One-Loop Off-Shell Amplitudes from Classical Equations of Motion

Humberto Gomez,[1,2] Renann Lipinski Jusinskas,[3] Cristhiam Lopez-Arcos,[4] and Alexander Quintero Vélez[4]

[1]*Department of Mathematical Sciences, Durham University, Stockton Road, DH1 3LE Durham, United Kingdom*
[2]*Facultad de Ciencias Basicas, Universidad Santiago de Cali, Calle 5 No. 62-00 Barrio Pampalinda, Cali, Valle, Colombia*
[3]*Institute of Physics of the Czech Academy of Sciences and CEICO, Na Slovance 2, 18221 Prague, Czech Republic*
[4]*Escuela de Matemáticas, Universidad Nacional de Colombia, Sede Medellín, Carrera 65 No. 59A–110, Medellín, Colombia*

(Received 15 August 2022; revised 14 October 2022; accepted 17 January 2023; published 24 February 2023)

In this Letter, we present a recursive method for computing one-loop off-shell integrands in colored quantum field theories. First, we generalize the perturbiner method by recasting the multiparticle currents as generators of off-shell tree-level amplitudes. After, by taking advantage of the underlying color structure, we define a consistent sewing procedure to iteratively compute the one-loop integrands. When gauge symmetries are involved, the whole procedure is extended to multiparticle solutions involving ghosts, which can then be accounted for in the full loop computation. Since the required input here is equations of motion and gauge symmetry, our framework naturally extends to one-loop computations in certain non-Lagrangian field theories.

DOI: 10.1103/PhysRevLett.130.081601

Introduction.—Scattering amplitudes are central objects of study in quantum field theory. More than convenient physical observables, they are deeply rooted in the very way we intuit particle interactions. And nothing captures this statement more clearly than Feynman diagrams and their beautiful simplicity.

We soon learn, however, that Feynman diagrams are far from being the most efficient way of computing scattering amplitudes. There has been an impressive progress over the years in tree- and loop-level computations. Most of these developments involve so-called on-shell methods (see, e.g., [1–6] for reviews on the always increasing number of techniques). On the other hand, off-shell methods are very scarce (see, e.g., [7] and references therein). Off-shell amplitudes have, in general, a richer structure. In addition to encoding the full on-shell information of a given process, off-shell results can be used in the computation of form factors (related to higher-derivative terms in effective actions), in the study of quantum corrections of propagators and vertices of a theory, renormalization group analysis, and as building blocks for higher-loop corrections.

In between on- and off-shell methods, the Berends-Giele (BG) currents [8] stand out for their elegant and recursive character. Simply put, BG currents represent tree-level amplitudes with one off-shell leg, which are naturally interpreted as branches of higher-point trees. Even more interesting is the fact that the BG prescription can be used to compute quantum effects. Indeed, one-loop integrands are obtained by sewing tree-level amplitudes with two off-shell legs. Following this idea, the matter contribution to one-loop amplitudes in QCD was computed in [9] by making an on-shell matter leg of the BG current off shell. This construction, however, cannot be easily extended to gluon loops because of the gauge symmetry: lifting the on-shell condition in gauge fields is a nontrivial task. In this case, a concrete solution has only recently been found in [10], following an extensive procedure using graphic rules in pure Yang-Mills theory. Numerical recursions for one-loop integrands are also known in the literature, see, e.g., [11,12].

We would like to offer a fresh perspective on this subject, and show how to obtain quantum off-shell currents via classical equations of motion [13]. The key ingredient is the perturbiner [15,16], a well-known method to recursively obtain Berends-Giele currents via formal multiparticle solutions of the field equations. The perturbiner method has proven to be a versatile technique to compute tree-level amplitudes in different theories [17–30]. What we propose, instead, is to view it as a generator of off-shell tree-level diagrams. This is achieved via multiparticle *Ansätze* that solve the interacting part of the field equations, while leaving single-particle states off shell. The standard notion of Berends-Giele currents is then generalized to a fully off-shell version. In turn, we are able to define a consistent sewing procedure to recursively generate one-loop off-shell currents, dubbed "one-loop preintegrands."

In this Letter, we focus on color-ordered theories, having the biadjoint scalar and Yang-Mills theories as working examples. The sewing procedure has to be supplied by a "cyclic completion," which is neatly implemented using the multi-index structure of the one-loop preintegrands. It restores the cyclicity of the partial amplitude while solving the combinatorial challenge of the Feynman approach. We start by illustrating the construction in the biadjoint scalar theory with the derivation of the off-shell recursion and the one-loop integrands. In Yang-Mills theory, we first perform the gauge fixing of the action with the introduction of Faddeev-Popov ghosts. We then extend the previous analysis to ghost loop contributions, finally proposing the full n-point one-loop integrand with off-shell external gluons.

