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Basic principles of fluid mechanics

Abstract

This chapter introduces some basic principles of fluid mechanics. Equations are derived to describe hydrostatic forces on inclined plates. The concept of Archimedes' principle is explained and an analytical solution for assessing the stability of floating of objects is developed. The continuity, momentum and Bernoulli equations are derived using a control-volume within a streamtube, located within a moving incompressible and inviscid fluid. The Bernoulli equation is used to derive the Torricelli equation and equations describing flow over sharp crested weirs. Practical applications are presented through a problem sheet with worked solutions.

Notation

a	Acceleration [LT^{-2}].
a_1	Depth of low density fluid in a u-tube manometer [L].
a_2	Elevation difference between fluid interfaces in a u-tube manometer [L].
A	Area [L^2].
$\hat{A}$	Mean cross-sectional area [L^2].
B	Breadth [L].
E	Total head [L].
F	Force [MLT^{-2}].
F_b	Buoyant force [MLT^{-2}].
F_g	Gravitational force [MLT^{-2}].
g	Gravitational acceleration [LT^{-2}].
h	Hydraulic head [L].
h_v	Velocity head [L].
H	Depth [L].

H_F	Depth of a hydrostatic force [L].
I_{yy}	Second moment of area about a centroid in the y-direction [L^4].
L	Length [L].
L_{BG}	Distance between the centre of buoyancy and the centre of gravity [L].
L_{BM}	Distance between the centre of buoyancy and the metacentre [L].
M	Mass [M].
p	Fluid pressure [$ML^{-1}T^{-2}$].
p_0	Atmospheric pressure [$ML^{-1}T^{-2}$].
p_a	Absolute pressure [$ML^{-1}T^{-2}$].
Q	Flow rate [L^3T^{-1}].
s	Distance along a streamtube [L].
t	Time [T].
U	Moment [ML^2T^{-2}].
U_b	Moment due to a buoyancy force [ML^2T^{-2}].
U_b'	Moment due to a buoyancy force following a rotation [ML^2T^{-2}].
V_I	Immersed volume [L^3].
v	Fluid velocity [LT^{-1}].
y	Distance from a liquid surface in the plane of a submerged and inclined plate [L].
y_F	Distance from a liquid surface to the location where a hydrostatic force applies, in the plane of a submerged and inclined plate [L].
y_c	Distance to a centroid from a datum [L].
y_c'	Distance to a centroid from a datum following a rotation [L].
z	Elevation [L].
θ	Inclination angle [-].
ρ	Mass density of a fluid [ML^{-3}].
ψ	Pressure head [L].
CoB	Centre of buoyancy.
CoG	Centre of gravity.

1.1 Introduction

This chapter provides a brief introduction to some basic principles of fluid mechanics. The first section focuses on hydrostatics. The second section focuses on moving fluids in the absence of frictional forces. All the results derived in this chapter are classic. Similar results with different derivations can be found in many existing textbooks concerning the mechanics of fluids. The reader is directed to [1] for alternative and sometimes more detailed explanations.

1.2 Hydrostatics

The study of forces in static fluid is referred to as hydrostatics. In this section we will develop the concept of hydrostatic pressure and use it to derive methods for measuring pressure, determining hydrostatic forces on inclined plates and assessing the stability of floating objects.

1.2.1 Hydrostatic pressure

Consider a tank of static liquid with a free upper surface open to the atmosphere (Fig. 1.1). Let A [L^2] be the plan area of the tank, H [L] be the depth of the liquid within the tank, p_0 [$ML^{-1}T^{-2}$] be the atmospheric pressure, ρ [ML^{-3}] be the mass density of the liquid and g [LT^{-2}] be gravitational acceleration. The downward force applied to the base of the tank will be $A(p_0+\rho gH)$. The absolute pressure, p_a [$ML^{-1}T^{-2}$], applied to the base of the tank is therefore $p_0+\rho gH$.

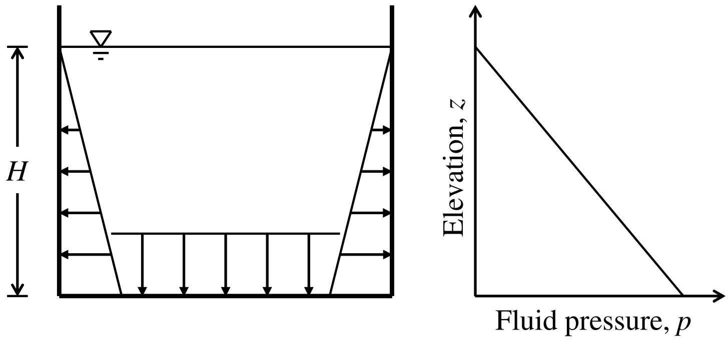

Figure 1.1: Hydrostatic fluid pressure in a tank of liquid.

In hydraulics, we are more concerned with a relative pressure, which represents the difference between absolute pressure and atmospheric pressure. Hereafter, the term fluid pressure, p [$ML^{-1}T^{-2}$], is taken to mean the relative pressure of a fluid, $p \equiv p_a - p_0$ [$ML^{-1}T^{-2}$].

Referring back to the tank of liquid in Fig. 1.1, $p = 0$, at the liquid surface and $p = \rho gH$ at the base of the tank. Furthermore, if z [L] represents elevation above the base of the tank

$$p = \rho g(H - z) \tag{1.1}$$

Eq. (1.1) is often referred to as a hydrostatic pressure profile. Interestingly, it will apply irrespective of the geometry of the tank of concern. Furthermore, pressure is isotropic, which means that its magnitude is independent of direction. This implies that the liquid applies a uniform force distribution to the base of the tank and a linear force distribution to the side walls of the tank (as indicated by the arrows in the tank shown in Fig. 1.1).

1.2.2 Measuring pressure

Note that hydrostatic pressure is independent of the quantity of fluid present. Instead, it is dependent on the elevation of the hydraulically connected free surface exposed to the atmosphere. For example, consider Fig. 1.2. The fluid pressure at the base of each of the first three devices (Figs. 1.2 a, b and c) should be the same because the elevation of the exposed free surface, in each device, is the same distance above the base of the underlying tanks.

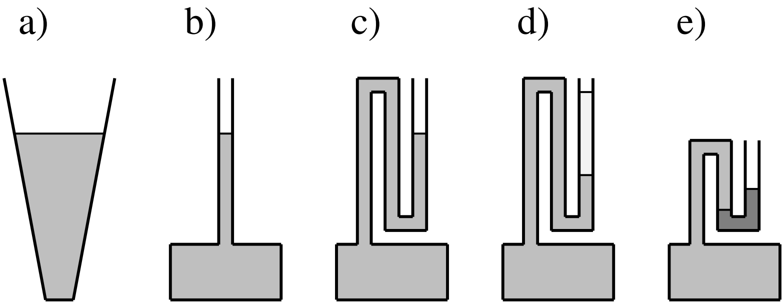

Figure 1.2: Examples of different vessels that have the same fluid pressure at the base. a) Inverted conical vase. b) Tank with a piezometer. c) Tank with a u-tube manometer containing common fluid. d) Tank with a u-tube manometer containing less dense fluid (the lighter shade). e) Tank with a u-tube manometer containing more dense fluid (the darker shade).

The device shown in Fig. 1.2b is in fact a measuring device referred to as a piezometer. The fluid pressure at the top of the tank can be measured by multiplying the elevation difference between the top of the tank and the free surface in the piezometer by ρg.

The device shown in Fig. 1.2c is an alternative measuring device referred to as a u-tube manometer. The fluid pressure at the top of the tank can be measured by multiplying the elevation difference between the top of the tank and the free surface in the manometer by ρg. The advantage of the u-tube manometer, over the piezometer, is that fluids with different densities to that being measured can be used to either provide more or less sensitivity. Furthermore, it is possible to measure absolute pressures, which are less than atmospheric pressure.

Fig. 1.2d shows a tank with the same pressure as that in Fig. 1.2c but with a u-tube manometer containing a less dense fluid (the lighter shade). Consequently, the free surface exposed to the atmosphere is higher in Fig. 1.2d as compared to in Fig. 1.2c. Small changes in pressure will lead to larger changes in fluid level when using a less dense fluid, hence it is more sensitive (as compared to when using a common fluid).

Fig. 1.2e shows a tank with the same pressure as that in Fig. 1.2c but with a u-tube manometer containing a more dense fluid (the darker shade). Consequently, the free surface exposed to the atmosphere is lower in Fig. 1.2e as compared to in Fig. 1.2c. Large pressures lead to smaller fluid levels when using a more dense fluid, hence a smaller u-tube can be used (as compared to when using a common fluid).

Note that given all five vessels in Fig. 1.2 are hydrostatic, exactly the same fluid levels would be observed if they were each linked to the same tank, such as shown in Fig. 1.3.

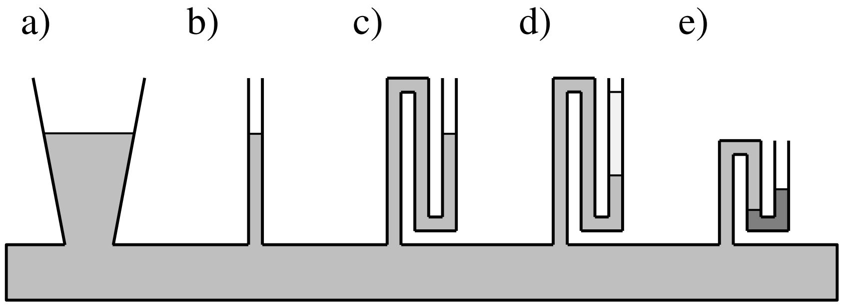

Figure 1.3: Examples of different fluid pressure measuring devices in a common tank of liquid. a) Inverted conical vase. b) Piezometer. c) U-tube manometer containing common fluid. d) U-tube manometer containing less dense fluid (the lighter shade). e) U-tube manometer containing more dense fluid (the darker shade).

