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# Torus counting and self-joinings of Kleinian groups

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**Abstract.** For any integer $d \geq 1$, we obtain counting and equidistribution results for tori with small volume for a class of $d$-dimensional torus packings, invariant under a self-joining $\Gamma_\rho < \prod_{i=1}^{d} \mathrm{PSL}_2(\mathbb{C})$ of a Kleinian group $\Gamma$ formed by a $d$-tuple of convex-cocompact representations $\rho = (\rho_1, \dots, \rho_d)$. More precisely, if $\mathcal{P}$ is a $\Gamma_\rho$-admissible $d$-dimensional torus packing, then for any bounded subset $E \subset \mathbb{C}^d$ with $\partial E$ contained in a proper real algebraic subvariety, we have

$$\lim_{s \to 0} s^{\delta_{L^1}(\rho)} \cdot \#\{T \in \mathcal{P} : \mathrm{Vol}(T) > s,\ T \cap E \neq \emptyset\} = c_{\mathcal{P}} \cdot \omega_\rho(E \cap \Lambda_\rho).$$

Here $\delta_{L^1}(\rho)$, $0 < \delta_{L^1}(\rho) \leq 2/\sqrt{d}$, denotes the critical exponent of the self-joining $\Gamma_\rho$ with respect to the $L^1$-metric on the product $\prod_{i=1}^{d} \mathbb{H}^3$, $\Lambda_\rho \subset (\mathbb{C} \cup \{\infty\})^d$ is the limit set of $\Gamma_\rho$, and $\omega_\rho$ is a locally finite Borel measure on $\mathbb{C}^d \cap \Lambda_\rho$ which can be explicitly described. The class of admissible torus packings we consider arises naturally from the Teichmüller theory of Kleinian groups. Our work extends previous results of [*H. Oh* and *N. Shah*, The asymptotic distribution of circles in the orbits of Kleinian groups, Invent. Math. **187** (2012), no. 1, 1–35] on circle packings (i.e., one-dimensional torus packings) to $d$-torus packings.

## 1. Introduction

In this paper, we obtain counting and equidistribution results for a certain class of $d$-dimensional torus packings invariant under self-joinings of Kleinian groups for any $d \geq 1$. One-dimensional torus packings are precisely circle packings. To motivate the formulation of our main results, we begin by reviewing counting results for circle packings that are invariant under Kleinian groups ([15, 22–24, 26], etc).

**Circle counting.** A circle packing in the complex plane $\mathbb{C}$ is simply a non-empty family of circles in $\mathbb{C}$, for which we allow intersections among themselves. In the whole paper, lines

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are also considered as circles of infinite radii. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ be a Zariski-dense convex-cocompact discrete subgroup. We call a circle packing $\mathcal{P}$ $\Gamma$-admissible if

- $\mathcal{P}$ consists of finitely many $\Gamma$-orbits of circles,
- $\mathcal{P}$ is locally finite, in the sense that no infinite sequence of circles in $\mathcal{P}$ converges to a circle.

We denote by $0 < \delta_\Gamma \le 2$ the critical exponent of $\Gamma$, i.e., the abscissa of convergence for the Poincaré series

$$\mathsf{P}(s) := \sum_{g \in \Gamma} e^{-s\, \mathsf{d}_{\mathbb{H}^3}(gp, p)},$$

where $p \in \mathbb{H}^3$ is any point and $\mathsf{d}_{\mathbb{H}^3}$ is the hyperbolic metric so that $(\mathbb{H}^3, \mathsf{d}_{\mathbb{H}^3})$ has constant curvature $-1$. The extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can be regarded as the geometric boundary of $\mathbb{H}^3$. The limit set of $\Gamma$ is the set of all accumulation points of the orbit $\Gamma(z)$ of $z \in \widehat{\mathbb{C}}$; we denote it by $\Lambda_\Gamma \subset \widehat{\mathbb{C}}$.

**Theorem 1.1** ([23]). *For any $\Gamma$-admissible circle packing $\mathcal{P}$, there exists a constant $c_{\mathcal{P}} > 0$ such that for any bounded measurable subset $E \subset \mathbb{C}$ whose boundary is contained in a proper real algebraic subvariety of $\mathbb{C}$,*

$$\lim_{s \to 0} s^{\delta_\Gamma} \#\{C \in \mathcal{P} : \mathrm{radius}(C) \ge s,\ C \cap E \ne \emptyset\} = c_{\mathcal{P}} \omega_\Gamma(E \cap \Lambda_\Gamma);$$

*here $\omega_\Gamma$ is the $\delta_\Gamma$-dimensional Hausdorff measure on $\mathbb{C} \cap \Lambda_\Gamma$ with respect to the Euclidean metric on $\mathbb{C}$.*

This theorem holds for a more general class of circle packings invariant by geometrically finite Kleinian groups, which includes the famous Apollonian circle packings for which the relevant counting result was first obtained in [15] (see [23] for more details and examples).

**Torus counting.** The main goal of this paper is to prove a higher dimensional analogue of Theorem 1.1. Let $d \ge 1$. By a torus in $\mathbb{C}^d$ we mean a Cartesian product of $d$-number of circles $C_1, \dots, C_d \subset \mathbb{C}$. However, it will be convenient to consider it as a $d$-tuple of circles

$$T = (C_1, \dots, C_d) \tag{1.2}$$

rather than a subset $C_1 \times \cdots \times C_d \subset \mathbb{C}^d$. A $d$-dimensional torus packing in $\mathbb{C}^d$ is simply a non-empty family of $d$-tori in $\mathbb{C}^d$.

The volume of $T$ is given by

$$\mathrm{Vol}(T) = \prod_{i=1}^{d} 2\pi\, \mathrm{radius}\, C_i.$$

Figure 1 shows some image of a 2-torus packing. Although the torus $T = C_1 \times C_2$ in Figure 1 *appears* to be in $\mathbb{R}^3$, it should be understood as a subset of $\mathbb{R}^4$, representing the Cartesian product of the boundary circles of two disks.

We are interested in understanding the asymptotic counting and distribution of tori with small volumes in a torus packing that is invariant under a self-joining of a convex-cocompact Kleinian group.

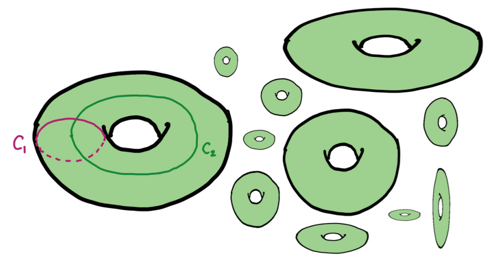


Figure 1. A torus packing.

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a convex-cocompact discrete subgroup and let

$$\rho = (\rho_1 = \mathrm{id}, \rho_2, \ldots, \rho_d)$$

be a $d$-tuple of faithful convex-cocompact representations of $\Gamma$ into $\mathrm{PSL}_2(\mathbb{C})$. Let

$$G = \prod_{i=1}^{d} \mathrm{PSL}_2(\mathbb{C}).$$

The *self-joining* of $\Gamma$ via $\rho$ is defined as the following discrete subgroup of $G$:

$$\Gamma_\rho = \{(\rho_1(g), \ldots, \rho_d(g)) : g \in \Gamma\}.$$

Throughout the paper we will always assume that $\Gamma_\rho$ is Zariski-dense in $G$. Each $\rho_i$ induces a unique equivariant homeomorphism $f_i : \Lambda_\Gamma \to \Lambda_{\rho_i(\Gamma)}$, which is called the $\rho_i$-boundary map [35]. In this paper, we define the limit set of $\Gamma_\rho$ by

$$\Lambda_\rho = \{(f_1(\xi), \ldots, f_d(\xi)) \in \widehat{\mathbb{C}}^d : \xi \in \Lambda_\Gamma\}.$$

We call a torus $T = (C_1, \ldots, C_d)$ $\Gamma_\rho$-admissible if for each $1 \le i \le d$,

- $\rho_i(\Gamma)C_i$ is a locally finite circle packing,
- $f_i(C_1 \cap \Lambda_\Gamma) = C_i \cap \Lambda_{\rho_i(\Gamma)}$.

The second condition is equivalent to

$$T \cap \Lambda_\rho = \{(\xi_1, \ldots, \xi_d) \in \Lambda_\rho : \xi_1 \in C_1 \cap \Lambda_\Gamma\},$$

that is, the circular slice $C_1 \cap \Lambda_\Gamma$ completely determines the toric slice $T \cap \Lambda_\rho$.

**Definition 1.3.** A torus packing $\mathscr{P}$ is called $\Gamma_\rho$-admissible if

- $\mathscr{P}$ consists of finitely many $\Gamma_\rho$-orbits of $\Gamma_\rho$-admissible tori,
- $\mathscr{P}$ is locally finite in the sense that no infinite sequence of tori in $\mathscr{P}$ converges to a torus.

**Remark 1.4.** We remark that when $\#(C_1 \cap \Lambda_\Gamma) \ge 3$, the locally finiteness hypotheses in the above definition can be reduced to the local-finiteness of the circle packing $\Gamma C_1$ (see Proposition 3.11).

We denote by $\delta_{L^1}(\rho)$ the abscissa of convergence of the series

$$s \mapsto \mathsf{P}_{L^1}(s) := \sum_{g \in \Gamma} e^{-s \sum_{i=1}^{d} \mathsf{d}_{\mathbb{H}^3}(\rho_i(g)p, p)}$$

for $p \in \mathbb{H}^3$, which is the critical exponent of $\Gamma_\rho$ with respect to the $L^1$ product metric on $\prod_{i=1}^{d}(\mathbb{H}^3, \mathsf{d}_{\mathbb{H}^3})$.

We first state the following special case of the main result of this paper.

**Theorem 1.5.** *Let $\mathcal{P}$ be a $\Gamma_\rho$-admissible torus packing. There exists a constant $c_{\mathcal{P}} > 0$ such that for any bounded measurable subset $E \subset \mathbb{C}^d$ with boundary contained in a proper real algebraic subvariety, we have*

$$\lim_{s \to 0} s^{\delta_{L^1}(\rho)} \#\{T \in \mathcal{P} : \mathrm{Vol}(T) > s,\ T \cap E \neq \emptyset\} = c_{\mathcal{P}} \omega_{\Gamma_\rho}(E \cap \Lambda_\rho),$$

*where $\omega_{\Gamma_\rho}$ is a locally finite Borel measure on $\mathbb{C}^d \cap \Lambda_\rho$ which can be explicitly described. In particular, if $\mathcal{P}$ is bounded, then*

$$\lim_{s \to 0} s^{\delta_{L^1}(\rho)} \#\{T \in \mathcal{P} : \mathrm{Vol}(T) > s\} = c_{\mathcal{P}} |\omega_{\Gamma_\rho}|.$$

**Remark 1.6.** (1) Since $\delta_{L^1}(\rho)$ is bounded above by the usual critical exponent $\delta_{\Gamma_\rho}$ of $\Gamma_\rho$ with respect to the Riemannian metric (which equals the $L^2$ product metric) on $\prod_{i=1}^{d} \mathbb{H}^3$, we have

$$0 < \delta_{L^1(\rho)} \le \delta_{\Gamma_\rho} \le \frac{1}{\sqrt{d}} \max_i (\dim(\Lambda_{\rho_i(\Gamma)})) \le \frac{2}{\sqrt{d}}$$

by [13, Corollary 3.6]; here the notation $\dim(\cdot)$ means the Hausdorff dimension of a measurable subset of $\widehat{\mathbb{C}} \simeq \mathbb{S}^2$ with respect to the spherical metric.

(2) If all $\rho_i : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ are quasiconformal deformations of $\Gamma$ and

$$(*) \qquad \infty \notin \bigcup_{i=1}^{d} \Lambda_{\rho_i(\Gamma)},$$

then for any bounded torus packing $\mathcal{P} = \Gamma_\rho T$ with $T = (C_1, \dots, C_d)$, $\mathcal{P}$ is locally finite if and only if $\{\rho_i(\gamma)C_i : \gamma \in \Gamma\}$ is a locally finite circle packing for all $1 \le i \le d$. This is because the boundary map $f_i$ is the restriction to $\Lambda_{\rho_i(\Gamma)}$ of the quasiconformal homeomorphism $F_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ associated to $\rho_i$, and under the hypothesis $(*)$, the $F_i$ are bi-Hölder maps on any compact subset of $\mathbb{C}$ (see [7, 35]).

**More general torus-counting theorems.** In order to present a more general torus-counting theorem, we define the length vector of a torus $T = (C_1, \dots, C_d)$ by

$$\boldsymbol{v}(T) = -(\log \mathrm{radius}(C_1), \dots, \log \mathrm{radius}(C_d)) \in \mathbb{R}^d,$$

where we used the negative sign so that the $i$-th coordinate of $\boldsymbol{v}(T)$ tends to $+\infty$ as $C_i$ shrinks to a point. The following result is the main theorem of this paper.

**Theorem 1.7.** *Let $\psi$ be any linear form on $\mathbb{R}^d$ such that $\psi > 0$ on $(\mathbb{R}_{\ge 0})^d - \{0\}$. There exist $\delta_\psi > 0$ and a locally finite Borel measure $\omega_\psi$ on $\Lambda_\rho \cap \mathbb{C}^d$ depending only on $\Gamma_\rho$ and $\psi$ for which the following hold: for any $\Gamma_\rho$-admissible torus packing $\mathcal{P}$, there exists a con-*

*stant* $c_\psi = c_{\mathcal{P},\psi} > 0$ *such that for any bounded measurable subset* $E \subset \mathbb{C}^d$ *with boundary contained in a proper real algebraic subvariety, we have, as* $R \to \infty$,

$$\lim_{R\to\infty} \frac{1}{e^{\delta_\psi R}} \#\{T \in \mathcal{P} : \psi(\boldsymbol{v}(T)) < R,\ T \cap E \neq \emptyset\} = c_\psi \omega_\psi(E \cap \Lambda_\rho). \tag{1.8}$$

The description of the measure $\omega_\psi$ (Definition 6.1) depends on the higher rank Patterson–Sullivan theory. In fact, it is equivalent to the unique $(\Gamma_\rho, \psi_0)$-conformal measure on $\Lambda_\rho$, where $\psi_0$ is the unique $\Gamma_\rho$-critical linear form (Definition 2.8) proportional to $\psi$. We refer to Definition 2.6 for the definition of $\delta_\psi$.

**Remark 1.9.** (1) Theorem 1.5 can be deduced from this theorem by considering the linear form $\psi : (t_1, \ldots, t_d) \mapsto t_1 + \cdots + t_d$ (see Example 8.3).

(2) Our approach can also handle the case where $\psi(\boldsymbol{v}(T))$ is replaced by the Euclidean norm of $\boldsymbol{v}(T)$ in (1.8); indeed, the analysis involved in that case is easier due to the strict convexity of the Euclidean balls in $\mathbb{R}^d$ (see the last subsection of Section 8).

(3) The fact that the sublevel sets $\{t \in \mathbb{R}^d : \psi(t) < c\}$ are linear (hence not strictly convex) presents new technical difficulties which were not dealt with in related previous works such as [23] and [6].

We now discuss examples of admissible torus packings arising naturally from the Teichmüller theory of Kleinian groups.

**Example 1.10.** (1) Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a Zariski-dense and convex-cocompact subgroup whose domain of discontinuity $\Omega_\Gamma := \widehat{\mathbb{C}} - \Lambda_\Gamma$ has a connected component which is a round open disk $B$. Let $C_1 := \partial B$ and $d \geq 2$. By the Teichmüller theory of $\Gamma$, which relates the Teichmüller space of the Riemann surface $\Gamma\backslash\Omega_\Gamma$ and the quasi-conformal deformation space of $\Gamma$ (see [19, Theorem 5.27] and [18]) we may choose quasi-conformal deformations $\rho_i : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$, $2 \leq i \leq d$, whose associated quasiconformal maps $f_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ map $C_1$ to a circle, say, $C_i$. Then $T = (C_1, \ldots, C_d)$ is a $\Gamma_\rho$-admissible torus for $\rho = (\mathrm{id}, \rho_2, \ldots, \rho_d)$ and hence $\mathcal{P} = \Gamma_\rho T$ is a $\Gamma_\rho$-admissible torus packing (see Figure 2 for an example when $d = 2$). Note also that $\mathcal{P}$ consists of *disjoint* tori, and hence gives rise to a genuine packing.

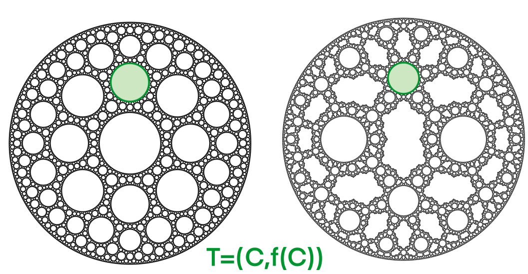


Figure 2. The left-hand side is the limit set of a convex-cocompact Kleinian group $\Gamma$ and the right-hand side is the limit set of a quasi-conformal deformation, say, $\rho_0$, of $\Gamma$. Denoting by $f$ the associated quasiconformal map, $f$ maps the first green circle, say $C$, to the second green circle. Hence the torus $T = (C, f(C))$ is a $(\mathrm{id} \times \rho_0)(\Gamma)$-admissible torus. (Image credit: Curtis McMullen and Yongquan Zhang.)

