\sup_{0 \le s \le T}|Y_s|^2\right] \le C(r_0; ; ; ;T)\left(\frac{1}{\varepsilon}E|\xi|^2 + \left(\int_0^T |f(s;0;0)|ds\right)^2\right) + \varepsilon E\left[\sup_{0 \le s \le T}|Y_s|^2\right]:$$

Taking ε small enough so that $C(r_0; ; ; ;T)\varepsilon = \frac{1}{2}$, we can get the desired result, which completes the proof. □

Corollary 3.6. Assume that the same conditions hold as in Lemma 3.5. Then for any $p \ge 2$,

$$\begin{aligned}
&E\left[\sup_{0 \le s \le T}|Y_s|^p + \left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] + \left(|\xi|_T^0\right)^p \\
&\le C(p; r_0; ; ; ;T)E\left[|\xi|^p + \left(\int_0^T |f(s;0;0)|ds\right)^p\right]:
\end{aligned}$$

Proof. Without loss of generality, assume that the right-side hand of the above inequality is nite. Since is a deterministic function, it follows from Lemma 3.5 that

$$\| \|_{\mathcal{T}}^{0,p} \quad C(p; r_0; ; ; ; T \ )E \ | \ |^p + \int_0^T |f(s;0;0)|ds \Big)^p : \tag{3.10}$$

Note that $(Y_t + \ _t \quad _T ; Z_t)_{0 \ t \ T}$ solves the following standard BSDE:

$$\tilde{Y}_t = \ + \int_t^T \tilde{f}(s; \tilde{Y}_s; \tilde{Z}_s)ds \quad \int_t^T \tilde{Z}_s dB_s; \ 8t \ 2 \ [0;T];$$

whose generator is given by $\tilde{f}(s;y;z) = f(s;y + \ _T \quad _s;z)$. In view of [41 , Theorem 4.4.4], we deduce that

$$\begin{aligned}
&E \Big(\sup_{0 \ s \ T} |Y_s|^p + \int_0^T |Z_s|^2 ds \Big)^{\frac{p}{2}} \\
&2^{p \ 1} E \Big(\sup_{0 \ s \ T} |\tilde{Y}_s|^p + \| \|_{\mathcal{T}}^{0,p} + \int_0^T |\tilde{Z}_s|^2 ds \Big)^{\frac{p}{2}} \\
&C(p; \ ;T)E \ | \ |^p + \int_0^T |f(s; \ _T \quad _s;0)|ds \Big)^p + \| \|_{\mathcal{T}}^{0,p} \\
&C(p; \ ;T)E \ | \ |^p + \int_0^T |f(s;0;0)|ds \Big)^p + \| \|_{\mathcal{T}}^{0,p} ;
\end{aligned}$$

which together with (3.10) indicates the desired result. The proof is complete. □

Lemma 3.7. Assume that assumptions (H1) , (H3) and (H4) hold. Let $(Y^i; Z^i; \ ^i)$ be a square integrable solution to (1.1) corresponding to the data $(\ ^i; f^i)$, $i = 1;2$. Then, there exists a constant $C(\ ;T)$ such that

$$\begin{aligned}
&E \sup_{t \, 2 \, [0;T]} |\hat{Y}_t|^2 + \int_0^T |\hat{Z}_s|^2 ds \\
&C(\ ;T)e^{\frac{2}{r_0}(| \ ^1|^0_T + | \ ^2|^0_T)} E \ | \ ^1 \quad ^2| + \int_0^T |f^1 \ s; Y_s^2; Z_s^2 \quad f^2 \ s; Y_s^2; Z_s^2 \ |ds \Big)^2 ;
\end{aligned}$$

where $\hat{\ }_t = \hat{`}^1_t \quad \hat{`}^2_t$ for $\hat{`}_t = Y_t; Z_t; \ _t$.

Proof. Denote by

$$I = E \ | \ ^1 \quad ^2| + \int_0^T |f^1 \ s; Y_s^2; Z_s^2 \quad f^2 \ s; Y_s^2; Z_s^2 \ |ds \Big)^2 ;$$

and

$$a_t = \ 2 \ + 2 \ ^2 \ t + r_0^{\ 1} \ | \ ^1|^0_t + | \ ^2|^0_t :$$

Applying Itô's formula to $e^{a_t}|\hat{Y}_t|^2$ on $[t;T]$, we have

$$\begin{aligned}
e^{a_T}|\hat{Y}_T|^2 \quad e^{a_t}|\hat{Y}_t|^2 = &\int_t^T e^{a_s}|\hat{Y}_s|^2 da_s \quad 2\int_t^T e^{a_s} \hat{Y}_s; f^1 \ s; Y_s^1; Z_s^1 \quad f^1 \ s; Y_s^2; Z_s^2 \ ds \\
&+ \int_t^T e^{a_s}|\hat{Z}_s|^2 ds \quad 2\int_t^T e^{a_s} \hat{Y}_s; f^1 \ s; Y_s^2; Z_s^2 \quad f^2 \ s; Y_s^2; Z_s^2 \ ds \\
&2\int_t^T e^{a_s} \hat{Y}_s; d\hat{b}_s i + 2\int_t^T e^{a_s} \hat{Y}_s; \hat{Z}_s dB_s i ;
\end{aligned}$$

which together with

$$2\,\langle Y_s^1 - Y_s^2, f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)\rangle + 2\,\|Y_s^1 - Y_s^2\|^2 + \frac{1}{2}|Z_s^1 - Z_s^2|^2$$

indicates

$$\begin{aligned} &e^{\alpha_t}|\psi_t|^2 + \frac{1}{2}\int_t^T e^{\alpha_s}|\phi_s|^2 ds \\ &\quad e^{\alpha_T}|\xi^1 - \xi^2|^2 + 2\int_t^T e^{\alpha_s}|\psi_s||f^1(s,Y_s^2,Z_s^2) - f^2(s,Y_s^2,Z_s^2)|ds \\ &\quad \frac{1}{r_0}\int_t^T e^{\alpha_s}|\psi_s|^2 d(|\xi^1|_s^0 + |\xi^2|_s^0) + 2\int_t^T e^{\alpha_s}\langle\psi_s, db_s\rangle - 2\int_t^T e^{\alpha_s}\langle\psi_s, \phi_s dB_s\rangle. \end{aligned} \tag{3.11}$$

