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A càdlàg rough path foundation for robust finance

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Abstract
Using rough path theory, we provide a pathwise foundation for stochastic Itô integration which covers most commonly applied trading strategies and mathematical models of financial markets, including those under Knightian uncertainty. To this end, we introduce the so-called property (RIE) for càdlàg paths, which is shown to imply the existence of a càdlàg rough path and of quadratic variation in the sense of Föllmer. We prove that the corresponding rough integrals exist as limits of left-point Riemann sums along a suitable sequence of partitions. This allows one to treat integrands of non-gradient type and gives access to the powerful stability estimates of rough path theory. Additionally, we verify that (path-dependent) functionally generated trading strategies and Cover's universal portfolio are admissible integrands, and that property (RIE) is satisfied by both (Young) semimartingales and typical price paths.

Keywords Föllmer integration · Model uncertainty · Semimartingale · Pathwise integration · Rough path · Functionally generated portfolios · Universal portfolio

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1 Introduction

A fundamental pillar of mathematical finance is the theory of stochastic integration initiated by K. Itô in the 1940s. Itô's stochastic integration not only allows a well-posedness theory for most probabilistic models of financial markets, but also comes with invaluable properties, such as having an integration by parts formula and chain rule, and that of being a continuous operator (with respect to suitable spaces of random variables) which is essential for virtually all applications. However, despite the elegance and success of Itô integration, it also admits some significant drawbacks from both theoretical and practical perspectives.

The construction of the Itô integral requires one to fix a probability measure a priori and is usually based on a limiting procedure of approximating Riemann sums in probability. While in mathematical finance, the Itô integral usually represents the capital gains process from continuous-time trading in a financial market, it lacks a robust pathwise meaning. That is, the stochastic Itô integral does not have a well-defined value on a given "state of the world", e.g. a realised price trajectory of a liquidly traded asset on a stock exchange. This presents a gap between probabilistic models and their financial interpretation. Addressing the pathwise meaning of stochastic integration has led to a stream of literature beginning with the classical works of Bichteler [7] and Willinger and Taqqu [53]; see also Karandikar [30] and Nutz [41].

The requirement of fixing a probability measure to have access to Itô integration becomes an even more severe obstacle when one wants to develop mathematical finance under model risk—also known as Knightian uncertainty. Starting from the seminal works of Avellaneda et al. [5] and Lyons [38], there has been an enormous and ongoing effort to treat the challenges posed by model risk in mathematical finance, that is, the risk stemming from the possible misspecification of an adopted stochastic model, typically represented by a single fixed probability measure. The majority of the existing robust treatments of financial modelling replace the single probability measure by a family of (potentially singular) probability measures, or even take so-called model-free approaches where no probabilistic structure of the underlying price trajectories is assumed; see for example Hobson [27] for classical lecture notes on robust finance. In particular, the latter model-free approaches often require a purely deterministic integration theory sophisticated enough to handle the irregular sample paths of standard continuous-time financial models and commonly employed functionally generated trading strategies.

In the seminal paper [19], Föllmer provided the first deterministic analogue to stochastic Itô integration which had the desired properties required by financial applications. Indeed, assuming that a càdlàg path $S\colon [0,T] \to \mathbb{R}^d$ possesses a suitable notion of quadratic variation along a sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions of the interval $[0,T]$, Föllmer proved that the limit

$$\int_0^t \mathrm{D}f(S_u)\,\mathrm{d}S_u := \lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}^n} \mathrm{D}f(S_u)(S_{v\wedge t} - S_{u\wedge t}), \qquad t \in [0,T],$$

where $\mathrm{D}f$ denotes the gradient of f, exists for all twice continuously differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$. The resulting pathwise integral $\int_0^t \mathrm{D}f(S_u)\,\mathrm{d}S_u$ is often called the Föllmer integral and has proved to be a valuable tool in various applications in model-free finance; for some recent examples, we refer to Föllmer and Schied [21], Davis et al. [14], Schied et al. [46] and Cuchiero et al. [13]. In fact, even classical Riemann–Stieltjes integration has been successfully used as a substitute for Itô integration in model-free finance; see e.g. Dolinsky and Soner [16] or Hou and Obłój [28].

By now arguably the most general pathwise (stochastic) integration theory is provided by the theory of rough paths, as introduced by Lyons [39], and its recent extension to càdlàg rough paths by Friz and Shekhar [25], Friz and Zhang [26] and Chevyrev and Friz [9]. Rough integration can be viewed as a generalisation of Young integration which is able to handle paths of lower regularity. While rough integration allows one to treat the sample paths of numerous stochastic processes as integrators and offers powerful pathwise stability estimates, it comes with a pitfall from a financial perspective: the rough integral is defined as a limit of so-called compensated Riemann sums and thus apparently does not correspond to the canonical financial interpretation as the capital gains process generated by continuous-time trading. Even worse, choosing a rough path without care might lead to an anticipating integral, corresponding e.g. to Stratonovich integration, thus introducing undesired arbitrage when used as a capital process.

To overcome these issues, we introduce the so-called property (RIE) for a càdlàg path $S\colon [0,T] \to \mathbb{R}^d$ and a sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions of the interval $[0,T]$. This property is very much in the same spirit as Föllmer's assumption of quadratic variation along a sequence of partitions. Indeed, we show that property (RIE) implies the existence of quadratic variation in the sense of Föllmer, and even the existence of a càdlàg rough path $\mathbf{S}$ above S, which loosely speaking corresponds to an "Itô" rough path in a probabilistic setting. Assuming property (RIE), we prove that the corresponding rough integrals exist as limits of left-point Riemann sums along the sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions. This result restores the canonical financial interpretation for rough integration and links it to Föllmer integration for càdlàg paths. Property (RIE) was previously introduced by Perkowski and Prömel [42] for continuous paths, though we emphasise that the present more general càdlàg setting requires quite different techniques compared to the continuous setting of [42].

Given the aforementioned results, a càdlàg path which satisfies property (RIE) permits the path-by-path existence of rough integrals with their desired financial interpretation, and moreover maintains access to their powerful stability results which ensure that the integral is a continuous operator. This appears to be a significant advantage compared to the classical notions of pathwise stochastic integration in Bichteler [7], Willinger and Taqqu [53], Karandikar [30], Nutz [41] which do not come with such stability estimates. In particular, the pathwise stability results of rough path theory allow one to prove a model-free version of the so-called fundamental theorem of derivative trading—see Armstrong et al. [4]—and may be of interest when investigating discretisation errors of continuous-time trading in model-free finance; see Riga [44]. Furthermore, in contrast to Föllmer integration, rough integration allows one to consider general functionally generated integrands $(g(S_t))$, where g is a general (sufficiently smooth) function $g\colon \mathbb{R}^d \to \mathbb{R}^d$ and *not* necessarily the gradient of another vector field $f\colon \mathbb{R}^d \to \mathbb{R}$. For instance, model-free portfolio theory constitutes a research direction in which it is beneficial to consider non-gradient trading strategies; see Allan et al. [1]. Even more generally, rough integration allows one to treat path-dependent functionally generated options in the sense of Dupire [17], and pathwise versions of Cover's universal portfolio, as discussed in Sect. 3.

Of course, it remains to verify that property (RIE) is a reasonable modelling assumption in mathematical finance, in the sense that it is fulfilled almost surely by sample paths of the commonly used probabilistic models of financial markets. Since it seems natural that continuous-time trading takes place when the underlying price process fluctuates, we employ sequences of partitions based on such a "space discretisation". For such sequences of partitions, we show that the sample paths of càdlàg semimartingales almost surely satisfy property (RIE). This result is then extended to so-called Young semimartingales, which are stochastic possesses given by the sum of a càdlàg local martingale and an adapted càdlàg process of finite q-variation for some $q < 2$. Finally, we prove that property (RIE) is satisfied by typical price paths in the sense of Vovk [49], which correspond to a model-free version of "no unbounded profit with bounded risk".

The paper is structured as follows. In Sect. 2, we introduce property (RIE) and verify the properties of the associated rough integration as described above. In Sect. 3, we exhibit functionally generated trading strategies and generalisations thereof which provide valid integrands for rough integration. In Sect. 4, we prove that (Young) semimartingales and typical price paths satisfy property (RIE).

2 Rough integration under property (RIE)

In this section, we develop pathwise integration under property (RIE). We set up the essential ingredients from rough path theory in Sect. 2.2 and show in Sect. 2.3 that paths satisfying (RIE) serve as suitable integrators in mathematical finance. Finally, in Sect. 2.4, we connect property (RIE) with the existence of quadratic variation in the sense of Föllmer.

2.1 Basic notation

Let $(\mathbb{R}^d, |\cdot|)$ denote the standard Euclidean space and $D([0,T];\mathbb{R}^d)$ the space of all càdlàg (i.e., right-continuous with left limits) functions $[0,T] \to \mathbb{R}^d$. A partition $\mathcal{P} = \mathcal{P}([s,t])$ of the interval $[s,t]$ is a finite subset of $[s,t]$ which includes the endpoints s and t, i.e., $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$ for some $N \in \mathbb{N}$. We also identify such a partition with the induced collection of intervals between the successive points of $\mathcal{P}$, i.e., with a slight abuse of notation, $\mathcal{P} = \{[t_i, t_{i+1}] : i = 0, 1, \ldots, N-1\}$. The mesh size of a partition $\mathcal{P}$ is given by $|\mathcal{P}| := \max\{t_{i+1} - t_i : i = 0, \ldots, N-1\}$, and for a partition $\mathcal{P}$ of the interval $[s,t]$ and a subinterval $[u,v] \subseteq [s,t]$, we write $\mathcal{P}([u,v]) := (\mathcal{P} \cup \{u,v\}) \cap [u,v] = \{r \in \mathcal{P} \cup \{u,v\} : u \leq r \leq v\}$ for the restriction of the partition $\mathcal{P}$ to the interval $[u,v]$. A sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions is called *nested* if $\mathcal{P}^n \subseteq \mathcal{P}^{n+1}$ for all $n \in \mathbb{N}$.