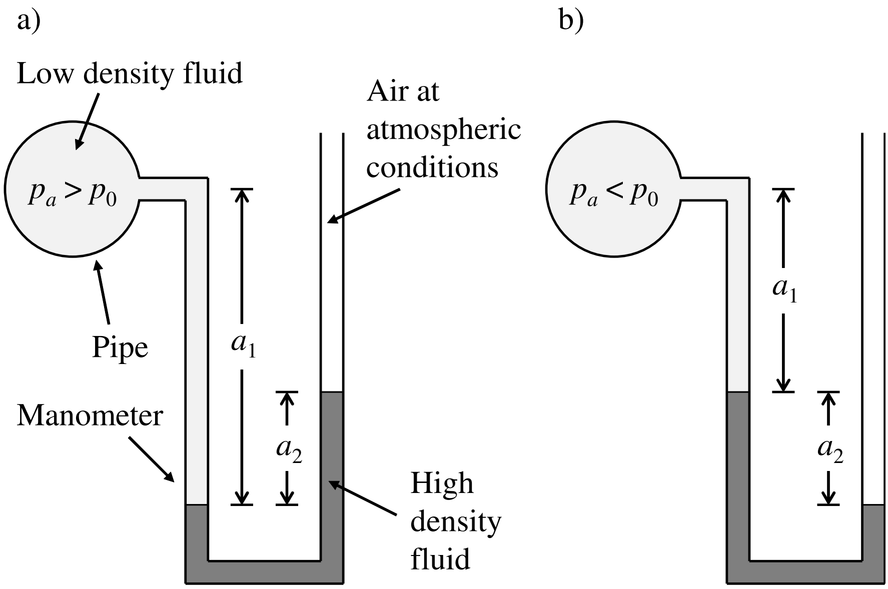

Figure 1.4: High density fluid u-tube manometers connected to a pipe. The circles represent a cross-section through the pipe. a) Measurement of an absolute fluid pressure greater than atmospheric pressure. b) Measurement of an absolute fluid pressure greater than atmospheric.

Fig. 1.4a shows a u-tube manometer measuring the absolute pressure, p_a, of a fluid flowing in a pipe, where the absolute pressure is greater than atmospheric pressure, p_0. The mass density of the low density fluid in the pipe is ρ_1 [ML^{-3}] and the mass density of the high density fluid in the u-tube is ρ_2 [ML^{-3}].

Challenge 1.1 Use the concept of hydrostatic pressure to determine an expression for the absolute pressure in the pipe shown in Fig. 1.4a.

The absolute pressure in the pipe, shown in Fig. 1.4a, can be obtained by equating two expressions for the absolute pressure at the interface between the low and high density fluids in the u-tube.

The hydrostatic pressure contribution due to the low density fluid is $\rho_1 a_1 g$. There is also the absolute pressure in the pipe, p_a. The absolute pressure at the interface between low density and high density fluids in the u-tube manometer is therefore $p_a + \rho_1 a_1 g$.

On the other hand, the hydrostatic pressure contribution due to the high density fluid is $\rho_2 a_2 g$. There is also the atmospheric pressure, p_0. It can therefore also be said that the absolute pressure at the interface between low density and high density fluids in the u-tube manometer is $p_0 + \rho_2 a_2 g$.

Equating the two expressions above and solving for p_a gives us

$$p_a = p_0 + \rho_2 a_2 g - \rho_1 a_1 g \tag{1.2}$$

Fig. 1.4b shows a u-tube manometer measuring the absolute pressure, p_a, of a fluid flowing in a pipe were the absolute pressure is less than atmospheric pressure, p_0. Again, ρ_1 [ML^{-3}] is the mass density of the low density fluid in the pipe and ρ_2 [ML^{-3}] is the mass density of the high density fluid in the u-tube.

Challenge 1.2 Use the concept of hydrostatic pressure to determine an expression for the absolute pressure in the pipe shown in Fig. 1.4b.

The absolute pressure in the pipe, shown in Fig. 1.4b, can be obtained by determining an expression for the absolute pressure at the high density fluid free surface and equating this with the atmospheric pressure.

The hydrostatic pressure contribution due to the low density fluid is $\rho_1 a_1 g$. The hydrostatic pressure contribution due to the high density fluid is $\rho_2 a_2 g$. There is also the absolute pressure in the pipe, p_a. The absolute pressure at the high density fluid free surface is therefore $p_a + \rho_1 a_1 g + \rho_2 a_2 g$. Equating this with the atmospheric pressure, p_0, and solving for p_a gives us

$$p_a = p_0 - \rho_2 a_2 g - \rho_1 a_1 g \tag{1.3}$$

1.2.3 Hydrostatic forces on inclined plates

Here we will derive expressions for the total force applied to a submerged inclined plate due to hydrostatic pressure. We will then use moment matching to determine the depth at which the force applies. To begin with, we will focus on a rectangular plate. The same theory will then be extended to account for plates of arbitrary geometries.

1.2.3.1 Rectangular plate

Consider an immersed rectangular plate of breadth, B [L], and length, L [L], inclined at an angle, θ [-], to the horizontal axis (see Fig. 1.5). The depth of fluid above the top and bottom of the plate are denoted H_0 [L] and H_1 [L], respectively. Let y [L] be a distance from the liquid surface in the plane of the plate. The origin of the y-axis at the liquid surface is denoted O. The distance, along the y-axis, from O to the top and bottom of the plate are denoted y_0 [L] and y_1 [L], respectively. Note that $L = y_1 - y_0$, $H_0 = y_0 \sin\theta$ and $H_1 = y_1 \sin\theta$.

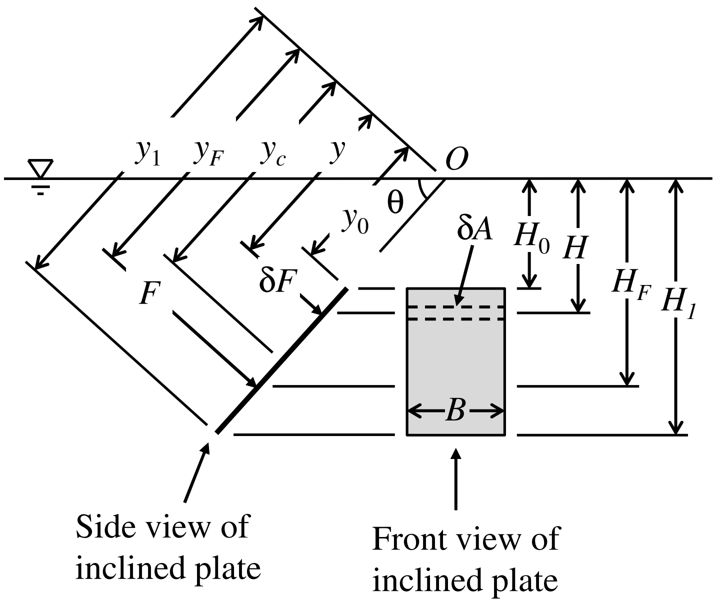

Figure 1.5: Hydrostatic pressure on an inclined rectangular plate.

Challenge 1.3 Determine an expression for the force, δF [MLT^{-2}], applied to a thin strip of area, δA [L^2], located at a distance, y, along the y-axis due to hydrostatic pressure. Your final expression should be in terms of ρ, g, θ, y and δA.

Consider a thin strip of area, δA [L^2], located at a distance, y, along the y-axis. The fluid depth at this location is $y\sin\theta$. Therefore the hydrostatic pressure at this point is $\rho g y \sin\theta$. It follows that the force applied to the strip, δF [MLT^{-2}], due to hydrostatic pressure is found from

$$\delta F = \rho g y \sin\theta \delta A \tag{1.4}$$

Challenge 1.4 Determine an expression for the total force, F [MLT^{-2}], applied to the inclined plate due to hydrostatic pressure. Your final expression should be in terms of ρ, g, B, L, H_0 and H_1.

The total force, F [MLT^{-2}], applied to the inclined plate is found by integrating Eq. (1.4)

$$F = \rho g \sin\theta \int y dA \tag{1.5}$$

Noting that the plate exists for $y \in [y_0, y_1]$ and $\frac{dA}{dy} = B$, it follows that

$$\begin{aligned} F &= \rho g B \sin\theta \int_{y_0}^{y_1} y dy \\ &= \rho g B \sin\theta \left(\frac{y_1^2 - y_0^2}{2} \right) \end{aligned}$$

Given that $L = y_1 - y_0$, $H_0 = y_0 \sin\theta$ and $H_1 = y_1 \sin\theta$, it can be further stated that

$$F = \rho g B L \left(\frac{H_0 + H_1}{2} \right) \tag{1.6}$$

Challenge 1.5 Determine an expression for the moment about O, δU [ML^{2}T^{-2}], applied to a thin strip of area, δA [L^2], located at a distance, y, along the y-axis, due to hydrostatic pressure. Your final expression should be in terms of ρ, g, θ, y and δA.

The hydrostatic force applied to a thin strip, gives rise to a moment about O, $\delta U = \delta F y$, where δF is given by Eq. (1.4). It follows that

$$\delta U = \rho g y^2 \sin\theta \delta A \tag{1.7}$$

Challenge 1.6 Determine an expression for the total moment about O, U [ML^{2}T^{-2}], applied to the inclined plate due to hydrostatic pressure.