Biadjoint theory: Off-shell recursion.—We will work here in d-dimensional Minkowski space with metric $\eta_{\mu\nu}$ ($\mu,\nu = 0,\ldots,d-1$) and negative time signature. We also use the shorthand notation $a\cdot b = a^\mu b^\nu \eta_{\mu\nu}$, $k^2 = k\cdot k$, and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Consider a massless biadjoint scalar with cubic self-interaction and classical equation of motion [31],

$$\Box\phi = \frac{1}{2}[\![\phi,\phi]\!]. \tag{1}$$

Here we have $\phi = \phi_{a\tilde{a}}T^a \otimes \tilde{T}^{\tilde{a}}$, where a and $\tilde{a}$ are adjoint indices associated with two different quadratic Lie algebras, with generators T^a and $\tilde{T}^{\tilde{a}}$, and

$$[\![T^a\otimes\tilde{T}^{\tilde{a}}, T^b\otimes\tilde{T}^{\tilde{b}}]\!] = [T^a,T^b]\otimes[\tilde{T}^{\tilde{a}},\tilde{T}^{\tilde{b}}]. \tag{2}$$

We are interested in building a generator of tree-level multiparticle currents *à la* Berends-Giele currents [8], but through a modified perturbiner with off-shell external legs. To this end, we first look for a multiparticle solution to (1) of the form

$$\phi(x) = \sum_{P,Q}\Phi_{P|Q}e^{ik_P\cdot x}T^{a_P}\otimes\tilde{T}^{\tilde{a}_Q}. \tag{3}$$

The sum ranges over all words $P = p_1,\ldots,p_n$ and $Q = q_1,\ldots,q_n$ of length $|P| = |Q| = n$, where p_i and q_i are single-particle labels. Furthermore, we have $k_P = k_{p_1}+\cdots+k_{p_n}$, $T^{a_P} = T^{a_1}\ldots T^{a_n}$, and $\tilde{T}^{\tilde{a}_Q} = \tilde{T}^{\tilde{a}_1}\ldots\tilde{T}^{\tilde{a}_n}$. Inserting this back into (1) leads to $k_p^2 = 0$ for single-particle states and the following recursion relation:

$$\Phi_{P|Q} = \frac{1}{s_P}\sum_{P=RS}\sum_{Q=TU}[\Phi_{R|T}\Phi_{S|U} - (R\leftrightarrow S)], \tag{4}$$

where $s_P = k_P^2$ are the Mandelstam variables. The sums over $P = RS$ and $Q = TU$ denote deconcatenations of the word P into R and S, and the word Q into T and U, respectively. For example, for $P = ijk$ we have $(R,S) = (i,jk),(ij,k)$. By construction, the coefficients $\Phi_{P|Q}$ automatically vanish unless the words P and Q are related via permutation.

The multiparticle currents given by (4) are identified with Berends-Giele currents at tree level and can then be used to compute double-partial amplitudes [20]. On the other hand, by dropping the on-shell condition for the single-particle states, equation (4) defines an off-shell recursion that only solves the equation of motion (1) at the multiparticle level. For instance, this can be expressed as

$$\Box\phi - \frac{1}{2}[\![\phi,\phi]\!] = \sum_p k_p^2\Phi_{p|q}e^{ik_p\cdot x}T^{a_p}\otimes\tilde{T}^{\tilde{a}_q}, \tag{5}$$

where the sum is taken over the single-particle states and $k_p^2 \neq 0$. It is therefore fair to say that the recursion relations in (4) solve the interacting part of the biadjoint scalar theory while leaving the single-particle states off shell. In other words, they can be interpreted as a generator of off-shell trees. Bearing this in mind, we may refer to the coefficients $\Phi_{P|Q}$ as off-shell Berends-Giele double currents.

We can then establish the connection between $\Phi_{P|Q}$ and the off-shell scattering amplitudes in biadjoint scalar theory. This is realized by a direct extrapolation of the Berends-Giele prescription, such that the off-shell tree-level double-partial amplitudes are determined through the formula

$$m(Pn|Qn) = \lim_{k_P\to -k_n} s_P\Phi_{P|Q}\Phi_{n|n}, \tag{6}$$

where the limit enforces momentum conservation.

Biadjoint theory: One-loop integrands.—Off-shell trees are the building blocks of loop amplitudes via a sewing procedure. We will now show that the off-shell perturbiner expansion leads to a simple algorithm for computing one-loop integrands in the biadjoint scalar theory.