Setting $\Delta_{[0,T]} := \{(s,t) \in [0,T]^2 : s \leq t\}$, a *control function* is a function $w \colon \Delta_{[0,T]} \to [0,\infty)$ which is superadditive, i.e., $w(s,u) + w(u,t) \leq w(s,t)$ for $0 \leq s \leq u \leq t \leq T$.

Throughout this section, we fix a finite time interval $[0, T]$ and the dimension $d \in \mathbb{N}$. We also adopt the convention that given a path A defined on $[0, T]$, we write $A_{s,t} := A_t - A_s$ for the increment of A over the interval $[s, t]$. Note, however, that whenever A is a two-parameter function defined on $\Delta_{[0,T]}$, the notation $A_{s,t}$ simply denotes the value of A evaluated at the pair of times $(s, t) \in \Delta_{[0,T]}$.

If A denotes either a path $[0, T] \to E$ or a two-parameter function $\Delta_{[0,T]} \to E$ for some normed vector space E, then for any $p \in [1, \infty)$, the p-variation of A over the interval $[s, t]$ is defined by

$$\|A\|_{p,[s,t]} := \left(\sup_{\mathcal{P}([s,t])} \sum_{[u,v] \in \mathcal{P}([s,t])} |A_{u,v}|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all partitions $\mathcal{P}([s, t])$ of the interval $[s, t] \subseteq [0, T]$. If $\|A\|_{p,[0,T]} < \infty$, then A is said to have *finite p-variation*.

We write $D^p = D^p([0, T]; E)$ for the space of all càdlàg paths $A \colon [0, T] \to E$ of finite p-variation, and similarly $D_2^p = D_2^p(\Delta_{[0,T]}; E)$ for the space of two-parameter functions $A \colon \Delta_{[0,T]} \to E$ of finite p-variation which are such that the maps $s \mapsto A_{s,t}$ for fixed t, and $t \mapsto A_{s,t}$ for fixed s, are both càdlàg. Note that a function A having finite p-variation is equivalent to the existence of a control function w such that $|A_{s,t}|^p \le w(s, t)$ for all $(s, t) \in \Delta_{[0,T]}$. For instance, one may take $w(s, t) = \|A\|_{p,[s,t]}^p$.

2.2 Càdlàg rough path theory and property (RIE)

While rough path theory has by now been well studied in the case of continuous paths, as exhibited in a number of books, notably Friz and Hairer [24], its extension to càdlàg paths appeared only recently, starting with Friz and Shekhar [25]. In this section we mainly rely on results regarding forward integration with respect to càdlàg rough paths as presented in Friz and Zhang [26].

In the following, we fix $p \in (2, 3)$ and $q \ge p$ such that

$$\frac{2}{p} + \frac{1}{q} > 1,$$

and define $r > 1$ by the relation

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

This means in particular that $1 < p/2 \le r < p \le q < \infty$.

Throughout the paper, we use the symbol $\lesssim$ to denote inequality up to a multiplicative constant which depends only on the numbers p, q and r chosen above.

We begin by recalling the definition of a càdlàg rough path as well as the corresponding notion of controlled paths. **In the following**, we write $A \otimes B$ for the tensor product of two vectors $A, B \in \mathbb{R}^d$, i.e., the $d \times d$-matrix with (i, j)-component given by $[A \otimes B]^{ij} = A^i B^j$ for $1 \le i, j \le d$.

Definition 2.1 We say that a triplet $\mathbf{X} = (X, Z, \mathbb{X})$ is a *(càdlàg) rough path* (over $\mathbb{R}^d$) if $X \in D^p([0,T];\mathbb{R}^d)$, $Z \in D^p([0,T];\mathbb{R}^d)$, $\mathbb{X} \in D_2^{p/2}(\Delta_{[0,T]};\mathbb{R}^{d\times d})$ and if Chen's relation

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + Z_{s,u} \otimes X_{u,t} \tag{2.1}$$

holds for all times $0 \leq s \leq u \leq t \leq T$. We denote the space of càdlàg rough paths by $\mathcal{V}^p$. Note that p is fixed throughout.

The reader is encouraged to check that given càdlàg paths X and Z of bounded variation, setting $\mathbb{X}_{s,t} = \int_s^t Z_{s,u} \otimes \mathrm{d}X_u = \int_s^t Z_u \otimes \mathrm{d}X_u - Z_s \otimes X_{s,t}$ for all $(s,t) \in \Delta_{[0,T]}$, with the integral defined as a limit of left-point Riemann sums, gives a rough path. Although the integral $\int_s^t Z_{s,u} \otimes \mathrm{d}X_u$ is not in general well defined when X and Z are not of bounded variation, given a rough path $(X, Z, \mathbb{X})$, we may think of $\mathbb{X}$ as postulating a "candidate" for the value of such integrals.

Remark 2.2 The definition of rough paths introduced above looks slightly different from the standard definition in which one takes $X = Z$. Our definition is slightly more general, but the corresponding theory works in exactly the same way and turns out to be more convenient in the context of property (RIE) as we shall see later.

More precisely, the matrix $\mathbb{X}_{s,t}$ will later represent for us the (a priori ill-defined) 'integral' $\int_s^t S_{s,u} \otimes \mathrm{d}S_u$ which will be defined as the limit as $n \to \infty$ of the Riemann sums $(\int_s^t S^n_{s,u} \otimes \mathrm{d}S_u)_{n\in\mathbb{N}}$ appearing in property (RIE) below. In the continuous-path setting of Perkowski and Prömel [42], a linear interpolation is used to provide a continuous approximation of S^n, leading to a Stratonovich-type integral in the limit, which is subsequently converted back into an Itô-type integral. Thanks to the recently developed theory of càdlàg rough paths, we can use here a more direct argument which avoids this detour. This means working directly with the integral $\int_s^t S^n_{s,u} \otimes \mathrm{d}S_u$, which corresponds to taking $X = S$ and $Z = S^n$ in Definition 2.1, thus requiring $X \neq Z$.

For two rough paths $\mathbf{X} = (X, Z, \mathbb{X})$ and $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Z}, \tilde{\mathbb{X}})$, we use the notation

$$\|\mathbf{X}\|_{p,[s,t]} := \|X\|_{p,[s,t]} + \|Z\|_{p,[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]}$$

and define the pseudometric

$$\|\mathbf{X};\tilde{\mathbf{X}}\|_{p,[s,t]} := \|X - \tilde{X}\|_{p,[s,t]} + \|Z - \tilde{Z}\|_{p,[s,t]} + \|\mathbb{X} - \tilde{\mathbb{X}}\|_{\frac{p}{2},[s,t]},$$

for $[s,t] \subseteq [0,T]$.

In the following, we write $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$ for the space of linear maps $\mathbb{R}^d \to \mathbb{R}^d$.

Definition 2.3 Let $Z \in D^p([0,T];\mathbb{R}^d)$. A pair (F, F') is called a *controlled path* (with respect to Z) if we have $F \in D^p([0,T];\mathbb{R}^d)$, $F' \in D^q([0,T];\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d))$ and $R^F \in D_2^r(\Delta_{[0,T]};\mathbb{R}^d)$, where the remainder R^F is defined implicitly by the relation

$$F_{s,t} = F'_s Z_{s,t} + R^F_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

We refer to F' as the *Gubinelli derivative* of F (with respect to Z) and denote the space of such controlled paths by $\mathcal{V}_Z^{q,r}$.

Given a path $Z \in D^p([0,T];\mathbb{R}^d)$, the space $\mathcal{V}_Z^{q,r}$ of controlled paths becomes a Banach space when equipped with the norm $(F, F') \mapsto |F_0| + \|F, F'\|_{\mathcal{V}_Z^{q,r},[0,T]}$, where

$$\|F, F'\|_{\mathcal{V}_Z^{q,r},[0,T]} := |F'_0| + \|F'\|_{q,[0,T]} + \|R^F\|_{r,[0,T]}.$$

With the concepts of rough paths and controlled paths at hand, we are ready to introduce rough integration. The following result is a straightforward extension of Allan et al. [1, Lemma 2.6] and its proof follows almost verbatim.

Proposition 2.4 *Let $\mathbf{X} = (X, Z, \mathbb{X}) \in \mathcal{V}^p$ be a càdlàg rough path. Let $(F, F') \in \mathcal{V}_Z^{q,r}$ and $(G, G') \in \mathcal{V}_X^{q,r}$ be controlled paths with respect to Z and X, respectively, with remainders R^F and R^G. Then for each $t \in [0,T]$, the limit*

$$\int_0^t F_u \, \mathrm{d}G_u := \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} (F_u G_{u,v} + F'_u G'_u \mathbb{X}_{u,v}) \tag{2.2}$$

exists along every sequence of partitions $\mathcal{P}$ of the interval $[0,t]$ with mesh size $|\mathcal{P}|$ tending to zero, and the limit does not depend on the choice of sequence of partitions. We call this limit the rough integral *of (F, F') against (G, G') (relative to the rough path $\mathbf{X}$). It moreover comes with the estimate*

$$\begin{aligned}
&\left| \int_s^t F_u \, \mathrm{d}G_u - F_s G_{s,t} - F'_s G'_s \mathbb{X}_{s,t} \right| \\
&\leq C\big(\|F'\|_\infty (\|G'\|^q_{q,[s,t]} + \|Z\|^p_{p,[s,t]})^{\frac{1}{r}} \|X\|_{p,[s,t]} + \|F\|_{p,[s,t]} \|R^G\|_{r,[s,t]} \\
&\qquad + \|R^F\|_{r,[s,t]} \|G'\|_\infty \|X\|_{p,[s,t]} + \|F'G'\|_{q,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]}\big)
\end{aligned} \tag{2.3}$$

for all $(s,t) \in \Delta_{[0,T]}$, where the constant C depends only on p, q and r.

We interpret the product of vectors $F_u G_{u,v}$ appearing in (2.2) as the Euclidean inner product, and the rough integral itself is real-valued. However, we remark that in general the path F may take values in a space of linear maps, and rough integrals may be defined to take values in a space of vectors, or even an infinite-dimensional Banach space.