Note that $|\xi^i|_t^0 = \int_0^t 1_{\{E[Y_s^i]\neq 0\}} d|\xi^i|_s^0$ and $\xi_t^i = \int_0^t \eta_s^i d|\xi^i|_s^0$ for some Borel measurable function η^i satisfying $\eta_s^i \in K_{E[Y_s^i]}$ $d|\xi^i|_s^0$-a.e. Then recalling assumption (H1) and assertion (i) of Remark 2.2, we obtain

$$\begin{aligned} E\int_t^T e^{\alpha_s}\langle\psi_s, d\hat{\kappa}_s\rangle &= \int_t^T e^{\alpha_s}\, E\,Y_s^1 \wedge E\,Y_s^2\,;\,\eta_s^1\, d|\xi^1|_s^0 \\ &\quad + \int_t^T e^{\alpha_s}\, E\,Y_s^2 \wedge E\,Y_s^1\,;\,\eta_s^2\, d|\xi^2|_s^0 \\ &= \int_t^T e^{\alpha_s}\, E\,Y_s^1 \wedge E\,Y_s^2\,;\,\eta_s^1\, 1_{\{E[Y_s^1]\neq 0\}} d|\xi^1|_s^0 \\ &\quad + \int_t^T e^{\alpha_s}\, E\,Y_s^2 \wedge E\,Y_s^1\,;\,\eta_s^2\, 1_{\{E[Y_s^2]\neq 0\}} d|\xi^2|_s^0 \\ &\quad \frac{1}{2r_0}\int_t^T e^{\alpha_s} E|\psi_s|^2\, d(|\xi^1|_s^0 + |\xi^2|_s^0). \end{aligned} \tag{3.12}$$

In view of (3.11), we derive that for any $t \in [0,T]$

$$\begin{aligned} &E\,e^{\alpha_t}|\psi_t|^2 + \int_t^T e^{\alpha_s}|\phi_s|^2 ds \\ &\quad 4e^{\alpha_T} E|\xi^1 - \xi^2|^2 + \int_t^T |\psi_s||f^1(s,Y_s^2,Z_s^2) - f^2(s,Y_s^2,Z_s^2)|ds. \end{aligned} \tag{3.13}$$

On the other hand, it follows from

$$b_t = \psi_0 - E[\psi_t] - E\int_0^t f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\,ds$$

and (3.13) that

$$\begin{aligned} &\sup_{t\in[0,T]}|b_t|^2 \\ &\quad C(\alpha,T)\Big\{\sup_{t\in[0,T]} E|\psi_t|^2 + \int_0^T|\phi_s|^2 ds + \int_0^T|f^1(s,Y_s^2,Z_s^2) - f^2(s,Y_s^2,Z_s^2)|ds\Big\}^2 \\ &\quad C(\alpha,T)e^{\alpha_T}\Big\{I + E\int_0^T|\psi_s||f^1(s,Y_s^2,Z_s^2) - f^2(s,Y_s^2,Z_s^2)|ds\Big\}. \end{aligned}$$

Recalling the de nition of (1.1) and using BDG's inequality and (3.13), we have

$$\begin{aligned}
&E\Big[\sup_{0\le s\le T}|\hat{Y}_s|^2\Big]\\
&\le C(\lambda,T)e^{\alpha_T}\Big\{I+E\int_0^T|\hat{Y}_s||f^1(s,Y_s^2,Z_s^2)-f^2(s,Y_s^2,Z_s^2)|ds\Big\}\\
&\le C(\lambda,T)e^{\alpha_T}\Big\{I+\frac{1}{2}C(\lambda,T)e^{\alpha_T}E\Big[\int_0^T|f^1(s,Y_s^2,Z_s^2)-f^2(s,Y_s^2,Z_s^2)|ds\Big]^2\\
&\quad+\frac{1}{2}E\Big[\sup_{0\le s\le T}|\hat{Y}_s|^2\Big]\Big\}.
\end{aligned}$$

It follows that

$$E\Big[\sup_{0\le s\le T}|\hat{Y}_s|^2\Big]\le C(\lambda,T)e^{2\alpha_T}I,$$

which is the desired result. □

The following lemma is a direct consequence of Lemma 3.7.

Lemma 3.8. Suppose assumptions (H1) , (H3) and (H4) hold. Then the BSDE (1.1) with mean re ection has at most one square integrable solution.

Proof. The proof is immediate from Lemma 3.7 by taking $(\xi^i,f^i)=(\xi,f),i=1,2.$ □

Compared with the uniqueness, the existence of solution to the mean re ected BSDE (1.1) is much more complicated, which will be stated in the next section. In what follows, we deal with the case of convex re ection domains to illustrate our main idea.

Lemma 3.9. Suppose that assumptions (H1) –(H4) are ful lled. Assume also that the generator f is independent of the rst unknown y. Then the BSDE (1.1) with mean re ection admits a unique square integrable solution.