(2) Let $\Gamma$ be a rigid acylindrical convex-cocompact Kleinian group, that is, $\Omega_\Gamma$ is a union of infinitely many round disks with mutually disjoint closures. Let $\rho_0 : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be a quasiconformal deformation of $\Gamma$ which is not a conjugation, and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the associated quasiconformal map. Denoting by $\mathcal{C}$ the space of all round circles in $\widehat{\mathbb{C}}$, it follows from [2, 20, 21] that the set of all circles $C \in \mathcal{C}$ such that $\#C \cap \Lambda_\Gamma \geq 2$ and $f(C)$ is a circle is a finite union of closed $\Gamma$-orbits in $\mathcal{C}$. Indeed, if $C \in \mathcal{C}$ meets $\Lambda_\Gamma$ at more than one point, then either $C$ separates $\Lambda_\Gamma$ or $C \subset \Lambda_\Gamma$. Since the set of circles contained in $\Lambda_\Gamma$ is a finite union of closed $\Gamma$-orbits, it suffices to note that the set of all separating circles such that $f(C)$ is a circle is a finite union of closed $\Gamma$-orbits. This follows from [20, Theorem 1.5] and [2, Theorem 1.6], since otherwise such a set must be dense in the space $\mathcal{C}_{\Lambda_\Gamma}$ of all circles meeting $\Lambda_\Gamma$, and hence $f$ must map all circles in $\mathcal{C}_{\Lambda_\Gamma}$ to circles. That implies that $f$ is conformal [19] and hence $\rho$ is a conjugation, a contradiction.

Therefore the following 2-dimensional torus packing

$$\mathcal{P} := \{(C, f(C)) : C, f(C) \text{ are circles and } \#C \cap \Lambda_\Gamma \geq 2\}$$

is $(\mathrm{id} \times \rho_0)(\Gamma)$-admissible.

**On the proof of Theorem 1.7.** First of all, the self-joining group $\Gamma_\rho$ is an *Anosov subgroup* of $G$ introduced in [10] (see Definition (2.2)), which enables us to apply the general ergodic theory developed for Anosov subgroups. While certain types of counting problems for orbits of Anosov subgroups in affine symmetric spaces were studied in our earlier paper [6] using higher rank Patterson–Sullivan theory, there were certain serious technical restrictions imposed in [6] which made it unclear what kind of torus packing counting problems could be approached using techniques there. One of the main novelties of this paper is to have isolated a natural class of torus packings (which are provided by the Teichmüller theory of Kleinian groups) for which we can apply the counting machinery of [6].

It is not hard to reduce the proof of Theorem 1.7 to the case where $\mathcal{P}$ is of the form $\Gamma_\rho T_0$, where $T_0$ is the product of the unit circles centered at the origin and $\psi$ is a so-called $\Gamma_\rho$-critical linear form (see Definition 2.8). As in [23], we first translate the counting problem for torus packings into an orbital counting problem in $H\backslash G$, where $H = \mathrm{Stab}_G(T_0)$; by introducing a suitable bounded measurable subset $B_\psi(E, R) \subset H\backslash G$ in (4.12), we are led to consider the asymptotic of

$$\#([e]\Gamma_\rho \cap B_\psi(E, R))$$

as $R \to \infty$. The key ingredient for obtaining (1.8) as $R \to \infty$ is a description of the asymptotic behavior of

$$\int_{B_\psi(E,R)} \left( \int_{\Gamma_\rho \cap H\backslash H} f([h]g)\, d[h] \right) d[g] \tag{1.11}$$

for $f \in C_c(\Gamma_\rho\backslash G)$, as $R$ tends to infinity, as given in Theorem 7.1. The $\Gamma_\rho$-admissibility assumption on $\mathcal{P} = \Gamma_\rho T_0$ is used to guarantee

- the existence of some compact subset $S \subset \Gamma_\rho \cap H\backslash H$, *independent of* $R$, such that the integral (1.11) can be expressed as

$$\int_{[g]\in B_\psi(E,R)} \left( \int_{[h]\in S} f([h]g)\, d[h] \right) d[g], \tag{1.12}$$

- the finiteness of the skinning constant of $\Gamma_\rho \cap H\backslash H$ (see (5.5)).

With this information, as well as the analysis of the asymptotic shape of the family of the subsets $\{B_\psi(E, R) : R > 0\}$, we are able to apply the mixing result from [5, Theorem 3.4] and [4, Theorem 1.3 & Theorem 1.4], and the equidistribution result from [6] which describes the asymptotic of the integral (1.11) in terms of the Burger–Roblin measures introduced in [6]. We emphasize that due to the higher rank nature of the subsets $B_\psi(E, R)$, combined with the linear nature of $\psi$, whose sublevel sets are not strictly concave, the uniformity aspect in these results (see Propositions 5.6 and 5.8 for the nature of the uniformity that is required) is crucial for our analysis. In fact, working on this article led us to conjecture the precise uniformity formulation of the mixing results in [4], which were verified and appeared in an updated version by the authors. Finally, we remark that the measure $\omega_\psi$ is the leafwise measure of the Burger–Roblin measure on the strict upper triangular subgroup of $G$ ($\simeq \mathbb{C}^d$) (see Proposition 6.3).

**Organization.**

- In Section 2, we start by recalling the basic higher rank Patterson–Sullivan theory of self-joining groups.
- In Section 3, we discuss an important property of $\Gamma_\rho$-admissible torus packings and its consequences.
- In Section 4, we define the family $\{B_\psi(E, R) \subset H\backslash G : R > 0\}$ and explain how Theorem 1.5 can be translated into an orbital-counting problem for a $\Gamma_\rho$-orbit in $H\backslash G$ with respect to the family $\{B_\psi(E, R) : R > 0\}$.
- In Section 5, mixing and equidistribution results from [4] [6] will be recalled with an emphasis on their uniformity aspects.
- In Section 6, the measure $\omega_\psi$ will be given explicitly and analyzed.
- In Section 7, we prove the key technical ingredient (Theorem 7.1) of the paper, which accounts for the asymptotic distribution of the average of translates of the $H$-orbit over the set $B_\psi(E, R)$ as $R \to \infty$.
- In Section 8, we prove the main theorem (Theorem 1.5).
- In Section 9, we prove that every proper subvariety of $\mathbb{C}^d$ has zero Patterson–Sullivan measure and hence zero $\omega_\psi$ measure; this is shown for a general Anosov subgroup of a semisimple real algebraic group.

**Acknowledgement.** We would like to thank Dongryul Kim for useful conversations on a related topic.

## 2. Self-joinings and higher rank Patterson–Sullivan theory

Let $\mathbb{H}^3 = \{(z, r) : z \in \mathbb{C},\ r > 0\}$ denote the upper halfspace model of hyperbolic 3-space with constant curvature $-1$, $\mathsf{d}$ the hyperbolic metric on $\mathbb{H}^3$ and $o = (0, 1) \in \mathbb{H}^3$. The geometric boundary of $\mathbb{H}^3$ is the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, which is the Riemann sphere. The Möbius transformation action of the group $\mathrm{PSL}_2(\mathbb{C})$ on $\widehat{\mathbb{C}}$ extends to the action on the compactification $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$, and gives rise to the identification

$$\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^\circ(\mathbb{H}^3),$$

the identity component of the isometry group of $\mathbb{H}^3$. Similarly, the product group

$$G = \prod_{i=1}^{d} \mathrm{PSL}_2(\mathbb{C})$$

acts on $\widehat{\mathbb{C}}^d$ component-wise, giving rise to an isomorphism of $G$ with $\mathrm{Isom}^\circ(\prod_{i=1}^d \mathbb{H}^3)$, the identity component of the isometry group of the Riemannian product $(\mathbb{H}^3)^d$.

**Self-joinings of convex-cocompact subgroups.** Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a torsion-free convex-cocompact subgroup, that is, the convex core of the associated hyperbolic manifold $\Gamma\backslash\mathbb{H}^3$ is compact.

Let $\rho = (\rho_1 = \mathrm{id}, \rho_2, \dots, \rho_d)$ be a $d$-tuple of faithful convex-cocompact representations of $\Gamma$ into $\mathrm{PSL}_2(\mathbb{C})$, i.e., each $\rho_i(\Gamma)$ is a convex-cocompact subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

**Definition 2.1.** The self-joining of $\Gamma$ by $\rho$ is defined as the following discrete subgroup of $G$:

$$\Gamma_\rho = \{(\rho_1(g), \dots, \rho_d(g)) \in G : g \in \Gamma\}.$$

Recall that throughout the entire paper we assume that

$$\Gamma_\rho \text{ is Zariski-dense in } G.$$

**Anosov subgroups.** Let $|\cdot|$ denote the word length on $\Gamma$ with respect to a fixed finite generating set. Since each $\rho_i$ is convex-cocompact, there exists $C > 0$ such that

$$\mathsf{d}(\rho_i(g)o, o) > C|g| - C^{-1} \quad \text{for all } g \in \Gamma \text{ and } 1 \le i \le d. \tag{2.2}$$

In other words, $\Gamma_\rho$ is an *Anosov subgroup* (with respect to a minimal parabolic subgroup) (see [12] and [10]). This is the most important feature of the self-joining $\Gamma_\rho$ which will be used in this paper. We remark that any Anosov subgroup of $G$ arises in this way in view of the characterization [12, Theorem 1.5].

**Limit set.** The product $\mathcal{F} = \widehat{\mathbb{C}}^d$ is equal to the Furstenberg boundary of $G$; note that for $d > 1$, $\mathcal{F}$ is not the geometric boundary of $\prod_{i=1}^d \mathbb{H}^3$. Let $P < G$ be the product of the upper triangular subgroups of the $\mathrm{PSL}_2(\mathbb{C})$ components of $G$, i.e., $P = \mathrm{Stab}_G(\infty, \dots, \infty)$. Then

$$\mathcal{F} \simeq G/P.$$

The *limit set* of $\Gamma_\rho$ in $\mathcal{F}$ is defined as the set of all accumulation points of any $\Gamma_\rho$-orbits in $\prod_{i=1}^d \mathbb{H}^3$ on $\mathcal{F} = \widehat{\mathbb{C}}^d$:

$$\Lambda_\rho := \Big\{\lim_{j\to\infty} (\rho_1(g_j)o, \dots, \rho_d(g_j)o) \in \widehat{\mathbb{C}}^d : g_j \in \Gamma,\ g_j \to \infty\Big\}.$$

This definition coincides with the definition of the limit set given by Benoist (see [17, Lemma 2.13] and [1]). Note that for $d = 1$, this is the usual limit set $\Lambda_\Gamma$ of the Kleinian group $\Gamma$. Let $\Lambda_{\rho_i(\Gamma)} \subset \widehat{\mathbb{C}}$ denote the usual limit set of $\rho_i(\Gamma)$.

By the convex-cocompact assumption on $\rho_i$, there exists a unique $\rho_i$-equivariant homeomorphism $f_i : \Lambda_\Gamma \to \Lambda_{\rho_i(\Gamma)}$:

$$f_i(g\xi) = \rho_i(g) f_i(\xi) \quad \text{for all } g \in \Gamma \text{ and } \xi \in \Lambda_\Gamma. \tag{2.3}$$

In particular, we have

$$\Lambda_\rho = \{(f_1(\xi), \dots, f_d(\xi)) : \xi \in \Lambda_\Gamma\}.$$

**Cartan projection.** For $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, set

$$a_t = \left( \begin{pmatrix} e^{\frac{t_1}{2}} & 0 \\ 0 & e^{-\frac{t_1}{2}} \end{pmatrix}, \dots, \begin{pmatrix} e^{\frac{t_d}{2}} & 0 \\ 0 & e^{-\frac{t_d}{2}} \end{pmatrix} \right). \tag{2.4}$$

We let

$$A = \{a_t : t \in \mathbb{R}^d\} < G \quad \text{and} \quad A^+ = \{a_t : t_i \geq 0 \text{ for all } 1 \leq i \leq d\}.$$

We respectively identify $\mathbb{R}^d$ and $\mathbb{R}^d_{\geq 0}$ with the Lie algebra $\mathfrak{a} = \log A$ and its positive Weyl chamber $\mathfrak{a}^+ = \log A^+$ via the map $t \mapsto \log a_t$. For $g = (g_1, \dots, g_d) \in G$, the *Cartan projection* of $g$ is defined as

$$\mu(g) = (\mathsf{d}(g_1 o, o), \dots, \mathsf{d}(g_d o, o)) \in \mathfrak{a}^+.$$

**Limit cone and its dual cone.**

**Definition 2.5.** The *limit cone* of the discrete subgroup $\Gamma_\rho$ is the asymptotic cone of $\{\mu(\gamma) \in (\mathbb{R}_{\geq 0})^d : \gamma \in \Gamma_\rho\}$, which we denote by $\mathcal{L}_\rho$. Alternatively, it is the smallest closed cone in $\mathfrak{a}^+$ containing $\{(\ell_1(g), \dots, \ell_d(g)) : g \in \Gamma\}$, where $\ell_i(g)$ denotes the length of the closed geodesic representing the conjugacy class of $\rho_i(g)$ (see [3] and [1, Theorem 1.2]).

Since $\sup_{g \in \Gamma}(\ell_i(g)/\ell_j(g)) < \infty$ for all $i, j$ by the convex-cocompactness assumption, we have

$$\mathcal{L}_\rho - \{0\} \subset \operatorname{int} \mathfrak{a}^+,$$

where $\operatorname{int} \mathcal{C}$ denotes the interior of a cone $\mathcal{C}$. We denote by $\mathfrak{a}^*$ the space of all linear forms on $\mathfrak{a}$. The dual cone of $\mathcal{L}_\rho$ is given by

$$\mathcal{L}_\rho^* := \{\psi \in \mathfrak{a}^* : \psi|_{\mathcal{L}_\rho} \geq 0\}.$$

Note that

$$\psi|_{\mathcal{L}_\rho - \{0\}} > 0 \quad \text{if and only if} \quad \psi \in \operatorname{int} \mathcal{L}_\rho^*.$$

**Definition 2.6.** For $\psi \in \operatorname{int} \mathcal{L}_\rho^*$, let $\delta_\psi \in [0, \infty]$ denote the abscissa of convergence for the series

$$s \mapsto \sum_{\gamma \in \Gamma_\rho} e^{-s\psi(\mu(\gamma))}.$$

**Critical linear forms.** Let $\|\cdot\|$ be the Euclidean norm on $\mathfrak{a} = \mathbb{R}^d$. The *growth indicator function* $\Phi_\rho : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ (see [29, Section 4.2]) is defined as follows: $\Phi_\rho(0) = 0$ and for any vector $u \in \mathfrak{a}^+ - \{0\}$,

$$\Phi_\rho(u) := \|u\| \inf_{\substack{\text{open cones } \mathcal{D} \subset \mathfrak{a}^+ \\ u \in \mathcal{D}}} \tau_{\mathcal{D}}, \tag{2.7}$$

where $\tau_{\mathcal{D}}$ is the abscissa of convergence of the series

$$\mathsf{P}_{\mathcal{D}}(s) = \sum_{\gamma \in \Gamma_\rho, \, \mu(\gamma) \in \mathcal{D}} e^{-s\|\mu(\gamma)\|}.$$

**Definition 2.8.** A linear form $\psi \in \mathfrak{a}^*$ is said to be $\Gamma_\rho$-*critical* if

- $\psi \geq \Phi_\rho$ on $\mathfrak{a}^+$,
- $\psi(u) = \Phi_\rho(u)$ for some $u \in \mathfrak{a}^+ - \{0\}$.

The following lemma is due to Quint.