Remark 2.5 In the case when $G = X$ (so that G' is the identity map and $R^G = 0$), the integral defined in Proposition 2.4 reduces to the more classical notion of the rough integral of the controlled path (F, F') against the rough path $\mathbf{X}$, given by

$$\int_0^t F_u \, \mathrm{d}\mathbf{X}_u = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F_u X_{u,v} + F'_u \mathbb{X}_{u,v}.$$

Remark 2.6 Combining the estimate in (2.3) with the relation $G_{s,t} = G'_s X_{s,t} + R^G_{s,t}$, it follows that the rough integral $\int_0^\cdot F_u \,\mathrm{d}G_u$ is itself a controlled path with respect to X with Gubinelli derivative FG', so that $(\int_0^\cdot F_u \,\mathrm{d}G_u, FG') \in \mathcal{V}^{q,r}_X$.

Notice that the construction of the rough integral in (2.2) is based on so-called compensated Riemann sums $\sum_{[u,v]\in\mathcal{P}}(F_u G_{u,v} + F'_u G'_u \mathbb{X}_{u,v})$ instead of classical left-point Riemann sums $\sum_{[u,v]\in\mathcal{P}} F_u G_{u,v}$. While the classical Riemann sums come with a natural interpretation as capital gains processes in the context of mathematical finance, the financial interpretation of compensated Riemann sums is by no means obvious. However, we show later in Theorem 2.15 that the integral can be given a natural formulation as a limit of suitable left-point Riemann sums. Moreover, an advantage of rough integration is that it provides rather powerful stability estimates, for instance as presented in the next result.

Proposition 2.7 *Let $\mathbf{X} = (X, Z, \mathbb{X})$, $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Z}, \tilde{\mathbb{X}}) \in \mathcal{V}^p$ be càdlàg rough paths and $(F, F') \in \mathcal{V}^{q,r}_Z$, $(\tilde{F}, \tilde{F}') \in \mathcal{V}^{q,r}_{\tilde{Z}}$ controlled paths with remainders R^F and $R^{\tilde{F}}$, respectively. Then*:

(i) *We have the estimate*

$$\begin{aligned}
&\left\| \int_0^\cdot F_u \,\mathrm{d}\mathbf{X}_u - \int_0^\cdot \tilde{F}_u \,\mathrm{d}\tilde{\mathbf{X}}_u \right\|_{p;[0,T]} \\
&\leq C\big((|\tilde{F}_0| + \|\tilde{F}, \tilde{F}'\|_{\mathcal{V}^{q,r}_{\tilde{Z}},[0,T]})(1 + \|X\|_{p;[0,T]} + \|\tilde{Z}\|_{p;[0,T]})\|\mathbf{X}; \tilde{\mathbf{X}}\|_{p;[0,T]} \\
&\qquad + (|F_0 - \tilde{F}_0| + |F'_0 - \tilde{F}'_0| + \|F' - \tilde{F}'\|_{q;[0,T]} + \|R^F - R^{\tilde{F}}\|_{r;[0,T]}) \\
&\qquad\quad \times (1 + \|Z\|_{p;[0,T]})\|\mathbf{X}\|_{p;[0,T]}\big),
\end{aligned}$$

where the constant C depends only on p, q and r.

(ii) *Let $(G, G') \in \mathcal{V}^{q,r}_X$ and $(\tilde{G}, \tilde{G}') \in \mathcal{V}^{q,r}_{\tilde{X}}$ also be controlled paths with remainders R^G and $R^{\tilde{G}}$, respectively. Let $M > 0$ be a constant such that*

$$\begin{aligned}
&\|\mathbf{X}\|_{p,[0,T]}, |F_0|, \|F, F'\|_{\mathcal{V}^{q,r}_Z,[0,T]}, \|G, G'\|_{\mathcal{V}^{q,r}_X,[0,T]} \leq M, \\
&\|\tilde{\mathbf{X}}\|_{p,[0,T]}, |\tilde{F}_0|, \|\tilde{F}, \tilde{F}'\|_{\mathcal{V}^{q,r}_{\tilde{Z}},[0,T]}, \|\tilde{G}, \tilde{G}'\|_{\mathcal{V}^{q,r}_{\tilde{X}},[0,T]} \leq M.
\end{aligned}$$

We then have the estimate

$$\begin{aligned}
&\left\| \int_0^\cdot F_u \,\mathrm{d}G_u - \int_0^\cdot \tilde{F}_u \,\mathrm{d}\tilde{G}_u \right\|_{p,[0,T]} \\
&\leq C\big(|F_0 - \tilde{F}_0| + |F'_0 - \tilde{F}'_0| + \|F' - \tilde{F}'\|_{q,[0,T]} + \|R^F - R^{\tilde{F}}\|_{r,[0,T]} \\
&\qquad + |G'_0 - \tilde{G}'_0| + \|G' - \tilde{G}'\|_{q,[0,T]} + \|R^G - R^{\tilde{G}}\|_{r,[0,T]} + \|\mathbf{X}; \tilde{\mathbf{X}}\|_{p,[0,T]}\big),
\end{aligned}$$

where the new constant C depends on p, q, r and M.

Proof We present here only the proof of (ii) since the proof of (i) follows almost verbatim. Here, the multiplicative constant implied by the symbol $\lesssim$ is allowed to depend on the numbers p, q and r as usual, and additionally on the constant M. Following the proof of Friz and Zhang [26, Lemma 3.4], one deduces in our more general setting the estimates

$$\begin{aligned}\|F - \tilde{F}\|_{p,[0,T]} \lesssim |F'_0 - \tilde{F}'_0| + \|F' - \tilde{F}'\|_{q,[0,T]} + \|R^F - R^{\tilde{F}}\|_{r,[0,T]} \\ + \|Z - \tilde{Z}\|_{p,[0,T]} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&\|R^{\int_0^\cdot F_u \,\mathrm{d}G_u} - R^{\int_0^\cdot \tilde{F}_u \,\mathrm{d}\tilde{G}_u}\|_{r,[0,T]} \\ &\lesssim |F_0 - \tilde{F}_0| + \|F - \tilde{F}\|_{p,[0,T]} + |F'_0 - \tilde{F}'_0| + \|F' - \tilde{F}'\|_{q,[0,T]} \\ &\quad + \|R^F - R^{\tilde{F}}\|_{r,[0,T]} + |G'_0 - \tilde{G}'_0| + \|G' - \tilde{G}'\|_{q,[0,T]} + \|R^G - R^{\tilde{G}}\|_{r,[0,T]} \\ &\quad + \|\mathbf{X}; \tilde{\mathbf{X}}\|_{p,[0,T]}. \end{aligned} \tag{2.5}$$

Recalling Remark 2.6, we find by using the controlled path structure of the rough integrals that

$$\begin{aligned}\left\| \int_0^\cdot F_u \,\mathrm{d}G_u - \int_0^\cdot \tilde{F}_u \,\mathrm{d}\tilde{G}_u \right\|_{p,[0,T]} &\lesssim |F_0 - \tilde{F}_0| + \|F - \tilde{F}\|_{p,[0,T]} + |G'_0 - \tilde{G}'_0| \\ &\quad + \|G' - \tilde{G}'\|_{q,[0,T]} + \|X - \tilde{X}\|_{p,[0,T]} \\ &\quad + \|R^{\int_0^\cdot F_u \,\mathrm{d}G_u} - R^{\int_0^\cdot \tilde{F}_u \,\mathrm{d}\tilde{G}_u}\|_{r,[0,T]}. \end{aligned} \tag{2.6}$$

The result then follows upon substituting (2.4) and (2.5) into (2.6). □

In the spirit of Föllmer's [19] assumption of quadratic variation along a sequence of partitions, we introduce the following property.

Property 2.8 (RIE) Let $p \in (2,3)$ and let $\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, be a sequence of nested partitions of the interval $[0,T]$ such that $|\mathcal{P}^n| \to 0$ as $n \to \infty$. For $S \in D([0,T];\mathbb{R}^d)$, we define $S^n \colon [0,T] \to \mathbb{R}^d$ by

$$S_t^n = S_T \mathbf{1}_{\{T\}}(t) + \sum_{k=0}^{N_n - 1} S_{t_k^n} \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t), \qquad t \in [0,T],$$

for each $n \in \mathbb{N}$. We assume that

- the sequence $(S^n)_{n \in \mathbb{N}}$ of paths converges uniformly to S as $n \to \infty$;
- the Riemann sums $\int_0^t S_u^n \otimes \mathrm{d}S_u := \sum_{k=0}^{N_n-1} S_{t_k^n} \otimes S_{t_k^n \wedge t, t_{k+1}^n \wedge t}$ converge uniformly

as $n \to \infty$ to a limit, which we denote by $\int_0^t S_u \otimes \mathrm{d}S_u$, $t \in [0,T]$;

- there exists a control function w such that

$$\sup_{(s,t)\in\Delta_{[0,T]}} \frac{|S_{s,t}|^p}{w(s,t)} + \sup_{n\in\mathbb{N}} \sup_{0\le k<\ell\le N_n} \frac{|\int_{t_k^n}^{t_\ell^n} S_u^n \otimes \mathrm{d}S_u - S_{t_k^n} \otimes S_{t_k^n,t_\ell^n}|^{\frac{p}{2}}}{w(t_k^n, t_\ell^n)} \le 1. \quad (2.7)$$

In (2.7) and hereafter, we adopt the convention that $\frac{0}{0} := 0$.

Definition 2.9 A path $S \in D([0,T];\mathbb{R}^d)$ is said to *satisfy (RIE)* with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$ if p, $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and S together satisfy property (RIE).

The name "RIE" is an abbreviation for "Riemann" as we assume the convergence of the Riemann sums $\int S_u^n \otimes \mathrm{d}S_u$ instead of the discrete quadratic variations as in [19]. Indeed, property (RIE) is a stronger assumption than the existence of quadratic variation in the sense of Föllmer, and it is even enough to allow us to lift S in a canonical way to a rough path (see Lemma 2.13 below), giving us access to the powerful stability results of rough path theory such as those in Proposition 2.7. Moreover, property (RIE) can be verified for most typical stochastic processes in mathematical finance, as we shall see in Sect. 4.