Proof. Let $(\hat{Y},\hat{Z})$ be the $S^2(R^m)\times H^2(R^{m\times d})$-solution to the following standard BSDE:

$$\hat{Y}_t=\xi+\int_t^T f(s,\hat{Z}_s)ds-\int_t^T\hat{Z}_s dB_s,\ \forall t\in[0,T]. \tag{3.14}$$

Set $\eta_t=E[\xi+\int_t^T f(s,\hat{Z}_s)ds]$. It is easy to check that $\eta\in C(R^m)$ with $\eta_T=E[\xi]\in\overline{D}$. By Theorem 2.7, there exists a unique solution $(\hat{\gamma};\hat{\alpha})$ to BSP $(D;\Phi)$. Set $Y_t=\hat{Y}_t-\hat{\gamma}_t+\eta_T$ and $Z_t=\hat{Z}_t$. It follows that $(Y,Z)\in S^2(R^m)\times H^2(R^{m\times d})$ solves the following BSDE

$$Y_t=\xi+\int_t^T f(s,Z_s)ds-\int_t^T Z_s dB_s+\hat{\gamma}_T-\hat{\gamma}_t,\ \forall t\in[0,T].$$

Note also that

$$E[Y_t]=E\Big[\xi+\int_t^T f(s,Z_s)ds\Big]+\eta_T-\eta_t=\eta_t+\eta_T-\eta_t=\eta_t\in\overline{D}.$$

Therefore, $(Y,Z,\hat{\alpha})$ is a square integrable solution of the BSDE (1.1) . The uniqueness follows from Lemma 3.8 and the proof is complete. □

Note that a non-empty convex set satis es assumption (H1) with any $r_0>0$. Then it follows from assertion (i) of Remark 2.2 that for any $x\in\partial D$ and $n\in K_x$

$$\langle x-y,n\rangle\ge 0,\ \forall y\in\overline{D}. \tag{3.15}$$

Theorem 3.10. Suppose that assumptions (H2) –(H4) are satis ed. If the domain D is convex, then the mean re ected BSDE (1.1) has a unique square integrable solution.

Proof. For any given $(U;V) \in H^2(R^m) \times H^{-2}(R^{m\times d})$, it follows from Lemma 3.9 that the following BSDE with mean re ection

$$\begin{cases} Y_t = \xi + \int_t^T f(s;U_s;V_s)ds - \int_t^T Z_s dB_s + K_T - K_t, & \forall t \in [0;T], \\ \mathbb{E}[Y_t] \in \overline{D}, & \forall t \in [0;T] \end{cases} \tag{3.16}$$

has a unique solution $(Y;Z;K) \in S^2(R^m) \times H^{-2}(R^{m\times d}) \times V(R^m)$. Now we de ne a map

$$\begin{aligned} \Phi : H^{-2}(R^m) \times H^{-2}(R^{m\times d}) &\to H^{-2}(R^m) \times H^{-2}(R^{m\times d}), \\ (U;V) &\mapsto (Y;Z). \end{aligned} \tag{3.17}$$

Then it suf ces to show that the map Φ is a contraction.

For each $i \in \{1;2\}$, denote by $(Y^i;Z^i;K^i)$ the solution to (3.16) corresponding to the data $(U^i;V^i) \in H^{-2}(R^m) \times H^{-2}(R^{m\times d})$. Denote by $\hat{\ell} = \ell^1 - \ell^2$ for $\ell = Y;Z;K;U;V$ and $a = 4\beta^2 + 1$. Then applying Itô's formula to $e^{at}|\hat{Y}_t|^2$ and using a similar analysis as in Lemma 3.7, we obtain

$$\begin{aligned} |\hat{Y}_0|^2 = a\int_0^T e^{as}|\hat{Y}_s|^2 ds &- 2\int_0^T e^{as}\langle \hat{Y}_s, f(s;U_s^1;V_s^1) - f(s;U_s^2;V_s^2)\rangle ds \\ &+ \int_0^T e^{as}|\hat{Z}_s|^2 ds - 2\int_0^T e^{as}\langle \hat{Y}_s, d\hat{K}_s\rangle + 2\int_0^T e^{as}\hat{Y}_s\hat{Z}_s dB_s \\ &\le \int_0^T e^{as}\left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds - \frac{1}{2}\int_0^T e^{as}\left(|\hat{U}_s|^2 + |\hat{V}_s|^2\right) ds \\ &\quad - 2\int_0^T e^{as}\langle \hat{Y}_s, d\hat{K}_s\rangle + 2\int_0^T e^{as}\hat{Y}_s\hat{Z}_s dB_s, \end{aligned} \tag{3.18}$$

where we have used the fact that

$$2\langle \hat{Y}_s, f(s;U_s^1;V_s^1) - f(s;U_s^2;V_s^2)\rangle \le 4\beta^2|\hat{Y}_s|^2 + \frac{1}{2}|\hat{U}_s|^2 + |\hat{V}_s|^2$$

in the last inequality. In view of (3.15), we get

$$\begin{aligned} \mathbb{E}\int_0^T e^{as}\langle \hat{Y}_s, d\hat{K}_s\rangle &= \int_0^T e^{as}\,\mathbb{E}\left[\left(Y_s^1 - \mathbb{E}[Y_s^1] - (Y_s^2 - \mathbb{E}[Y_s^2])\right)\mathbb{1}_{\{\mathbb{E}[Y_s^1]\in\partial D\}}\right] dK_s^1 \\ &\quad + \int_0^T e^{as}\,\mathbb{E}\left[\left(Y_s^2 - \mathbb{E}[Y_s^2] - (Y_s^1 - \mathbb{E}[Y_s^1])\right)\mathbb{1}_{\{\mathbb{E}[Y_s^2]\in\partial D\}}\right] dK_s^2 \\ &\le 0. \end{aligned} \tag{3.19}$$

Putting (3.18) and (3.19) together yields that

$$\mathbb{E}\int_0^T e^{as}\left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds \le \frac{1}{2}\mathbb{E}\int_0^T e^{as}\left(|\hat{U}_s|^2 + |\hat{V}_s|^2\right) ds.$$

Therefore, the map Φ has a unique fixed point $(Y;Z) \in H^{-2}(R^m) \times H^{-2}(R^{m\times d})$. We denote by

$$K_t = Y_0 - Y_t - \int_0^t f(s;Y_s;Z_s)ds + \int_0^t Z_s dB_s,\ t \in [0;T].$$

It follows from the de nition of the map Φ that $K \in V(R^m)$ satis es (i) and (ii) in De nition 3.1. Finally, with the help of BDG's inequality, we have $Y \in S^2(R^m)$, which ends the proof. □

Remark 3.11. Note that (3.15) is crucial to construct a contraction map in the proof of Theorem 3.10. In the non-convex re ection domains case, (3.19) should be replaced by (3.12), and then we need to estimate $d\|j^0_s$, which results in the main dif culty.