The total moment about O, U [ML^{2}T^{-2}], applied to the inclined plate due to hydrostatic pressure is found by integrating Eq. (1.7)

$$U = \rho g \sin\theta \int y^2 dA \tag{1.8}$$

Noting again that the plate exists for $y \in [y_0, y_1]$ and $\frac{dA}{dy} = B$, it follows that

$$\begin{aligned} U &= \rho g B \sin\theta \int_{y_0}^{y_1} y^2 dy \\ &= \rho g B \sin\theta \left(\frac{y_1^3 - y_0^3}{3} \right) \\ &= \rho g B \sin\theta (y_1 - y_0) \left(\frac{y_0^2 + y_0 y_1 + y_1^2}{3} \right) \\ &= \rho g B L \csc\theta \left(\frac{H_0^2 + H_0 H_1 + H_1^2}{3} \right) \end{aligned} \tag{1.9}$$

Challenge 1.7 Use moment matching to determine the depth, H_F [L], at which the total force, F, applies.

Let H_F [L] be the depth at which the total force, F, applies. The location at which the total force acts on the y-axis, $y_F = H_F \csc\theta$. Consequently, it can also be stated that

$$U = F H_F \csc\theta \tag{1.10}$$

Substituting Eqs. (1.6) and (1.9) into Eq. (1.10) and solving for H_F leads to

$$H_F = \frac{2}{3}\left(\frac{H_0^2 + H_0 H_1 + H_1^2}{H_0 + H_1}\right) \tag{1.11}$$

Note that

$$\lim_{H_0 \to 0} H_F = \frac{2H_1}{3} \tag{1.12}$$

1.2.3.2 Plates with arbitrary geometries

Note that Eqs. (1.5) and (1.8) also apply to inclined plates with arbitrary geometries (see Fig. 1.6).

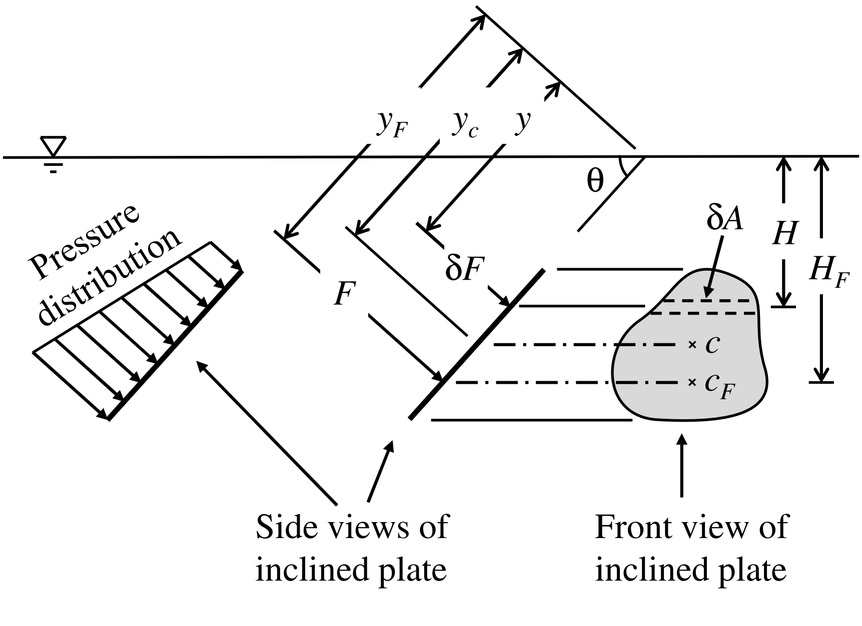

Figure 1.6: Hydrostatic pressure on an inclined plate with arbitrary geometry.

Let y_c [L] be the location (along the y-axis) of the centroid (analogous to the centre of gravity or centre of mass) for the plate of concern, defined by

$$y_c A = \int y dA \tag{1.13}$$

Substituting Eq. (1.13) into Eq. (1.5) leads to

$$F = \rho g y_c A \sin\theta \tag{1.14}$$

Let I_{yy} [L^4] be the second moment of area in the y-direction, about the centroid, for the plate of concern, defined by

$$I_{yy} = \int y(y - y_c)dA \tag{1.15}$$

Expanding the brackets in Eq. (1.15), substituting Eq. (1.13) and rearranging leads to

$$\int y^2 dA = I_{yy} + y_c^2 A \tag{1.16}$$

which is often referred to as the parallel axis theorem [2, p. 90].

Substituting Eq. (1.16) into Eq. (1.8) leads to

$$U = \rho g (I_{yy} + y_c^2 A)\sin\theta \tag{1.17}$$

Substituting Eqs. (1.14) and (1.17) into Eq. (1.10) and solving for H_F then gives us [1, p. 62]

$$H_F = \left(\frac{I_{yy}}{y_c A} + y_c\right)\sin\theta \tag{1.18}$$

For the aforementioned inclined rectangular plate, $A = BL$, $y_c = \left(\frac{H_0+H_1}{2}\right)\csc\theta$ and $I_{yy} = \frac{BL^3}{12}$ [2, p. 97], which on substitution into Eqs. (1.14) and (1.18) leads to Eqs. (1.6) and (1.11), respectively. Expressions for A, y_c and I_{yy} are available (e.g. [3]) or can be determined for any alternative geometry.

1.2.4 Buoyancy

Consider a vertically prismatic solid object of length, L [L], and cross-sectional area, A [L^2]. The solid object is fully immersed within a hydrostatic fluid of mass density, ρ [ML^{-3}]. The depth of fluid above the base of the object is H [L].

Challenge 1.8 Determine the net upward force exerted by the hydrostatic liquid on the solid object.

The hydrostatic fluid exerts a downward force on the object of $F_1 = \rho g(H-L)A$ and an upward force on the object of $F_2 = \rho g H A$. The net upward force, F_b [MLT^{-2}], exerted by the hydrostatic liquid is therefore

$$F_b = F_2 - F_1 = \rho g L A \tag{1.19}$$

This net-upward force, F_b, is commonly referred to as the buoyant force. Providing the weight of a solid object, F_g [MLT^{-2}], is less than its associated buoyant force, the object will float at the surface of a liquid. Furthermore, the depth of immersion, H [L], can be determined by recognising that, for floating objects, $F_g = F_b$.

Challenge 1.9 Determine the depth of immersion, H [L], for the previously mentioned vertically prismatic object, assuming it has a mass density of ρ_s [ML^{-3}] where $\rho_s < \rho$.

Suppose the previously mentioned vertically prismatic object has a mass density of ρ_s [ML^{-3}], where $\rho_s < \rho$. In this case we would expect the object to float at the liquid surface. Let H [L] represent the depth of liquid above the base of the object (i.e., the depth of immersion). Given that the object is floating, the top surface of the object is not exposed to the fluid. Therefore the downward force acting on the object due to the hydrostatic fluid, $F_1 = 0$. The upward force acting on the object due to the hydrostatic fluid, $F_2 = \rho g H A$. It follows that the buoyant force, $F_b = F_2 - F_1 = \rho g H A$. The weight of the object, $F_g = \rho_s g L A$. Setting $F_g = F_b$ and solving for H then leads to

$$H = \frac{\rho_s L}{\rho} \tag{1.20}$$

1.2.4.1 Archimedes' principle

Archimedes' principle states more generally that the buoyant force, F_b, acting on a fully or partially immersed object is equal to the weight of fluid displaced by the immersed volume V_I [L^3], for the object of concern, i.e.,

$$F_b = \rho g V_I \tag{1.21}$$

In the case of the fully immersed vertically prismatic object (recall Eq. (1.19)) $V_I = LA$. In the case of the partially immersed vertically prismatic object (recall Eq. (1.20)) $V_I = HA$. Archimedes' principle can also be shown to apply to solid objectives of any arbitrary and irregular geometry.

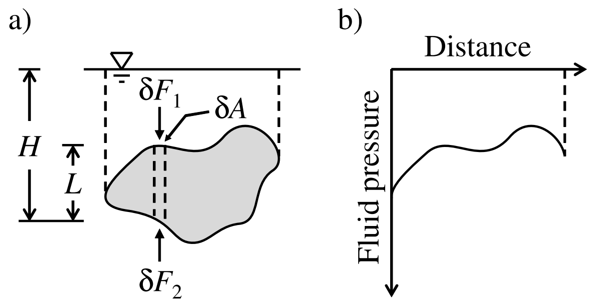

Figure 1.7: a) Vertical hydrostatic forces applied to a prismatic element within an immersed solid object (the shaded region). b) Hydrostatic pressure distribution applied downwards on the upper surface of the immersed solid object. c) Hydrostatic pressure distribution applied upwards on the lower surface of the immersed solid object.

Fig. 1.7a shows a vertical section through a fully immersed irregular solid object. The fluid pressure acting downwards on the upper surface of the object is linearly proportional to the depth of water above the upper surface (see Fig. 1.7b). The fluid pressure acting upwards on the lower surface of the object is linearly proportional to the depth of water above the lower surface (see Fig. 1.7c).

Consider a vertically prismatic element, within the solid object, of plan view area, δA [L^2], and thickness, L [L]. The depth of water above the base of the element is H [L]. The downward force due to hydrostatic pressure acting on the top of the element is

$$\delta F_1 = \rho g (H - L) \delta A$$

The upward force due to hydrostatic pressure acting on the base of the element is

$$\delta F_2 = \rho g H \delta A$$

The buoyant force acting on the element is

$$\delta F_b = \delta F_2 - \delta F_1 = \rho g L \delta A$$

Noting that L is spatially variable due to the irregularity of the object, the total buoyant force acting on the object is found from

$$F_b = \rho g \int L dA$$

But of course, in this case, $\int L dA = V_I$ and so Eq. (1.21) applies.

1.2.4.2 Stability of floating objects

Whether or not a floating object is stable depends on the locations of both its centre of gravity (CoG) and centre of buoyancy (CoB). Consider the vertical section through a floating object shown in Fig. 1.8a. The CoG represents the location at which the weight of the object, F_g [MLT^{-2}], acts. The CoB represents the location at which the associated buoyant force, F_b [MLT^{-2}], acts.