**Lemma 2.9** ([29, Theorem 4.2.2, Lemmas 3.1.3 and 3.1.7]). *The following statements hold:*

- *For each $\psi \in \operatorname{int} \mathcal{L}_\rho^*$, there exists $s > 0$ such that $s\psi$ is a $\Gamma_\rho$-critical linear form.*
- *If $\psi$ is $\Gamma_\rho$-critical, then $\delta_\psi = 1$.*

*Proof.* Set $s_0 := \inf\{s \geq 0 : s\psi \geq \Phi_\rho\}$; we have $s_0 \in (0, \infty)$ by [29, Theorem 4.2.2]. It follows that $s_0\psi \geq \Phi_\rho$ and $s_0\psi(u) = \Phi_\rho(u)$ for some $u \in \mathfrak{a}^+$ with $\|u\| = 1$, by the upper semi-continuity of $\Phi_\rho$ (see [29, Lemma 3.1.7]). In particular, $s_0\psi$ is $\Gamma_\rho$-critical and the first assertion follows. The second assertion follows from [29, Lemma 3.1.3]. □

**Patterson–Sullivan measures.** Fix $o = (0, 1) \in \mathbb{H}^3$. By abuse of notation, we set

$$o = (o, \dots, o) \in \prod_{i=1}^{d} \mathbb{H}^3.$$

For $\xi = (\xi_1, \dots, \xi_d) \in \widehat{\mathbb{C}}^d$ and $g = (g_1, \dots, g_d) \in G$, the *vector-valued Busemann map* is defined as

$$\beta_\xi(go, o) = (\beta_{\xi_1}(g_1 o, o), \dots, \beta_{\xi_d}(g_d o, o)) \in \mathfrak{a},$$

where $\{\xi_i(t) : t \geq 0\}$ is a geodesic ray in $\mathbb{H}^3$ with $\lim_{t\to+\infty} \xi_i(t) = \xi_i$ and

$$\beta_{\xi_i}(g_i o, o) = \lim_{t\to+\infty} \mathsf{d}(g_i o, \xi_i(t)) - \mathsf{d}(o, \xi_i(t)).$$

Given a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure $\nu$ supported on $\Lambda_\rho$ is called a $(\Gamma_\rho, \psi)$-*Patterson–Sullivan* (PS) measure if for all $\gamma \in \Gamma_\rho$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{-\psi(\beta_\xi(\gamma o, o))}.$$

We will say that $\nu$ is a $\Gamma_\rho$-PS measure if it is a $(\Gamma_\rho, \psi)$-PS measure for some $\psi \in \mathfrak{a}^*$. Extending the Patterson–Sullivan theory for rank one groups ([27, 33]), Quint [30] constructed a $(\Gamma_\rho, \psi)$-PS measure for each $\Gamma_\rho$-critical linear form $\psi \in \mathfrak{a}^*$ (see [3] for earlier works on this). As $\Gamma_\rho$ is a Zariski-dense Anosov subgroup of $G$, the following is a special case of [17].

**Lemma 2.10** ([17, Theorem 1.1 and Theorem 4.3]). *For each $u \in \operatorname{int} \mathcal{L}_\rho$, there exists a unique $\Gamma_\rho$-critical linear form $\psi_u \in \mathfrak{a}^*$ such that $\psi_u(u) = \Phi_\rho(u)$, and a unique $(\Gamma_\rho, \psi_u)$-PS measure $\nu_{\psi_u}$. The maps $u \mapsto \psi_u$ and $u \mapsto \nu_{\psi_u}$ give bijections among*

$$\begin{aligned} \{u \in \operatorname{int} \mathcal{L}_\rho : \|u\| = 1\} &\leftrightarrow \{\Gamma_\rho\text{-}\textit{critical linear forms}\} \\ &\leftrightarrow \{\Gamma_\rho\text{- PS}\ \textit{measures}\}. \end{aligned}$$

## 3. Properties of admissible torus packings

**Notations.** We will be using the following notations throughout the paper. For $z = (z_i)_{i=1}^d \in \mathbb{C}^d$, set

$$n_z = \left( \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & z_d \\ 0 & 1 \end{pmatrix} \right) \in G. \tag{3.1}$$

We also define the following subgroups:

$$N = \{n_z : z \in \mathbb{C}^d\}, \quad \check{N} = \{n_z^t : z \in \mathbb{C}^d\},$$

$$K = \prod_{i=1}^d \mathrm{PSU}(2), \qquad H = \prod_{i=1}^d \left(\mathrm{PSU}(1,1) \cup \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \mathrm{PSU}(1,1)\right),$$

where

$$\mathrm{PSU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

and

$$\mathrm{PSU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}.$$

We set

$$M = \left\{ \left( \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}, \dots, \begin{pmatrix} e^{i\theta_d} & 0 \\ 0 & e^{-i\theta_d} \end{pmatrix} \right) : \theta_1, \dots, \theta_d \in \mathbb{R} \right\};$$

note that $M$ is equal to the centralizer of $A$ in $K$.

Let $\mathcal{C}$ denote the space of all circles in $\hat{\mathbb{C}}$ (recall that a union of line and $\{\infty\}$ is considered as a circle with infinite radius) and $\mathcal{T} = \mathcal{C} \times \cdots \times \mathcal{C}$ the space of all tori in $\prod_{i=1}^d \hat{\mathbb{C}}$. Under the identification made in (1.2), we may consider a torus as an element of $\mathcal{T}$, and a torus packing with a subset of $\mathcal{T}$.

**$H$-orbits corresponding to admissible torus packings.** Throughout the paper, we fix the following torus:

$$T_0 = (C_0, \dots, C_0) \in \mathcal{T},$$

where $C_0 = \{|z| = 1\}$ is the unit circle centered at the origin. Note that

$$H = \mathrm{Stab}_G(T_0) \quad \text{and} \quad K = \mathrm{Stab}_G(o).$$

Since $G$ acts transitively on $\mathcal{T}$, we can endow $\mathcal{T} \simeq G/H$ with the quotient topology on $G/H$. Similarly, the topology on $\mathcal{C}$ will be induced from $\mathrm{PSL}(2,\mathbb{C})/\mathrm{PSU}(1,1)$.

We call a torus $T = (C_1, \dots, C_d)$ $\Gamma_\rho$-admissible if for each $1 \le i \le d$,

- $\{\rho_i(\gamma) C_i \in \mathcal{C} : \gamma \in \Gamma\}$ is a locally finite circle packing,
- $f_i(C_1 \cap \Lambda_\Gamma) = C_i \cap \Lambda_{\rho_i(\Gamma)}$.

**Definition 3.2.** A torus packing $\mathcal{P} \subset \mathcal{T}$ is called $\Gamma_\rho$-admissible if

- $\mathcal{P}$ consists of finitely many $\Gamma_\rho$-orbits of $\Gamma_\rho$-admissible tori,
- $\mathcal{P}$ is locally finite in the sense that no infinite sequence of tori in $\mathcal{P}$ converges to a torus.

The following lemma is rather standard (see for instance [23, Lemma 3.2]).

**Lemma 3.3.** *The followings are equivalent:*

(1) *The torus packing $\Gamma_\rho T_0 \subset \mathcal{T}$ is locally finite.*

(2) *The inclusion map $f : \Gamma_\rho \cap H \backslash H \to \Gamma_\rho \backslash G$ is proper.*

(3) *$\Gamma_\rho \backslash \Gamma_\rho H$ is closed in $\Gamma_\rho \backslash G$.*

**Proposition 3.4.** *If $\mathcal{P} = \Gamma_\rho T_0$ is $\Gamma_\rho$-admissible, for any bounded subset $\mathcal{O} \subset \Gamma_\rho \backslash G$, the subset*

$$\{[h] \in \Gamma_\rho \cap H \backslash H : [h]A^+ \cap \mathcal{O} \neq \emptyset\} \tag{3.5}$$

*is bounded.*

*Proof.* Suppose not. Then there exist three sequences $g_i \in \Gamma$, $(h_{i,1}, \dots, h_{i,d}) \in H$, and $(t_{i,1}, \dots, t_{i,d}) \in \mathfrak{a}^+$ such that $(\Gamma_\rho \cap H)(h_{i,1}, \dots, h_{i,d}) \to \infty$ in $\Gamma_\rho \cap H \backslash H$ as $i \to \infty$ and for each $1 \le j \le d$,

$$s_{i,j} := \rho_j(g_i) h_{i,j} \begin{pmatrix} e^{\frac{t_{i,j}}{2}} & 0 \\ 0 & e^{-\frac{t_{i,j}}{2}} \end{pmatrix} \tag{3.6}$$

is a bounded sequence in $\mathrm{PSL}(2, \mathbb{C})$.

Let $H_0 = \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{C})}(C_0)$ and let $\mathsf{D}$ be a Dirichlet fundamental domain for the action of $\Gamma \cap H_0$ on the convex hull $\widehat{C}_0 \subset \mathbb{H}^3$ of $C_0$. By the admissibility hypothesis, $\Gamma C_0$ is a locally finite circle packing. Hence the inclusion map $\Gamma \cap H_0 \backslash \widehat{C}_0 \to \Gamma \backslash \mathbb{H}^3$ is a proper map. Since $\Gamma$ is convex-cocompact, it follows that

$$\partial \mathsf{D} \cap \Lambda_\Gamma = \emptyset$$

(see [25, Proposition 5.1]), where $\partial D := D \cap C_0 \subset \widehat{\mathbb{C}}$ denotes the boundary at infinity of $D$.

By replacing $h_{i,1}$ with an element of $(\Gamma \cap H_0) h_{i,1}$ and modifying $g_i$ if necessary, we may assume that $h_{i,1} o \in \mathsf{D}$. Since $(\Gamma_\rho \cap H)(h_{i,1}, \dots, h_{i,d}) \to \infty$ in $\Gamma_\rho \cap H \backslash H$ as $i \to \infty$, we must have $h_{i,\ell} \to \infty$ in $H_0$ for some $1 \le \ell \le d$. By (3.6) and by the assumption that the sequence $\{s_{i,j} : i = 1, 2, \dots\}$ is bounded for each $1 \le j \le d$, we have

$$\xi_j := \lim_{i \to \infty} h_{i,j} \begin{pmatrix} e^{\frac{t_{i,j}}{2}} & 0 \\ 0 & e^{-\frac{t_{i,j}}{2}} \end{pmatrix} o = \lim_{i \to \infty} \rho_j(g_i^{-1}) s_{i,j} o \in \Lambda_{\rho_j(\Gamma)}. \tag{3.7}$$

It follows from the $\rho_j$-equivariance of $f_j$ that $\xi_j = f_j(\xi_1)$ for each $1 \le j \le d$.

We will need the following general fact from hyperbolic geometry: *for any sequence $h_i \in H_0$ and $t_i \ge 0$ $(i \in \mathbb{N})$, the sequence*

$$\left\{ h_i \begin{pmatrix} e^{\frac{t_i}{2}} & 0 \\ 0 & e^{-\frac{t_i}{2}} \end{pmatrix} o \in \mathbb{H}^3 : i \in \mathbb{N} \right\} \tag{3.8}$$

*accumulates on $C_0$ if and only if $\{h_i \in H_0 : i \in \mathbb{N}\}$ is unbounded. In this case,* (3.8) *shares the same limit point with $\{h_i o \in \mathbb{H}^3 : i \in \mathbb{N}\}$ along any of its convergent subsequence.*

Now, since $h_{i,\ell} \to \infty$, it follows from (3.7) and the above fact that $\xi_\ell \in C_0 \cap \Lambda_{\rho_\ell(\Gamma)}$.

Since $C_0 \cap \Lambda_\Gamma = f_\ell^{-1}(C_0 \cap \Lambda_{\rho_\ell(\Gamma)})$ by the assumption that $\mathcal{P}$ is $\Gamma_\rho$-admissible, we have $\xi_1 = f_\ell^{-1}(\xi_\ell) \in C_0 \cap \Lambda_\Gamma$. By (3.7) and the previous fact from hyperbolic geometry, this implies that $h_{i,1}$ is unbounded and $h_{i,1}o \to \xi$ as $i \to \infty$. On the other hand, since $h_{i,1}o \in \mathsf{D}$, we have $\xi_1 \in \partial\mathsf{D}$. Hence $\xi_1 \in \partial\mathsf{D} \cap \Lambda_\Gamma$; this yields a contradiction since $\partial\mathsf{D} \cap \Lambda_\Gamma = \emptyset$. □

**Proposition 3.9.** *If $\mathcal{P} = \Gamma_\rho T_0$ is $\Gamma_\rho$-admissible, then the following hold:*

(1) *the set*

$$\{[h] \in \Gamma_\rho \cap H\backslash H : hP \in \Lambda_\rho\}$$

*is compact,*

(2) *for any bounded subset $S \subset G$ and any closed cone $\mathscr{E} \subset \mathfrak{a}^+$ such that $\mathscr{E} \cap \mathscr{L}_\rho = \{0\}$, we have*

$$\#((H\backslash H\Gamma_\rho) \cap (H\backslash H \exp(\mathscr{E})S)) < \infty.$$

To prove the proposition, we use the following lemma, which is equivalent to [17, Proposition 7.4] in view of the characterization of the limit cone $\mathscr{L}_\rho$ as an asymptotic cone of $\{\mu(\gamma) : \gamma \in \Gamma_\rho\}$ given in [1, Theorem 1.2].

**Lemma 3.10** (Uniform conicality of $\Lambda_\rho$, see [17, Proposition 7.4]). *There exists a compact subset $\mathcal{Q} \subset G$ such that the following holds: for any $g \in G$ with $gP \in \Lambda_\rho$ and any closed convex cone $\mathcal{D} \subset \operatorname{int}\mathfrak{a}^+ \cup \{0\}$ whose interior contains $\mathscr{L}_\rho - \{0\}$, we can find sequences $\gamma_i \in \Gamma_\rho$ and $\log a_i \to \infty$ in $\mathcal{D}$ such that*

$$\gamma_i g a_i \in \mathcal{Q} \quad \textit{for all } i \geq 1.$$

**Proof of Proposition 3.9.** Let $\mathcal{Q} \subset G$ be as in Lemma 3.10. Choose any closed convex cone $\mathcal{D} \subset \operatorname{int}\mathfrak{a}^+ \cup \{0\}$ whose interior contains $\mathscr{L}_\rho - \{0\}$. Since the inclusion map

$$\Gamma_\rho \cap H\backslash H \to \Gamma_\rho\backslash G$$

is a proper map, Lemma 3.10 implies that

$$\{[h] \in \Gamma_\rho \cap H\backslash H : hP \in \Lambda_\rho\} \subset \{[h] \in \Gamma_\rho \cap H\backslash H : [h]\exp\mathcal{D} \cap \mathcal{Q} \neq \emptyset\}.$$

By Proposition 3.4, the subset on the right-hand side is bounded. Therefore (1) follows.

Suppose (2) is false. Then there exists a bounded subset $S \subset G$ and infinite sequences $t_i \in \mathscr{E}$, $t_i \to \infty$, $\gamma_i \in \Gamma_\rho$, $h_i \in H$, and $s_i \in S$ such that

$$\gamma_i = h_i a_{t_i} s_i,$$

and $H\gamma_i \neq H\gamma_j$ for $i \neq j$. Since the image of $\gamma_i^{-1} h_i a_{t_i} = s_i^{-1} \in S^{-1}$ under the projection $G \to \Gamma_\rho\backslash G$ is bounded, it follows again from Proposition 3.4 that there exists a sequence $\delta_i \in \Gamma_\rho \cap H$ such that the sequence $\widetilde{h}_i := \delta_i h_i$ is bounded. Set $\widetilde{\gamma}_i := \delta_i\gamma_i$. Note that

$$H\widetilde{\gamma}_i = H\gamma_i$$

and $\widetilde{\gamma}_i = \widetilde{h}_i a_{t_i} s_i \in \Gamma_\rho$. Since both $\widetilde{h}_i$ and $s_i$ are bounded, the sequences $t_i$ and $\mu(\widetilde{\gamma}_i)$ are within bounded distance of each other. Now using the fact that $\mathscr{L}_\rho$ is the asymptotic cone of $\{\mu(\gamma) : \gamma \in \Gamma_\rho\}$, and $\mathscr{E} \cap \mathscr{L}_\rho = \{0\}$, we have $t_i \notin \mathscr{E}$ for all sufficiently large $i$, which is a contradiction.

**Closedness of $\Gamma_\rho T_0$.** The following proposition says that local finiteness of $\Gamma_\rho T_0 \subset \mathcal{T}$ is a consequence of the local finiteness of $\Gamma C_0 \subset \mathcal{C}$ when $T_0$ is an admissible torus with $\#(C_0 \cap \Lambda_\Gamma) \geq 3$.

**Proposition 3.11.** *Let $\Gamma C_0$ be closed in $\mathcal{C}$ with the property that $\#(C_0 \cap \Lambda_\Gamma) \geq 3$. If $f_i(C_0 \cap \Lambda_\Gamma) = C_0 \cap \Lambda_{\rho_i(\Gamma)}$ for each $1 \leq i \leq d$, then $\Gamma_\rho T_0$ is closed in $\mathcal{T}$ and $\rho_i(\Gamma) C_0$ is closed in $\mathcal{C}$ for all $2 \leq i \leq d$.*

*Proof.* Suppose that a sequence $T_n = (\rho_1(g_n)C_0, \rho_2(g_n)C_0, \dots, \rho_d(g_n)C_0)$ converges to some torus $T = (C_1, C_2, \dots, C_d)$ for $g_n \in \Gamma$. We need to show that $T \in \Gamma_\rho T_0$. Since $\Gamma C_0$ is closed and hence locally finite by Lemma 3.3, we may assume that for all $n \geq 1$, $g_n C_0 = C_1$ by throwing away finitely many $g_n$ (recall $\rho_1 = \mathrm{id}$). Observe that

$$\rho_i(g_n) f_i(C_0 \cap \Lambda_\Gamma) = f_i(g_n(C_0 \cap \Lambda_\Gamma)) = f_i(C_1 \cap \Lambda_\Gamma)$$

by (2.3). On the other hand

$$f_i(C_0 \cap \Lambda_\Gamma) = C_0 \cap \Lambda_{\rho_i(\Gamma)}$$

and it contains at least three distinct points. Since two circles sharing three distinct points must be equal to each other, we get $\rho_i(g_n) C_0 = C_1$ for all $1 \leq i \leq d$ and all $n$. It follows that $T_n = T = T_0$ for all $n$, proving the first claim. The second claim can be proved similarly. □

Although we will not be using the following proposition in the rest of our paper, it is of independent interest and extends the analogous fact for convex-cocompact groups for $d = 1$.