Remark 3.12. In the case of convex re ection domains, the solution to the mean re ected BSDE (1.1) can be also constructed through a penalization approach as in [24], see Theorem A.5 in the appendix.

4 The existence

This section is devoted to the study of the existence results of multi-dimensional mean re ected BSDEs with non-convex re ection domains. In what follows, we shall combine a xed-point argument and a truncation technique to deal with it through a priori estimates established in Section 3.

Firstly, we employ a xed-point argument to show the existence of solutions under an additional condition on the generator.

Lemma 4.1. Suppose that assumptions (H1) –(H4) are satis ed. Assume in addition that there exists a nonnegative process $g \in H^{1;1}(R)$ such that

$$|f(t;y;z)| \le g_t;\ 8(t;y;z) \in [0;T] \times R^m \times R^{m\times d}:$$

Then the BSDE (1.1) with mean re ection admits a unique square integrable solution.

Proof. Using the same notations as in the proof of Theorem 3.10. In particular, we can also de ne the map as in (3.17). Setting

$$a_t = 4\beta^2 + 1)t + b\int_0^t E[g_s]ds$$

for some constant $b > 0$ which is to be determined later. According to the derivation of (3.18), we have

$$\begin{aligned}
|\psi_0|^2 =& \int_0^T e^{a_s}|\psi_s|^2 da_s - 2\int_0^T e^{a_s}\langle\psi_s, f(s,U^1_s,V^1_s) - f(s,U^2_s,V^2_s)\rangle ds \\
&+ \int_0^T e^{a_s}|\phi_s|^2 ds - 2\int_0^T e^{a_s}\langle\psi_s,\phi_s\rangle dB_s \\
&- \int_0^T e^{a_s}|\psi_s|^2 + |\phi_s|^2)\, ds - \frac{1}{2}\int_0^T e^{a_s}|\theta_s|^2 + |\psi_s|^2)\, ds \\
&+ b\int_0^T e^{a_s}|\psi_s|^2 E[g_s]ds - 2\int_0^T e^{a_s}\langle\psi_s, d\beta_s\rangle + 2\int_0^T e^{a_s}\langle\psi_s,\phi_s\rangle dB_s:
\end{aligned} \tag{4.1}$$

Taking expectations on both sides of (4.1) yields that

$$\begin{aligned}
&E\int_0^T e^{a_s}|\psi_s|^2 + |\phi_s|^2)\, ds \\
&\le \frac{1}{2}E\int_0^T e^{a_s}|\theta_s|^2 + |\psi_s|^2)\, ds + 2E\int_0^T e^{a_s}\langle\psi_s, d\beta(s)\rangle \\
&\quad - b\int_0^T e^{a_s}E|\psi_s|^2 E[g_s]ds:
\end{aligned} \tag{4.2}$$

In view of (3.12), we derive

$$E\int_0^T e^{a_s}\langle\psi_s, d\beta(s)\rangle \le \frac{1}{2r_0}\int_0^T e^{a_s}E|\psi_s|^2 d(|\gamma^1|^0_s + |\gamma^2|^0_s): \tag{4.3}$$

Next we need to estimate the right-hand side of (4.3) , where Lemma 2.9 plays an important role. Indeed, denote by $^i_t = E[Y^i_t]$ and $^i_t = E[\ + \int_t^T f(s;U^i_s;V^i_s)ds]$ for each $t \ 2\ [0;T]$, $i = 1;2$. Then we see that $(\ ^i;\ ^i)$ is the unique solution of BSP $(D;\ ^i)$. It is easy to check that ^{i} has nite total variation with

$$j\ ^i j^s_t \quad E\int_s^t jf\ r;U^i_r;V^i_r\ jds \quad \int_s^t E[g_r]dr;\ 0 \quad s \quad t \quad T;\ i = 1;2: \tag{4.4}$$

Thus it follows from (4.4) and Lemma 2.9 that there exists a constant $C(r_0;\ ;\ )$ such that

$$dj\ ^i j^0_s \quad C(r_0;\ ;\ )E[g_s]ds;\ ds\text{-a.e.};\ i = 1;2:$$

Consequently, setting $b = \frac{2C(r_0;\ ;\ )}{r_0}$ and recalling (4.2), (4.3), we deduce that

$$E\int_0^T e^{a_s}\ j\Psi_s j^2 + j\mathcal{Z}_s j^2\ ds \quad \frac{1}{2}E\int_0^T e^{a_s}\ j\mathcal{U}_s j^2 + j\mathcal{V}_s j^2\ ds\ ;$$

which implies that the map de ned in (3.17) has a unique xed point $(Y;Z)$. Finally, by a similar analysis as in the proof of Theorem 3.10, we can get the desired result. □

Remark 4.2. In Lemma 4.1, we assume an additional bounded condition on the driver, which implies that

$$dj\ ^n j^0_s \quad C(r_0;\ ;\ )E\ jf\ s;Y^n_s;Z^n_s\ j\ ds \quad C(r_0;\ ;\ )E[g_s]ds;\ ds\text{-a.e.} \tag{4.5}$$

Here, $(Y^n;Z^n;\ ^n)$ is Picard iteration sequence with respect to the mean re ected BSDE (1.1) . Then, we are able to prove the convergence of $(Y^n;Z^n)$ under the following norm:

$$E\int_0^T \exp\ \ 4\ ^2 + 1\ t + \frac{2C(r_0;\ ;\ )}{r_0}\int_0^t E[g_s]ds \quad jY^n_t j^2 + jZ^n_t j^2\ dt\ ^{\frac{1}{2}}:$$

In general, it is dif cult to establish a priori estimate: $E[jf(s;Y^n_s;Z^n_s)j] \quad E[g_s]$ for some process $g \ 2\ H^{1;1}(R)$.