We start with the double current $\Phi_{lP|lQ}$, in which the single-particle label l plays a special role. Using (4), such current can be explicitly expressed as

$$\Phi_{lP|lQ} = \frac{1}{s_{lP}}\left(\Phi_{l|l}\Phi_{P|Q} + \sum_{P=RS}\sum_{Q=TU}\Phi_{lR|lT}\Phi_{S|U}\right). \tag{7}$$

We can then factor out the single-particle polarization $\Phi_{l|l}$ on the right-hand side and recast $\Phi_{lP|lQ}$ as

$$\Phi_{lP|lQ} = \Phi_{l|l}\Lambda_{P|Q}(\ell), \tag{8}$$

where $\ell^\mu \equiv k_l^\mu$, and

$$\Lambda_{P|Q}(\ell) = \frac{1}{(\ell+k_P)^2} \times\left[\Phi_{P|Q} + \sum_{P=RS}\sum_{Q=TU}\Lambda_{R|T}(\ell)\Phi_{S|U}\right]. \tag{9}$$

The double current $\Lambda_{P|Q}(\ell)$ is the fundamental ingredient for defining the one-loop integrands. It needs but a small upgrade.

In order to see this, observe that (6) and (8) yield

$$m(lPn|lQn) = \lim_{k_P \to -\ell - k_n} (\ell + k_P)^2 \Phi_{l|l} \Phi_{n|n} \Lambda_{P|Q}(\ell). \tag{10}$$

The sewing procedure $\Phi_{l|l}\Phi_{n|n} \to 1/\ell^2$, with $k_n = -\ell$, leads to what looks like an on-shell one-loop integrand $I_{\text{1-loop}}(P|Q) \approx \Lambda_{P|Q}(\ell)$. However, such an integrand is not cyclic in the words P and Q, for the singling out of the leg l has not been symmetrically done. Fortunately, the perturbiner framework enables a neat solution to this problem via a cyclic completion of the combinatorial sums defining the recursion.

We then introduce the modified double current,

$$\tilde{\Lambda}_{P|Q}(\ell) \equiv \frac{1}{(\ell + k_P)^2} \times \left[\Phi_{P|Q} + \frac{1}{2} \sum_{P=[RS]} \sum_{Q=[TU]} \Lambda_{R|T}(\ell) \Phi_{S|U} \right], \tag{11}$$

such that the one-loop integrand is expressed as

$$I_\ell^{\text{1-loop}}(P|Q) = \lim_{k_P \to 0} \tilde{\Lambda}_{P|Q}(\ell). \tag{12}$$

The words P and Q encode the (double) color ordering of the one-loop integrand. The first term inside the square brackets in (11) only yields tadpole diagrams, so it can be removed for convenience since their regularized contribution vanishes. The cyclic completion affects the remaining terms, with sums over $P = [RS]$ and $Q = [TU]$ consisting of all cyclic permutations of a given deconcatenation of P and Q. For example, the word $P = 1234$ has the usual deconcatenations

$$(R, S) = (1, 234), (12, 34), (123, 4). \tag{13}$$

The operation $P = [RS]$ leads instead to

$$\begin{aligned}(R, S) = &(1, 234), (2, 341), (3, 412), (4, 123), \\ &(12, 34), (23, 41), (34, 12), (41, 23), \\ &(123, 4), (234, 1), (341, 2), (412, 3).\end{aligned} \tag{14}$$

The factor of $1/2$ in Eq. (11) is needed because contributions of the type (R, S) and (S, R) lead to the same one-loop diagrams.

As an example, we will present the off-shell three-point one-loop integrands. There are only two of them, namely, $I_\ell^{\text{1-loop}}(123|123)$ and $I_\ell^{\text{1-loop}}(123|321)$. Using the rules outlined above, we obtain

$$I_\ell^{\text{1-loop}}(123|123) + I_\ell^{\text{1-loop}}(123|321) = \frac{1}{\ell^2 (\ell + k_1)^2 (\ell + k_1 + k_2)^2}, \tag{15}$$

and

$$I_\ell^{\text{1-loop}}(123|321) = -\frac{1}{\ell^2} \left(\frac{1}{s_{12} (\ell + k_1 + k_2)^2} + \frac{1}{k_1^2 (\ell + k_1)^2} + \frac{1}{k_2^2 (\ell + k_2)^2} \right), \tag{16}$$

with normalization $\Phi_{p|q} = \delta_{pq}$. These match the results obtained in, e.g., [32,33].

Yang-Mills theory: Off-shell recursion.—The main difference between the biadjoint model and the Yang-Mills theory is the gauge symmetry. While at tree level, the gauge fixing procedure is trivial, the loop construction naturally involves Faddeev-Popov ghosts. The covariant gauge fixed action can then be cast as

$$S = \int d^d x \mathrm{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + \partial^\mu b (\partial_\mu c - i[A_\mu, c]) \right\}, \tag{17}$$

where ξ is an arbitrary parameter. The gauge field A_μ and the ghost pair (b, c) are Lie algebra valued. The field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, with $A_\mu = A_\mu^a T^a$.