Remark 2.10 We highlight that rather than simply being a property of a path, property (RIE) is actually a property of a path together with a given sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions. Indeed, such a path S will in general not satisfy (RIE) with respect to a different sequence of partitions, and even if it does, the limit of Riemann sums $\int_0^t S_u \otimes \mathrm{d}S_u$ specified in property (RIE) may depend on the choice of sequence of partitions. However, in practice, there is often a natural choice for the sequence of partitions; see Remark 4.2. For clarity, hereafter, whenever we claim that a path satisfies property (RIE), we always make explicit the partition with respect to which the path satisfies (RIE) in the sense of Definition 2.9.

Remark 2.11 In Proposition 2.14 below, it is actually shown that it is sufficient in property (RIE) to assume that the sequence $(S^n)_{n\in\mathbb{N}}$ converges only pointwise to S, since the uniformity of this convergence then immediately follows.

Next we verify that property (RIE) ensures the existence of a càdlàg rough path. For this purpose, we consider a suitable approximating sequence for the so-called 'area process', which is represented by $\mathbb{X}$ in Definition 2.1.

Lemma 2.12 *Suppose $S \in D([0,T];\mathbb{R}^d)$ satisfies property* (RIE) *with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$ (as in Definition 2.9). If for each $n \in \mathbb{N}$, we define $A^n\colon \Delta_{[0,T]} \to \mathbb{R}^{d\times d}$ by*

$$A_{s,t}^n := \int_s^t S_{s,u}^n \otimes \mathrm{d}S_u = \int_s^t S_u^n \otimes \mathrm{d}S_u - S_s^n \otimes S_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}, \quad (2.8)$$

where $\int_s^t S_{s,u}^n \otimes \mathrm{d}S_u$ is defined as in property (RIE), *then there exists a constant C depending only on p such that*

$$\|A^n\|_{\frac{p}{2},[0,T]} \le C w(0,T)^{\frac{2}{p}} \qquad \textit{for every } n \in \mathbb{N}. \quad (2.9)$$

Proof Let $n \in \mathbb{N}$ and $(s,t) \in \Delta_{[0,T]}$. If there exists a k such that $t_k^n \leq s < t \leq t_{k+1}^n$, then we simply have $A_{s,t}^n = S_{t_k^n} \otimes S_{s,t} - S_{t_k^n} \otimes S_{s,t} = 0$. Otherwise, let k_0 be the smallest k such that $t_k^n \in (s,t)$ and k_1 the largest such k. It is easy to see that the triplet (S, S^n, A^n) satisfies Chen's relation (2.1), from which it follows that

$$A_{s,t}^n = A_{s,t_{k_0}^n}^n + A_{t_{k_0}^n,t_{k_1}^n}^n + A_{t_{k_1}^n,t}^n + S_{s,t_{k_0}^n}^n \otimes S_{t_{k_0}^n,t_{k_1}^n} + S_{s,t_{k_1}^n}^n \otimes S_{t_{k_1}^n,t}.$$

As we have already observed, we have $A_{s,t_{k_0}^n}^n = A_{t_{k_1}^n,t}^n = 0$. By (2.7), we have

$$|A_{t_{k_0}^n,t_{k_1}^n}^n|^{\frac{p}{2}} \leq w(t_{k_0}^n, t_{k_1}^n) \leq w(t_{k_0-1}^n, t).$$

We estimate the remaining terms as

$$\begin{aligned}
&|S_{s,t_{k_0}^n}^n \otimes S_{t_{k_0}^n,t_{k_1}^n}|^{\frac{p}{2}} + |S_{s,t_{k_1}^n}^n \otimes S_{t_{k_1}^n,t}|^{\frac{p}{2}} \\
&\lesssim |S_{s,t_{k_0}^n}^n|^p + |S_{t_{k_0}^n,t_{k_1}^n}|^p + |S_{s,t_{k_1}^n}^n|^p + |S_{t_{k_1}^n,t}|^p \\
&= |S_{t_{k_0-1}^n,t_{k_0}^n}|^p + |S_{t_{k_0}^n,t_{k_1}^n}|^p + |S_{t_{k_0-1}^n,t_{k_1}^n}|^p + |S_{t_{k_1}^n,t}|^p \\
&\leq w(t_{k_0-1}^n, t_{k_0}^n) + w(t_{k_0}^n, t_{k_1}^n) + w(t_{k_0-1}^n, t_{k_1}^n) + w(t_{k_1}^n, t) \\
&\leq 2w(t_{k_0-1}^n, t)
\end{aligned}$$

so that putting this all together yields the existence of a constant $\tilde{C} > 0$ such that $|A_{s,t}^n|^{\frac{p}{2}} \leq \tilde{C} w(t_{k_0-1}^n, t)$. Taking an arbitrary partition $\mathcal{P}$ of the interval $[0,T]$, it follows that $\sum_{[s,t]\in\mathcal{P}} |A_{s,t}^n|^{\frac{p}{2}} \leq 2\tilde{C} w(0,T)$. We thus conclude that (2.9) holds with $C = (2\tilde{C})^{\frac{2}{p}}$. □

Lemma 2.13 *Suppose that $S \in D([0,T];\mathbb{R}^d)$ satisfies property* (RIE) *with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$. With the natural notation $\int_s^t S_u \otimes \mathrm{d}S_u := \int_0^t S_u \otimes \mathrm{d}S_u - \int_0^s S_u \otimes \mathrm{d}S_u$, we define $A\colon \Delta_{[0,T]} \to \mathbb{R}^{d\times d}$ by*

$$A_{s,t} = \int_s^t S_u \otimes \mathrm{d}S_u - S_s \otimes S_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

Then the triplet $\mathbf{S} = (S, S, A)$ is a càdlàg rough path.

Proof It is straightforward to verify Chen's relation (2.1), i.e., that

$$A_{s,t} = A_{s,u} + A_{u,t} + S_{s,u} \otimes S_{u,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

By property (RIE), we know that $\lim_{n\to\infty} A_{s,t}^n = A_{s,t}$, where the convergence is uniform in (s,t), and thus A is itself càdlàg as a uniform limit of càdlàg functions. By the lower semi-continuity of the $\frac{p}{2}$-variation norm and Lemma 2.12, we have

$$\|A\|_{\frac{p}{2},[0,T]} \leq \liminf_{n\to\infty} \|A^n\|_{\frac{p}{2},[0,T]} \leq C w(0,T)^{\frac{2}{p}} < \infty.$$

It follows that (S, S, A) is a càdlàg rough path. □

2.3 The rough integral as a limit of Riemann sums

While the rough integral in (2.2) is a powerful tool to study various differential equations, it lacks the natural interpretation as the capital gains process in the context of mathematical finance. The aim of this subsection is to restore this interpretation by showing that the rough integral can be obtained as the limit of left-point Riemann sums provided that the integrator satisfies property (RIE). As preparation, we need the following approximation result.

Proposition 2.14 *Let* $\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, *be a sequence of nested partitions with vanishing mesh size, so that* $\mathcal{P}^n \subseteq \mathcal{P}^{n+1}$ *for all* n *and* $|\mathcal{P}^n| \to 0$ *as* $n \to \infty$ (*as in the setting of property* (RIE)). *Let* $F\colon [0,T] \to \mathbb{R}^d$ *be a càdlàg path and define*

$$F_t^n = F_T \mathbf{1}_{\{T\}}(t) + \sum_{k=0}^{N_n-1} F_{t_k^n} \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t), \qquad t \in [0,T]. \tag{2.10}$$

Let

$$J_F := \{t \in (0,T] : F_{t-,t} \neq 0\} \tag{2.11}$$

be the set of jump times of F. *The following are equivalent*:

(i) $J_F \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$.

(ii) *The sequence* $(F^n)_{n\in\mathbb{N}}$ *converges pointwise to* F.

(iii) *The sequence* $(F^n)_{n\in\mathbb{N}}$ *converges uniformly to* F.

Proof We first show that (i) and (ii) are equivalent. To this end, suppose that we have $J_F \subseteq \bigcup_{n\geq 1} \mathcal{P}^n$ and let $t \in (0,T]$. If $t \in J_F$, then there exists $m \geq 1$ such that $t \in \mathcal{P}^n$ for all $n \geq m$. Then we have $F_t^n = F_t$ for all $n \geq m$. If $t \notin J_F$, then $F_{t-} = F_t$, and since $|\mathcal{P}^n| \to 0$, it follows that $F_t^n \to F_{t-} = F_t$ as $n \to \infty$.

Conversely, if there exists $t \in J_F$ with $t \notin \bigcup_{n\geq 1} \mathcal{P}^n$, then $F_t^n \to F_{t-} \neq F_t$ so that $F_t^n \nrightarrow F_t$. This establishes the equivalence of (i) and (ii).

Since (iii) clearly implies (ii), it only remains to show that (ii) implies (iii). By Fraňková [22, Theorem 3.3], it is enough to show that the family of paths $\{F^n : n \geq 1\}$ is equiregulated in the sense of [22, Definition 3.1].

Step 1. Let $t \in (0,T]$ and $\varepsilon > 0$. Since the left limit F_{t-} exists, there exists $\delta > 0$ with $t - \delta > 0$ such that

$$|F_{s,t-}| < \frac{\varepsilon}{2} \qquad \text{for all } s \in (t-\delta, t).$$

Let

$$m = \min\{n \geq 1 : \exists\, k \text{ such that } t_k^n \in (t-\delta, t)\}.$$

Since $|\mathcal{P}^n| \to 0$ as $n \to \infty$, we know that $m < \infty$. Moreover, since the sequence of partitions is nested, we immediately have that for all $n \geq m$, there exists a k such that

$t_k^n \in (t-\delta, t)$. We define

$$u = \min\{t_k^m \in \mathcal{P}^m : t_k^m \in (t-\delta, t)\} = \min\big(\mathcal{P}^m \cap (t-\delta, t)\big)$$

and let $s \in [u, t)$ and $n \geq 1$.

If $n < m$, then there does not exist a k such that $t_k^n \in (t-\delta, t)$, which implies that F^n is constant on the interval $(t-\delta, t)$ and hence $F_s^n = F_{t-}^n$.