Next, we utilize an approximation approach to remove the above additional condition when the terminal value has a nite moment of order $p > 2$. For this purpose, we introduce the following approximating sequence $(f^n)_{n\ 1}$:

$$f^n(t;y;z) = f\ \ t;\ _{B_n}(y);\ _{B_n}(z)\ ;\ 8(t;y;z) \ 2\ [0;T] \quad R^m \quad R^{m\ d}; \tag{4.6}$$

where $_{B_n}$ is the projection on $B_n := f x \ 2\ E : jxj \quad ng$ for each Euclidian space E.

Remark 4.3. Note that $j\ _{B_n}(x) \quad _{B_n}(y)j \quad j\ x \quad yj$ for any $x;y \ 2\ E$. Hence under assumption (H4) , f^n also satis es (H4) with the same Lipschitz constant .

Lemma 4.4. Suppose that assumptions (H1) , (H2) and (H4) are satis ed. Assume in addition that $\ 2\ L^p(R^m)$ and $f(\ ;0;0) \ 2\ H^{1;p}(R^m)$ for some $p > 2$. Then the mean re ected BSDE (1.1) has a unique square integrable solution.

Proof. Without loss of generality, assume that $p < 4$. It is easy to check that for any $(t;y;z) \ 2\ [0;T] \quad R^m \quad R^{m\ d}$,

$$jf^n(t;y;z)j \quad j\ f(t;0;0)j + 2\ n\ \ 2\ H^{1;p}\ R^m \quad H^{\ 1;1}\ R^m\ ;$$

where f^n is given by (4.6) . Thus, it follows from Lemma 4.1 that the following BSDE with mean re ection

$$\begin{cases} Y^n_t = \ + \int_t^T f^n\ s;Y^n_s;Z^n_s\ ds \quad \int_t^T Z^n_s dB_s + \ ^n_T \quad \ ^n_t;\ \ 8t \ 2\ [0;T]; \\ E\ Y^n_t\ \ 2\ \overline{D}; \qquad\qquad\qquad\

admits a unique square integrable solution $(Y^n; Z^n; {}^n)$ for each $n \quad 1$. In view of Corollary 3.6 and noting that $f^n(t; 0; 0) = f(t; 0; 0)$, we have

$$\sup_{n \quad 1} E \Big[\sup_{0 \quad t \quad T} |Y_t^n|^p + \Big(\int_0^T |Z_s^n|^2 ds \Big)^{\frac{p}{2}} + \sup_{n \quad 1} |{}^n|_T^{0 \quad p} \Big] \quad M \tag{4.8}$$

with

$$M := C(p; r_0; ; ; ; T \quad) E \Big[|\ |^p + \Big(\int_0^T |f(s; 0; 0)| ds \Big)^p \Big] :$$

In the rest of the proof, we will prove that the limit of $(Y^n; Z^n; {}^n)$ is the desired solution, which will be divided into two steps.

Step 1. The convergence. For any xed $k \quad n$, set $\hat{b}_t^{k;n} = \hat{`}_t^k \quad \hat{`}_t^n$ for $\hat{`}_t = Y_t; Z_t; {}_t$. Note that

$$|f^k(t; Y_t^n; Z_t^n) \quad f^n(t; Y_t^n; Z_t^n)| \quad |Y_t^n| 1_{\{|Y_t^n| > n\}} + |Z_t^n| 1_{\{|Z_t^n| > n\}} :$$

Then with the help of Lemma 3.7, we derive that

$$\begin{aligned} & E \Big[\sup_{s \in [0;T]} |\hat{Y}_s^{k;n}|^2 + \int_0^T |\hat{Z}_s^{k;n}|^2 ds \Big] \\ & C(\ ; T\ ) e^{\frac{2}{r_0}(|\ |_T^{0\ k} + |\ |_T^{0\ n})} E \Big[\int_0^T |Y_s^n| 1_{\{|Y_s^n| > n\}} + |Z_s^n| 1_{\{|Z_s^n| > n\}} ds \Big]^2 : \end{aligned} \tag{4.9}$$

Applying Hölder's inequality yields that

$$\begin{aligned} & E \Big[\int_0^T |Y_s^n| 1_{\{|Y_s^n| > n\}} + |Z_s^n| 1_{\{|Z_s^n| > n\}} ds \Big]^2 \\ & C(p; T) E \Big[\int_0^T |Y_s^n|^{\frac{4}{p}} 1_{\{|Y_s^n| > n\}} + |Z_s^n|^{\frac{4}{p}} 1_{\{|Z_s^n| > n\}} ds \Big]^{\frac{p}{2}} \\ & \frac{C(p; T)}{n^{p \quad 2}} E \Big[\sup_{0 \quad t \quad T} |Y_t^n|^p + \Big(\int_0^T |Z_s^n|^2 ds \Big)^{\frac{p}{2}} \Big] \\ & \frac{C(p; T) M}{n^{p \quad 2}}; \end{aligned} \tag{4.10}$$

where we have used (4.8) in the last inequality. Since $|\ |_T^{0\ n}$ is uniformly bounded, it follows from (4.9) that

$$\lim_{k;n \to 1} E \Big[\sup_{0 \quad s \quad T} |Y_s^n \quad Y_s^k|^2 + \int_0^T |Z_s^n \quad Z_s^k|^2 ds \Big] = 0 : \tag{4.11}$$