Our next step is to build the generator of tree-level multiparticle currents with off-shell external legs, just like in the biadjoint case. Even though ghosts do not appear in classical configurations, they will be treated for now as scalars with the wrong statistics. The equations of motion derived from (17) are given by

$$\Box A_\mu = (1 - 1/\xi) \partial_\mu (\partial_\nu A^\nu) - i[A^\nu, F_{\mu\nu}] + i\partial_\nu [A^\nu, A_\mu] - i\{\partial_\mu b, c\}, \tag{18a}$$

$$\Box b = i[A_\mu, \partial^\mu b], \tag{18b}$$

$$\Box c = i[A_\mu, \partial^\mu c] + i[\partial^\mu A_\mu, c]. \tag{18c}$$

Note that the solutions of (18) match classical Yang-Mills solutions when $\partial^\mu A_\mu = b = c = 0$. Multiparticle solutions are obtained via the *Ansatz*

$$A_\mu = \sum_P \mathcal{A}_{P\mu} e^{ik_P \cdot x} T^{a_P}, \tag{19a}$$

$$b = \sum_P b_P e^{ik_P \cdot x} T^{a_P}, \tag{19b}$$

$$c = \sum_P c_P e^{ik_P \cdot x} T^{a_P}. \tag{19c}$$

The currents $\mathcal{A}_P^\mu$ reduce to ordinary vector polarizations ϵ_p^μ for one-lettered words. The multiparticle *Ansatz* can then be plugged back into (18), leading to the following recursions:

$$\left[\eta_{\mu\nu}s_P + \frac{(1-\xi)}{\xi}k_{P\mu}k_{P\nu}\right]\mathcal{A}_P^\nu = \sum_{P=QR}[k_{R\mu}b_Rc_Q + \mathcal{A}_Q^\nu\mathcal{A}_R^\rho(k_{P\nu}\eta_{\mu\rho} + k_{R\nu}\eta_{\mu\rho} + k_{Q\mu}\eta_{\nu\rho}) - (Q\leftrightarrow R)] + \sum_{P=QRS}[\mathcal{A}_Q^\nu\mathcal{A}_R^\rho\mathcal{A}_S^\sigma(\eta_{\nu\sigma}\eta_{\mu\rho} - \eta_{\nu\rho}\eta_{\mu\sigma}) + (Q\leftrightarrow S)], \tag{20a}$$

$$b_P = -\frac{1}{s_P}\sum_{P=QR}[b_Q(k_Q\cdot\mathcal{A}_R) - (Q\leftrightarrow R)], \tag{20b}$$

$$c_P = -\frac{1}{s_P}\sum_{P=QR}[c_Q(k_P\cdot\mathcal{A}_R) - (Q\leftrightarrow R)]. \tag{20c}$$

The sum over $P = QRS$ denotes deconcatenations of the word P into Q, R, and S.

Following the Faddeev-Popov procedure, all *physical* single-particle polarizations are transversal ($k_p\cdot\mathcal{A}_p = 0$). Therefore, $\mathcal{A}_P^\mu$ is identified with Berends-Giele currents and can then be used to compute color-ordered amplitudes via the usual prescription [8]. Just like in the biadjoint case, we can determine a generator of off-shell trees by dropping the on-shell condition for the single-particle states, therefore solving the equations of motion (18) only at the multiparticle level. For instance,

$$\partial^\nu F_{\mu\nu} - \frac{1}{\xi}\partial^\mu(\partial^\nu A_\nu) - i[A^\nu, F_{\mu\nu}] = \sum_p\left[k_p^2\mathcal{A}_{p\mu} + \frac{(1-\xi)}{\xi}k_{p\mu}(k_p\cdot\mathcal{A}_p)\right]e^{ik_p\cdot x}T^{a_p}, \tag{21}$$

where the sum is now restricted to single-particle states. Interestingly, this is enough to show that $\mathcal{A}_P^\mu$ still satisfies the shuffle identity $\mathcal{A}_{Q\sqcup\!\sqcup R}^\mu = 0$, with $Q\sqcup\!\sqcup R$ yielding the sum over all possible shuffles between the words Q and R. At tree level it leads to the Kleiss-Kuijf relations [34]. At one loop, when we sew two legs and sum over specific cyclic permutations of the remaining single-particle labels, the shuffle identity indirectly leads to the Bern-Dixon-Dunbar-Kosower (BDDK) relations [35].

Yang-Mills model: Gluon loop.—From now on we will work with $\xi = 1$, which helps to implement the recursion (20) through a scalarlike propagator $1/s_P$.