**Proposition 3.12.** *Let $T$ be a torus and $H_T$ be the stabilizer of $T$ in $G$. Suppose $\Gamma_\rho T$ is closed with $\#(T \cap \Lambda_\rho) \geq 3$. Then $\Gamma_\rho \cap H_T$ is a non-elementary Anosov subgroup and*

$$T \cap \Lambda_\rho = \Lambda_{\Gamma_\rho \cap H_T}.$$

*Proof.* Without loss of generality, we assume that $H_T$ is the product of copies of the group $\mathrm{PSL}_2(\mathbb{R})$. We use the characterization of an Anosov subgroup as a subgroup of $G$ satisfying the properties of Regularity, Conicality, Antipodality, shown in [11, Theorem 1.1]. Since $H_T \cap \Gamma_\rho$ is a subgroup of an Anosov subgroup $\Gamma_\rho$, it follows that $H_T$ contains $A$ and $H_T/(H_T \cap P) \subset G/P$ is the Furstenberg boundary of $H_T$, the regularity and antipodality are immediate.

We deduce the conicality as follows. Let $\xi \in T \cap \Lambda_\rho$. We can choose $h \in H_T$ such that $h(H_T \cap P) = \xi$. Since $\Gamma_\rho$ is Anosov, $\xi$ is a radial limit point of $\Gamma_\rho$, that is, there exist $a_n \to \infty$ in $A^+$ and $\delta_n \in \Gamma_\rho$ such that $\delta_n h a_n$ is bounded. Since the map $\Gamma_\rho \cap H_T \backslash H_T \to \Gamma_\rho \backslash G$ is proper by Lemma 3.3 and $h a_n \in H_T$, it follows that there exists $\widetilde{\delta}_n \in \Gamma_\rho \cap H_T$ that $\widetilde{\delta}_n h a_n$ is bounded. This implies that $\xi = h(H_T \cap P)$ is a radial limit point of $\Gamma_\rho \cap H_T$ in $H_T/(H_T \cap P)$. Hence we have shown that $T \cap \Lambda_\rho$ is equal to the set $\Lambda^{\mathrm{rad}}_{\Gamma_\rho \cap H_T}$ of all radial limit points of $\Gamma_\rho \cap H_T$. Since $\Lambda_{\Gamma_\rho \cap H_T} \subset T \cap \Lambda_\rho$, it follows that

$$\Lambda_{\Gamma_\rho \cap H_T} = \Lambda^{\mathrm{rad}}_{\Gamma_\rho \cap H_T}.$$

Thus, $\Gamma_\rho \cap H_T$ is conical. This proves that $\Gamma_\rho \cap H_T$ is Anosov. The hypothesis $\#(T \cap \Lambda_\rho) \geq 3$ now implies that $\Gamma_\rho \cap H_T$ is non-elementary. □

## 4. Torus counting function for admissible torus packings

We write $\mathsf{r}(C)$ for the radius of a circle $C$. Given a torus $T = (C_1, \dots, C_d) \in \mathcal{T}$, we define its *length vector* $\boldsymbol{v}(T) \in \mathfrak{a} \cup \{\infty\}$ by

$$\boldsymbol{v}(T) = -(\log \mathsf{r}(C_1), \dots, \log \mathsf{r}(C_d))$$

if $\mathsf{r}(C_i) < \infty$ for all $1 \le i \le d$, and $\boldsymbol{v}(T) = \infty$ otherwise.

We will call a linear form $\psi \in \mathfrak{a}^*$ *positive* if $\psi > 0$ on $\mathfrak{a}^+ - \{0\}$.

In the rest of this section, we fix

- a $\Gamma_\rho$-admissible torus packing $\mathcal{P} = \Gamma_\rho T_0$,
- a positive $\Gamma_\rho$-critical linear form $\psi \in \mathfrak{a}^*$.

**Definition 4.1** (Counting function). For a bounded subset $E \subset \mathbb{C}^d$ and $R > 0$, we set

$$N_R(\mathcal{P}, \psi, E) = \#\{T \in \mathcal{P} : \psi(\boldsymbol{v}(T)) < R,\ T \cap E \ne \emptyset\}. \tag{4.2}$$

The local finiteness assumption on $\mathcal{P}$ together with the positivity hypothesis on $\psi$ guarantees that:

**Lemma 4.3.** *For any bounded subset $E \subset \mathbb{C}^d$ and $R > 0$, $N_R(\mathcal{P}, \psi, E) < \infty$.*

*Proof.* It follows from the local-finiteness of $\rho_i(\Gamma)C_0$ that there are only finitely many circles in $\rho_i(\Gamma)C_0$ of radius bounded from below intersecting a fixed bounded set. In particular,

$$n_0 := \#\{T \in \mathcal{P} : \boldsymbol{v}(T) \notin \mathfrak{a}^+ \text{ and } T \cap E \ne \emptyset\} < \infty.$$

By the positivity hypothesis on $\psi$, we have

$$c := \inf_{v \in \mathfrak{a}^+, \|v\|=1} \psi(v) > 0$$

and hence $\psi(v) \ge c\|v\|$ for all $v \in \mathfrak{a}^+$. Hence

$$\begin{aligned} N_R(\mathcal{P}, \psi, E) - n_0 &\le \#\left\{T = (C_1, \dots, C_d) \in \mathcal{P} : \sum_{i=1}^d |\log \mathsf{r}(C_i)|^2 \le \frac{R^2}{c^2} \text{ and } T \cap E \ne \emptyset\right\} \\ &\le \# \sum_{i=1}^d \{C \in \rho_i(\Gamma)C_0 : e^{-\frac{R}{c}} \le \mathsf{r}(C_i) \text{ and } C \cap \pi_i(E) \ne \emptyset\}, \end{aligned}$$

where $\pi_i(E)$ denotes the projection of $E$ to the $i$-th factor $\hat{\mathbb{C}}$. The last quantity is finite by the local-finiteness of $\rho_i(\Gamma)C_0$. This proves the claim. □

We will introduce a subset $\widetilde{B}_\psi(E, R) \subset H\backslash G$ and explain how $N_R(\mathcal{P}, \psi, E)$ is related to the number of $\Gamma_\rho$-orbits in the set $\widetilde{B}_\psi(E, R)$.

**Definition of $\widetilde{B}_\psi(E, R)$.** For $R > 0$, we define

$$A^+_{\psi,R} = \{a_t \in A^+ : \psi(t) < R\},$$

where $a_t$ is defined as in (2.4). As $\psi$ is positive, $A^+_{\psi,R}$ is bounded.

For any subset $E \subset \mathbb{C}^d$, we define

$$N_E = \{n_z \in N : z \in E\}$$

where $n_z$ is defined as in (3.1). For any $\varepsilon > 0$, set

$$E_\varepsilon^- := \bigcap_{\|w\|<\varepsilon} E + w \quad \text{and} \quad E_\varepsilon^+ := \bigcup_{\|w\|<\varepsilon} E + w. \tag{4.4}$$

**Definition 4.5.** For any bounded $E \subset \mathbb{C}^d$ and $R > 0$, we define the following bounded subset of $H\backslash G$:

$$\widetilde{B}_\psi(E, R) := H\backslash HKA^+_{\psi,R}N_{-E} \subset H\backslash G. \tag{4.6}$$

The following proposition allows us to reformulate the counting problem in terms of the sets $\widetilde{B}_\psi(E_\varepsilon^\pm, R)$ (cf. [23, Proposition 3.7]): For $\varepsilon > 0$, set

$$q_0(\mathcal{P}, E, \varepsilon) := \#\left\{T = (C_1, \dots, C_d) \in \mathcal{P} : \sum_{i=1}^d \mathsf{r}(C_i)^2 > \frac{\varepsilon^2}{4} \text{ and } T \cap E \neq \emptyset\right\}. \tag{4.7}$$

The finiteness of $q_0(\mathcal{P}, E, \varepsilon)$ can be seen as in the proof of Lemma 4.3.

**Proposition 4.8.** *Let $E \subset \mathbb{C}^d$ be a bounded subset. For any $\varepsilon > 0$ small enough and any $R > 0$, we have*

$$\#([e]\Gamma_\rho \cap \widetilde{B}_\psi(E_\varepsilon^-, R)) - q_0 \le N_R(\mathcal{P}, \psi, E) \le \#([e]\Gamma_\rho \cap \widetilde{B}_\psi(E_\varepsilon^+, R)) + q_0$$

*where $q_0 = q_0(\mathcal{P}, E, \varepsilon)$.*

*Proof.* Let $\widehat{T}_0 = \widehat{C}_0 \times \cdots \times \widehat{C}_0$. Note that

$$\begin{aligned} &\#([e]\Gamma_\rho \cap H\backslash HKA^+_{\psi,R}N_{-E_\varepsilon^\pm}) \\ &= \#\{\gamma \in \Gamma_\rho \cap H\backslash\Gamma_\rho : H\gamma \cap KA^+_{\psi,R}N_{-E_\varepsilon^\pm} \neq \emptyset\} \\ &= \#\{\gamma \in \Gamma_\rho/\Gamma_\rho \cap H : \gamma HK \cap N_{E_\varepsilon^\pm}(A^+_{\psi,R})^{-1}K \neq \emptyset\} \\ &= \#\{\gamma T_0 \in \mathcal{P} : \gamma\widehat{T}_0 \cap N_{E_\varepsilon^\pm}(A^+_{\psi,R})^{-1}o \neq \emptyset\}. \end{aligned} \tag{4.9}$$

Observe that for $z = (z_i)_{i=1}^d \in \mathbb{C}^d$, $t = (t_i)_{i=1}^d \in \mathbb{R}^d$ and $o = (0,1)_{i=1}^d \in \prod_{i=1}^d \mathbb{H}^3$, we have

$$n_z a_t o = (z_i, t_i)_{i=1}^d \in \prod_{i=1}^d \mathbb{H}^3.$$

Hence, if $\gamma T_0 \in \mathcal{P}$, $\gamma\widehat{T}_0 \cap N_{E_\varepsilon^-}(A^+_{\psi,R})^{-1}o \neq \emptyset$ and $\sum_{i=1}^d \mathsf{r}(\rho_i(\gamma)C_0)^2 \le \frac{\varepsilon^2}{4}$, then

$$\gamma T_0 \cap E \neq \emptyset \quad \text{and} \quad \psi(\boldsymbol{v}(\gamma T_0)) < R.$$

This observation combined with (4.9) gives the lower bound in the statement of the proposition. Similarly, if $\gamma T_0 \in \mathcal{P}$ satisfies $\gamma T_0 \cap E \neq \emptyset$, $\psi(\boldsymbol{v}(\gamma T_0)) < R$ and $\sum_{i=1}^d \mathsf{r}(\rho_i(\gamma)C_0)^2 \le \frac{\varepsilon^2}{4}$, then $\gamma T_0 \subset E_{\varepsilon^+}$ and hence $\gamma\widehat{T}_0 \cap N_{E_\varepsilon^+}(A^+_{\psi,R})^{-1}o \neq \emptyset$. This combined with (4.9) gives the upper bound, proving the proposition. □

**Definition of $B_\psi(E,R)$.** Let $\mathcal{D} \subset \operatorname{int}\mathfrak{a}^+$ be any closed cone such that

$$\operatorname{int}\mathcal{D} \supset \mathcal{L}_\rho - \{0\}. \tag{4.10}$$

Throughout the section we fix one such $\mathcal{D}$ and set, for any $R > 0$,

$$D := \exp\mathcal{D} \quad \text{and} \quad D_{\psi,R} = D \cap A^+_{\psi,R}. \tag{4.11}$$

Analogously to $\widetilde{B}_\psi(E,R)$, we now define

$$B_\psi(E,R) = B_{D,\psi}(E,R) := H\backslash HKD_{\psi,R}N_{-E} \subset H\backslash G. \tag{4.12}$$

**$B_\psi(E,R)$ in terms of $G = HA^+K$ decomposition.** We will now express the set $B_\psi(E,R)$ in terms of the generalized Cartan decomposition $G = HA^+K$ (cf. [9, p. 439]). Given $\varepsilon > 0$ and a subset $W \subset G$, let $W_\varepsilon$ denote the intersection of $W$ and the $\varepsilon$-ball around $e$ in $G$.

**Lemma 4.13** ([23, Proposition 4.2]). *For $d = 1$, we have*

(1) *If $a_t \in HKa_sK$ for some $s \geq 0$, then $|t| \leq s$.*

(2) *For any $\varepsilon > 0$, there exists $R_1(\varepsilon) > 0$ such that*

$$\{k \in K : a_tk \in HKA^+ \text{ for some } t > R_1(\varepsilon)\} \subset K_\varepsilon M.$$

We set

$$\mathcal{X}_\varepsilon := \left\{a_t \in A^+ : \min_{1\leq i\leq d} t_i \leq R_1\left(\frac{\varepsilon}{\sqrt{d}}\right)\right\}, \tag{4.14}$$

that is, the closed $R_1(\varepsilon/\sqrt{d})$-neighborhood of $\partial A^+$, where $R_1(\varepsilon/\sqrt{d}) > 0$ is the constant as given in Lemma 4.13 (2).

We deduce the following.

**Lemma 4.15.** *For any $\varepsilon > 0$ and $R > 0$,*

$$KA^+_{\psi,R} \subset H(A^+_{\psi,R} - \mathcal{X}_\varepsilon)K_\varepsilon \cup H\mathcal{X}_\varepsilon K.$$

*Proof.* For any $k = (k_1,\ldots,k_d) \in K$ and $a_t \in A^+$, using the decomposition

$$G = HA^+K,$$

we can find $h = (h_1,\ldots,h_d) \in H$, $a_s \in A^+$ and $\ell = (\ell_1,\ldots,\ell_d) \in K$ such that

$$k_i \begin{pmatrix} e^{\frac{t_i}{2}} & 0 \\ 0 & e^{-\frac{t_i}{2}} \end{pmatrix} = h_i \begin{pmatrix} e^{\frac{s_i}{2}} & 0 \\ 0 & e^{-\frac{s_i}{2}} \end{pmatrix} \ell_i \tag{4.16}$$

for all $i = 1,\ldots,d$, where $t = (t_i)_{1\leq i\leq d}$ and $s = (s_i)_{1\leq i\leq d}$. From Lemma 4.13 (1), we then have $s_i \leq t_i$. Since $\psi|_{\mathfrak{a}^+} \geq 0$, we have

$$\psi(s) \leq \psi(t).$$

Hence if $a_t \in A^+_{\psi,R}$, we have $a_s \in A^+_{\psi,R}$. Furthermore, if $a_s \notin \mathcal{X}_\varepsilon$, we have $s_i > R_1(\varepsilon/\sqrt{d})$ for each $i$ and hence $\ell \in K_\varepsilon M$ by Lemma 4.13 (2). Since $K_\varepsilon M = MK_\varepsilon$ and $M \subset H$, this proves the lemma. □

**Further refinement.** The following lemma appears in [23, Proposition 4.7] for the case $d = 1$, and this implies the general $d$-case as the computations can be reduced to each component.

**Lemma 4.17** ([23, Proposition 4.7]). *There exists $\ell' \geq 1$ such that for all small enough $\varepsilon > 0$, and $a \in A^+$*

$$aK_\varepsilon M \subset H(aA_{\ell'\varepsilon})N_{\ell'\varepsilon}.$$

Using Lemma 4.17, we obtain:

**Lemma 4.18.** *For any $\varepsilon > 0$ and a bounded subset $E \subset \mathbb{C}^d$, there exists a compact subset $Z = Z(E, D, \varepsilon) \subset H\backslash G$ such that*

$$\widetilde{B}_\psi(E, R) \subset H\backslash HD_{\psi,R}A_{\ell'\varepsilon}N_{-E^+_{\ell'\varepsilon}} \cup H\backslash H(A^+ - D)KN_{-E} \cup Z.$$

*Proof.* Since $\mathcal{D}$ is a closed cone contained in int $\mathfrak{a}^+$, it follows that $D \cap \mathcal{X}_\varepsilon$ is a compact subset. Therefore $Z := H\backslash H(D \cap \mathcal{X}_\varepsilon)KN_{-E}$ is a bounded subset of $H\backslash G$.