Note that

$$\hat{b}_t^{k;n} = \hat{Y}_0^{k;n} \quad E\big[\hat{Y}_t^{k;n}\big] \quad E \Big[\int_0^t f^k(s; Y_s^k; Z_s^k) \quad f^n(s; Y_s^n; Z_s^n) ds \Big] :$$

Recalling (4.10) and (4.11), we obtain

$$\begin{aligned} & \lim_{k;n \to 1} \sup_{0 \quad t \quad T} |\hat{b}_t^{k;n}|^2 \\ & C(\ ; T\ ) \lim_{k;n \to 1} E \Big[\sup_{0 \quad s \quad T} |\hat{Y}_s^{k;n}|^2 + \int_0^T |\hat{Z}_s^{k;n}|^2 ds \Big] + \frac{C(p; T) M}{n^{p \quad 2}} \\ & = 0 : \end{aligned} \tag{4.12}$$

Consequently, putting (4.11) and (4.12) together, we can nd a triple of processes $(Y;Z;\ ) \ 2\ S^2(R^m)\ \ H^{\ 2}(R^{m\ \ d})\ \ V\ (R^m)$ so that

$$\lim_{n!1}\ \ E\ \sup_{0\ \ s\ \ T} jY_s^n\ \ \ Y_sj^2 + \int_0^{\ T} jZ_s^n\ \ \ Z_sj^2ds\ +\ \sup_{0\ \ t\ \ T} j\ _t^n\ \ \ _tj\ \ = 0: \tag{4.13}$$

Step 2. The solution. Recalling the de nition of $f^{\ n}$, we have

$$jf^{\ n}\ \ t;Y_t^n;Z_t^n\ \ \ f\ (t;Y_t;Z_t)j\ \ \ jY_t^n\ \ \ Y_tj + jZ_t^n\ \ \ Z_tj\ +\ \ (jY_tj1_{fj\ Y_tj>n\ \ g} + jZ_tj1_{fj\ Z_tj>n\ \ g});$$

which together with (4.13) indicates that

$$\begin{aligned}
&\lim_{n!1}\ E\ \int_0^{\ T} jf^{\ n}\ \ s;Y_s^n;Z_s^n\ \ \ f\ (s;Y_s;Z_s)jds \\
&\quad \lim_{n!1}\ \ C(\ ;T\ )E\ \sup_{0\ \ s\ \ T} jY_s^n\ \ \ Y_sj^2 + \int_0^{\ T} jZ_s^n\ \ \ Z_sj^2ds\ \ ^{\frac{1}{2}} \\
&\qquad\qquad + \frac{C(\ ;T\ )}{n}E\ \sup_{0\ \ s\ \ T} jY_sj^2 + \int_0^{\ T} jZ_sj^2ds \\
&\quad = 0:
\end{aligned} \tag{4.14}$$

Thus, letting $n\ !\ 1$ in (4.7) , we conclude that $(Y;Z;\ )$ satis es equation (1.1) . It remains to show that the component satis es (i) and (ii) in De nition 3.1, which is equivalent to prove that $(\ ;\ )$ is the solution of BSP $(D;\ )$ with $_t := E[Y_t]$ and $_t := E[\ + \int_t^T f\ (s;Y_s;Z_s)ds]$.

Set

$$_t^n := E\ Y_t^n\ \ \text{and}\ \ _t^n := E\ \ + \int_t^{\ T} f^{\ n}\ \ s;Y_s^n;Z_s^n\ \ ds\ \ \text{for all}\ \ n\ \ \ 1:$$

It is clear that $(\ ^n;\ ^n)$ is the solution of BSP $(D;\ ^n)$ for each $n\ \ \ 1$. Recalling (4.8) , (4.13) and (4.14), we have

$$\sup_{n\ \ 1} j\ ^nj_T^0 < 1\ \ \text{and}\ \ \lim_{n!1}\ \sup_{0\ \ t\ \ T} j\ _t^n\ \ \ _tj + j\ _t^n\ \ \ _tj + j\ _t^n\ \ \ _tj\ \ = 0;$$

which together with Lemma 2.8 indicates the desired result. The proof is complete. □

Finally, we are ready to complete the proof of Theorem 3.3.

The proof of Theorem 3.3. It suf ces to prove the existence. Set

$$^{(n)} :=\ \ 1_{fj\ \ j\ \ \ ng} + E[\ 1_{fj\ \ j>n\ \ g}];\ f^{(n)}(t;y;z) := f\ (t;y;z)\ \ \ f\ (t;0;0)1_{fj\ f\ (t;0;0)j>n\ \ g};\ 8n\ \ \ 1: \tag{4.15}$$

It is easy to check that the data $(\ ^{(n)};f^{(n)})$ satis es the following condition:

$^{(n)}$ is bounded and $E[\ ^{(n)}] = E[\ ]\ 2\ \overline{D}$.
$f^{(n)}(t;0;0)$ is bounded and $f^{\ n}$ satis es (H4) .

By Lemma 4.4, there exists a unique square integrable solution $(Y^{(n)};Z^{(n)};\ ^{(n)})$ to the following BSDE with mean re ection:

$$\begin{cases} Y_t^{(n)} =\ \ ^{(n)} + \int_t^{\ T} f^{(n)}\ \ s;Y_s^{(n)};Z_s^{(n)}\ \ ds\ \ \int_t^{\ T} Z_s^{(n)}dB_s + \ _T^{(n)}\ \ \ _t^{(n)};\ \ 8t\ 2\ [0;T]; \\ E\ Y_t^{(n)}\ \ 2\ \overline{D}; \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 8t\ 2\ [0;T]: \end{cases} \tag{4.16}$$

It is obvious that

$$jf^{(n)}(t;0;0)j\ \ \ j\ f\ (t;0;0)j;\ E\ j\ ^{(n)}j^2\ \ \ 4E\ j\ j^2\ ;\ n\ \ \ 1;$$

which together with Lemma 3.5 implies

$$\begin{aligned}\sup_{n\ 1}E\ \sup_{0\ t\ T}|Y_t^{(n)}|^2+\int_0^{Z\ T}|Z_s^{(n)}|^2ds\ +\sup_{n\ 1}|\ ^{(n)}|_T^{0\ 2}\\ C(r_0;\ ;\ ;\ ;T\ )E\ |\ |^2+\int_0^{Z\ T}|f\ (s;0;0)|ds\ \ :\end{aligned}\tag{4.17}$$

The rest of the proof is similar to that of Lemma 4.4. For readers' convenience, we give the sketch of the proof.