Toward the one-loop construction, let us consider the word $\hat{P} = lP$, explicitly factorizing the single-particle label l, with associated polarization ϵ_l^μ and momentum k_l^μ. In this case, we define $\mathcal{A}_{lP\mu} = \epsilon_l^\nu\mathcal{J}_{P\mu\nu}$, with

$$s_{lP}\mathcal{J}_{P\mu\nu} = \mathcal{A}_{P\rho}[\delta_\mu^\rho(k_{lP}+k_P)_\nu + \delta_\nu^\rho(k_l - k_P)_\mu - \eta_{\mu\nu}(k_l + k_{lP})^\rho] + \sum_{P=QR}(2\delta_\mu^\rho\delta_\nu^\sigma - \delta_\mu^\sigma\delta_\nu^\rho - \eta_{\mu\nu}\eta^{\rho\sigma})\mathcal{A}_{Q\rho}\mathcal{A}_{R\sigma} + \sum_{P=QR}[\delta_\mu^\sigma(k_{lP}+k_R)^\rho - \delta_\mu^\rho(k_{lP}+k_{lQ})^\sigma + \eta^{\rho\sigma}(k_l + k_Q - k_R)_\mu]\mathcal{J}_{Q\rho\nu}\mathcal{A}_{R\sigma} + (2\delta_\gamma^\rho\delta_\mu^\sigma - \eta^{\rho\sigma}\eta_{\mu\gamma} - \delta_\gamma^\sigma\delta_\mu^\rho)\sum_{P=QRS}\mathcal{J}_{Q\rho\nu}\mathcal{A}_{R\sigma}\mathcal{A}_S^\gamma. \tag{22}$$

This is but a recasting of Eq. (20a) that singles out the particle l, though physically meaningful: the current $\mathcal{J}_{P\mu\nu}$ is the one-loop preintegrand for a gluon loop.

Let us first examine the following object:

$$A(l,P,n) = \lim_{k_{lPn}\to 0}s_{lP}(\epsilon_l^\nu\mathcal{J}_{P\mu\nu})\epsilon_n^\mu, \tag{23}$$

where ϵ_n^μ is the polarization of an off-shell leg with momentum k_n^μ. The analogy with (10) is clear. The sewing procedure is simply $\epsilon_l^\mu\epsilon_n^\nu \to \eta^{\mu\nu}/k_l^2$, with $k_l = -k_n^\mu$, yielding a look-alike one-loop integrand $I^{\text{1-loop}}(P) \approx \eta^{\mu\nu}\mathcal{J}_{P\mu\nu}$ for a single-trace color-ordered correlator. Once more, the issue with this construction is that the current $\mathcal{J}_{P\mu\nu}$ is no longer cyclic in the word P, so its cyclic completion has to be introduced by hand.

As in the biadjoint case, we take the loop momentum to be $\ell^\mu \equiv k_l^\mu$, with $k_P^\mu = 0$. If we explicitly remove tadpole contributions, given by the first line in (22), the one-loop integrand can be cast as

$$I_{\text{gluon}}^{\text{1-loop}}(P;\ell) \equiv \eta^{\mu\nu}\tilde{\mathcal{J}}_{P\mu\nu}(\ell), \tag{24}$$

with

$$\begin{aligned}\tilde{\mathcal{J}}_{P\mu\nu} = {}& \frac{1}{2\ell^2}\sum_{P=[QR]}\mathcal{J}_{Q\rho\mu}\mathcal{A}_{R\sigma}[\delta^\sigma_\nu(k_R+\ell)^\rho \\ & -\delta^\rho_\nu(k_Q+2\ell)^\sigma+\eta^{\rho\sigma}(2k_Q+\ell)_\nu] \\ & +\frac{1}{\ell^2}\sum_{P=[QRS]}\mathcal{J}_{Q\rho\mu}\mathcal{A}_{R\sigma}\mathcal{A}_{S\gamma} \\ & \times(2\delta^\sigma_\nu\eta^{\gamma\rho}-\eta^{\sigma\gamma}\delta^\rho_\nu-\delta^\gamma_\nu\eta^{\rho\sigma}). \end{aligned} \tag{25}$$

The sum over $P=[QR]$ is analogous to the biadjoint case. In particular, the factor of $1/2$ accounts for the equivalent contributions mentioned after Eq. (14). The sum over $P=[QRS]$ is simpler, related to quartic vertices in Yang-Mills theory: the deconcatenations of all cyclic permutations in P are inequivalent and have to be included.

Because of the color structure, as in the biadjoint construction, (24) reproduces each of the diagrams contributing to the one-loop off-shell integrand without repetitions. The only subtlety missed by (24) is the automorphism of the two-point integrand with $P=12$ and $P=21$, which has to be taken into account separately. It corresponds to the color-stripped one-loop bubble diagram with external moment $k_2^\mu=-k_1^\mu=k^\mu$. From (22) and (25), we obtain

$$I^{\text{1-loop}}_{\text{gluon}}(12;\ell)=\frac{1}{2}\frac{1}{\ell^2(\ell-k)^2}N_{\mu\nu}\epsilon_1^\mu\epsilon_2^\nu, \tag{26}$$

with

$$\begin{aligned}N^{\mu\nu}={}&[\delta^\mu_\rho(\ell-2k)_\sigma+\eta_{\rho\sigma}(k-2\ell)^\mu+\delta^\mu_\sigma(\ell+k)_\rho] \\ &\times[\eta^{\nu\sigma}(\ell+k)^\rho+\eta^{\rho\sigma}(k-2\ell)^\nu+\eta^{\nu\rho}(\ell-2k)^\sigma],\end{aligned} \tag{27}$$

matching the textbook computation using Feynman diagrams. The extra factor $1/2$ cancels the automorphism over counting.