Suppose instead that $n \geq m$. Let $i = \max\{k : t_k^n \leq s\}$ and $j = \max\{k : t_k^n < t\}$. By the definition of u, we see that $t_i^n \in [u, t)$ and $t_j^n \in [u, t)$. Then

$$|F_s^n - F_{t-}^n| = |F_{t_i^n} - F_{t_j^n}| \leq |F_{t_i^n, t-}| + |F_{t_j^n, t-}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus we have $|F_s^n - F_{t-}^n| < \varepsilon$ for all $s \in [u, t)$ and all $n \geq 1$.

Step 2. Let $t \in (J_F \cup \{0\}) \setminus \{T\}$ and $\varepsilon > 0$. Since F is right-continuous, there exists a $\delta > 0$ with $t + \delta < T$ such that

$$|F_{t,s}| < \varepsilon \qquad \text{for all } s \in [t, t+\delta).$$

By part (i), we know that $t \in \bigcup_{n\geq 1} \mathcal{P}^n$. Let

$$m = \min\{n \geq 1 : \exists\, k \text{ such that } t_k^n = t\}.$$

Since $t \in \bigcup_{n\geq 1} \mathcal{P}^n$, it is clear that $m < \infty$. We define

$$u = \min\{t_k^m \in \mathcal{P}^m : t_k^m > t\} = \min\big(\mathcal{P}^m \cap (t, T]\big).$$

We then let $v \in (t, u \wedge (t+\delta))$, $s \in (t, v]$, and $n \geq 1$.

If $n < m$, then since $v < u$, there does not exist a k such that $t_k^n \in [t, v]$. Hence F^n is constant on the interval $[t, v]$ so that in particular $F_s^n = F_t^n$.

Suppose instead that $n \geq m$. By the definition of m, there exists a j with $t_j^n = t$. Let $i = \max\{k : t_k^n \leq s\}$. In particular, we then have $t = t_j^n \leq t_i^n \leq s \leq v < t + \delta$ and hence

$$|F_s^n - F_t^n| = |F_{t_i^n} - F_{t_j^n}| = |F_{t, t_i^n}| < \varepsilon.$$

Thus we have $|F_s^n - F_t^n| < \varepsilon$ for all $s \in (t, v]$ and all $n \geq 1$.

Step 3. Let $t \in (0, T) \setminus J_F$ and $\varepsilon > 0$. Since F is continuous at time t, there exists a $\delta > 0$ with $0 < t - \delta$ and $t + \delta < T$ such that

$$|F_{s,t}| < \frac{\varepsilon}{2} \qquad \text{for all } s \in (t-\delta, t+\delta).$$

Let

$$m = \min\{n \geq 1 : \exists\, k \text{ such that } t_k^n \in (t-\delta, t]\}.$$

Since $|\mathcal{P}^n| \to 0$ as $n \to \infty$, we know that $m < \infty$. We define

$$u = \min\{t_k^m \in \mathcal{P}^m : t_k^m > t\} = \min\big(\mathcal{P}^m \cap (t, T]\big).$$

We then let $v \in (t, u \wedge (t+\delta))$, $s \in (t, v]$ and $n \geq 1$.

If $n < m$, then since $v < u$, there does not exist a k such that $t_k^n \in (t, v]$. Hence F^n is constant on the interval $[t, v]$ so that in particular $F_s^n = F_t^n$.

Suppose instead that $n \geq m$. Let $i = \max\{k : t_k^n \leq s\}$ and $j = \max\{k : t_k^n \leq t\}$. Since by the definition of m, there exists at least one k such that $t_k^n \in (t - \delta, t]$, and since $t < s \leq v < t + \delta$, it follows that $t_i^n \in (t - \delta, t + \delta)$ and $t_j^n \in (t - \delta, t]$. Then

$$|F_s^n - F_t^n| = |F_{t_i^n} - F_{t_j^n}| \leq |F_{t_i^n, t}| + |F_{t_j^n, t}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus we have $|F_s^n - F_t^n| < \varepsilon$ for all $s \in (t, v]$ and all $n \geq 1$. It follows that the family of paths $\{F^n : n \geq 1\}$ is indeed equiregulated. $\square$

The next theorem is the main result of this section, stating that the rough integral can be approximated by left-point Riemann sums along a suitable sequence of partitions, in the spirit of Föllmer's pathwise integration.

Theorem 2.15 *Let $q \geq p$ be such that $\frac{2}{p} + \frac{1}{q} > 1$ and $r > 1$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Suppose that $S \in D([0, T]; \mathbb{R}^d)$ satisfies property* (RIE) *with respect to p and $(\mathcal{P}^n)_{n \in \mathbb{N}}$. Let $(F, F') \in \mathcal{V}_S^{q,r}$ and $(G, G') \in \mathcal{V}_S^{q,r}$ be controlled paths with respect to S, and assume that $J_F \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$, where J_F is the set of jump times of F as in* (2.11)*. Then the rough integral of (F, F') against (G, G') relative to the rough path $\mathbf{S} = (S, \mathbb{S}, A)$ as defined in* (2.2) *is given by*

$$\int_0^t F_u \, \mathrm{d}G_u = \lim_{n \to \infty} \sum_{k=0}^{N_n - 1} F_{t_k^n} G_{t_k^n \wedge t, t_{k+1}^n \wedge t},$$

where the convergence is uniform in $t \in [0, T]$.

Proof We recall from Lemma 2.13 that $\mathbf{S} = (S, \mathbb{S}, A)$ is a rough path so that by Proposition 2.4, the rough integral of (F, F') against (G, G') (relative to $\mathbf{S}$) exists. It is also clear that $\mathbf{S}^n := (S, \mathbb{S}^n, A^n)$ is a rough path, where A^n was defined in (2.8). Moreover, by property (RIE), we immediately have that $\mathbb{S}^n$ and A^n converge uniformly to $\mathbb{S}$ and A, respectively, as $n \to \infty$.

For each $n \geq 1$, let F^n be the path defined in (2.10). We consider the pair (F^n, F') as a controlled path with respect to S^n, defining the remainder term R^n by the usual relation

$$F_{s,t}^n = F_s' S_{s,t}^n + R_{s,t}^n, \qquad (s, t) \in \Delta_{[0,T]}.$$

Since S^n converges uniformly to S and, by Proposition 2.14, F^n converges uniformly to F, it follows that R^n also converges uniformly to the remainder term R corresponding to the S-controlled path (F, F').

We observe that $\|S^n\|_{p,[0,T]} \leq \|S\|_{p,[0,T]}$ and $\|F^n\|_{p,[0,T]} \leq \|F\|_{p,[0,T]}$, and we have from Lemma 2.12 that $\|A^n\|_{\frac{p}{2},[0,T]} \leq Cw(0, T)^{\frac{2}{p}}$ for every $n \geq 1$. It remains to show that R^n is bounded in r-variation, uniformly in n.

Let $n \geq 1$ and $(s,t) \in \Delta_{[0,T]}$. If there exists a k such that $t_k^n \leq s < t < t_{k+1}^n$, then

$$R_{s,t}^n = F_{s,t}^n - F_s' S_{s,t}^n = F_{t_k^n, t_k^n} - F_s' S_{t_k^n, t_k^n} = 0.$$

If there exists a k such that $t_k^n \leq s < t = t_{k+1}^n$, then

$$\begin{aligned} |R_{s,t}^n|^r &= |F_{s,t}^n - F_s' S_{s,t}^n|^r \\ &= |F_{t_k^n, t_{k+1}^n} - F_s' S_{t_k^n, t_{k+1}^n}|^r \\ &\lesssim |F_{t_k^n, t_{k+1}^n} - F_{t_k^n}' S_{t_k^n, t_{k+1}^n}|^r + |F_{t_k^n, s}' S_{t_k^n, t_{k+1}^n}|^r \\ &\lesssim |R_{t_k^n, t_{k+1}^n}|^r + |F_{t_k^n, s}'|^q + |S_{t_k^n, t_{k+1}^n}|^p, \end{aligned}$$

where in the last line we used Young's inequality, recalling that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Otherwise, let k_0 be the smallest k with $t_k^n \in [s,t]$ and k_1 the largest such k. After a short calculation, we find that

$$R_{s,t}^n = R_{s,t_{k_0}^n}^n + R_{t_{k_0}^n, t_{k_1}^n}^n + R_{t_{k_1}^n, t}^n + F_{s,t_{k_0}^n}' S_{t_{k_0}^n, t_{k_1}^n} + F_{s,t_{k_1}^n}' S_{t_{k_1}^n, t}^n.$$

We observe that $S_{t_{k_1}^n, t}^n = 0$ and $R_{t_{k_0}^n, t_{k_1}^n}^n = R_{t_{k_0}^n, t_{k_1}^n}$. We can deal with the terms $R_{s,t_{k_0}^n}^n$ and $R_{t_{k_1}^n, t}^n$ using the above, and we bound $|F_{s,t_{k_0}^n}' S_{t_{k_0}^n, t_{k_1}^n}|^r \lesssim |F_{s,t_{k_0}^n}'|^q + |S_{t_{k_0}^n, t_{k_1}^n}|^p$. Putting this all together, we have that

$$|R_{s,t}^n|^r \leq C\big(|R_{t_{k_0-1}^n, t_{k_0}^n}|^r + |F_{t_{k_0-1}^n, s}'|^q + |S_{t_{k_0-1}^n, t_{k_0}^n}|^p + |R_{t_{k_0}^n, t_{k_1}^n}|^r + |F_{s,t_{k_0}^n}'|^q + |S_{t_{k_0}^n, t_{k_1}^n}|^p\big),$$

where the constant C depends only on p, q and r. Taking an arbitrary partition $\mathcal{P}$ of the interval $[0,T]$, we deduce that

$$\sum_{[s,t] \in \mathcal{P}} |R_{s,t}^n|^r \leq 2C(\|R\|_{r,[0,T]}^r + \|F'\|_{q,[0,T]}^q + \|S\|_{p,[0,T]}^p).$$

Thus $\|R^n\|_{r,[0,T]}$ is bounded uniformly in $n \geq 1$.