Note that by Lemma 4.15 and Lemma 4.17

$$\begin{aligned} KA^+_{\psi,R} &\subset H(A^+_{\psi,R} - \mathcal{X}_\varepsilon)K_\varepsilon \cup H\mathcal{X}_\varepsilon K \\ &\subset HD_{\psi,R}K_\varepsilon \cup H(A^+ - D)K \cup H(D \cap \mathcal{X}_\varepsilon)K \\ &\subset HD_{\psi,R}A_{\ell'\varepsilon}N_{\ell'\varepsilon} \cup H(A^+ - D)K \cup H(D \cap \mathcal{X}_\varepsilon)K. \end{aligned} \tag{4.19}$$

The claim now follows from the definition of $\widetilde{B}_\psi(E, R)$. □

**Corollary 4.20.** *For any $\varepsilon > 0$, there exist $q_1 = q_1(E, D, \varepsilon) > 0$ and $\ell' = \ell'(\psi)$ such that*

$$\begin{aligned} \#([e]\Gamma_\rho \cap B_\psi(E, R)) &\leq \#([e]\Gamma_\rho \cap \widetilde{B}_\psi(E, R)) \\ &\leq \#([e]\Gamma_\rho \cap B_\psi(E^+_{\ell'\varepsilon}, R + \ell'\varepsilon)) + q_1. \end{aligned}$$

*Proof.* The first inequality is trivial. For the second inequality, choose a slightly smaller closed cone $\mathcal{D}_0 \subset \text{int}\, \mathcal{D}$ such that $\mathcal{L}_\rho - \{0\} \subset \text{int}\, \mathcal{D}_0$ and set

$$\mathcal{E} = \overline{\mathfrak{a}^+ - \mathcal{D}_0}.$$

Note that $D_{\psi,R} - (D_0)_{\psi,R}A_{\ell'\varepsilon}$ is a bounded set and hence applying Lemma 4.18 to the cone $\mathcal{D}_0$ shows

$$\widetilde{B}_\psi(E, R) \subset H\backslash HD_{\psi,R}N_{-E^+_{\ell'\varepsilon}} \cup H\backslash H\mathcal{E}KN_{-E} \cup Z'$$

for some compact set $Z' \subset H\backslash G$. Applying Proposition 3.9 with $S = KN_{-E}$ gives the desired conclusion. □

## 5. Mixing and equidistribution with uniform bounds

We fix a positive $\Gamma_\rho$-critical linear form $\psi \in \mathfrak{a}^*$ and the $(\Gamma_\rho, \psi)$-PS measure $\nu_\psi$ given by Lemma 2.10. In this section, we recall the results of [4] and [6] on mixing (Proposition 5.6) and equidistribution (Proposition 5.8), with emphasis placed on their uniformity aspects that are crucial in our application.

**Burger–Roblin measures $m^{\mathrm{BR}}$ and $m^{\mathrm{BR}*}$.** Recall that $P = \mathrm{Stab}_G(\infty, \dots, \infty)$ denotes the product of upper triangular subgroups. We also denote by $\check{P} = \mathrm{Stab}_G(0, \dots, 0)$ the product of lower triangular subgroups.

For $g \in G$, its visual images are defined by

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := g\check{P} \in \mathcal{F}.$$

Let $\mathcal{F}^{(2)}$ denote the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$ under the diagonal action, that is,

$$\mathcal{F}^{(2)} = \{(g^+, g^-) : g \in G\}.$$

The map

$$gM \mapsto (g^+, g^-, b = \beta_{g^-}(o, go))$$

gives a homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times \mathfrak{a}$, called the Hopf parametrization of $G/M$. We define a locally finite Borel measure $\tilde{m}^{\mathrm{BR}}_{\psi}$ on $G/M$ as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{m}^{\mathrm{BR}}_{\psi}(g) = e^{\psi(\beta_{g^+}(o,go)) + 2\sigma(\beta_{g^-}(o,go))}\, d\nu_{\psi}(g^+) dm_o(g^-) db, \tag{5.1}$$

where $m_o$ is the unique $K$-invariant probability measure on $\mathcal{F}$, $db$ is the Lebesgue measure on $\mathfrak{a}$, and $\sigma$ is the linear form on $\mathfrak{a}$ defined by

$$\sigma(t_1, \dots, t_d) = t_1 + \cdots + t_d. \tag{5.2}$$

By abusing notation slightly, we also use $\tilde{m}^{\mathrm{BR}}_{\psi}$ to denote the corresponding $M$-invariant measure on $G$ induced by $\tilde{m}^{\mathrm{BR}}_{\psi}$. The measure $\tilde{m}^{\mathrm{BR}}_{\psi}$ is left $\Gamma_\rho$-invariant and induces an $\check{N}$-invariant locally finite measure on $\Gamma_\rho \backslash G$, which we denote by $m^{\mathrm{BR}}_{\psi}$.

Similarly, but with a different parameterization $g = (g^+, g^-, b = \beta_{g^+}(o, go))$, we define the following $N$-invariant locally finite Borel measure on $G$:

$$d\tilde{m}^{\mathrm{BR}*}_{\psi}(g) = e^{2\sigma(\beta_{g^+}(o,go)) + \psi(\beta_{g^-}(o,go))}\, dm_o(g^+)\, d\nu_{\psi}(g^-)\, db. \tag{5.3}$$

We have the following decomposition (see [6, (4.8)]).

**Lemma 5.4.** *For $f \in C_c(P\check{N})$,*

$$\tilde{m}^{\mathrm{BR}}_{\psi}(f) = \int_{NAM} \left( \int_{\check{N}} f(nam\check{n})\, d\check{n} \right) e^{-\psi(\log a)} e^{\psi(\beta_{n^-}(o,no))}\, dm\, da\, d\nu_{\psi}(n^-),$$

*where $dm$, $da$, $d\check{n}$ denote the Haar measures for $M$, $A$, $\check{N}$, respectively.*

We note that in Lemma 5.4, $dm$ is normalized to be a probability measure on $M$, $da$ is normalized to be compatible with the restriction of the Killing form on the lie algebra of $A$, and $d\check{n}$ is equivalently given by the density $\check{n} \mapsto e^{2\rho(\beta_{\check{n}^+}(o,\check{n}o))} d\nu_0(\check{n}^+)$, where $\nu_0$ denotes the unique $K$-invariant probability measure on $\mathcal{F}$.

**Patterson–Sullivan measure $\mu^{\mathrm{PS}}_{\Gamma_\rho \cap H \backslash H, \psi}$ (see [6, Definition 8.7]).** Define a measure $\mu^{\mathrm{PS}}_{H,\psi}$ on $H$ as follows: for $\phi \in C_c(H)$, let

$$\mu^{\mathrm{PS}}_{H,\psi}(\phi) = \int_{h \in H/H\cap P} \int_{p \in H \cap P} \phi(hp) e^{\psi(\beta_{h^+}(o,hpo))}\, dp\, d\nu_{\psi}(h^+),$$

where $dp$ is a right-Haar probability measure on $H \cap P$ (note that $H \cap P$ is compact for the pair $(G, H)$ we are considering); for $h \in H/H \cap P$, $h^+$ is well-defined and independent of the choice of a representative. The measure defined above is $\Gamma_\rho \cap H$-invariant: for any $\gamma \in \Gamma_\rho \cap H$,

$$\gamma_* \mu^{\mathrm{PS}}_{H,\psi} = \mu^{\mathrm{PS}}_{H,\psi}.$$

Therefore, if $\Gamma_\rho \backslash \Gamma_\rho H$ is closed in $\Gamma_\rho \backslash G$, then $d\mu^{\mathrm{PS}}_{H,\psi}$ induces a locally finite Borel measure on $\Gamma_\rho \backslash \Gamma_\rho H \simeq \Gamma_\rho \cap H \backslash H$, denoted by $\mu^{\mathrm{PS}}_{\Gamma_\rho \cap H \backslash H,\psi}$. The *skinning constant* of $\Gamma_\rho \cap H \backslash H$ with respect to $\nu_\psi$ is defined as the total mass:

$$\mathrm{sk}_{\Gamma_\rho,\psi}(H) := \|\mu^{\mathrm{PS}}_{\Gamma_\rho \cap H \backslash H,\psi}\| \in [0, \infty]. \tag{5.5}$$

**Uniform mixing.** We fix the unique unit vector $u = u_\psi \in \operatorname{int} \mathcal{L}_\rho$ such that

$$\psi(u) = \Phi_\rho(u)$$

provided by Lemma 2.10.

Since the cone $\mathfrak{a}^+$ is contained in the closed half space $\{\psi \geq 0\}$ and $\psi(u) > 0$, it follows that $\mathfrak{a}^+$ can be parameterized by the map

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \ker \psi &\to \mathfrak{a}, \\ (s, w) &\mapsto su + \sqrt{s} w. \end{aligned}$$

The following mixing result is due to [5, Theorem 3.4] and [4, Theorems 1.4 and 1.5]: the uniform bound as stated in the second part is crucial in our application as remarked before.

**Theorem 5.6.** *There exists an inner product $\langle \cdot, \cdot \rangle_*$ on $\mathfrak{a}$ and $\kappa, \ell > 0$ such that for any $f_1, f_2 \in C_c(\Gamma_\rho \backslash G)$ and $w \in \ker \psi$,*

$$\begin{aligned} &\lim_{s \to +\infty} s^{\frac{d-1}{2}} e^{(2\sigma - \psi)(su_\psi + \sqrt{s} w)} \int_{\Gamma_\rho \backslash G} f_1(x \exp(su_\psi + \sqrt{s} w)) f_2(x)\, dx \\ &\quad = \kappa\, e^{-\ell I(w)} m^{\mathrm{BR}}_\psi(f_1) m^{\mathrm{BR}_*}_\psi(f_2), \end{aligned}$$

*where $I : \ker \psi \to \mathbb{R}_{\geq 0}$ is given by*

$$I(w) = \frac{\|w\|_*^2 - \langle w, u_\psi \rangle_*^2}{\|u_\psi\|_*^2}.$$

*Moreover, there are $s_0, \ell > 0$ and $C' = C'(f_1, f_2) > 0$ such that for all $(s, w) \in (s_0, \infty) \times \ker \psi$ with $su_\psi + \sqrt{s} w \in \mathfrak{a}^+$, we have*

$$\left| s^{\frac{d-1}{2}} e^{(2\sigma - \psi)(su_\psi + \sqrt{s} w)} \int_{\Gamma_\rho \backslash G} f_1(x \exp(su_\psi + \sqrt{s} w)) f_2(x)\, dx \right| \leq C' e^{-\ell I(w)}. \tag{5.7}$$

**Uniform equidistribution.** Using Theorem 5.6, the following equidistribution result can be obtained as in [6, Proposition 8.11] and using a partition of unity argument for $\phi$.

**Proposition 5.8.** *Assume $\Gamma_\rho H$ is closed, let $f \in C_c(\Gamma_\rho \backslash G)$ and $\phi \in C_c(\Gamma_\rho \cap H \backslash H)$. For any $w \in \ker \psi$, we have*

$$\begin{aligned} &\lim_{s \to +\infty} s^{\frac{d-1}{2}} e^{(2\sigma - \psi)(su_\psi + \sqrt{s} w)} \int_{\Gamma_\rho \backslash \Gamma_\rho H} f([h] \exp(su_\psi + \sqrt{s} w)) \phi(h)\, dh \\ &\quad = \kappa\, e^{-\ell I(w)}\, m^{\mathrm{BR}}_\psi(f)\, \mu^{\mathrm{PS}}_{\Gamma_\rho \cap H \backslash H,\psi}(\phi), \end{aligned} \tag{5.9}$$

*where* $\kappa, \ell > 0$ *and* $I : \ker\psi \to \mathbb{R}_{\geq 0}$ *are given by Theorem* 5.6. *Moreover, there exists a constant* $C'' = C''(f, \phi), s_0 > 0$ *such that for all* $(s, w) \in (s_0, \infty) \times \ker\psi$ *with* $su + \sqrt{s}w \in \mathfrak{a}^+$,

$$\left| s^{\frac{d-1}{2}} e^{(2\sigma - \psi)(su_\psi + \sqrt{s}w)} \int_H f([h]\exp(su_\psi + \sqrt{s}w))\phi(h)\, dh \right| < C'' e^{-\ell I(w)}. \tag{5.10}$$

## 6. The measure $\omega_\psi$

Fix a positive $\Gamma_\rho$-critical linear form $\psi \in \mathfrak{a}^*$ and the $(\Gamma_\rho, \psi)$-PS measure $\nu_\psi$ given by Lemma 2.10.

**Definition 6.1.** We define a locally finite Borel measure $\omega_\psi = \omega_{\Gamma_\rho, \psi}$ on $\mathbb{C}^d$ as follows: for all $f \in C_c(\mathbb{C}^d)$,

$$\omega_\psi(f) = \int_{\mathbb{C}^d} f(z) e^{\psi(\beta_z(o, n_z \cdot o))}\, d\nu_\psi(z).$$

For each small $\varepsilon > 0$, let $\phi^\varepsilon \in C_c(N_\varepsilon A_\varepsilon M_\varepsilon \check{N}_\varepsilon)$ be a non-negative function such that $\int_G \phi^\varepsilon\, dg = 1$, where $dg$ is a Haar measure on $G$ and for any $z \in \mathbb{C}^d$, set

$$\phi^\varepsilon_z(g) := \int_M \phi^\varepsilon(gmn_z)\, dm,$$

where $dm$ is a probability Haar measure on $M$.

The main goal of this section is to establish Corollary 6.5, which roughly says

$$\int_{-E} m^{\mathrm{BR}}_\psi(\phi^\varepsilon_z)\, dz \approx \omega_\psi(E).$$

Let $E \subset \mathbb{C}^d$ be a fixed bounded Borel set and let $\varepsilon > 0$ be small enough so that

$$A_\varepsilon M_\varepsilon \check{N}_\varepsilon N_{-E} N_1 \subset NAM\check{N}.$$

For all $z \in \mathbb{C}^d$, define $\Phi^\varepsilon_E \in C_c(\mathbb{C}^d)$ by

$$\Phi^\varepsilon_E(z) := \int_{N_{-E}MA\check{N}} \phi^\varepsilon(n_z g)\, dg = \int_{N_{-E}A_\varepsilon M_\varepsilon \check{N}_\varepsilon} \phi^\varepsilon(n_z g)\, dg.$$

Recalling the definition of $E^\pm_\varepsilon$ from (4.4), we have:

**Lemma 6.2.** *For all* $z \in \mathbb{C}^d$,

$$\mathbb{1}_{E^-_\varepsilon}(n_z) \leq \Phi^\varepsilon_E(z) \leq \mathbb{1}_{E^+_\varepsilon}(n_z).$$

*Proof.* Trivially, $0 \leq \Phi^\varepsilon_E(z) \leq 1$. If $z \in E^-_\varepsilon$, then $n_z^{-1} N_\varepsilon \subset N_{-E}$, hence

$$\Phi^\varepsilon_E(z) = \int_{N_{-E}MA\check{N}} \phi^\varepsilon(n_z g)\, dg \geq \int_{n_z^{-1}N_\varepsilon MA\check{N}} \phi^\varepsilon(n_z g)\, dg = \int_G \phi^\varepsilon(g)\, dg = 1.$$

On the other hand, if $z \notin E^+_\varepsilon$, then $n_z^{-1} N_\varepsilon \cap N_{-E} = \emptyset$, hence we have $\phi^\varepsilon(n_z g) = 0$ for all $g \in N_{-E}MA\check{N}$ by uniqueness of the $NAM\check{N}$ decomposition, giving $\Phi^\varepsilon_E(z) = 0$. □

We now relate the Burger–Roblin measure of $\phi^\varepsilon$ and Patterson–Sullivan measure of $\Phi^\varepsilon$.