If the CoG is below the CoB, the object will be unconditionally stable. However, it is commonly the case that the CoG is above the CoB. Providing the CoG is directly above the CoB, the object will remain upright in the liquid. However, a minor rotation of the object of concern will lead to both the CoG and CoB to form a turning moment, causing the object to rotate until the CoG and the CoB reside in the same vertical plane again. This may lead to the object returning to its upright position. Alternatively, the object may continue to rotate in the direction of the perturbation, causing it capsize.

Consider Figs. 1.8a and b. When the object is upright, F_g and F_b act in the same vertical plane and therefore do not form a turning moment (Fig. 1.8a). However, tilting the object clockwise, by a small angle, θ [-], leads to the submerged portion of the object being predominantly on the right-hand-side. This in turn leads to the CoB moving to the right. In contrast, providing the load within the object is not redistributed (which is reasonable assumption, given that θ is small), the location of the CoG, relative to the object, remains unchanged. But importantly, the two loads, F_g and F_b, no longer act in the same vertical plane and therefore form a counter-clockwise turning moment. However, the effect of this turning moment is to return the object back to the original upright position. In this case, the object can be described as being buoyantly stable.

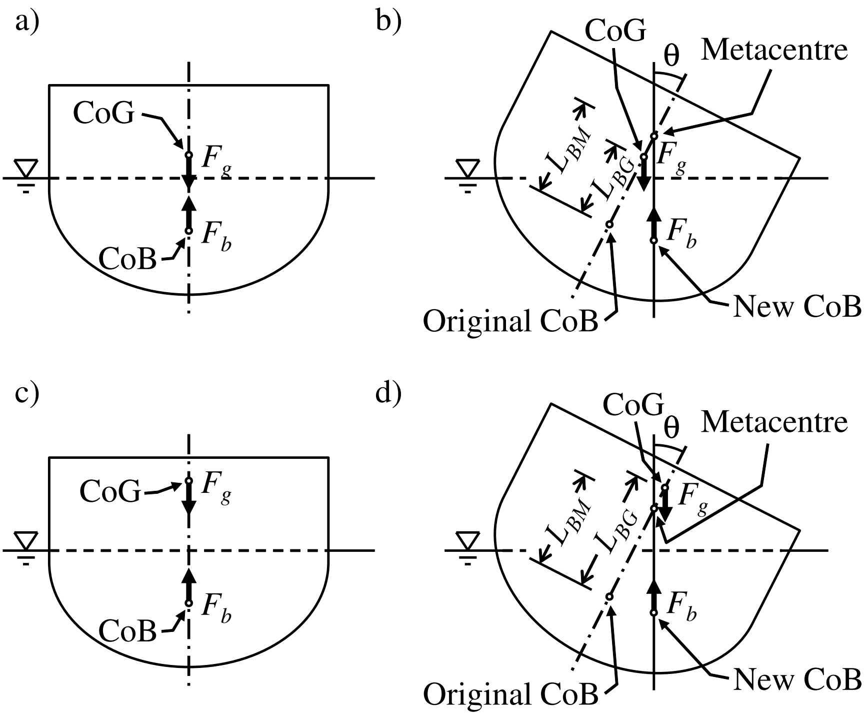

Figure 1.8: Schematic diagrams showing locations of the centre of gravity (CoG), centre of buoyancy (CoB) and metacentre for a floating object that is tilted clockwise from an upright position. a) and b) represent an object that is buoyantly stable. c) and d) represent and object that is buoyantly unstable.

Consider Figs. 1.8c and d. The only difference between the objects in Figs. 1.8a and c is that the CoG for the object is higher in Fig. 1.8c. Consequently, when the object is tilted clockwise, the new CoB is to the left of the CoG (see Fig. 1.8 d) such that the two associated forces form a clockwise turning moment, which will not return the boat back to the original upright position. Instead, this clockwise turning moment will cause the boat to carry on rotating clockwise such that it will capsize. In this case, the object can be described as being buoyantly unstable.

Whether or not an object is buoyantly stable is of particular interest to the maritime community. Interestingly, it is possible to derive an expression that predicts the buoyant stability of an objective, which can be written purely in terms of its geometrical properties. The starting point is to consider a vertical line that crosses the new CoB due to rotation and intersects the line connecting the original CoB and the CoG of the object of concern. This intersection point is referred to as the metacentre (see Figs. 1.8b and d).

If the metacentre is to the right of the CoG, then the rotated CoB is also to the right of the CoG and the object is buoyantly stable (as in Fig. 1.8b). In this case, the distance between original CoB and the CoG, L_{BG} [L], is less than the distance between the original CoB and the metacentre, L_{BM} [L].

If the metacentre is to the left of the CoG, then the rotated CoB is also to the left of the CoG and the object is buoyantly unstable (as in Fig. 1.8d). In this case, the distance between original CoB and the CoG, L_{BG} [L], is greater than the distance between the original CoB and the metacentre, L_{BM} [L].

To summarise:

- $L_{BG} < L_{BM}$ means buoyantly stable.
- $L_{BG} > L_{BM}$ means buoyantly unstable.

The challenge is therefore to derive an expression for L_{BM}.

1.2.4.3 Distance between centre of buoyancy and metacentre

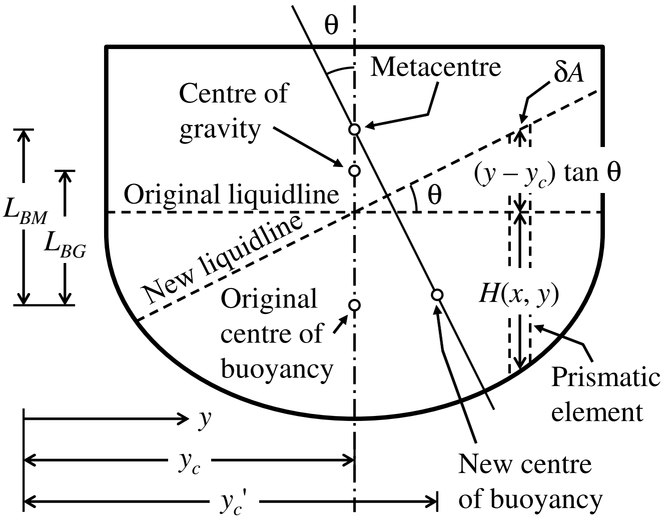

Figure 1.9: Detailed annotation of Fig. 1.8b.

Fig. 1.9 shows a more detailed annotation of Fig. 1.8b, which will aid in the determination of an expression for the distance, L_{BM} [L], between the original CoB and the metacentre.

Let y [L] be a horizontal distance from an arbitrary point to the left of the floating object. Consider a prismatic element of plan area, δA [L^2], located at a distance,

y, along the y-axis. Let $H(x,y)$ [L] be the depth of liquid above the base of the element. The buoyant force associated with this prismatic element is found from $\delta F_b = \rho g H \delta A$ [MLT^{-2}]. This buoyant force gives rise to a moment about the origin of the y-axis, of $\delta U_b = \delta F_b y$ [ML^{2}T^{-2}]. The moment, U_b [ML^{2}T^{-2}], due to the total buoyant force of the object, is therefore found from

$$U_b = \rho g \int yHdA \tag{1.22}$$

Let y_c [L] denote the location of the CoB (prior to rotation) on the y-axis. It also therefore follows that

$$U_b = F_b y_c \tag{1.23}$$

where F_b is the total buoyant force defined by Archimedes' principle (Eq. (1.21)).

Substituting Eqs. (1.21) and (1.22) into Eq. (1.23) and solving for y_c gives us

$$y_c = \frac{\int yHdA}{V_I} \tag{1.24}$$

where V_I [L^3] is the total immersed volume of the object.

Consider the aforementioned prismatic element following a small clockwise rotation of θ [-]. The depth of liquid above the base of the element is now $H + (y - y_c)\tan\theta$ (see Fig. 1.9). Considering Eq. (1.22), the moment, U_b' [ML^{2}T^{-2}], due to the total buoyant force of the object after rotation, is found from

$$U_b' = \rho g \int y[H + (y - y_c)\tan\theta]dA \tag{1.25}$$

Let y_c' [L] denote the new location of the CoB (after rotation) on the y-axis. It also therefore follows that

$$U_b' = F_b y_c' \tag{1.26}$$

where again, F_b is the total buoyant force defined by Archimedes' principle (Eq. (1.21)).

Substituting Eqs. (1.21) and (1.25) into Eq. (1.26) and solving for y_c' gives us

$$y_c' = \frac{\int y[H + (y - y_c)\tan\theta]dA}{V_I} \tag{1.27}$$

Furthermore, substituting Eqs. (1.15) and (1.24) into Eq. (1.27) leads to

$$y_c' - y_c = \frac{I_{yy}\tan\theta}{V_I} \tag{1.28}$$

where I_{yy} [L^4] is the second moment of area.

Moreover, it can be said that (see Fig. 1.9)

$$y_c' - y_c = L_{BM}\tan\theta + O(\theta) \tag{1.29}$$

Equating Eqs. (1.28) and (1.29) and imposing that $\theta \ll 1$ therefore gives us [1, p. 75]

$$L_{BM} = \frac{I_{yy}}{V_I} \tag{1.30}$$

So it can be said that a floating object should be buoyantly stable providing

$$L_{BG} < \frac{I_{yy}}{V_I} \tag{1.31}$$

1.3 Moving fluids

Fluid movement is a three-dimensional process with fluid particles moving along complicated and varying fluid pathways. A simple method for reducing a three-dimensional system to a one-dimensional problem is to focus on individual fluid pathways. Each pathway can be thought of as a streamtube. The rule about streamtubes is that fluid particles can only travel along a streamtube and cannot travel across a streamtube. Consequently, we only need to consider a one-dimensional distance along a streamtube. Below we will use this important concept to derive three extensively used hydraulic equations to describe moving fluids:

1. The continuity equation.
2. The momentum equation.
3. The Bernoulli equation.

It will then be demonstrated how to use these concepts to measure fluid velocity with a Pitot tube, predict the rate at which a tank of liquid drains through an orifice and measure flow rate in an open channel using a sharp crested weir.