Step 1. The convergence. For any xed k n, set $b_t^{(k;n)}=\ _t^{(k)}\ \ _t^{(n)}$ for $\ _t=Y_t;Z_t;\ _t$. Note that

$$|f^{(k)}\ t;Y_t^{(n)};Z_t^{(n)}\ \ f^{(n)}\ t;Y_t^{(n)};Z_t^{(n)}|\ \ |\ \ f\ (t;0;0)|1_{\{|f\ (t;0;0)|>n\ g}:$$

Applying Lemma 3.7 yields that

$$\begin{aligned}&E\ \sup_{0\ s\ T}|b_s^{(k;n)}|^2+\int_0^{Z\ T}|b_s^{(k;n)}|^2ds\\ &C(\ ;T\ )e^{\frac{2}{r_0}(|\ ^{(k)}|_T^0+|\ ^{(n)}|_T^0)}E\ |\ ^{(k)}\ \ ^{(n)}|^2+\int_0^{Z\ T}|f\ (s;0;0)|1_{\{|f\ (s;0;0)|>n\ g}ds\ \ :\end{aligned}\tag{4.18}$$

Since $|\ ^{(n)}|\ \ |\ +E[|\ |]$, we can use dominated convergence theorem to obtain

$$\lim_{n!1}\ E\ |\ ^{(n)}\ \ |^2+\int_0^{Z\ T}|f\ (s;0;0)|1_{\{|f\ (s;0;0)|>n\ g}ds\ \ ^2=0:\tag{4.19}$$

Then it follows from (4.17)–(4.19) that

$$\lim_{k;n\ !1}\ \ E\ \sup_{0\ s\ T}|b_s^{(k;n)}|^2\ +E\int_0^{Z\ T}|b_s^{(k;n)}|^2ds\ \ =0:$$

Consequently, in view of the derivation of (4.12), we also have

$$\lim_{k;n\ !1}\ \sup_{0\ t\ T}|b_t^{(k;n)}|=0:$$

Step 2. The solution. It is clear that there exists a triple of processes $(Y;Z;\ )\ 2$ $S^2(R^m)\ \ H^{\ 2}(R^{m\ \ d})\ \ V\ (R^m)$ such that

$$\lim_{n!1}\ \ E\ \sup_{0\ s\ T}|Y_s^n\ \ Y_s|^2\ +E\int_0^{Z\ T}|Z_s^n\ \ Z_s|^2ds\ +\sup_{0\ t\ T}|\ _t^n\ \ \ _t|\ \ =0:\tag{4.20}$$

By a similar analysis as in the proof of Lemma 4.4 (Step 2), we obtain

$$\lim_{n!1}\ E\int_0^{Z\ T}|f^{(n)}\ s;Y_s^{(n)};Z_s^{(n)}\ \ f\ (s;Y_s;Z_s)|ds\ \ =0;\tag{4.21}$$

which indicates that that $(Y;Z;\ )$ satis es (1.1) by letting $n\ !\ 1$ in (4.16) . Moreover, satis es (i) and (ii) in De nition 3.1. The proof is complete. □

A Approximation by penalization method

In this appendix, we shall use a penalization method to construct the unique solution to the mean re ected BSDE (1.1) when the re ection domain is convex inspired by the results of [24]. Indeed, the unique solution can be represented as the limit of a sequence

of penalized mean- led BSDEs. Moreover, we can remove assumption (H2) in the convex case.

Assume that the domain D is convex. Then, for any $x \in R^m$, there exists a unique point $\Pi(x) \in \overline{D}$ such that

$$|x - \Pi(x)| = d(x; D) := \inf\{|x - y| : y \in D\}.$$

The following properties of the inward normal re ection is important for our subsequent discussions.

Lemma A.1 ([24, 38]). (i) For any $x \in R^m$, $x^0 \in \overline{D}$,

$$\langle x^0 - \Pi(x), x - \Pi(x) \rangle \leq 0. \tag{A.1}$$

(ii) For any $x, x^0 \in R^m$,

$$\langle x^0 - \Pi(x), x - \Pi(x) \rangle \leq |x^0 - \Pi(x^0)| \, |x - \Pi(x)|. \tag{A.2}$$

(iii) There exist a point $x_0 \in D$ and a constant $\gamma > 0$ such that for any $x \in R^m$,

$$\langle x - x_0, x - \Pi(x) \rangle \geq \gamma |x - \Pi(x)|. \tag{A.3}$$

Since ∂D is not regular, let us recall the approximation of D in [24]: For any $\varepsilon > 0$, there exists a convex regular domain D^ε (with smooth boundary) such that

$$\sup_{x \in D} d(x; D^\varepsilon) < \varepsilon \quad \text{and} \quad \sup_{x \in D^\varepsilon} d(x; D) < \varepsilon.$$

It is easy to check that $|d(x; D) - d(x; D^\varepsilon)| < \varepsilon$. Denote by Π^ε the projection from R^m to D^ε.