Let $p' > p$, $q' > q$ and $r' > r$ such that $p' \in (2,3)$, $q' \geq p'$, $\frac{2}{p'} + \frac{1}{q'} > 1$ and $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'}$. Since the sequence $(S^n)_{n \geq 1}$ has uniformly bounded p-variation and S^n converges uniformly to S as $n \to \infty$, it follows by interpolation that S^n converges to S with respect to the p'-variation norm, i.e., $\|S^n - S\|_{p',[0,T]} \to 0$ as $n \to \infty$. It follows similarly that $\|A^n - A\|_{\frac{p'}{2},[0,T]} \to 0$ and $\|R^n - R\|_{r',[0,T]} \to 0$, and hence also that $\|\mathbf{S}^n; \mathbf{S}\|_{p',[0,T]} \to 0$ as $n \to \infty$. It thus follows from part (ii) of Proposition 2.7 that

$$\int_0^t F_u^n \, \mathrm{d}G_u \longrightarrow \int_0^t F_u \, \mathrm{d}G_u \qquad \text{as } n \to \infty, \tag{2.12}$$

where the convergence is uniform in $t \in [0,T]$. Note that in (2.12), the rough integral $\int_0^t F_u^n \, \mathrm{d}G_u$ is defined relative to the rough path $\mathbf{S}^n = (S, S^n, A^n)$, whilst the limiting rough integral $\int_0^t F_u \, \mathrm{d}G_u$ is defined relative to $\mathbf{S} = (S, S, A)$.

We recall from Proposition 2.4 that the integral of (F^n, F') against (G, G') relative to $\mathbf{S}^n = (S, S^n, A^n)$ is given by the limit

$$\int_0^t F_u^n \,\mathrm{d}G_u = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} (F_u^n G_{u,v} + F_u' G_u' A_{u,v}^n),$$

where the limit is taken over any sequence of partitions of the interval $[0, t]$ with vanishing mesh size. Take any refinement $\tilde{\mathcal{P}}$ of the partition $(\mathcal{P}^n \cup \{t\}) \cap [0, t]$ (where as usual $\mathcal{P}^n$ is the partition given in property (RIE)) and let $[u, v] \in \tilde{\mathcal{P}}$. By the choice of the partition $\tilde{\mathcal{P}}$, there exists a k such that $t_k^n \le u < v \le t_{k+1}^n$ which, recalling (2.8), implies that $A_{u,v}^n = 0$. Since the mesh size of $\tilde{\mathcal{P}}$ may be arbitrarily small, it follows that

$$\lim_{|\tilde{\mathcal{P}}|\to 0} \sum_{[u,v]\in\tilde{\mathcal{P}}} F_u' G_u' A_{u,v}^n = 0.$$

To conclude, we then simply recall (2.12) and note that

$$\int_0^t F_u^n \,\mathrm{d}G_u = \lim_{|\tilde{\mathcal{P}}|\to 0} \sum_{[u,v]\in\tilde{\mathcal{P}}} F_u^n G_{u,v} = \sum_{k=0}^{N_n-1} F_{t_k^n} G_{t_k^n\wedge t, t_{k+1}^n\wedge t}.$$ □

We can actually generalise the result of Theorem 2.15 to a slightly larger class of integrands.

Corollary 2.16 *Recall the assumptions of Theorem 2.15 and let $\gamma \in D^r([0, T]; \mathbb{R}^d)$. Then $(H, H') := (F + \gamma, F')$ is a controlled path with respect to S, and the rough integral of (H, H') against (G, G') is given by*

$$\int_0^t H_u \,\mathrm{d}G_u = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} H_{t_k^n} G_{t_k^n\wedge t, t_{k+1}^n\wedge t}$$

for every $t \in [0, T]$.

The point here is that the path γ may have jump times which do not belong to the set $\bigcup_{n\in\mathbb{N}} \mathcal{P}^n$.

Proof of Corollary 2.16 Since γ has finite r-variation, we immediately have that γ is a controlled path with Gubinelli derivative simply given by $\gamma' = 0$. By linearity, it is then clear that $(H, H') = (F, F') + (\gamma, 0)$ is indeed a controlled path with respect to S. Since $\gamma' = 0$, we have from Proposition 2.4 that

$$\int_0^t \gamma_u \,\mathrm{d}G_u = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \gamma_u G_{u,v} = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \gamma_{t_k^n} G_{t_k^n\wedge t, t_{k+1}^n\wedge t}.$$

By linearity, we have $\int_0^t H_u \,\mathrm{d}G_u = \int_0^t F_u \,\mathrm{d}G_u + \int_0^t \gamma_u \,\mathrm{d}G_u$, and the result then follows from Theorem 2.15. □

2.4 Link to Föllmer integration

In his seminal paper, Föllmer [19] introduced a notion of pathwise integration based on the concept of quadratic variation and derived a corresponding pathwise Itô formula, which have both proved to be useful tools in robust approaches to mathematical finance.

In the following, we write $\mathcal{B}[0,T]$ for the Borel σ-algebra on $[0,T]$.

Definition 2.17 Let $S \in D([0,T];\mathbb{R})$ and $\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \geq 1$, be a sequence of partitions with vanishing mesh size. We say that S has *quadratic variation along* $(\mathcal{P}^n)_{n\in\mathbb{N}}$ *in the sense of Föllmer* if the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ on $([0,T],\mathcal{B}[0,T])$ defined by

$$\mu_n := \sum_{k=0}^{N_n-1} |S_{t_k^n,t_{k+1}^n}|^2 \delta_{t_k^n}$$

converges weakly to a measure μ such that the map

$$t \mapsto [S]_t^c := \mu([0,t]) - \sum_{0<s\leq t} |S_{s-,s}|^2$$

is continuous and increasing. We then call the function $[S]$ given by $[S]_t = \mu([0,t])$ the *quadratic variation* of S along $(\mathcal{P}^n)_{n\in\mathbb{N}}$. We say that a path $S \in D([0,T];\mathbb{R}^d)$ has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ in the sense of Föllmer if the condition above holds for S^i and $S^i + S^j$ for every (i,j), and in this case, we write

$$[S^i, S^j] := \frac{1}{2}([S^i + S^j] - [S^i] - [S^j]). \tag{2.13}$$

Assuming that a path $S \in D([0,T];\mathbb{R}^d)$ has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and $f \in C^2(\mathbb{R}^d;\mathbb{R})$, Föllmer showed that the limit

$$\int_0^T \mathrm{D}f(S_u)\,\mathrm{d}S_u := \lim_{n\to\infty} \sum_{[s,t]\in\mathcal{P}^n} \mathrm{D}f(S_s)S_{s,t}$$

exists and that the resulting integral $\int_0^T \mathrm{D}f(S_u)\,\mathrm{d}S_u$ satisfies a pathwise Itô formula; see [19, Théorème]. This result can also be explained via the language of rough path theory; see Friz and Hairer [24, Chap. 5.3]. Let us remark that the Föllmer integral $\int_0^T \mathrm{D}f(S_u)\,\mathrm{d}S_u$ is only well defined for gradients $\mathrm{D}f$ and not for general functions, as its existence is given by the corresponding pathwise Itô formula.

In the following, we relate property (RIE) to the existence of quadratic variation in the sense of Föllmer. To this end, for each $i = 1,\dots,d$, we introduce

$$S_t^{n,i} = S_T^i \mathbf{1}_{\{T\}}(t) + \sum_{k=0}^{N_n-1} S_{t_k^n}^i \mathbf{1}_{[t_k^n,t_{k+1}^n)}(t)$$

and the discrete quadratic variation $[S^i, S^j]^n$ by

$$[S^i, S^j]^n_t = \sum_{k=0}^{N_n-1} S^i_{t^n_k \wedge t, t^n_{k+1} \wedge t} S^j_{t^n_k \wedge t, t^n_{k+1} \wedge t}, \qquad t \in [0, T].$$

Proposition 2.18 *Let $\mathcal{P}^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\}$, $n \in \mathbb{N}$, be a sequence of nested partitions with vanishing mesh size and $S \in D([0, T]; \mathbb{R}^d)$. The following conditions are equivalent*:

(i) *For every pair (i, j), the Riemann sums $\int_0^t S^{n,i}_u \,\mathrm{d}S^j_u + \int_0^t S^{n,j}_u \,\mathrm{d}S^i_u$ converge uniformly to a limit, which we denote by $\int_0^t S^i_u \,\mathrm{d}S^j_u + \int_0^t S^j_u \,\mathrm{d}S^i_u$.*

(ii) *For every pair (i, j), the discrete quadratic variation $[S^i, S^j]^n$ converges uniformly to a càdlàg path, which we denote by $[S^i, S^j]^{\mathcal{P}}$.*

(iii) *The path S has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ in the sense of Föllmer.*

Moreover, if these conditions hold, then the path $[S^i, S^j]^{\mathcal{P}}$ has finite total variation and for every (i, j), we have $[S^i, S^j] = [S^i, S^j]^{\mathcal{P}}$ and the equality

$$S^i_t S^j_t = S^i_0 S^j_0 + \int_0^t S^i_u \,\mathrm{d}S^j_u + \int_0^t S^j_u \,\mathrm{d}S^i_u + [S^i, S^j]^{\mathcal{P}}_t \tag{2.14}$$

for every $t \in [0, T]$.