**Proposition 6.3.** *There exist $C, c > 0$ such that for all sufficiently small $\varepsilon > 0$, we have*

$$(1 - C\varepsilon)\,\omega_\psi(\Phi^\varepsilon_{E^-_{c\varepsilon}}) \le \int_{-E} \widetilde{m}^{\mathrm{BR}}_\psi(\phi^\varepsilon_z)\,dz \le (1 + C\varepsilon)\,\omega_\psi(\Phi^\varepsilon_{E^+_{c\varepsilon}}).$$

*Proof.* By Lemma 5.4, we have

$$\begin{aligned}\int_{-E} \widetilde{m}^{\mathrm{BR}}_\psi(\phi^\varepsilon_z)\,dz &= \int_{-E}\int_M\int_N\int_{AM\check{N}} \phi^\varepsilon(n_{z'}am\check{n}\widetilde{m}n_z)\\ &\qquad \times e^{-\psi(\log a)}e^{\psi(\beta_{z'}(o,n_{z'}o))}\,d\check{n}\,d\widetilde{m}\,da\,d\nu_\psi(z')\,dm\,dz\\ &= \int_{\mathbb{C}^d}\left(\int_{AM\check{N}N_{-E}} \phi^\varepsilon(n_{z'}am\check{n}n_z)e^{-\psi(\log a)}dz\,d\check{n}\,dm\,da\right)d\omega_\psi(z'),\end{aligned}$$

where all the densities appearing in the expression are those of the corresponding Haar measures, except for $d\nu_\psi$ and $d\omega_\psi$. Note that if $\phi^\varepsilon(nam\check{n}n_z) \neq 0$, then $nam\check{n}n_z \in N_\varepsilon A_\varepsilon M_\varepsilon \check{N}_\varepsilon$, hence

$$\begin{aligned}(6.4)\qquad am\check{n} &\in AM\check{N} \cap \big(n^{-1}N_\varepsilon n_z \exp(\mathrm{Ad}_{n_{-z}}(\log(A_\varepsilon M_\varepsilon \check{N}_\varepsilon)))\big)\\ &\subset AM\check{N} \cap (n^{-1}n_z N_{c'\varepsilon}A_{c'\varepsilon}M_{c'\varepsilon}\check{N}_{c'\varepsilon}) = A_{c'\varepsilon}M_{c'\varepsilon}\check{N}_{c'\varepsilon}\end{aligned}$$

for some $c' \ge 1$ depending only on $E$. Decomposing the Haar measure $dg$ on $G$ according to $AM\check{N}N$ and then restricting to $A_{c'\varepsilon}M_{c'\varepsilon}\check{N}_{c'\varepsilon}N_{-E}$ gives

$$e^{-\psi(\log a)}dz\,d\check{n}\,dm\,da = (1 + O(\varepsilon))\,dg$$

since $a \in A_{c'\varepsilon}$ and $dg = dz\,d\check{n}\,dm\,da$ for $g = am\check{n}n_z$ (see [14, Chapter 8]). Hence

$$\int_{-E} m^{\mathrm{BR}}_\psi(\phi^\varepsilon_z)\,dz = (1 + O(\varepsilon))\int_{\mathbb{C}^d}\int_{AM\check{N}N_{-E}} \phi^\varepsilon(n_{z'}g)\,dg\,d\omega_\psi(z'),$$

with the implied constant depending only on $E$. Using the maximum of $\|\mathrm{Ad}_{n_z}\|$ over $z \in \pm E$ together with the $NAM\check{N}$ decomposition of $\exp(\mathrm{Ad}_{n_{-z}}(\log(A_{c'\varepsilon}M_{c'\varepsilon}\check{N}_{c'\varepsilon})))$ as above gives the existence of $c \ge c'$ such that

$$N_{-E^-_{c\varepsilon}}A_\varepsilon M_\varepsilon \check{N}_\varepsilon \subset A_{c'\varepsilon}M_{c'\varepsilon}\check{N}_{c'\varepsilon}N_{-E} \subset N_{-E^+_{c\varepsilon}}A_{c\varepsilon}M_{c\varepsilon}\check{N}_{c\varepsilon}.$$

Combined with (6.4), for every $z' \in \mathbb{C}^d$ we have

$$\int_{N_{-E^-_{c\varepsilon}}AM\check{N}} \phi^\varepsilon(n_{z'}g)\,dg \le \int_{AM\check{N}N_{-E}} \phi^\varepsilon(n_{z'}g)\,dg \le \int_{N_{-E^+_{c\varepsilon}}AM\check{N}} \phi^\varepsilon(n_{z'}g)\,dg,$$

giving the desired inequality. □

Combining Lemma 6.2 and Proposition 6.3 gives the following result.

**Corollary 6.5.** *There exist $C, c > 0$ such that for all $\varepsilon > 0$ sufficiently small,*

$$(1 - C\varepsilon)\,\omega_\psi(E^-_{(1+c)\varepsilon}) \le \int_{-E} m^{\mathrm{BR}}_\psi(\phi^\varepsilon_z)\,dz \le (1 + C\varepsilon)\omega_\psi(E^+_{(1+c)\varepsilon}).$$

## 7. Equidistribution in average

We fix a positive $\Gamma_\rho$-critical $\psi \in \mathfrak{a}^*$, $\nu_\psi$ and $u = u_\psi$, continuing the notations from Sections 4 and 5. We also fix a closed cone $\mathcal{D} \subset \operatorname{int} \mathfrak{a}^+$ such that $\operatorname{int} \mathcal{D} \supset \mathcal{L}_\rho - \{0\}$ and set $D := \exp \mathcal{D}$ as in (4.10) and (4.11). Recall the notation $B_\psi(E, R) = H \backslash HKD_{\psi,R}N_{-E}$ for a bounded subset $E \subset \mathbb{C}^d$, and $\kappa, \ell > 0$ given by Theorem 5.6.

The main goal of this section is to prove the following main technical ingredient of the proof of Theorem 1.7, using Proposition 5.8.

**Theorem 7.1.** *For any $f \in C_c(\Gamma_\rho \backslash G)$ and a bounded measurable subset $E \subset \mathbb{C}^d$ such that $\omega_\psi(\partial E) = 0$,*

$$\lim_{R \to \infty} e^{-R} \int_{B_\psi(E,R)} \int_{\Gamma_\rho \cap H \backslash H} f(\Gamma_\rho h g)\, d[h]\, d[g] = c_{\Gamma_\rho, \psi} \int_{-E} m_\psi^{\mathrm{BR}}(f_z)\, dz,$$

*where*

$$c_{\Gamma_\rho,\psi} := \frac{\kappa \operatorname{sk}_{\Gamma_\rho,\psi}(H)}{\Phi_\rho(u)} \left( \int_{\ker \psi} e^{-\ell I(w)}\, dw \right)$$

*and $f_z \in C_c(\Gamma_\rho \backslash G)$ is defined by*

$$f_z(x) := \int_M f(x m n_z)\, dm.$$

In the above, $d[g]$ denotes the $G$-invariant measure on $H \backslash G$ which is compatible to Haar measures $dg$ and $dh$ on $G$ and $H$ respectively, that is, for any $f \in C_c(G)$,

$$\int_G f(g)\, dg = \int_{H \backslash G} \left( \int_H f(hg)\, dh \right) d[g].$$

**Integral computation.** For each $w \in \ker \psi$, let $Q_R(w) \subset (0, \infty)$ be defined as

$$Q_R(w) := \{ s \in \mathbb{R}_{>0} : su + \sqrt{s} w \in A^+_{\psi,R} \}. \tag{7.2}$$

Since $\psi(w) = 0$, we compute that for all $R > 0$, $Q_R(w)$ is an interval of the form

$$Q_R(w) = (0, \tfrac{1}{\Phi_\rho(u)} R).$$

The uniform bound in Proposition 5.8 enables us to use the dominated convergence theorem to prove the following result.

**Lemma 7.3.** *For $f \in C_c(\Gamma_\rho \backslash G)$, $\phi \in C_c(H)$ and a bounded measurable subset $E$ of $\mathbb{C}^d$, define for each $w \in \ker \psi$,*

$$p_R(w) = e^{-R} \int_E \int_{Q_R(w)} s^{\frac{d-1}{2}} e^{2\sigma(su + \sqrt{s}w)} \int_H f_z(\Gamma_\rho h \exp(su + \sqrt{s}w)) \phi(h)\, dh\, ds\, dz.$$

*Then:*

1. $\lim_{R \to \infty} p_R(w) = \frac{\kappa \mu^{\mathrm{PS}}_{H,\psi}(\phi)}{\Phi_\rho(u)} e^{-\ell I(w)} \int_E m_\psi^{\mathrm{BR}}(f_z)\, dz$ *and*
2. $p_R(w) \le C e^{-\ell I(w)}$ *for some $C = C(f, E, \phi) > 0$.*

*Proof.* For simplicity, set $c_u = \Phi_\rho(u)$ in this proof. For all sufficiently large $R > 0$, we may rewrite $p_R(w)$ as

$$p_R(w) = e^{-R} \int_E \int_0^{R/c_u} e^{c_u s} J(s, w, z)\, ds\, dz$$
$$= \int_E \int_{-R/c_u}^0 e^{c_u s'} J(s' + R/c_u, w, z)\, ds'\, dz$$

where

$$J(s, w, z) = s^{\frac{d-1}{2}} e^{(2\sigma - \psi)(su + \sqrt{s} w)} \int_H f_z(\Gamma_\rho h \exp(su + \sqrt{s} w))\phi(h)\, dh.$$

By Proposition 5.8, $J(s, w, z) \to \kappa e^{-\ell I(w)} m_\psi^{\mathrm{BR}}(f_z)\mu_H^{\mathrm{PS}}(\phi)$ as $s \to \infty$ and

$$J(s, w, z) \le C'' e^{-\ell I(w)},$$

where $C'' = C''(\sup_{z\in E} f_z, \phi)$ is as in Proposition 5.8. Hence (1) follows from the dominated convergence theorem as $R \to \infty$. Assertion (2) follows from the bound

$$p_R(w) \le \mathrm{Vol}(E) \int_{-\frac{R}{c_u}}^0 e^{c_u s} J(s + \tfrac{R}{c_u}, w, z)\, ds$$

by setting $C = \frac{1}{c_u} \mathrm{Vol}(E) C''$. □

**Proof of Theorem 7.1.** Without loss of generality, we may assume that $f \ge 0$. For $[g] \in H\backslash G$, set

$$f^H([g]) := \int_{\Gamma_\rho \cap H \backslash H} f(\Gamma_\rho h g)\, dh.$$

By Proposition 3.4, and using the expression of $g$ with respect to the generalized Cartan decomposition $G = HA^+K$, we can choose $\phi \in C_c(\Gamma_\rho \cap H\backslash H)$ depending only on the support of $f$ and $E$ such that

$$f_z^H([g]) = \int_{\Gamma_\rho \cap H \backslash H} f_z(\Gamma_\rho h g)\phi(h)\, dh \tag{7.4}$$

for all $z \in E$. This will allow us to apply Proposition 5.8 directly to $f_z^H$. Furthermore, by Proposition 3.9(1), the support of $\mu^{\mathrm{PS}}_{\Gamma_\rho\cap H\backslash H,\psi}$ is compact, so we may additionally assume that $\phi = 1$ on the support of $\mu^{\mathrm{PS}}_{\Gamma_\rho\cap H\backslash H,\psi}$ and hence

$$\mathrm{sk}_{\Gamma_\rho,\psi}(H) = \mu^{\mathrm{PS}}_{\Gamma_\rho\cap H\backslash H,\psi}(\phi).$$

By Proposition 3.9 (2),

$$\int_{H\backslash H(A^+ - D)KN_{-E}} f^H([g])\, d[g] < \infty.$$

Since $d[a_t n_z] = e^{2\sigma(t)} dt\, dz$, where $\sigma(t) = \sum_{i=1}^d t_i$, we deduce from Lemma 4.18 and the inclusion $B_\psi(E, R) \subset \widetilde{B}_\psi(E, R)$ that

$$\limsup_{R\to\infty} e^{-R} \int_{B_\psi(E,R)} f^H([g])\, d[g] \tag{7.5}$$
$$\le \limsup_{R\to\infty} e^{-R} \int_{-E^+_{\ell\varepsilon}} \int_{A^+_{\psi, R+\ell\varepsilon}} f^H([a_t n_z]) e^{2\sigma(t)}\, dt\, dz.$$

Since $M \subset H$, we have

$$f^H([a_t n_z]) = f^H([m a_t n_z]) = f^H([a_t m n_z]) = \int_M f^H([a_t m n_z])\,dm = f_z^H([a_t]).$$

We now compute the upper limit in (7.5).

Using (7.2) and (7.4) together with the fact that $t = su + \sqrt{s}w$ on $\mathfrak{a}^+$ hence

$$dt = s^{\frac{d-1}{2}}\,ds\,dw,$$

we first rewrite (7.5) as $\limsup_{R\to\infty} \int_{\ker\psi} p_R(w)\,dw$, where

$$\begin{aligned} p_R(w) &:= e^{-R} \int_{-E^+_{\ell\varepsilon}} \int_{Q_{R+\ell\varepsilon}(w)} s^{\frac{d-1}{2}} e^{2\sigma(su+\sqrt{s}w)} f_z^H([\exp(su+\sqrt{s}w)])\,ds\,dz \\ &= e^{-R} \int_{-E^+_{\ell\varepsilon}} \int_{Q_{R+\ell\varepsilon}(w)} s^{\frac{d-1}{2}} e^{2\sigma(su+\sqrt{s}w)} \\ &\quad\times \int_{\Gamma_\rho\cap H\backslash H} f_z(\Gamma_\rho h \exp(su+\sqrt{s}w))\phi(h)\,dh\,ds\,dz. \end{aligned}$$

Applying Lemma 7.3 by replacing $R$ with $R+\ell\varepsilon$ and $E$ with $-E^+_{\ell\varepsilon}$, by the dominated convergence theorem,

$$\begin{aligned} \lim_{R\to\infty} \int_{\ker\psi} p_R(w)\,dw &= \int_{\ker\psi} \lim_{R\to\infty} p_R(w)\,dw \\ &= \frac{\kappa\,\mu^{\mathrm{PS}}_{\Gamma_\rho\cap H\backslash H,\psi}(\phi) e^{\ell\varepsilon}}{\Phi_\rho(u)} \int_{\ker\psi} e^{-\ell I(w)}\,dw \int_{-E^+_{\ell\varepsilon}} m^{\mathrm{BR}}_\psi(f_z)\,dz. \end{aligned}$$

Altogether, we have thus obtained

$$\limsup_{R\to\infty} e^{-R} \int_{B_\psi(E,R)} f^H([g])\,d[g] \le c_{\Gamma_\rho,\psi}\, e^{\ell\varepsilon} \int_{-E^+_{\ell\varepsilon}} m^{\mathrm{BR}}_\psi(f_z)\,dz.$$

Similarly, but applying Lemma 4.18 to $\widetilde{B}_\psi(E^-_{\ell\varepsilon}, R-\ell\varepsilon)$ and $D_0 = \exp\mathcal{D}_0$, where $\mathcal{D}_0$ is a cone such that $\mathcal{L}_\rho - \{0\} \subset \operatorname{int}\mathcal{D}_0 \subset \overline{\mathcal{D}_0} \subset \mathcal{D}$, we have

$$\liminf_{R\to\infty} e^{-R} \int_{B_\psi(E,R)} f^H([g])\,d[g] \ge c_{\Gamma_\rho,\psi}\, e^{-\ell\varepsilon} \int_{-E^-_{\ell\varepsilon}} m^{\mathrm{BR}}_\psi(f_z)\,dz.$$

Note that by Corollary 6.5, we have

$$(1-C\varepsilon)\,\omega_\psi(E^-_{(1+c)\varepsilon}) \le \int_{-E} m^{\mathrm{BR}}_\psi(\phi^\varepsilon_z)\,dz \le (1+C\varepsilon)\,\omega_\psi(E^+_{(1+c)\varepsilon})$$

for all sufficiently small $\varepsilon > 0$. Since $\omega_\psi(\partial E) = 0$, taking $\varepsilon \to 0^+$ completes the proof.

## 8. Proof of the main counting theorem

In this section, we prove the following main theorem of this paper.

**Theorem 8.1.** *Let $\mathcal{P}$ be a $\Gamma_\rho$-admissible torus packing. For any positive linear form $\psi \in \mathfrak{a}^*$, there exist a constant $c_\psi = c_{\mathcal{P},\psi} > 0$ such that for any bounded measurable subset $E \subset \mathbb{C}^d$ with boundary contained in a proper real algebraic subvariety, we have*

$$\lim_{R\to\infty} e^{-\delta_\psi R} N_R(\mathcal{P}, \psi, E) = c_\psi \omega_\psi(E). \tag{8.2}$$

**Example 8.3.** Note that $\mathrm{Vol}(T) = (2\pi)^d e^{-\sigma(\mathbf{v}(T))}$ since $\sigma(t_1, \dots, t_d) = t_1 + \dots + t_d$. Hence, we have

$$N_R(\mathcal{P}, \sigma, E) = \#\{T \in \mathcal{P} : \mathrm{Vol}(T) \geq (2\pi)^d e^{-R},\ T \cap E \neq \emptyset\}.$$

Since $\sigma \in \mathfrak{a}^*$ is positive, Theorem 1.5 is a special case of Theorem 8.1, with $\delta_{L^1}(\rho) = \delta_\sigma$, $c_{\mathcal{P}} = (2\pi)^{d\delta} c_{\mathcal{P},\sigma}$ and $\omega_\psi = \omega_{\Gamma_\rho,\sigma}$.

The proof of the following lemma is postponed until the final section (Theorem 9.2).

**Lemma 8.4.** *For any bounded measurable subset $E \subset \mathbb{C}^d$ with $\partial E$ contained in a proper real algebraic subvariety, we have $\omega_\psi(\partial E) = 0$.*

Since every homothety class of a positive linear form can be represented by a positive $\Gamma_\rho$-critical linear form (Lemma 2.9) and $\delta_\psi = 1$ for critical linear forms, Theorem 8.1 follows from Lemma 8.4 and the following.