We have also checked that Eq. (24) reproduces the three- and four-point one-loop integrands obtained using Feynman rules. These are of course only consistency checks, since our construction goes far beyond. Just like at tree level, the recursive implementation of (20a) and (22) greatly simplifies the involved computational work and can be easily implemented via commonly available software for symbolic computation.

Yang-Mills model: Ghost loop.—The steps for the definition of a ghost loop are very similar to the ones taken before. In this case, we have to consider off-shell multiparticle currents with one ghost external leg labeled by l, either b_l or c_l, such that $b_{lP}=b_l\mathcal{B}_P$ and $c_{lP}=c_l\mathcal{C}_P$, with

$$\mathcal{B}_P=-\frac{1}{s_{lP}}\left[(k_l\cdot\mathcal{A}_P)+\sum_{P=QR}\mathcal{B}_Q(k_{lQ}\cdot\mathcal{A}_R)\right], \tag{28a}$$

$$\mathcal{C}_P=-\frac{1}{s_{lP}}\left[(k_{lP}\cdot\mathcal{A}_P)+\sum_{P=QR}\mathcal{C}_Q(k_{lP}\cdot\mathcal{A}_R)\right]. \tag{28b}$$

The currents $\mathcal{B}_P$ and $\mathcal{C}_P$ involve only gluons, since the ghost polarization has been explicitly stripped off.

We then define the one-loop integrands

$$\tilde{\mathcal{B}}_P(\ell)=-\frac{1}{\ell^2}\sum_{P=[QR]}\mathcal{B}_Q(k_{lQ}\cdot\mathcal{A}_R), \tag{29a}$$

$$\tilde{\mathcal{C}}_P(\ell)=-\frac{1}{\ell^2}\sum_{P=[QR]}\mathcal{C}_Q(k_l\cdot\mathcal{A}_R), \tag{29b}$$

where $k_P=0$, and the tadpole contributions have been removed. The cyclic completion is being enforced in the sums following the same recipe of the pure gluon case, yielding the full diagrammatic expansion without redundant contributions. Note that $\tilde{\mathcal{B}}_P=\tilde{\mathcal{C}}_P$ when $(k_Q\mathcal{A}_Q)=0$, i.e., when the external gluons are on shell.

The full one-loop integrand with off-shell external gluons is

$$I^{\text{1-loop}}_\ell(P)=I^{\text{1-loop}}_{\text{gluon}}(P;\ell)-\tilde{\mathcal{C}}_P(\ell), \tag{30}$$

where the ghost contribution appears with a minus sign (fermionic loop).

As an example, we present the one-loop gluon self-energy,

$$\Sigma(k)=\int d^d\ell\left(\frac{1}{2}\eta^{\mu\nu}\tilde{\mathcal{J}}_{12\mu\nu}-\tilde{\mathcal{C}}_{12}\right)\equiv\epsilon_1^\mu\epsilon_2^\nu\Pi_{\mu\nu}(k), \tag{31}$$

where $k\equiv k_2=-k_1$, $\Pi_{\mu\nu}=\Pi(k^2)(k^2\eta_{\mu\nu}-k_\mu k_\nu)$, and

$$\begin{aligned}\Pi(k^2)={}&\frac{i}{2}\pi^{d/2}(-k^2)^{d/2-2}(5-7d)\\ &\times\frac{\Gamma(d/2)^2\Gamma(1-d/2)}{\Gamma(d)}.\end{aligned} \tag{32}$$

The loop integral was performed via an analytic continuation from $k^2<0$ and the dimensional regularization is implicit. The poles of $\Gamma(1-d/2)$ encode infrared as well as ultraviolet divergences.

Final remarks.—We have established here a robust framework for computing one-loop off-shell integrands in color-ordered theories from classical equations of motion. The main ingredient in this construction is the observation that multiparticle solutions can be viewed as a generator of off-shell trees in a given field theory. Because of the color structure, it is straightforward to identify the cyclic completion necessary to obtain the full one-loop integrand.

By construction, the one-loop integrands in (11) for the biadjoint scalar and in (30) for Yang-Mills theory lead to single-trace partial amplitudes, which we denote by $A_{n;0}$ (with $n=|P|$). Therefore, the full one-loop amplitudes in these theories are more conveniently represented by the Del

Duca–Dixon-Maltoni color decomposition [36]. For instance, in Yang-Mills theory, we have

$$A_{\text{tot}}^{\text{1-loop}} = \sum_{\sigma \in S_{n-1}/\mathcal{R}} c_n(\sigma) A_{n;0}(\sigma_1, \ldots, \sigma_n), \tag{33}$$

where σ is the color order (i.e., the word P in our formulas), and $c_n(\sigma)$ is the color basis defined by nested commutators of the group generators,

$$c_n(\sigma) \equiv \mathrm{Tr}(T^a[T^{\sigma_1}, [\ldots, [T^{\sigma_{n-1}}, [T^{\sigma_n}, T^a]]\ldots]]), \tag{34}$$

with shorthand $T^\sigma = T^{a_\sigma}$. In the sum, S_{n-1} denotes permutations of $(n-1)$ legs, and $\mathcal{R}$ denotes reflection. Although (33) is expanded in a single-trace basis, it encodes also the double-trace (nonplanar) contributions, which can be determined via the BDDK relations (see, e.g., [37] for a nice summary of different color decompositions).