Proof We have

$$\begin{aligned} S^i_t S^j_t - S^i_0 S^j_0 &= \sum_{k=0}^{N_n-1} (S^i_{t^n_{k+1} \wedge t} S^j_{t^n_{k+1} \wedge t} - S^i_{t^n_k \wedge t} S^j_{t^n_k \wedge t}) \\ &= \sum_{k=0}^{N_n-1} (S^i_{t^n_k \wedge t} S^j_{t^n_k \wedge t, t^n_{k+1} \wedge t} + S^j_{t^n_k \wedge t} S^i_{t^n_k \wedge t, t^n_{k+1} \wedge t}) \\ &\quad + \sum_{k=0}^{N_n-1} S^i_{t^n_k \wedge t, t^n_{k+1} \wedge t} S^j_{t^n_k \wedge t, t^n_{k+1} \wedge t} \\ &= \int_0^t S^{n,i}_u \,\mathrm{d}S^j_u + \int_0^t S^{n,j}_u \,\mathrm{d}S^i_u + [S^i, S^j]^n_t, \end{aligned}$$

from which it follows that conditions (i) and (ii) are equivalent, and that (2.14) then also holds. In this case, we also have that

$$[S^i, S^j]^{\mathcal{P}}_t = \frac{1}{4}([S^i + S^j, S^i + S^j]^{\mathcal{P}}_t - [S^i - S^j, S^i - S^j]^{\mathcal{P}}_t),$$

so that as the difference of two non-decreasing functions, $[S^i, S^j]^{\mathcal{P}}$ has finite total variation.

For one-dimensional paths S, the equivalence of conditions (ii) and (iii) follows from Vovk [52, Propositions 3 and 4]. The extension of this to d-dimensional paths S

and the equality $[S^i, S^j] = [S^i, S^j]^{\mathcal{P}}$ then follow from the polarisation identity

$$[S^i, S^j]^n_t = \frac{1}{2}([S^i + S^j, S^i + S^j]^n_t - [S^i, S^i]^n_t - [S^j, S^j]^n_t)$$

and the definition of $[S^i, S^j]$ in (2.13). □

Remark 2.19 From a semimartingale perspective, in the integration by parts formula (2.14), one might expect to see left limits in the integrands. Since our integrals are defined as limits of left-point Riemann sums, taking such left limits is not necessary and would actually not change the value of the integrals; see Remark 3.3. Nevertheless, our rough integrals are also consistent with Itô integrals; see Proposition 4.8.

Remark 2.20 As an immediate consequence of Proposition 2.18, we have that if a path S satisfies (RIE) along $(\mathcal{P}^n)_{n\in\mathbb{N}}$, then it has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ in the sense of Föllmer, thus allowing one to apply all the known results regarding Föllmer integration.

In particular, if a vector field $f: \mathbb{R}^d \to \mathbb{R}$ is of class C^3, then by Theorem 2.15, the Föllmer integral $\int_0^{\cdot} \mathrm{D} f(S_u)\,\mathrm{d}S_u$ coincides with the rough integral $\int_0^{\cdot} \mathrm{D} f(S_u)\,\mathrm{d}\mathbf{S}_u$. We thus obtain the rough Itô formula

$$\begin{aligned}
&f(S_t) - f(S_0)\\
&= \int_0^t \mathrm{D} f(S_u)\,\mathrm{d}\mathbf{S}_u + \frac{1}{2}\int_0^t \mathrm{D}^2 f(S_u)\,\mathrm{d}[S]_u\\
&\quad + \sum_{0<u\le t}\bigg(f(S_u) - f(S_{u-}) - \mathrm{D} f(S_{u-})\Delta S_u - \frac{1}{2}\mathrm{D}^2 f(S_{u-})(\Delta S_u \otimes \Delta S_u)\bigg),
\end{aligned}$$

which holds for every $t \in [0, T]$, where $[S] = ([S^i, S^j])_{1\le i,j\le d}$ denotes the quadratic variation matrix and $\Delta S_u := S_{u-,u}$.

We note that the formula above is precisely the Itô formula for rough paths derived in Friz and Zhang [26]. Of course, this formula remains valid when f is only assumed to be of class C^2, in which case the first integral is interpreted in the sense of Föllmer. Taking $f \in C^3$ is only necessary to ensure that the integrand $\mathrm{D} f(S)$ is a controlled path with respect to S, so that this integral may be interpreted in the rough sense.

3 Functionally generated trading strategies and their generalisations

Given property (RIE), we can introduce a model-free framework for continuous-time financial markets with a possibly infinite time horizon. In this section, we verify that most relevant trading strategies from a practical perspective, such as delta-hedging strategies and functionally generated strategies, are admissible integrands for price paths satisfying property (RIE). Furthermore, the underlying rough integration allows us to deduce stability estimates for admissible strategies.

3.1 Price paths and admissible strategies

For a path $S\colon [0,\infty) \to \mathbb{R}^d$, we denote by $S|_{[0,T]}$ the restriction of S to the interval $[0,T]$.

Definition 3.1 For a fixed $p \in (2,3)$, we say that a path $S \in D([0,\infty);\mathbb{R}^d)$ is a *price path* if there exists a nested sequence of locally finite partitions $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of the interval $[0,\infty)$ with vanishing mesh size on compacts such that for all $T > 0$, the restriction $S|_{[0,T]}$ satisfies (RIE) with respect to p and $(\mathcal{P}^n([0,T]))_{n\in\mathbb{N}}$. We denote the family of all such price paths by Ω_p.

Note that the sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions may depend on the choice of price path $S \in \Omega_p$, consistent with the stochastic framework where this sequence is naturally defined in terms of (probabilistic) stopping times.

Having fixed the model-free structure of the underlying price paths, we can introduce the class of admissible strategies and the corresponding capital process.

Definition 3.2 Let $p \in (2,3)$ and let $S \in \Omega_p$ be a price path. We say that a càdlàg path $\varphi\colon [0,\infty) \to \mathbb{R}^d$ is an *admissible strategy (with respect to S)* if

- there exist $q \geq p$ and $r > 1$ with $2/p + 1/q > 1$ and $1/r = 1/p + 1/q$ such that for every $T > 0$, there exists a path $\varphi'\colon [0,T] \to \mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$ such that the pair $(\varphi,\varphi') \in \mathcal{V}_S^{q,r}$ is a controlled path with respect to S in the sense of Definition 2.3,
- and $J_\varphi \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$, where J_φ is the set of jump times of φ in $(0,\infty)$ and $(\mathcal{P}^n)_{n\in\mathbb{N}}$ is the sequence of partitions associated with the price path $S \in \Omega_p$.

We denote the space of all admissible strategies (with respect to S) by $\mathcal{A}_S$.

We define the *capital process* associated with φ and S as the $\mathbb{R}$-valued path $V^\varphi(S)$ given by

$$V_t^\varphi(S) := \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \sum_{i=1}^{d} \varphi^i_{t_k^n}(S^i_{t^n_{k+1}\wedge t} - S^i_{t^n_k\wedge t}), \qquad t \in [0,\infty), \tag{3.1}$$

where $\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, is the sequence of partitions specified in property (RIE).

Remark 3.3 One might think that left-continuity of integrands would be a natural assumption to capture the previsible nature of trading strategies in stochastic finance. However, in the present setting, this assumption is not necessary as the corresponding capital process $V^\varphi(S)$, which, as we shall see below, may be expressed as a rough integral, does not change when replacing φ by its left-continuous modification; see e.g. Friz and Shekhar [25, Theorem 31]. The reason for this is essentially the left-point Riemann sum construction of the integral. Indeed, suppose that S has a jump at a time $t > 0$. The contribution to the capital process $V^\varphi(S)$ at time t is then given by $\lim_{s\to t, s<t} \varphi_s S_{s,t}$, which is invariant to the choice of φ or its left-continuous modification. Furthermore, we shall see that $V^\varphi(S)$ coincides with the classical stochastic Itô integral whenever both the rough and stochastic integrals are defined; see Sect. 4.4 below.

The condition $J_\varphi \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$ means that one is allowed to use trading strategies whose jump points are included in the underlying sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions. On the one hand, many frequently used trading strategies such as delta-hedging satisfy this condition, and further examples are discussed later in this section. On the other hand, the sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions can be fixed a priori to allow a desired class of trading strategies, e.g. buy-and-hold strategies along the sequence of dyadic partitions.

Proposition 3.4 *Let $p \in (2,3)$, let $S \in \Omega_p$ be a price path and $\varphi \in \mathcal{A}_S$ an admissible strategy (in the sense of Definition 3.2). Then the capital process $V^\varphi(S)$ as defined in (3.1) exists as a locally uniform limit and is actually given by*

$$V_t^\varphi(S) = \int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s, \qquad t \in [0,\infty), \tag{3.2}$$

that is, as the rough integral of the controlled path $(\varphi,\varphi') \in \mathcal{V}_S^{q,r}$ against the rough path $\mathbf{S}$ defined in Lemma 2.13. Moreover, given another price path $\tilde{S} \in \Omega_p$ and an admissible strategy $\tilde{\varphi} \in \mathcal{A}_{\tilde{S}}$ with respect to $\tilde{S}$, we have for every $T > 0$ that

$$\begin{aligned}
&|V_T^\varphi(S) - V_T^{\tilde{\varphi}}(\tilde{S})| \\
&\le C\Big((|\tilde{\varphi}_0| + \|\tilde{\varphi},\tilde{\varphi}'\|_{\mathcal{V}_{\tilde{S}}^{q,r}}) \\
&\qquad \times (1 + \|S\|_{p,[0,T]} + \|\tilde{S}\|_{p,[0,T]})\|\mathbf{S};\tilde{\mathbf{S}}\|_{p,[0,T]} \\
&\qquad + (|\varphi_0 - \tilde{\varphi}_0| + |\varphi_0' - \tilde{\varphi}_0'| + \|\varphi' - \tilde{\varphi}'\|_{q,[0,T]} + \|R^\varphi - R^{\tilde{\varphi}}\|_{r,[0,T]}) \\
&\qquad\quad \times (1 + \|S\|_{p,[0,T]})\|\mathbf{S}\|_{p,[0,T]}\Big),
\end{aligned}$$

where the constant C depends on p, q and r.

Proof Let $T > 0$, and let $(\mathcal{P}^n)_{n\in\mathbb{N}}$ be a sequence of nested partitions such that p, $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and S satisfy property (RIE) on the interval $[0,T]$. Recall from property (RIE) the existence of the limit $\int_0^t S_u \otimes \mathrm{d}S_u$ for every $t \in [0,T]$. By Lemma 2.13, defining the function $A\colon \Delta_{[0,T]} \to \mathbb{R}^{d\times d}$ by

$$A_{s,t} := \int_s^t S_u \otimes \mathrm{d}S_u - S_s \otimes S_{s,t},$$

we have that the triplet $\mathbf{S} = (S, S, A)$ is a càdlàg rough path (in the sense of Definition 2.1). Hence the rough integral in (3.2) is well defined by Proposition 2.4 (see also Remark 2.5) and satisfies (3.1) as a locally uniform limit by Theorem 2.15.