**Proposition 8.5.** *For any positive $\Gamma_\rho$-critical linear form $\psi \in \mathfrak{a}^*$ and any bounded measurable subset $E \subset \mathbb{C}^d$ with $\omega_\psi(\partial E) = \emptyset$, we have*

$$\lim_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E) = c_\psi \omega_\psi(E)$$

*for some constant $c_\psi > 0$.*

**Special case: $\mathcal{P} = \Gamma_\rho T_0$.** We will first prove Proposition 8.5 for the special case when $\mathcal{P} = \Gamma_\rho T_0$. This will allow us to apply the results obtained in previous sections.

Let $\mathcal{D}$ be as defined in (4.10), and for any $R > 0$, $A_R$ denote the $R$-neighborhood of $e$ in $A$. Fix closed cones $\mathcal{D}^\pm \subset \mathrm{int}\,\mathfrak{a}^+$ such that

$$\mathcal{L}_\rho - \{0\} \subset \mathrm{int}\,\mathcal{D}^-, \quad \mathcal{D}^- - \{0\} \subset \mathrm{int}\,\mathcal{D} \quad \text{and} \quad \mathcal{D} - \{0\} \subset \mathrm{int}\,\mathcal{D}^+.$$

Let $D^\pm = \exp \mathcal{D}^\pm$ and $R_0 > 0$ be such that

$$D^- - A_{R_0} \subset \bigcap_{a \in A_1} Da, \qquad \bigcup_{a\in A_1} (D - A_{R_0})a \subset D^+.$$

Recall the definitions of $D^\pm_{\psi,R}$ and $E^\pm_\varepsilon$ from (4.4) and (4.11). Now defining

$$\begin{aligned} B^0_\psi(E, R) &:= H\backslash HKD_{[R_0,R)}N_{-E}, \\ B^\varepsilon_\psi(E, R)^- &:= H\backslash HKD^-_{[R_0,R)}N_{-E^-_\varepsilon}, \\ B^\varepsilon_\psi(E, R)^+ &:= H\backslash HKD^+_{[R_0,R)}N_{-E^+_\varepsilon}, \end{aligned} \tag{8.6}$$

where $D^\pm_{[R_0,R)} = D^\pm_{\psi,R} - A_{R_0}$, we have the following inclusions.

**Lemma 8.7** ([23, Lemma 6.3]). *For all $\varepsilon > 0$ small enough, there exists a neighborhood $\mathcal{O}_\varepsilon \subset G$ of the identity such that for all $R > R_0$,*

$$B^{\varepsilon}_{\psi}(E, R-\varepsilon)^{-} \subset B^{0}_{\psi}(E,R)\mathcal{O}_\varepsilon \subset B^{\varepsilon}_{\psi}(E, R+\varepsilon)^{+}.$$

We now use the sets $B^{\varepsilon}_{\psi}(E, R \pm \varepsilon)^{\pm}$ to obtain the asymptotic of $\#([e]\Gamma_\rho \cap B^{0}_{\psi}(E,R))$. Let us define functions $F_R$, and $F_R^{\varepsilon,\pm}$ on $\Gamma_\rho \backslash G$ by

$$F_R([g]) := \sum_{\gamma \in (\Gamma_\rho \cap H)\backslash \Gamma_\rho} \mathbb{1}_{B^{0}_{\psi}(E,R)}(H\gamma g),$$

and

$$F_R^{\varepsilon,\pm}([g]) := \sum_{\gamma \in (\Gamma_\rho \cap H)\backslash \Gamma_\rho} \mathbb{1}_{B^{\varepsilon}_{\psi}(E,R\pm\varepsilon)^{\pm}}(H\gamma g). \tag{8.8}$$

Note that

$$F_R([e]) = \#([e]\Gamma_\rho \cap B^{0}_{\psi}(E,R)), \tag{8.9}$$

and by Lemma 8.7, we have

$$F_R^{\varepsilon,-}([g]) \le F_R([e]) \le F_R^{\varepsilon,+}([g])$$

for all $[g] \in [e]\mathcal{O}_\varepsilon$ and all $\varepsilon$ small enough and less than the injectivity radius of $[e] \in \Gamma_\rho \backslash G$. Now fix any non-negative function $\phi^\varepsilon \in C_c([e]\mathcal{O}_\varepsilon)$ such that $\int \phi^\varepsilon([g])\, d[g] = 1$, where $d[g]$ is a Haar measure on $\Gamma_\rho \backslash G$. Then

$$\langle F_R^{\varepsilon,-}, \phi^\varepsilon \rangle \le F_R([e]) \le \langle F_R^{\varepsilon,+}, \phi^\varepsilon \rangle, \tag{8.10}$$

where

$$\langle \psi_1, \psi_2 \rangle = \int_{\Gamma_\rho \backslash G} \psi_1([g])\psi_2([g])\, d[g]$$

whenever the integral converges. We will use Theorem 7.1 to estimate the integrals $\langle F_R^{\varepsilon,\pm}, \phi^\varepsilon \rangle$ (cf. [23, (6.6), p. 30] and [6, Proposition 9.10]).

**Proposition 8.11.** *For any $\varepsilon > 0$ small enough, we have*

$$\langle F_R^{\varepsilon,\pm}, \phi^\varepsilon \rangle \sim c_\psi e^{R\pm\varepsilon} \int_{-E_\varepsilon^{\pm}} m_\psi^{\mathrm{BR}}(\phi_z^\varepsilon)\, dz \quad \text{as } R \to \infty,$$

*where the constant $c_\psi = c_{\Gamma_\rho,\psi}$ is given in Theorem 7.1.*

*Proof.* Using unfolding, we have

$$\begin{aligned}\langle F_R^{\varepsilon,\pm}, \phi^\varepsilon \rangle &= \int_{\Gamma_\rho \backslash G} \Bigg( \sum_{\gamma \in (\Gamma_\rho \cap H)\backslash \Gamma_\rho} \mathbb{1}_{B^{\varepsilon}_{\psi}(E,R\pm\varepsilon)^{\pm}}(H\gamma g) \Bigg) \phi^\varepsilon([g])\, dg \\ &= \int_{\Gamma_\rho \cap H \backslash G} \mathbb{1}_{B^{\varepsilon}_{\psi}(E,R\pm\varepsilon)^{\pm}}(Hg)\phi^\varepsilon([g])\, dg \\ &= \int_{B^{\varepsilon}_{\psi}(E,R\pm\varepsilon)^{\pm}} \bigg( \int_{\Gamma_\rho \cap H \backslash H} \phi^\varepsilon(\Gamma_\rho hg)\, dh \bigg)\, d(Hg).\end{aligned}$$

Since the set difference between $B^{\varepsilon}_{\psi}(E, R \pm \varepsilon)^{\pm}$ and $B_{D^{\pm},\psi}(E_\varepsilon^{\pm}, R \pm \varepsilon)$ is bounded independent of $R$, Theorem 7.1 then gives the claimed identity. □

*Proof of Proposition* 8.5 *when* $\mathcal{P} = \Gamma_\rho T_0$. Let us note that the set difference between $B^0_\psi(E, R)$ and $B_{D,\psi}(E, R)$ is bounded independent of $R$. Hence, by Proposition 4.8 and Corollary 4.20,

$$\begin{aligned}\liminf_{R\to\infty} e^{-R}\#([e]\Gamma_\rho \cap B^0_\psi(E^-_\varepsilon, R)) &\le \liminf_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E)\\ &\le \limsup_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E)\\ &\le \limsup_{R\to\infty} e^{-R}\#([e]\Gamma_\rho \cap B^0_\psi(E^+_{(\ell'+1)\varepsilon}, R + \ell'\varepsilon)).\end{aligned}$$

Let $\varepsilon_0 = (\ell' + 2)\varepsilon$. The above computation, combined with (8.9), (8.10) and Proposition 8.11 gives

$$\begin{aligned}c_\psi e^{-\varepsilon_0} \int_{-E^-_{\varepsilon_0}} m^{\mathrm{BR}}_\psi(\phi^{\varepsilon_0}_z)\, dz &\le \liminf_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E)\\ &\le \limsup_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E)\\ &\le c_\psi e^{\varepsilon_0} \int_{-E^+_{\varepsilon_0}} m^{\mathrm{BR}}_\psi(\phi^{\varepsilon_0}_z)\, dz.\end{aligned}$$

Corollary 6.5 now gives

$$\begin{aligned}c_\psi e^{-\varepsilon_0}(1 - C\varepsilon_0)\,\omega_\psi(E^-_{(2+c)\varepsilon_0}) &\le \liminf_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E)\\ &\le \limsup_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E)\\ &\le c_\psi e^{\varepsilon_0}(1 + C\varepsilon_0)\,\omega_\psi(E^+_{(2+c)\varepsilon_0}).\end{aligned}$$

Since $\omega_\psi(\partial E) = 0$ by Lemma 8.4, the regularity of $\omega_\psi$ gives

$$\lim_{\varepsilon\to 0^+} c_\psi e^{\pm\varepsilon}(1 \pm C\varepsilon)\,\omega_\psi(E^\pm_{(2+c)\varepsilon}) = c_\psi\, \omega_\psi(E),$$

completing the proof. □

**General case.** Without loss of generality, we may assume that $\mathcal{P}$ consists of a single $\Gamma_\rho$-orbit; hence let $\mathcal{P} = \Gamma_\rho T$ be a $\Gamma_\rho$-admissible torus packing. We write

$$T = g_0 T_0,$$

where $g_0 = n_{z_0} a_{t_0}$; here $z_0$ is the vector consisting of the centers of the circles of $T$ and $t_0 = \log r_0$, where $r_0 = (r_1, \dots, r_d)$ are the corresponding radii. Set

$$\Gamma^{g_0}_\rho := g_0^{-1}\Gamma_\rho g_0.$$

Note that

$$\begin{aligned}N_R(\mathcal{P}, \psi, E) &= \#\{T' \in \Gamma_\rho T : T' \cap E \ne \emptyset \text{ and } \psi(\boldsymbol{v}(T')) \le R\}\\ &= \#\{T' \in \Gamma_\rho g_0 T_0 : T' \cap E \ne \emptyset \text{ and } \psi(\boldsymbol{v}(T')) \le R\}\\ &= \#\{T' \in \Gamma^{g_0}_\rho T_0 : g_0 T' \cap E \ne \emptyset \text{ and } \psi(\boldsymbol{v}(g_0 T')) \le R\}.\\ &= \#\{\gamma \in (\Gamma^{g_0}_\rho \cap H)\backslash\Gamma^{g_0}_\rho : g_0\gamma^{-1} T_0 \cap E \ne \emptyset \text{ and } \psi(\boldsymbol{v}(g_0\gamma^{-1} T_0)) \le R\}.\end{aligned}$$

Similarly to Proposition 4.8, we can obtain the following estimate of $N_R(\mathcal{P}, \psi, E)$ in terms of $\widetilde{B}_\psi(E^\pm_\varepsilon, R)$.

**Proposition 8.12.** *For any $\varepsilon > 0$, there exists $q_0 = q_0(\mathcal{P}, \varepsilon) > 0$ such that for any $R > 0$ and any bounded measurable subset $E \subset \mathbb{C}^d$, we have*

$$\#([e]\Gamma_\rho^{g_0} \cap \widetilde{B}_\psi(E_\varepsilon^-, R)g_0) - q_0 \le N_R(\mathcal{P}, \psi, E) \le \#([e]\Gamma_\rho^{g_0} \cap \widetilde{B}_\psi(E_\varepsilon^+, R)g_0) + q_0.$$

Note that $\Gamma_\rho^{g_0}$ is also a self-joining of convex cocompact representations. Let $\Lambda_\rho^{g_0}$ and $\mathcal{L}_\rho^{g_0}$ denote its limit set and limit cone, respectively. It is immediate from the definition that

$$\Lambda_\rho^{g_0} = g_0^{-1}\Lambda_\rho \quad \text{and} \quad \mathcal{L}_\rho^{g_0} = \mathcal{L}_\rho.$$

Now, writing $g_0 = (g_{0,1}, \dots, g_{0,d})$, the homeomorphisms in (2.3) associated to $\Gamma_\rho^{g_0}$ can be written as $g_{0,i}^{-1} f_i g_{0,1}$, where $1 \le i \le d$. A direct computation shows that $T$ is $\Gamma_\rho$-admissible if and only if $T_0$ is $\Gamma_\rho^{g_0}$-admissible. Hence, we can apply the results obtained in previous sections for a new subgroup $\Gamma_\rho^{g_0}$.

**Transition from $\Gamma_\rho$ to $\Gamma_\rho^{g_0}$.** Let $\Phi_\rho^{g_0} = \Phi_{\Gamma_\rho^{g_0}}$ denote the growth indicator function associated to $\Gamma_\rho^{g_0}$. The following lemma is standard and can be proved using [1, Lemma 4.6], [29, Lemma 3.1.6] and the definition of $\Phi_\rho$.

**Lemma 8.13.** *We have*

$$\Phi_\rho^{g_0} = \Phi_\rho.$$

Since $\psi$ is $\Gamma_\rho$-critical, it follows from Lemma 8.13 that $\psi$ is $\Gamma_\rho^{g_0}$-critical. The unique unit vectors, as provided by Lemma 2.10 remain the same, regardless of whether we view $\psi$ as a $\Gamma_\rho$-critical linear form or $\Gamma_\rho^{g_0}$-critical linear form. We denote by $\nu_\psi^{g_0}$ the $(\Gamma_\rho^{g_0}, \psi)$-PS probability measure supported on $\Lambda_\rho^{g_0}$. Define a measure $\widetilde{\nu}_\psi^{g_0}$ on $\widehat{\mathbb{C}}^d$ via the formula

$$d\widetilde{\nu}_\psi^{g_0}(z) = e^{-\psi(\beta_z(o, g_0 \cdot o))} d((g_0)_* \nu_\psi^{g_0})(z).$$

**Lemma 8.14.** *We have*

$$\frac{\widetilde{\nu}_\psi^{g_0}}{|\widetilde{\nu}_\psi^{g_0}|} = \nu_\psi.$$

*Proof.* Since the support of $\nu_\psi^{g_0}$ is $\Lambda_\rho^{g_0} = g_0^{-1}\Lambda_\rho$, we have

$$(g_0)_* \nu_\psi^{g_0}(\Lambda_\rho) = \nu_\psi^{g_0}(g_0^{-1}\Lambda_\rho) = 1.$$

Therefore, $\widetilde{\nu}_\psi^{g_0}$ is also supported on $\Lambda_\rho$. Furthermore, for any $\gamma \in \Gamma_\rho$, we have

$$\begin{aligned} d\gamma_* \widetilde{\nu}_\psi^{g_0}(z) &= e^{-\psi(\beta_{\gamma^{-1}z}(o, g_0 o))} d\gamma_*(g_0)_* \nu_\psi^{g_0}(z) \\ &= e^{-\psi(\beta_z(\gamma o, \gamma g_0 o))} d(g_0)_*(g_0^{-1}\gamma g_0)_* \nu_\psi^{g_0}(z) \\ &= e^{-\psi(\beta_z(\gamma o, \gamma g_0 o))} e^{-\psi(\beta_{g_0^{-1}z}(g_0^{-1}\gamma g_0 o, o))} d(g_0)_* \nu_\psi^{g_0}(z) \\ &= e^{\psi(\beta_z(\gamma g_0 o, \gamma o))} e^{\psi(\beta_z(g_0 o, \gamma g_0 o))} e^{\psi(\beta_z(o, g_0 o))} d\widetilde{\nu}_\psi^{g_0}(z) \\ &= e^{\psi(\beta_z(o, g_0 o))} e^{\psi(\beta_z(g_0 o, \gamma g_0 o))} e^{\psi(\beta_z(\gamma g_0 o, \gamma o))} d\widetilde{\nu}_\psi^{g_0}(z) \\ &= e^{\psi(\beta_z(o, \gamma o))} d\widetilde{\nu}_\psi^{g_0}(z), \end{aligned}$$

i.e.,

$$\frac{d\gamma_* \tilde{\nu}_\psi^{g_0}}{d\tilde{\nu}_\psi^{g_0}}(z) = e^{-\psi(\beta_z(\gamma o, o))}.$$

This shows that $\tilde{\nu}_\psi^{g_0}/|\tilde{\nu}_\psi^{g_0}|$ is a $(\Gamma_\rho, \psi)$-PS probability measure. By [17, Theorem 1.3], $\nu_\psi$ is the unique $(\Gamma_\rho, \psi)$-PS probability measure, and hence the lemma is proved. □

Let $\omega_{g_0,\psi}$ denote the measure defined as in (6.1), associated to $\Gamma_\rho^{g_0}$ and $\psi$. Using Lemma 8.14, we can now show the following result.