Lemma A.2 (Lemma 2.2 and Corollary 2.3 in [24]). There exists a constant $\gamma > 0$ such that for any $\varepsilon < 1$ and $x \in R^m$,

$$|\Pi^\varepsilon(x) - \Pi(x)| \leq \sqrt{\varepsilon^2 + d(x; D^\varepsilon)},$$

and

$$|\Pi^\varepsilon(x) - \Pi(x)| 1_{\{d(x;D) > \varepsilon\}} \leq \frac{\varepsilon}{\sqrt{d(x; D^\varepsilon)}} 1_{\{d(x;D) > \varepsilon\}}.$$

Now, we introduce the following penalized mean- led BSDEs:

$$Y_t^n = \int_t^T f(s, Y_s^n, Z_s^n)\, ds - n \int_t^T (Y_s^n - \Pi^\varepsilon(Y_s^n))\, ds - \int_t^T Z_s^n \, dB_s, \; 0 \leq t \leq T. \tag{A.4}$$

According to the results of [10] up to a slight modi cation, the BSDE (A.4) has a unique solution $(Y^n, Z^n) \in S^2(R^m) \times H^2(R^{m \times d})$ under assumptions (H3) and (H4). Then, we de ne

$$K_t^n = n \int_0^t (Y_s^n - \Pi^\varepsilon(Y_s^n))\, ds, \; \forall t \in [0, T]. \tag{A.5}$$

The penalized term K^n forces the mean of the solution Y^n to stay within the domain $\overline{D}$. We will show that (Y^n, Z^n, K^n) converges as $n \to \infty$ and the limit is the unique solution to the BSDE (1.1) with mean re ection.

Lemma A.3. Suppose that assumptions (H3) and (H4) hold. Then,

$$E\Big[\sup_{0 \leq s \leq T} |Y_s^n|^2 + \int_0^T |Z_s^n|^2 ds\Big] \leq C(\gamma, T) E\Big[|x_0|^2 + |\varepsilon|^2 + \int_0^T |f(s, 0, 0)| ds\Big]^2;$$

$$|K_T^n|^2 \leq C(\gamma, T, \varepsilon) E\Big[|x_0|^2 + |\varepsilon|^2 + \int_0^T |f(s, 0, 0)| ds\Big]^2.$$

Proof. The main idea is from [24]. For reader's convenience, we give the sketch of the proof. By a standard calculus, applying Itô's formula to $jY_t^n \quad x_0j^2$ yields that for any $t \ 2 \ [0;T]$,

$$\begin{aligned} E \ jY_t^n \quad x_0j^2 + \int_t^T jZ_s^n j^2 ds \ = E \ j \quad x_0j^2 \ + 2E \int_t^T Y_s^n \quad x_0; f \ s; Y_s^n; Z_s^n \quad ds \\ 2n \int_t^T E \ Y_s^n \quad x_0; E \ Y_s^n \quad E \ Y_s^n \quad ds: \end{aligned} \tag{A.6}$$

Since the third term in the right-hand side of (A.6) is non-positive according to (A.3) , we have

$$\begin{aligned} E \ jY_t^n \quad x_0j^2 + \int_t^T jZ_s^n j^2 ds \quad E \ j \quad x_0j^2 \ + 2E \sup_{0 \ t \ T} jY_t^n \quad x_0j \int_0^T jf \ (s; x_0; 0)jds \\ + \ 2 \ + 2 \ ^2 E \int_t^T jY_t^n \quad x_0j^2 ds \ + \frac{1}{2} E \int_t^T jZ_s^n j^2 ds \ : \end{aligned}$$

In view of Gronwall's inequality, we obtain

$$\sup_{0 \ t \ T} E \ jY_t^n \quad x_0j^2 \quad C(\ ;T \ ) \ E \ j \quad x_0j^2 \ + E \sup_{0 \ t \ T} jY_t^n \quad x_0j \int_0^T jf \ (s; x_0; 0)jds \quad ; \tag{A.7}$$

and then

$$E \int_0^T jZ_s^n j^2 ds \quad C(\ ;T \ ) \ E \ j \quad x_0j^2 \ + E \sup_{0 \ t \ T} jY_t^n \quad x_0j \int_0^T jf \ (s; x_0; 0)jds \ : \tag{A.8}$$

On the other hand, recalling the de nition of (A.4) and noting that ^{n} is deterministic, we get

$$\begin{aligned} & j \ ^n_T \quad ^n_t j^2 \\ & = E \quad Y_t^n + \int_t^T f \ s; Y_s^n; Z_s^n \ ds \ ^2 \\ & \quad C(\ ;T \ ) \ jx_0j^2 + E \ j \ j^2 + \int_0^T jf \ (s; 0; 0)jds \ ^2 + \sup_{0 \ t \ T} jY_t^n \quad x_0j \int_0^T jf \ (s; x_0; 0)jds \quad ; \end{aligned} \tag{A.9}$$

where we have used (A.7) and (A.8) in the last inequality. Note that

$$\begin{aligned} & \sup_{0 \ t \ T} jY_t^n \quad x_0j^2 \\ & \quad 4 \ j \quad x_0j^2 + \int_0^T jf \ s; Y_s^n; Z_s^n \ jds \ ^2 + \sup_{0 \ t \ T} j \ ^n_T \quad ^n_t j^2 + \sup_{0 \ t \ T} \int_t^T Z_s^n dB_s \ ^2 : \end{aligned}$$

It follows from (A.7)–(A.9) and BDG's inequality that

$$\begin{aligned} & E \left[\sup_{0 \ t \ T} jY_t^n \quad x_0j^2 \right] \\ & \quad C(\ ;T \ )E \ jx_0j^2 + j \ j^2 + \int_0^T jf \ (s; 0; 0)jds \ ^2 + \sup_{0 \ t \ T} jY_t^n \quad x_0j \int_0^T jf \ (s; x_0; 0)jds \\ & \quad C(\ ;T \ )E \ jx_0j^2 + j \ j^2 + \int_0^T jf \ (s; 0; 0)jds \ ^2 + \frac{1}{2} E \left[\sup_{0 \ t \ T} jY_t^n \quad x_0j^2 \right] : \end{aligned} \tag{A.10}$$

Putting (A.8) and (A.10) together, we deduce that the rst inequality holds.