1.3.1 Control-volume in a streamtube

Consider steady-state, incompressible and inviscid flow through a streamtube (Fig. 1.10). Steady-state implies there are no changes in time. Incompressible flow implies that fluid density is constant. Inviscid flow implies that the fluid has no viscosity and frictional losses are negligible.

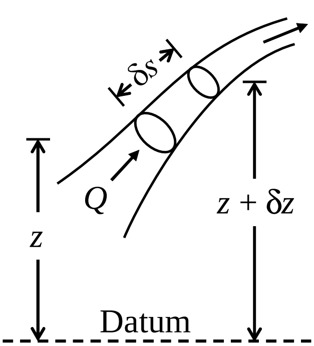

Figure 1.10: Schematic diagram of a streamtube.

A control-volume (CV) is isolated from the streamtube for further analysis (Fig. 1.11). The length of the CV is δs [L] and s [L] is a directional distance along the streamtube in the direction of moving fluid. The cross-sectional areas of the streamtube inlet and outlet are denoted A [L^2] and $A + \delta A$ [L^2], respectively. The elevation above a reference datum of the inlet and outlet are denoted z [L] and $z+\delta z$ [L], respectively. The fluid pressure at the inlet and outlet are denoted p [$ML^{-1}T^{-2}$] and $p+\delta p$ [$ML^{-1}T^{-2}$], respectively. The cross-sectional mean fluid velocity at the inlet and outlet are denoted v [LT^{-1}] and $v+\delta v$ [LT^{-1}], respectively. The rate of fluid volume movement flowing through the streamtube is denoted Q [L^3T^{-1}]. The density of the fluid is denoted ρ [ML^{-3}].

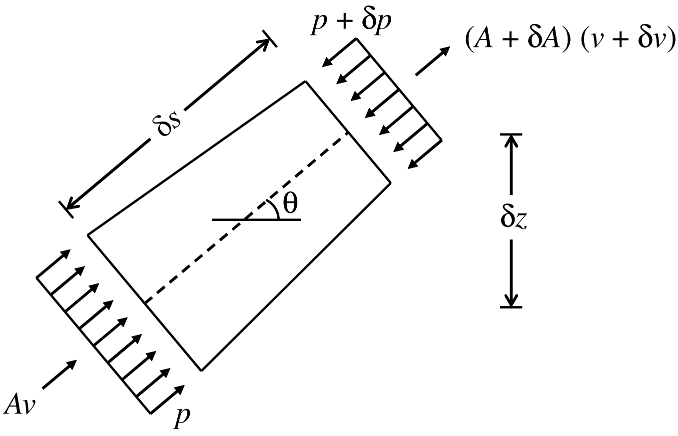

Figure 1.11: Schematic diagram of a control-volume within a streamtube.

1.3.2 The continuity equation

Challenge 1.10 Apply the principle of mass conservation to the CV described above and derive an expression relating v, δv, A, δA and Q.

Mass conservation dictates that the change in mass of fluid within a CV is equal to the mass of fluid entering a CV minus the mass of fluid leaving a CV. Under steady-state conditions there should be no change in mass. Therefore the mass of fluid entering a CV should be equal to the mass of fluid leaving a CV. For incompressible flows there is no change in density. It follows that for steady-state incompressible flow, the volume of fluid entering a CV should be equal to the volume of fluid leaving a CV such that [1, p. 91]

$$Av = (A+\delta A)(v+\delta v) = Q \tag{1.32}$$

which is widely referred to as the continuity equation.

1.3.3 The momentum equation

Challenge 1.11 Use Newton's second law to find an expression for the net-force acting on the CV in the s-direction in terms of ρ, Q and δv.

Newton's second law states that

$$F = Ma \tag{1.33}$$

where F [MLT^{-2}] is a force, M [M] is mass and a [LT^{-2}] is acceleration.

The acceleration, a, of fluid particles along the s-axis, is found from

$$a \equiv \frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{ds}{dt}\frac{\partial v}{\partial s}$$

where v [LT^{-1}] is the fluid velocity in the s-direction and t [T] is time.

It can also be said that

$$v = \frac{ds}{dt}$$

such that

$$a = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} \tag{1.34}$$

Furthermore, given that the flow is assumed to be steady-state, v is a constant with time and

$$a = v\frac{\partial v}{\partial s} \tag{1.35}$$

Recalling that $Q = Av$ it can be further stated that

$$a = \frac{Q}{A}\frac{\partial v}{\partial s}$$

and, assuming that a is constant across the CV,

$$a = \frac{Q}{\hat{A}}\frac{\delta v}{\delta s} \tag{1.36}$$

where $\hat{A}$ is the mean cross-sectional area of the CV.

The mass of fluid contained within the CV is found from

$$M = \hat{A}\delta s\rho \tag{1.37}$$

Substituting Eqs. (1.36) and (1.37) into Eq. (1.33) leads to [1, p. 136]

$$F = \rho Q\delta v \tag{1.38}$$

which is widely referred to as the momentum equation.

1.3.4 The Bernoulli equation

Challenge 1.12 Apply the momentum equation to the CV and derive a relationship between p, v and z.

To bring pressure into the system we need to quantify the net-force acting on the CV in the s-direction. This can be found from

$$F = \sum_{i=1}^{3} F_i$$

where F_1 is the force due to the fluid pressure, p, on the inlet of the CV, found from

$$F_1 = Ap$$

F_2 is the force due to the fluid pressure on the outlet of the CV, found from

$$F_2 = -(A + \delta A)(p + \delta p)$$

and F_3 is the force due to the self-weight of the fluid within the CV, found from (recall Eq. (1.37))

$$F_3 = -\hat{A}\delta s \rho g \sin\theta$$

where g [LT^{-2}] is gravitational acceleration and θ [-] is the angle of the s-axis to the horizontal axis.

Noting that $\sin\theta = \frac{\delta z}{\delta s}$, it also follows that

$$F_3 = -\hat{A}\rho g \delta z$$

and therefore

$$F = -\delta A p - (A + \delta A)\delta p - \hat{A}\rho g \delta z \tag{1.39}$$

Equating Eqs. (1.38) and (1.39) then leads to

$$\rho Q \delta v = -\delta A p - (A + \delta A)\delta p - \hat{A}\rho g \delta z$$

which can rearranged to get

$$\frac{p}{\hat{A}\rho g}\frac{\delta A}{\delta s} + \frac{(A + \delta A)}{\hat{A}\rho g}\frac{\delta p}{\delta s} + \frac{Q}{\hat{A} g}\frac{\delta v}{\delta s} + \frac{\delta z}{\delta s} = 0$$

If the CV is infinitesimally small (i.e., $\delta s \to 0$, $\delta A \to 0$ and $\hat{A} \to A$) it can be further stated that

$$\frac{p}{A\rho g}\frac{\partial A}{\partial s} + \frac{1}{\rho g}\frac{\partial p}{\partial s} + \frac{v}{g}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial s} = 0$$

Furthermore, because a streamtube should be smoothly varying in cross-sectional area, $\frac{\partial A}{\partial s} \ll A$ and

$$\frac{1}{\rho g}\frac{\partial p}{\partial s} + \frac{v}{g}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial s} = 0$$

Integrating both sides with respect to s then leads to [1, p. 94]

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = E \tag{1.40}$$

where E [L] is a term that is constant along the streamtube. Eq. (1.40) is widely referred to as the Bernoulli equation.

The constant, E [L], in Eq. (1.40) is often referred to as the total head, which represents the total mechanical energy per unit weight of fluid. Total head can be split into the following separate quantities:

$$E = \psi + h_v + z = h + h_v$$

where ψ [L], h_v [L] and h [L] are referred to as the pressure head, velocity head and hydraulic head, respectively, found from:

$$\psi = \frac{p}{\rho g}, \quad h_v = \frac{v^2}{2g}, \quad h = \psi + z$$

The Bernoulli equation, Eq. (1.40), gives that, for steady-state, incompressible and inviscid flow, the sum of the hydraulic head and velocity head is constant along a streamtube.

1.3.5 Summary of key results

Challenge 1.13 Set $A = A_1$, $A + \delta A = A_2$, $v = v_1$, $v + \delta v = v_2$, $p = p_1$, $p + \delta p = p_2$, $z = z_1$ and $z + \delta z = z_2$ in Eqs. (1.32), (1.38) and (1.40).

The continuity equation: $A_1 v_1 = A_2 v_2$

The momentum equation: $F = \rho Q(v_2 - v_1)$

The Bernoulli equation: $E = \dfrac{p_1}{\rho g} + \dfrac{v_1^2}{2g} + z_1 = \dfrac{p_2}{\rho g} + \dfrac{v_2^2}{2g} + z_2$

1.3.6 Measuring fluid velocity using a Pitot tube

Whereas hydraulic head can be measured using a piezometer, the measurement of total head requires a modified device known as a Pitot tube (see Fig. 1.12).

A piezometer is a vertical tube which should have an opening orientated normal to the fluid flow direction and located at the wall of a given pipe. Consequently, fluid particles within a piezometer should be static such that the elevation (above a given datum) of a liquid surface within its tube represents the hydraulic head, h.