The recursive character of the one-loop preintegrands (9), (22), and (28) represent an objective simplification over the traditional diagrammatic approach, since they can be algorithmically implemented without extra effort. More than that, their present form enables a transparent identification of the corresponding diagrams. For example, the external leg bubbles can be read off from (25) through the currents $\mathcal{A}_{R\mu}$ with $|R| = |P| - 1$. And finally, the one-loop preintegrands can be directly used as building blocks for higher-loop integrands via different sewings, since all external legs are off shell.

An interesting feature in our proposal is that *a priori* no Lagrangian is required. Therefore, we can compute the one-loop off-shell scattering of field theories that are known only at the level of equations of motion: for example, six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory, which is supposed to describe the low energy limit of $M5$-branes [38,39]. See also [40–42] for more details on the quantization of non-Lagrangian theories. More generally, our results can be applied to a variety of colored theories, including nonlinear sigma model, Chern-Simons, super Yang-Mills, etc. For gauge theories coupled to matter, like QCD, our method becomes slightly more involved since we lose the rigidity of the color structure. Preliminary results on this will be reported in a different work. Perhaps more noteworthy is the fact that we can generalize the results of [27] to off-shell graviton trees, including ghosts, therefore introducing a recursive tool for computing one-loop off-shell integrands in Einstein gravity [43].

We would like to thank Subhroneel Chakrabarti, Arthur Lipstein, Sebastian Mizera, and Christian Schubert for useful discussions and feedback on the manuscript. We would also like to thank Sebastian Mizera for sharing his perturbiner notebook with us. Finally, we would like to thank the anonymous referee for suggesting a simpler presentation for the cyclic completion procedure. H. G. is supported by the Royal Society via a PDRA grant.

[1] L. J. Dixon, Calculating scattering amplitudes efficiently, arXiv:hep-ph/9601359.

[2] Z. Bern, L. J. Dixon, and D. A. Kosower, On-shell methods in perturbative QCD, Ann. Phys. (Amsterdam) **322**, 1587 (2007).

[3] R. Roiban, M. Spradlin, and A. Volovich, Scattering amplitudes in gauge theories: Progress and outlook, J. Phys. A **44**, 450301 (2011).

[4] H. Elvang and Y. t. Huang, Scattering amplitudes, arXiv: 1308.1697.

[5] S. Weinzierl, Tales of 1001 gluons, Phys. Rep. **676**, 1 (2017).

[6] G. Travaglini, A. Brandhuber, P. Dorey, T. McLoughlin, S. Abreu, Z. Bern, N. E. J. Bjerrum-Bohr, J. Blümlein, R. Britto, and J. J. M. Carrasco *et al.*, The SAGEX review on scattering amplitudes, J. Phys. A **55**, 443001 (2022).

[7] C. Schubert, Perturbative quantum field theory in the string inspired formalism, Phys. Rep. **355**, 73 (2001).

[8] F. A. Berends and W. T. Giele, Recursive calculations for processes with n gluons, Nucl. Phys. **B306**, 759 (1988).

[9] G. Mahlon, Multigluon helicity amplitudes involving a quark loop, Phys. Rev. D **49**, 4438 (1994).

[10] K. Wu and Y. J. Du, Off-shell extended graphic rule and the expansion of Berends-Giele currents in Yang-Mills theory, J. High Energy Phys. 01 (2022) 162.

[11] A. van Hameren, Multi-gluon one-loop amplitudes using tensor integrals, J. High Energy Phys. 07 (2009) 088.

[12] S. Actis, A. Denner, L. Hofer, A. Scharf, and S. Uccirati, Recursive generation of one-loop amplitudes in the standard model, J. High Energy Phys. 04 (2013) 037.

[13] In principle, one can define multiparticle solutions to a quantum equation of motion (Dyson-Schwinger equation), as in [14], but a practical implementation is naturally more involved than their classical counterpart.

[14] K. Lee, Quantum off-shell recursion relation, J. High Energy Phys. 05 (2022) 051.

[15] A. A. Rosly and K. G. Selivanov, On amplitudes in the self-dual sector of Yang-Mills theory, Phys. Lett. B **399**, 135 (1997).

[16] A. A. Rosly and K. G. Selivanov, Gravitational SD perturbiner, arXiv:hep-th/9710196.