**Lemma 8.15.** *There exists $c_{g_0} > 0$ such that*

$$\omega_{g_0,\psi}(g_0^{-1}E) = c_{g_0}\omega_\psi(E)$$

*for all Borel sets $E \subset \mathbb{C}^d$.*

*Proof.* By Definition 6.1,

$$\omega_{g_0,\psi}(g_0^{-1}E) = \int_E d(g_0)_*\omega_{g_0,\psi}(z) = \int_E e^{\psi(\beta_{g_0^{-1}z}(o, n_{g_0^{-1}z}o))}\, d(g_0)_*\nu_\psi^{g_0}(z).$$

Now writing as seen above $g_0 = n_{z_0}a_{t_0}$, we thus have by Lemma 8.14

$$\begin{aligned}
\omega_{g_0,\psi}(g_0^{-1}E) &= \int_E e^{\psi(\beta_{g_0^{-1}z}(o, g_0^{-1}n_z a_{t_0}o))} e^{\psi(\beta_z(o, g_0 o))}\, d\tilde{\nu}_\psi^{g_0}(z) \\
&= |\tilde{\nu}_\psi^{g_0}| \int_E e^{\psi(\beta_z(g_0 o, n_z a_{t_0}o))} e^{\psi(\beta_z(o, g_0 o))}\, d\nu_\psi(z) \\
&= |\tilde{\nu}_\psi^{g_0}| \int_E e^{\psi(\beta_z(o, n_z a_{t_0}o))} e^{-\psi(\beta_z(o, n_z o))}\, d\omega_\psi(z) \\
&= |\tilde{\nu}_\psi^{g_0}| \int_E e^{\psi(\beta_z(n_z o, n_z a_{t_0}o))}\, d\omega_\psi(z) \\
&= |\tilde{\nu}_\psi^{g_0}| e^{\psi(\beta_0(o, a_{t_0}o))}\omega_\psi(E) \\
&= |\tilde{\nu}_\psi^{g_0}|\, e^{-\psi(t_0)}\omega_\psi(E),
\end{aligned}$$

as desired. □

Next, let

$$m^{\mathrm{BR}}_{\Gamma_\rho^{g_0},\psi} = m^{\mathrm{BR}}_{g_0,\psi}$$

denote the Burger–Roblin measure associated to $\Gamma_\rho^{g_0}$ and the linear form $\psi$. Denote the right $G$-action on functions on $\Gamma_\rho^{g_0}\backslash G$ by $(g \cdot f)([h]) = f([hg])$ and let $G_\varepsilon$ be the $\varepsilon$-neighborhood of $e$ in $G$. For any $\phi^\varepsilon \in C_c(\Gamma_\rho^{g_0}\backslash G)$ whose support is contained in $[e]G_\varepsilon$, we have the following.

**Lemma 8.16.** *For all small enough $\varepsilon > 0$,*

$$\int_{-E_\varepsilon^\pm} m^{\mathrm{BR}}_{g_0,\psi}((g_0 \cdot \phi^\varepsilon)_z)\, dz = e^{\psi(t_0)} \int_{-g_0^{-1}E_\varepsilon^\pm} m^{\mathrm{BR}}_{g_0,\psi}(\phi_z^\varepsilon)\, dz.$$

*Proof.* Denote the $\Gamma_\rho^{g_0}$ -invariant lift of $m^{\mathrm{BR}}_{g_0,\psi}$ to $G$ by $\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}$. We use the $G = KA\check{N}$ decomposition to write

$$d\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(kau) = e^{-\psi(\log a)}du\, da\, d\overline{\nu}^{g_0}_{\psi}(k) \tag{8.17}$$

by [6, Lemma 4.9], where the measure $\overline{\nu}^{g_0}_{\psi}$ on $K$ is defined by

$$\int_K f(k)\, d\overline{\nu}^{g_0}_{\psi}(k) = \int_{K/M}\int_M f(km)\, dm\, d\nu^{g_0}_{\psi}(k^+)$$

for all $f \in C(K)$. For any $f \in C_c(G)$, and measurable $L \subset \mathbb{C}^d$, using the fact that $g_0 = n_{z_0}a_{t_0}$, we have

$$\begin{aligned}\int_{-L}\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}((g_0\cdot f)_z)\, dz &= \int_{-L}\int_M\int_G f(gmn_z g_0)\, d\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(g)\, dm\, dz\\ &= \int_{z_0-L}\int_M\int_G f(gmn_z a_{t_0})\, d\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(g)\, dm\, dz\\ &= e^{2\sigma(t_0)}\int_{a_{-t_0}(z_0-L)}\int_M\int_G f(ga_{t_0}mn_z)\, d\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(g)\, dm\, dz.\end{aligned}$$

From (8.17), we obtain

$$\begin{aligned}\int_G f(ga_{t_0}mn_z)\, d\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(g) &= \int_{KA\check{N}} f(kaua_{t_0}mn_z)e^{-\psi(\log a)}du\, da\, d\overline{\nu}^{g_0}_{\psi}(k)\\ &= e^{-2\sigma(t_0)}\int_{KA\check{N}} f(kaa_{t_0}umn_z)e^{-\psi(\log a)}du\, da\, d\overline{\nu}^{g_0}_{\psi}(k)\\ &= e^{(\psi-2\sigma)(t_0)}\int_G f(gmn_z)\, d\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(g).\end{aligned}$$

This gives

$$\int_{-L}\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}((g_0\cdot f)_z)\, dz = e^{\psi(t_0)}\int_{-g_0^{-1}L}\widetilde{m}^{\mathrm{BR}}_{g_0,\psi}(f_z)\, dz,$$

proving the claim. □

*Proof of Proposition* 8.5 *for the general case.* We define a counting function that we again denote by $F_R$, on $\Gamma_\rho^{g_0}\backslash G$ by

$$F_R([g]) = \sum_{\gamma\in(\Gamma_\rho^{g_0}\cap H)\backslash\Gamma_\rho^{g_0}} \mathbb{1}_{B^0_\psi(E,R)}(\gamma g).$$

We have

$$\#([e]\Gamma^{g_0}\cap B^0_\psi(E,R)g_0) = (g_0^{-1}\cdot F_R)([e]). \tag{8.18}$$

Let $B^\varepsilon_\psi(E,R)^\pm$ be as in (8.6) and let $\mathcal{O}_\varepsilon$ be as in Lemma 8.7. Set now

$$F_R^{\varepsilon,\pm}([g]) := \sum_{\gamma\in(\Gamma_\rho^{g_0}\cap H)\backslash\Gamma_\rho^{g_0}} \mathbb{1}_{B^\varepsilon_\psi(E,R\pm\varepsilon)^\pm}(\gamma g). \tag{8.19}$$

Then

$$(g_0^{-1}\cdot F_R^{\varepsilon,-})([g]) \le (g_0^{-1}\cdot F_R)([e]) \le (g_0^{-1}\cdot F_R^{\varepsilon,+})([g])$$

for all $g \in \mathcal{O}_\varepsilon$. Thus, choosing a non-negative $\phi^\varepsilon \in C_c(\Gamma_\rho^{g_0}\backslash G)$ with support in $[e]\mathcal{O}_\varepsilon$ such that $\int \phi^\varepsilon([g])\, d[g] = 1$ gives

$$\langle F_R^{\varepsilon,-}, g_0 \cdot \phi^\varepsilon\rangle \le F_R([g_0]) \le \langle F_R^{\varepsilon,+}, g_0 \cdot \phi^\varepsilon\rangle. \tag{8.20}$$

Similarly as in Proposition 8.11, we have

$$\langle F_R^{\varepsilon,\pm}, g_0 \cdot \phi^\varepsilon\rangle \sim c e^{R\pm\varepsilon} \int_{-E_\varepsilon^\pm} m^{\mathrm{BR}}_{g_0,\psi}((g_0 \cdot \phi^\varepsilon)_z)\, dz \tag{8.21}$$

as $R \to \infty$, where $c = c_{\Gamma_\rho^{g_0},\psi}$.

Similarly to the proof for the special case, from Corollary 4.20, Proposition 8.12, (8.18), (8.20) and (8.21), we obtain for $\varepsilon_0 = (\ell' + 2)\varepsilon$,

$$\begin{aligned} c e^{-\varepsilon_0} \int_{-E_{\varepsilon_0}^-} m^{\mathrm{BR}}_{g_0,\psi}((g_0 \cdot \phi^{\varepsilon_0})_z)\, dz &\le \liminf_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E) \\ &\le \limsup_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E) \\ &\le c e^{-\varepsilon_0} \int_{-E_{\varepsilon_0}^+} m^{\mathrm{BR}}_{g_0,\psi}((g_0 \cdot \phi^{\varepsilon_0})_z)\, dz. \end{aligned}$$

Applying Lemma 8.16 and Corollary 6.5 gives

$$\begin{aligned} e^{\psi(t_0)}(1 - C\varepsilon_0)\, \omega_{g_0,\psi}(g_0^{-1} E^-_{(2+c)\varepsilon_0}) &\le \int_{-E_{\varepsilon_0}^\pm} m^{\mathrm{BR}}_{g_0,\psi}((g_0 \cdot \phi^{\varepsilon_0})_z)\, dz \\ &\le e^{\psi(t_0)}(1 + C\varepsilon_0)\, \omega_{g_0,\psi}(g_0^{-1} E^+_{(2+c)\varepsilon_0}). \end{aligned}$$

We now use Lemma 8.15 to change $\omega_{g_0,\psi}$ to $\omega_\psi$ and then taking $\varepsilon \to 0$ as in the case $T = T_0$ gives

$$\lim_{R\to\infty} e^{-R} N_R(\mathcal{P}, \psi, E) = c_0 \omega_\psi(E)$$

for some positive constant $c_0 > 0$. Since the left-hand side of the above equation does not depend on the choice of $g_0$, we in fact have that $c_0$ cannot depend on $g_0$ either, proving the theorem. □

**On Remark 1.9 (2).** There exists a unique vector $u = u_{\Gamma_\rho} \in (\mathbb{R}_{\ge 0})^d$ with the property that $\Phi_\rho(u) = \max\{\Phi_\rho(v) : \|v\| \le 1\}$, called the direction of the maximal growth of $\Gamma_\rho$. Moreover, $u \in \operatorname{int} \mathcal{L}_\rho$ (see [28, 32]). Let $\psi = \psi_u$ be as defined in Lemma 2.10. Then for all $w \in \ker \psi$, the subset

$$Q_R(w) := \{s \in \mathbb{R}_{>0} : \|su + \sqrt{s} w\| < R\} \tag{8.22}$$

is an interval of the form $(0, \frac{1}{2}(-\|w\|^2 + \sqrt{\|w\|^4 + 4R^2}))$. Then using [6, Lemma 9.4] substituting Lemma 7.3, our proof yields the following:

$$\lim_{R\to\infty} e^{-\delta_{\Gamma_\rho} R} \#\{T \in \mathcal{P} : \|\boldsymbol{v}(T)\| < R,\ T \cap E \ne \emptyset\} = c_\psi{}' \omega_\psi(E \cap \Lambda_\rho), \tag{8.23}$$

where $\delta_{\Gamma_\rho} = \Phi_\rho(u)$ and $c_\psi{}' > 0$.

We remark that whereas Lemma 7.3 relied on the uniformity in the mixing Theorem 5.6, [6, Lemma 9.4] did not need such uniformity. The reason is that for each $w \in \ker \psi$, the amount of time the trajectory $s \mapsto su + \sqrt{s} w$ spends in the set $\{v \in \mathfrak{a}^+ : \|v\| < R\}$ is much less than the time it spends in $\{v \in \mathfrak{a}^+ : \psi(v) < R\}$ to the extent that when viewed from a proper scale, it gives rise to an $L^1$-function on $\ker \psi$.

## 9. PS-measures are null on algebraic varieties

In this section, we prove that $\nu_\psi(S) = 0$ for any proper real algebraic subvariety $S$ of $\widehat{\mathbb{C}}^d$. Since $\omega_\psi$ is absolutely continuous with respect to $\nu_\psi$, it follows that $\omega_\psi(\partial E) = 0$ whenever $E$ has boundary contained in a proper real algebraic subvariety, and in particular, Lemma 8.4 follows.

We will in fact prove this in a more general setup, which we now explain.

Let $G$ be any connected semisimple linear real algebraic group. Let $P = MAN < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition. Let $\mathfrak{a}$ denote the Lie algebra of $A$. Let $\mathrm{i}$ denote the opposition involution on $\mathfrak{a}$. We remark that the opposition involution is non-trivial if and only if $G$ has a simple factor of type $A_n$ ($n \geq 2$), $D_{2n+1}$ ($n \geq 2$) and $E_6$ (see [34, Section 1.5.1]). For instance, when $G$ is a product of rank one groups, $\mathrm{i}$ is trivial.

A Borel probability measure $\nu$ on $\mathcal{F} = G/P$ is called a $(\Gamma, \psi)$-conformal measure for a linear form $\psi \in \mathfrak{a}^*$ if for all $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{-\psi(\beta_\xi(\gamma, e))},$$

where $\beta$ denotes the $\mathfrak{a}$-valued Busemann map [6, Definition 2.3]. When supported on the limit set $\Lambda$, it is called a $(\Gamma, \psi)$-PS-measure.

We recall the following result: consider the diagonal action of $\Gamma$ on $\mathcal{F} \times \mathcal{F}$.

**Proposition 9.1** ([16, Proposition 6.3]). *Let $\Gamma < G$ be a Zariski-dense Anosov subgroup of $G$ (with respect to $P$). Let $\psi \in \mathfrak{a}^*$ be a linear form. Let $\nu$ and $\nu_{\mathrm{i}}$ be respectively $(\Gamma, \psi)$ and $(\Gamma, \psi \circ \mathrm{i})$-PS measures. Then $(\mathcal{F} \times \mathcal{F}, \nu \times \nu_{\mathrm{i}},)$ is $\Gamma$-ergodic.*

**Theorem 9.2.** *Let $\Gamma < G$ be a Zariski-dense Anosov subgroup of $G$. For any $(\Gamma, \psi)$-PS measure $\nu$ for some $\psi \in \mathfrak{a}^*$ with $\psi \circ \mathrm{i} = \psi$, we have*

$$\nu(S) = 0$$

*for any proper real algebraic subvariety $S$ of $\mathcal{F}$.*

Theorem 9.2 follows from the following by Proposition 9.1.

**Theorem 9.3.** *Let $\Gamma < G$ be a discrete subgroup and let $\nu$ be a $(\Gamma, \psi)$-PS measure for some $\psi \in \mathfrak{a}^*$ such that the diagonal $\Gamma$-action on $(\mathcal{F} \times \mathcal{F}, \nu \times \nu)$ is ergodic. Then*

$$\nu(S) = 0$$

*for any proper real algebraic subvariety $S$ of $\mathcal{F}$.*

*Proof.* Suppose the theorem is false. Let $S$ be a proper subvariety of $\mathcal{F}$ with $\nu(S) > 0$ and of minimal dimension. We may assume without loss of generality that $S$ is irreducible.

As $(\nu \times \nu)(S \times S) = \nu(S) \times \nu(S) > 0$, it follows that the $\Gamma$-ergodicity of $\nu \times \nu$ implies that $\Gamma(S \times S)$ must have full $\nu \times \nu$-measure. Since for any $\gamma_0 \in \Gamma$, $(\nu \times \nu)(S \times \gamma_0 S) > 0$, there must exist $\gamma \in \Gamma$ such that $(S \cap \gamma_0 S) \cap (\gamma S \times \gamma S)$ has positive $\nu \times \nu$-measure. This implies $\nu(S \cap \gamma S) > 0$ and $\nu(\gamma_0 S \cap \gamma S) > 0$. Since $S$ is an irreducible variety, for any $\gamma \in \Gamma$, either $S = \gamma S$ or the dimension of $S \cap \gamma S$ is strictly smaller than that of $S$, and hence we have $\nu(S \cap \gamma S) = 0$. Therefore $S = \gamma S = \gamma_0 S$. Since $\gamma_0$ was arbitrary, it follows that $\Gamma S = S$; a contradiction to the Zariski-density of $\Gamma$. □

We deduce the following corollary when $G$ is of rank one. In this case, $G = \mathrm{Isom}^+(X)$ for a rank one symmetric space $X$ and $\mathcal{F}$ is equal to the geometric boundary of $X$. For a non-elementary discrete subgroup $\Gamma < G$ of divergence type (e.g., geometrically finite), there exists a unique $\Gamma$-conformal measure, say $\nu_\Gamma$, of dimension equal to the critical exponent $\delta_\Gamma$ and the diagonal $\Gamma$-action on $(\mathcal{F} \times \mathcal{F}, \nu_\Gamma \times \nu_\Gamma)$ is ergodic [31, Theorem 1.7]. Therefore we obtain

**Corollary 9.4.** *Let $G$ be of rank one and let $\Gamma < G$ be a Zariski-dense discrete subgroup of divergence type. Then $\nu_\Gamma(S) = 0$ for any proper real algebraic subvariety $S$ of $\mathcal{F}$.*

This corollary was obtained in [8] when $G = \mathrm{SO}(n, 1)$ and $\Gamma < G$ is geometrically finite.

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