A pitot tube is very similar to a piezometer except its opening should be orientated parallel to the flow direction and located directly where a velocity measurement is required. Consequently, fluid particles within the opening of a pitot tube should possess a similar kinetic energy to those particles outside of the Pitot tube. It follows that the elevation (above a given datum) of a liquid surface within its tube should represent the total head, E. A measurement of fluid velocity can be determined from

$$v = \sqrt{2g(E - h)}$$

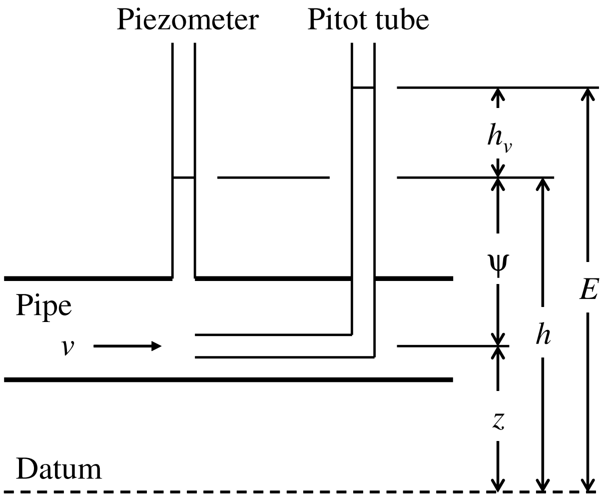

Figure 1.12: Schematic diagram of a piezometer and Pitot tube.

1.3.7 The Torricelli equation

Consider a streamtube within a large tank containing liquid draining through a small orifice at some vertical distance, H [L], below the liquid surface (Fig. 1.13). If the tank is big enough and the orifice is small enough, the liquid level in the tank will remain virtually unchanged. Let v [LT^{-1}] be the fluid velocity at the orifice. The fluid velocity at the liquid surface can be assumed zero.

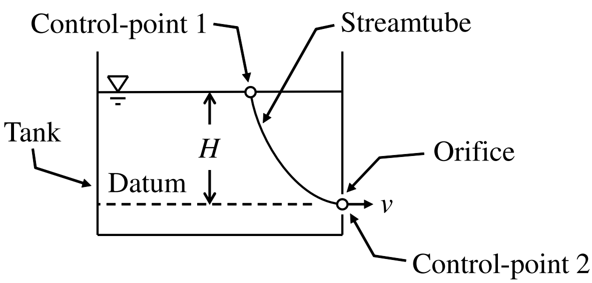

Figure 1.13: Schematic diagram of a tank of liquid draining through an orifice.

Challenge 1.14 Use the Bernoulli equation to determine a relationship between the fluid velocity at the orifice, v, and the depth of water above, H.

Consider two control-points along the streamtube. Control-point 1 is located at the liquid surface. Control-point 2 is located at the orifice. The Bernoulli equation states that

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2$$

In this context p_1, v_1 and z_1 are the pressure, fluid velocity and elevation above a datum, respectively, for Control-point 1 whereas p_2, v_2 and z_2 are the pressure, fluid velocity and elevation above a datum, respectively, for Control-point 2.

Both the liquid surface and the orifice are exposed to the atmosphere. Therefore $p_1 = p_2 = 0$. Given that the fluid velocity is zero at the liquid surface and v at the orifice, $v_1 = 0$ and $v_2 = v$. Taking the datum to be at the elevation of the orifice, $z_1 = H$ and $z_2 = 0$. Substituting the above information into the Bernoulli equation and solving for v leads to [1, p. 114]

$$v=\sqrt{2gH} \tag{1.41}$$

which is commonly referred to as the Torricelli equation.

1.3.8 Flow over a sharp crested weir

Sharp crested weirs (SCW) are often used to measure flow of liquid in open channels. An SCW typically comprises an obstruction with a sharp-edged opening, over which upstream liquid is forced to flow. Typically, the opening is of rectangular, triangular or trapezoidal section. The sharp edge is necessary to minimize energy losses as the liquid flows over the top. The sheet of liquid that flows out of the opening is referred to as the nappe. The underside of the nappe should be clear of an underlying solid bounding surface such that the nappe is close to atmospheric pressure throughout. For analysis the following assumptions are made:

1. Pressure throughout the nappe is at atmospheric pressure.
2. The upstream liquid surface is horizontal right up to the edge of the SCW.
3. Fluid pressure upstream of the SCW is hydrostatic.
4. Kinetic energy of the liquid upstream of the SCW is negligibly small compared to pressure energy and gravitational potential energy.

Challenge 1.15 Develop an expression for the flow rate, Q [$\mathrm{L}^3\mathrm{T}^{-1}$], through a rectangular SCW, of breadth B [L], in terms of g, B and the elevation of the liquid surface above the base of the SCW opening, denoted h [L] (Fig. 1.14).

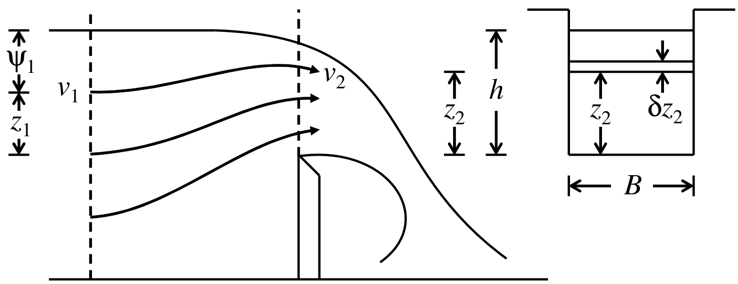

Figure 1.14: Streamtubes across a sharp crested weir.

Consider a rectangular SCW of breadth B. Liquid flows over the weir with a liquid depth of h (see Fig. 1.14). Let z be the elevation of a point above the base of the SCW opening. Now consider a control-point (CP) within a streamtube upstream of the SCW. Let p_1, v_1 and z_1 be the pressure, fluid velocity and elevation at the CP, respectively. Similarly, let p_2, v_2 and z_2 be the pressure, fluid velocity and elevation in the same streamtube but immediately above the SCW, respectively (Fig. 1.14). From Eq. (1.40) we have that

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2$$

Because of Assumption 1, $p_2=0$. Because of Assumptions 2 and 3, $\frac{p_1}{\rho g}+z_1=h$. Because of Assumption 4, $v_1=0$. It follows that

$$v_2=\sqrt{2g(h-z_2)} \tag{1.42}$$

from which it can be seen that the velocity varies with elevation within the SCW.

The total flow rate through the SCW is found from [1, p. 128]

$$\begin{aligned}Q&=\int v_2 dA\\&=B\int_0^h v_2 dz_2\\&=B\sqrt{2g}\int_0^h (h-z_2)^{1/2}dz_2\\&=B\sqrt{2g}\left[-\frac{2}{3}(h-z_2)^{3/2}\right]_0^h\\&=\frac{2B}{3}\sqrt{2gh^3}\end{aligned} \tag{1.43}$$

1.4 Problem sheet

Problem 1.1 (see Worked Solution 1.1)

A multi-fluid manometer is used to measure the pressure difference between two points in a water pipe network (see Fig. 1.15). Use hydrostatics to determine an expression for the pressure difference, $p_A - p_B$, in terms of the three fluid densities (ρ_w, ρ_m and ρ_o) and the five measured fluid depths (a_1, a_2, a_3, a_4 and a_5).

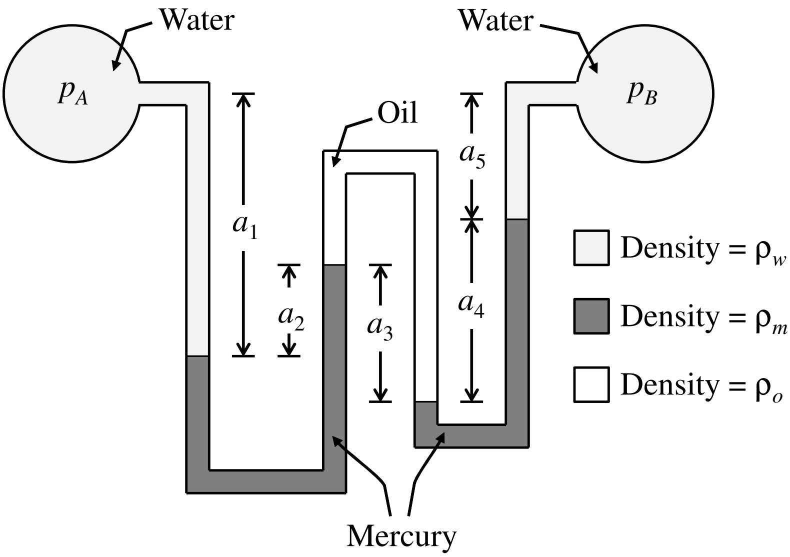

Figure 1.15: Two points in a water pipe network connected to a multi-fluid manometer.

Problem 1.2 (see Worked Solution 1.2)

A prismatic trough of isosceles triangular cross-section is filled with water with a density of 1000 kg m^{-3}. The isosceles triangle is of width, 0.5 m, and height, 0.5 m. The length of the trough is 3 m.

(a) Determine the magnitude and depth of the hydrostatic force on each of the inclined rectangular side walls.

(b) Determine the magnitude and depth of the hydrostatic force on each of the vertical triangular end walls.

Problem 1.3 (see Worked Solution 1.3)

A cylindrical buoy of 1 m diameter and 1.25 m height has a mass of 70 kg.

(a) Determine the depth of immersion when the buoy is floating in water with a density of 1000 kg m^{-3}.

(b) Check whether the buoy is buoyantly stable. Hint: The second moment of area of a circular section is $\frac{\pi D^4}{64}$ where D [L] is the diameter [2, p. 97].

(c) It is planned to place a flashing light on the top of the buoy. The mass of the light is 12 kg. Check whether the buoy will remain buoyantly stable.