[17] C. R. Mafra and O. Schlotterer, Solution to the nonlinear field equations of ten dimensional supersymmetric Yang-Mills theory, Phys. Rev. D **92**, 066001 (2015).

[18] S. Lee, C. R. Mafra, and O. Schlotterer, Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality, J. High Energy Phys. 03 (2016) 090.

[19] C. R. Mafra and O. Schlotterer, Berends-Giele recursions and the BCJ duality in superspace and components, J. High Energy Phys. 03 (2016) 097.

[20] C. R. Mafra, Berends-Giele recursion for double-color-ordered amplitudes, J. High Energy Phys. 07 (2016) 080.

[21] C. R. Mafra and O. Schlotterer, Non-Abelian Z-theory: Berends-Giele recursion for the α'-expansion of disk integrals, J. High Energy Phys. 01 (2017) 031.

[22] S. Mizera and B. Skrzypek, Perturbiner methods for effective field theories and the double copy, J. High Energy Phys. 10 (2018) 018.

[23] L. M. Garozzo, L. Queimada, and O. Schlotterer, Berends-Giele currents in Bern-Carrasco-Johansson gauge for

F^3- and F^4-deformed Yang-Mills amplitudes, J. High Energy Phys. 02 (2019) 078.
[24] C. Lopez-Arcos and A. Q. Vélez, L_∞-algebras and the perturbiner expansion, J. High Energy Phys. 11 (2019) 010.
[25] H. Gomez, R. L. Jusinskas, C. Lopez-Arcos, and A. Q. Velez, The L_∞ structure of gauge theories with matter, J. High Energy Phys. 02 (2021) 093.
[26] M. Guillen, H. Johansson, R. Lipinski Jusinskas, and O. Schlotterer, Scattering Massive String Resonances through Field-Theory Methods, Phys. Rev. Lett. **127**, 051601 (2021).
[27] H. Gomez and R. Lipinski Jusinskas, Multiparticle Solutions to Einstein's Equations, Phys. Rev. Lett. **127**, 181603 (2021).
[28] M. Ben-Shahar and M. Guillen, 10D super-Yang-Mills scattering amplitudes from its pure spinor action, J. High Energy Phys. 12 (2021) 014.
[29] V. G. Escudero, C. Lopez-Arcos, and A. Q. Velez, Homotopy double copy and the Kawai-Lewellen-Tye relations for the non-Abelian and tensor Navier-Stokes equations, arXiv:2201.06047.
[30] K. Cho, K. Kim, and K. Lee, The off-shell recursion for gravity and the classical double copy for currents, J. High Energy Phys. 01 (2022) 186.
[31] F. Cachazo, S. He, and E. Y. Yuan, Scattering of massless particles: Scalars, gluons and gravitons, J. High Energy Phys. 07 (2014) 033.
[32] S. He and E. Y. Yuan, One-loop scattering equations and amplitudes from forward limit, Phys. Rev. D **92**, 105004 (2015).
[33] H. Gomez, C. Lopez-Arcos, and P. Talavera, One-loop Parke-Taylor factors for quadratic propagators from massless scattering equations, J. High Energy Phys. 10 (2017) 175.
[34] R. Kleiss and H. Kuijf, Multi-gluon cross-sections and five jet production at hadron colliders, Nucl. Phys. **B312**, 616 (1989).
[35] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, One loop n point gauge theory amplitudes, unitarity and collinear limits, Nucl. Phys. **B425**, 217 (1994).
[36] V. Del Duca, L. J. Dixon, and F. Maltoni, New color decompositions for gauge amplitudes at tree and loop level, Nucl. Phys. **B571**, 51 (2000).
[37] Y. J. Du, B. Feng, and C. H. Fu, Dual-color decompositions at one-loop level in Yang-Mills theory, J. High Energy Phys. 06 (2014) 157.
[38] P. Claus, R. Kallosh, and A. Van Proeyen, M five-brane and superconformal (0,2) tensor multiplet in six-dimensions, Nucl. Phys. **B518**, 117 (1998).
[39] P. S. Howe, E. Sezgin, and P. C. West, Covariant field equations of the M theory five-brane, Phys. Lett. B **399**, 49 (1997).
[40] S. L. Lyakhovich and A. A. Sharapov, Schwinger-Dyson equation for non-Lagrangian field theory, J. High Energy Phys. 02 (2006) 007.
[41] S. L. Lyakhovich and A. A. Sharapov, Quantizing non-Lagrangian gauge theories: An augmentation method, J. High Energy Phys. 01 (2007) 047.
[42] S. L. Lyakhovich and A. A. Sharapov, Quantization of Donaldson-Uhlenbeck-Yau theory, Phys. Lett. B **656**, 265 (2007).
[43] H. Gomez, R. L. Jusinskas, C. Lopez-Arcos, and A. Q. Velez, One-loop off-shell recursions in Einstein gravity (to be published).