(d) Determine the maximum mass of flashing light that can be held in place whilst ensuring the buoy remains buoyantly stable.

Problem 1.4 (see Worked Solution 1.4)

A water pipe, of varying cross-sectional area, tapers from 0.3 m^2 at point A to 0.2 m^2 at point B. Point B is 5 m above point A. The velocity and pressure of the water at point A is 2.2 m s^{-1} and 120 kPa, respectively. Use the Bernoulli equation to determine the fluid pressure at point B.

Problem 1.5 (see Worked Solution 1.5)

A long bridge crosses a river and is supported by 1.4 m wide piers, each equally spaced by 7 m from centre to centre. The water depth upstream of the bridge is 1.8 m. The water depth between the piers is 1.55 m. Assuming the river bed is horizontal, use the Bernoulli equation to determine the volumetric flow rate under one arch.

Problem 1.6 (see Worked Solution 1.6)

A tapered pipe-bend of circular section has a 300 mm diameter inlet and 150 mm diameter outlet. The pipe-axis at the inlet is horizontal. The centre of outlet section is 1.4 m above the centre of the inlet section. The total volume of fluid contained within the bend is 0.085 m^3. Furthermore, the pipe-axis at the outlet is inclined at an angle of 60° to the horizontal axis. Determine the magnitude and direction of the net-force exerted on the pipe-bend by water flowing through it at 0.23 m^3 s^{-1} when the inlet gauge pressure is 140 kPa.

Problem 1.7 (see Worked Solution 1.7)

Consider a sharp crested weir (SCW) of symmetric trapezoidal section. The depth of the opening is 280 mm. The base of the opening is 100 mm wide and the top of the opening is 400 mm wide. Develop a relationship between the SCW discharge rate and the water depth upstream of the SCW and determine the discharge when the upstream liquid surface is 236 mm above the base of the SCW opening.

1.5 Worked solutions

Worked Solution 1.1 (see Problem 1.1)

The best way to approach this is to first develop expressions for p_A and p_B in terms of an intermediate pressure p_C, which represents the pressure at the mercury-oil interface in the left-hand-side set of manometer tubes (see Fig. 1.15). Note that this pressure also applies at an elevation of a_3 above the mercury-oil interface in the right-hand-side set of manometer tubes.

Considering the pressure at the mercury-water interface in the left-hand-set of manometer tubes we can say

$$p_A + \rho_w g a_1 = p_C + \rho_m g a_2$$

where ρ_w is the mass density of water and ρ_m is the mass density of mercury.

Solving for p_A gives us

$$p_A = p_C + \rho_m g a_2 - \rho_w g a_1 \tag{1.44}$$

Considering the pressure at the mercury-oil interface in the right-hand-set of manometer tubes we can say

$$p_C + \rho_o g a_3 = p_B + \rho_w g a_5 + \rho_m g a_4$$

where ρ_o is the mass density of the oil.

Solving for p_B gives us

$$p_B = p_C + \rho_o g a_3 - \rho_m g a_4 - \rho_w g a_5 \tag{1.45}$$

Subtracting Eq. (1.45) from Eq. (1.44) then leads to

$$p_A - p_B = \rho_m g(a_2 + a_4) - \rho_w g(a_1 - a_5) - \rho_o g a_3 \tag{1.46}$$

Worked Solution 1.2 (see Problem 1.2)

(a) The side-walls of the trough are analogous to inclined rectangular plates. The hydrostatic force, F, on an inclined rectangular plate is found from Eq. (1.6). In this case, $\rho = 1000$ kg $^{-3}$, $g = 9.81$ m s^{-2}, $B = 3$ m, $H_0 = 0$ m, $H_1 = 0.5$ m and $L = \sqrt{\frac{W^2}{4} + H_1^2}$ where $W = 0.5$ m. It follows that the hydrostatic force acting on each side wall, $F = 4110$ N.

The depth of the hydrostatic force, H_F, in this case, is found from Eq. (1.12). It follows that $H_F = 0.333$ m.

(b) The hydrostatic force, F, acting on the end walls can be found from Eq. (1.14) where y_c is found from Eq. (1.13). The area of a triangular end-wall is found from

$$A = \frac{WH_1}{2} \tag{1.47}$$

The area, δA, of a horizontal and elemental strip of thickness, δy, at a depth, y, is found from

$$\delta A = W\left(1 - \frac{y}{H_1}\right)\delta y \tag{1.48}$$

Substituting Eqs. (1.47) and (1.48) into Eq. (1.13) leads to

$$\begin{aligned} y_c &= \frac{2}{WH_1}\int_0^{H_1} W\left(y - \frac{y^2}{H_1}\right) dy \\ &= \frac{2}{H_1}\left[\frac{y^2}{2} - \frac{y^3}{3H_1}\right]_0^{H_1} \\ &= \frac{H_1}{3} \end{aligned} \tag{1.49}$$

Substituting Eqs. (1.47) and (1.49) into Eq. (1.14) leads to

$$F = \frac{\rho g W H_1^2 \sin\theta}{6} \tag{1.50}$$

Because the end wall is vertical, $\sin\theta = 1$. Letting $\rho = 1000$ kg $^{-3}$, $g = 9.81$ m s^{-2}, $B = 3$ m, $H_1 = 0.5$ m and $W = 0.5$, it follows that the hydrostatic force on each end wall, $F = 204$ N.

The depth of the hydrostatic force, H_F, is found from Eq. (1.18) where I_{yy} is found from Eq. (1.15). Substituting Eqs. (1.48) and (1.49) into Eq. (1.15) leads to [2, p. 94]

$$\begin{aligned} I_{yy} &= \int_0^{H_1} W\left(y - \frac{H_1}{3}\right)\left(y - \frac{y^2}{H_1}\right) dy \\ &= \int_0^{H_1} W\left(\frac{4y^2}{3} - \frac{y^3}{H_1} - \frac{H_1 y}{3}\right) dy \\ &= W\left[\frac{4y^3}{9} - \frac{y^4}{4H_1} - \frac{H_1 y^2}{6}\right]_0^{H_1} dy \\ &= \frac{WH_1^3}{36} \end{aligned} \tag{1.51}$$

Substituting Eqs. (1.47), (1.49) and (1.51) into Eq. (1.18) leads to

$$H_F = \frac{H_1}{2}\sin\theta \tag{1.52}$$

Given that $H_1 = 0.5$ m and $\sin\theta = 1$ it follows that follows that the hydrostatic force on each end wall is located at a depth, $H_F = 0.25$ m.

Worked Solution 1.3 (see Problem 1.3)

(a) For a floating object, the buoyant force is equal to the weight of the object of concern. Given Eq. (1.21), it can therefore be said that

$$M_1 g = \rho g V_I \tag{1.53}$$

where M_1 [M] is the mass of the buoy, g [LT^{-2}] is gravitational acceleration, ρ [ML^{-3}] is the fluid density and V_I [L^3] is the immersed volume of the buoy. It follows that

$$V_I = \frac{M_1}{\rho} \tag{1.54}$$

Given that the buoy is cylindrical

$$V_I = \frac{\pi D^2 H}{4} \tag{1.55}$$

where H [L] is the depth of immersion.

Equating Eqs. (1.54) and (1.55) and solving for H leads to

$$H = \frac{4M_1}{\pi D^2 \rho} \tag{1.56}$$

Letting $M_1 = 70$ kg, $D = 1$ m and $\rho = 1000$ kg m^{-3} it therefore follows that $H = 0.0891$ m.

(b) To determine whether the buoy is buoyantly stable we need to check that (recall Eq. (1.31))

$$L_{BM} - L_{BG} > 0$$

where L_{BG} [L] is the distance form the centre of buoyancy (CoB) to the centre of gravity (CoG) and L_{BM} [L] is the distance from the CoB to the metacentre.

For a circular section, the second moment of area, I_{yy} [L^4], is found from

$$I_{yy} = \frac{\pi D^4}{64} \tag{1.57}$$

Substituting Eqs. (1.54) and Eq. (1.57) into Eq. (1.30) leads to

$$L_{BM} = \frac{\pi D^4 \rho}{64 M_1} \tag{1.58}$$

Let L_{OB} and L_{OG} be the elevations of the CoB and CoG above the base of the buoy, respectively. It can therefore be said that $L_{BG} = L_{OG} - L_{OB}$. Given the simple geometrical nature of the cylindrical buoy it can be understood that (recall Eq. (1.56))

$$L_{OB} = \frac{H}{2} = \frac{2M_1}{\pi D^2 \rho} \tag{1.59}$$

$$L_{OG} = \frac{L}{2} \tag{1.60}$$

where L [L] is the height of the buoy. Consequently

$$L_{BM} - L_{BG} = \frac{\pi D^4 \rho}{64 M_1} - \frac{L}{2} + \frac{2M_1}{\pi D^2 \rho}$$

Letting $L = 1.25$ m, $L_{BM} - L_{BG} = 0.121$ m, which is > 0 so the buoy is buoyantly stable.

(c) In this case (compare with Eq. (1.61)),

$$V_I = \frac{M_1 + M_2}{\rho} \tag{1.61}$$

where M_2 [M] is the mass of light attached to the top of the buoy.

Substituting Eqs. (1.61) and Eq. (1.57) into Eq. (1.30) leads to

$$L_{BM} = \frac{\pi D^4 \rho}{64(M_1 + M_2)} \tag{1.62}$$

Given Eq. (1.59), it further follows that, in this case,

$$L_{OB} = \frac{2(M_1 + M_2)}{\pi D^2 \rho} \tag{1.63}$$

The elevation of the CoG for the combined buoy and light system is obtained from

$$(M_1 + M_2) L_{OG} = \frac{M_1 L}{2} + M_2 L$$

which recognises that the CoG of buoy is half-way up the buoy and the CoG of the light is at the top of the buoy. It therefore follows that

$$L_{OG} = \left(\frac{M_1 + 2M_2}{M_1 + M_2} \right) \frac{L}{2} \tag{1.64}$$

So in the presence of the light attached to the top of the buoy

$$L_{BM} - L_{BG} = \frac{\pi D^4 \rho}{64(M_1 + M_2)} - \left(\frac{M_1 + 2M_2}{M_1 + M_2} \right) \frac{L}{2} + \frac{2(M_1 + M_2)}{\pi D^2 \rho} \tag{1.65}$$

Letting $M_2 = 12$ kg, $L_{BM} - L_{BG} = -0.0656$ m, which is < 0 so the buoy is not buoyantly stable when the light is attached to it.

(d) The maximum mass of light that can be attached to the buoy can be determined by solving Eq. (1.65) for M_2 and setting $L_{BM} - L_{BG} = 0$.

Multiplying both sides of Eq. (1.65) by $(M_1 + M_2)$ leads to

$$(M_1 + M_2)(L_{BM} - L_{BG}) = \frac{\pi D^4 \rho}{64} - (M_1 + 2M_2) \frac{L}{2} + \frac{2(M_1 + M_2)^2}{\pi D^2 \rho}$$

Collecting terms of M_2 gives us

$$a M_2^2 + b M_2 + c = 0$$

where

$$a = \frac{2}{\pi D^2 \rho}$$

$$b = L_{BG} - L_{BM} - L + \frac{4M_1}{\pi D^2 \rho}$$

$$c = \frac{\pi D^4 \rho}{64} + \left(L_{BG} - L_{BM} - \frac{L}{2} \right) M_1 + \frac{2M_1^2}{\pi D^2 \rho}$$

Solving for M_2 then leads to

$$M_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.66}$$

Setting $L_{BM} - L_{BG} = 0$ then leads to $M_2 = 1820$ kg or 7.31 kg. The latter is obviously the relevant answer. So the maximum sized light for the buoy to remain buoyantly stable is 7.31 kg.

Worked Solution 1.4 (see Problem 1.4)

Using the Bernoulli equation we can say that

$$p_2 = p_1 + \rho\left[\frac{v_1^2 - v_2^2}{2} + g(z_1 - z_2)\right]$$

From the information provided we have that $A_1 = 0.3$ m^2, $A_2 = 0.2$ m^2, $z_1 = 0$ m, $z_2 = 5$ m, $v_1 = 2.2$ m s^{-1} and $p_1 = 120{,}000$ Pa. It is also assumed that the density of water, $\rho = 1000$ kg m^{-3}.

From the continuity equation we have that

$$v_2 = \frac{A_1}{A_2}v_1$$

therefore

$$p_2 = p_1 + \rho\left[\frac{v_1^2}{2A_2^2}(A_2^2 - A_1^2) + g(z_1 - z_2)\right]$$

So the pressure at point B is 67.9 kPa.

Worked Solution 1.5 (see Problem 1.5)

Consider a streamtube on the water surface. The pressure within the streamtube, both upstream of the bridge and under the arch, will be atmospheric. It follows from the Bernoulli equation that

$$v_2^2 - v_1^2 = 2g(z_1 - z_2)$$

where in this case $z_1 = 1.8$ m and $z_2 = 1.55$ m.

From the continuity equation we have that

$$v_1 = \frac{Q}{B_1 z_1} \quad \text{and} \quad v_2 = \frac{Q}{B_2 z_2}$$

where $B_1 = 7$ m and $B_2 = 7 - 1.4 = 5.6$ m.

Substituting the above equations into our rearranged form of the Bernoulli equation leads to

$$Q = \sqrt{2g(z_1 - z_2)\left[\frac{1}{(B_2 z_2)^2} - \frac{1}{(B_1 z_1)^2}\right]^{-1}}$$

Substituting the above values then leads to a flow rate of 26.5 m^{3}s^{-1}.

Worked Solution 1.6 (see Problem 1.6)

Given the information provided, it can be said that $\theta = 60°$, $D_1 = 0.3$ m, $D_2 = 0.15$ m, $p_1 = 140{,}000$ Pa, $Q = 0.23$ m^{3}s^{-1}, $z_1 = 0$ m, $z_2 = 1.4$ m and $V = 0.085$ m^3 (see Fig. 1.16). It is also assumed that the density of water, $\rho = 1000$ kg m^{-3}.

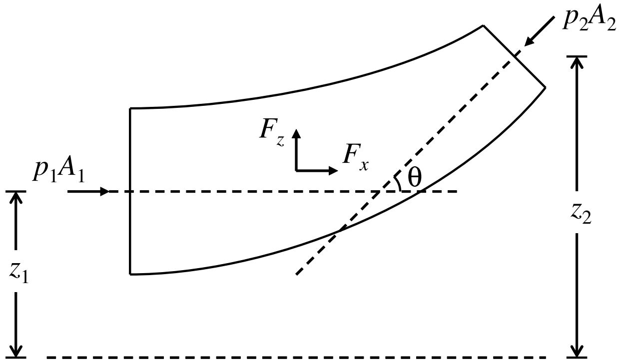

Figure 1.16: Schematic diagram of tapered pipe-bend.

Applying the momentum equation in x and z directions leads to

$$F_x + A_1 p_1 - A_2 p_2 \cos\theta = \rho Q(v_2 \cos\theta - v_1)$$

and

$$F_z - A_2 p_2 \sin\theta - \rho g V = \rho Q v_2 \sin\theta$$

where F_x and F_z are the net-forces exerted by the pipe-bend on the fluid in the x and z directions, respectively, with

$$A_1 = \frac{\pi D_1^2}{4} \quad \text{and} \quad A_2 = \frac{\pi D_2^2}{4}$$

From the continuity equation we have that

$$v_1 = \frac{4Q}{\pi D_1^2} \quad \text{and} \quad v_2 = \frac{4Q}{\pi D_2^2}$$

From the Bernoulli equation we then have that

$$p_2 = p_1 + \frac{8\rho Q^2}{\pi^2}\left(\frac{1}{D_1^4} - \frac{1}{D_2^4}\right) + \rho g(z_1 - z_2) = 46{,}860 \text{ Pa}$$

It follows from the momentum equations that

$$F_x = \frac{4\rho Q^2}{\pi}\left(\frac{\cos\theta}{D_2^2} - \frac{1}{D_1^2}\right) - \frac{\pi}{4}\left(D_1^2 p_1 - D_2^2 p_2 \cos\theta\right) = -8{,}734 \text{ N}$$

and

$$F_z = \left(\frac{4\rho Q^2}{\pi D_2^2} + \frac{\pi D_2^2 p_2}{4}\right)\sin\theta + \rho g V = 4144 \text{ N}$$

The resultant force and angle are then found from

$$F = \sqrt{F_x^2 + F_z^2} = 9{,}667 \text{ N}$$

and

$$\beta = \arctan(F_z/F_x) = -25.38^\circ$$

So the net-force acting on the pipe-bend is 9.67 kN acting downwards, away from the inlet, at an angle of 25.4° to the horizontal axis.

Worked Solution 1.7 (see Problem 1.7)

The discharge rate, Q, can be found from

$$Q = \int v dA = \int_0^h v \frac{dA}{dz} dz$$

where, from Eq. (1.42),

$$v = \sqrt{2g(h-z)}$$

with h [L] being the depth of water above the base of SCW opening, z [L] being elevation above the base of the SCW opening and A [L^2] being the cross-sectional area of the opening below z.

Considering the trapezoidal geometry of the opening, let $B_1 = 0.1$ m, $B_2 = 0.4$ m and $H = 0.28$ m such that (see Fig. 1.17)

$$A(z) = B_1 z + \left(\frac{B_2 - B_1}{H}\right)\frac{z^2}{2}$$

and

$$\frac{dA}{dz} = B_1 + \left(\frac{B_2 - B_1}{H}\right) z$$

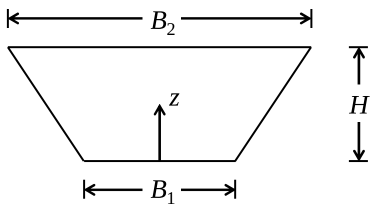

Figure 1.17: Schematic diagram of sharp crested weir of symmetric trapezoidal section.

It follows that

$$Q = \sqrt{2g}\int_0^h B_1(h-z)^{1/2} + \left(\frac{B_2 - B_1}{H}\right) z(h-z)^{1/2} dz$$

To aid integration, let $\xi = h - z$ such that

$$
\begin{aligned}
Q &= \sqrt{2g}\int_0^h B_1\xi^{1/2} + \left(\frac{B_2 - B_1}{H}\right)(h-\xi)\xi^{1/2}d\xi \\
&= \sqrt{2g}\int_0^h \left[B_1 + (B_2 - B_1)\frac{h}{H}\right]\xi^{1/2} - \left(\frac{B_2 - B_1}{H}\right)\xi^{3/2}d\xi \\
&= \sqrt{2g}\left[\frac{2}{3}\left(B_1 + (B_2 - B_1)\frac{h}{H}\right)\xi^{3/2} - \frac{2}{5}\left(\frac{B_2 - B_1}{H}\right)\xi^{5/2}\right]_0^h \\
&= \sqrt{2g}\left[\frac{2B_1h^{3/2}}{3} + \frac{4}{15}\left(\frac{B_2 - B_1}{H}\right)h^{5/2}\right] \\
&= \frac{2}{3}\sqrt{2gh^3}\left[B_1 + \frac{2}{5}(B_2 - B_1)\frac{h}{H}\right]
\end{aligned}
$$

For $h = 0.236$ m, it follows that $Q = 0.0681\ \mathrm{m^3s^{-1}}$.

1.6 References

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