For the stability estimate, we simply note that

$$|V_T^\varphi(S) - V_T^{\tilde{\varphi}}(\tilde{S})| = \left|\int_0^T \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^T \tilde{\varphi}_s \,\mathrm{d}\tilde{\mathbf{S}}_s\right| \le \left\|\int_0^\cdot \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^\cdot \tilde{\varphi}_s \,\mathrm{d}\tilde{\mathbf{S}}_s\right\|_{p,[0,T]}$$

and apply part (i) of Proposition 2.7. $\square$

Remark 3.5 Recall from Remark 2.5 that the rough integral in (3.2) is defined by the limit $\int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} (\varphi_u S_{u,v} + \varphi'_u A_{u,v})$, where the limit is taken over any sequence of partitions of the interval $[0,t]$ with vanishing mesh size. Here, φ_u and $S_{u,v}$ both take values in $\mathbb{R}^d$, and we interpret their multiplication as the Euclidean inner product. The derivative φ'_u takes values in $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$, which we can also identify with $\mathcal{L}(\mathbb{R}^{d\times d};\mathbb{R})$. Since $A_{u,v} \in \mathbb{R}^{d\times d}$, the product $\varphi'_u A_{u,v}$ also takes values in $\mathbb{R}$.

In the following, we show that the most relevant trading strategies from a practical viewpoint belong to the class of admissible strategies in the sense of Definition 3.2.

3.2 Functionally generated trading strategies

Having fixed the set Ω_p of underlying price paths, we start by introducing functionally generated portfolios. For this purpose, for some $d_A \in \mathbb{N}$, we fix a càdlàg path $A\colon [0,\infty) \to \mathbb{R}^{d_A}$ of locally bounded variation and assume that the jump times of A belong to the union of the partitions $(\mathcal{P}^n)_{n\in\mathbb{N}}$ appearing in property (RIE); that is, we assume that $J_A \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$, where $J_A := \{t \in (0,\infty) : A_{t-,t} \neq 0\}$. The path A is supposed to model additional information pertaining to the market which a trader would like to include in their trading decisions. For instance, the components of the path $A = (A^1, \dots, A^{d_A})$ could include time $t \mapsto t$, the running maximum $t \mapsto \max_{u\in[0,t]} S^i_u$ or the integral $t \mapsto \int_0^t S^i_u \,\mathrm{d}u$ for some (or all) $i = 1, \dots, d$. A more detailed discussion on practical choices of A can be found in Schied et al. [46].

For $\ell = d + d_A$, we denote by $C^2_b(\mathbb{R}^\ell;\mathbb{R}^d)$ the space of twice continuously differentiable functions $f\colon \mathbb{R}^\ell \to \mathbb{R}^d$ such that f and its derivatives up to order 2 are uniformly bounded, that is,

$$C^2_b(\mathbb{R}^\ell;\mathbb{R}^d) := \{f \in C^2(\mathbb{R}^\ell;\mathbb{R}^d) : \|f\|_{C^2_b} < \infty\}$$

with

$$\|f\|_{C^2_b} := \|f\|_\infty + \|\mathrm{D}f\|_\infty + \|\mathrm{D}^2 f\|_\infty.$$

For $S \in \Omega_p$, we introduce the set $\mathcal{G}^2_S$ of all generalised functionally generated trading strategies φ^f, which are all paths of the form

$$\varphi^f_t = (f^1_t, \dots, f^d_t) := f(S_t, A_t), \qquad t \in [0,\infty), \tag{3.3}$$

for some $f \in C^2_b(\mathbb{R}^\ell;\mathbb{R}^d)$. For $\varphi^f \in \mathcal{G}^2_S$, the corresponding capital process is given by

$$V^f_t(S) = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \sum_{i=1}^{d} f^i_{t^n_k}(S^i_{t^n_{k+1}\wedge t} - S^i_{t^n_k\wedge t}), \qquad t \in [0,\infty). \tag{3.4}$$

Proposition 3.6 *Let $p \in (2,3)$ and $S \in \Omega_p$, and let $\varphi^f, \varphi^{\tilde f} \in \mathcal{G}^2_S$. Then φ^f is an admissible strategy, i.e., $\varphi^f \in \mathcal{A}_S$, and the capital process $(V^f_t(S))_{t\in[0,\infty)}$ given in*

(3.4) *is well defined as a locally uniform limit in* $t \in [0, \infty)$. *Moreover, for every* $T \in [0, \infty)$, *we have the stability estimate*

$$|V_T^f(S) - V_T^{\tilde{f}}(S)| \leq C\|f - \tilde{f}\|_{C_b^2}(1 + \|S\|_{p,[0,T]}^2 + \|A\|_{1,[0,T]}) \\ \times (1 + \|S\|_{p,[0,T]})\|\mathbf{S}\|_{p,[0,T]}, \tag{3.5}$$

where the constant C *depends only on* p, *and the triplet* $\mathbf{S} = (S, \mathbb{S}, A)$ *is the càdlàg rough path defined in Lemma* 2.13.

Proof *(Admissibility)* Let $\varphi = \varphi^f \in \mathcal{G}_S^2$ be a functionally generated strategy of the form (3.3) for some $f \in C_b^2(\mathbb{R}^\ell; \mathbb{R}^d)$. Fix a $T \in [0, \infty)$. We claim that $(\varphi, \varphi') \in \mathcal{V}_S^{p,\frac{p}{2}}$ is a controlled path with respect to S in the sense of Definition 2.3 (with $q = p$ and $r = p/2$), where

$$\varphi'_t := \mathrm{D}_S f(S_t, A_t), \qquad t \in [0, T],$$

and $\mathrm{D}_S f$ denotes the derivative of f with respect to its first d components. To see this, we first note that

$$|\varphi'_{s,t}| = |\mathrm{D}_S f(S_t, A_t) - \mathrm{D}_S f(S_s, A_s)| \leq \|f\|_{C_b^2}(|S_{s,t}| + |A_{s,t}|)$$

so that

$$\|\varphi'\|_{p,[0,T]} \lesssim \|f\|_{C_b^2}(\|S\|_{p,[0,T]} + \|A\|_{1,[0,T]}) < \infty, \tag{3.6}$$

and hence $\varphi' \in D^p([0, T]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d))$. We moreover have that

$$\begin{aligned} R^\varphi_{s,t} &:= \varphi_{s,t} - \varphi'_s S_{s,t} \\ &= f(S_t, A_t) - f(S_s, A_s) - \mathrm{D}_S f(S_s, A_s) S_{s,t} \\ &= f(S_t, A_s) - f(S_s, A_s) - \mathrm{D}_S f(S_s, A_s) S_{s,t} + f(S_t, A_t) - f(S_t, A_s) \\ &= \int_0^1 \big(\mathrm{D}_S f(S_s + \tau S_{s,t}, A_s) - \mathrm{D}_S f(S_s, A_s)\big) S_{s,t} \,\mathrm{d}\tau + f(S_t, A_t) - f(S_t, A_s) \end{aligned}$$

so that $|R^\varphi_{s,t}| \leq \|f\|_{C_b^2}(|S_{s,t}|^2 + |A_{s,t}|)$. It follows that

$$\|R^\varphi\|_{\frac{p}{2},[0,T]} \lesssim \|f\|_{C_b^2}(\|S\|_{p,[0,T]}^2 + \|A\|_{1,[0,T]}) < \infty \tag{3.7}$$

so that $R^\varphi \in D^{p/2}(\Delta_{[0,T]}; \mathbb{R}^d)$, and thus the conditions of Definition 2.3 are satisfied. Thus by Proposition 2.4 (and Remark 2.5), we have the existence for each $t \in [0, T]$ of the ($\mathbb{R}$-valued) rough integral

$$\int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} (\varphi_u S_{u,v} + \varphi'_u A_{u,v}).$$

For a given càdlàg path $F\colon [0,T] \to \mathbb{R}^d$ (or $\mathbb{R}^{d_A}$), let $J_F = \{t \in (0,T] : F_{t-,t} \neq 0\}$ denote the jump times of F. It follows from property (RIE) and Proposition 2.14 that $J_S \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$. Since we also assumed that $J_A \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$, it then follows from (3.3) that $J_\varphi \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$. Thus by Theorem 2.15, we have that

$$\begin{aligned}\int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s &= \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \varphi_{t_k^n}(S_{t_{k+1}^n\wedge t} - S_{t_k^n\wedge t}) \\ &= \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \sum_{i=1}^{d} f^i_{t_k^n}(S^i_{t_{k+1}^n\wedge t} - S^i_{t_k^n\wedge t}) = V_t^f(S),\end{aligned}$$

and that this limit is uniform in $t \in [0,T]$.

(Stability estimate) Let φ and $\tilde{\varphi}$ be the strategies generated by f and $\tilde{f}$, respectively, as defined in (3.3). By part (i) of Proposition 2.7, we have the estimate

$$\begin{aligned}&|V_T^f(S) - V_T^{\tilde{f}}(S)| \\ &= \left|\int_0^T \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^T \tilde{\varphi}_s \,\mathrm{d}\mathbf{S}_s\right| \leq \left\|\int_0^\cdot \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^\cdot \tilde{\varphi}_s \,\mathrm{d}\mathbf{S}_s\right\|_{p,[0,T]} \\ &\lesssim (|\varphi_0 - \tilde{\varphi}_0| + |\varphi_0' - \tilde{\varphi}_0'| + \|\varphi' - \tilde{\varphi}'\|_{p,[0,T]} + \|R^\varphi - R^{\tilde{\varphi}}\|_{\frac{p}{2},[0,T]}) \\ &\quad \times (1 + \|S\|_{p,[0,T]})\|\mathbf{S}\|_{p,[0,T]}.\end{aligned} \tag{3.8}$$

As above, here

$$\varphi_t' = \mathrm{D}_S f(S_t, A_t) \qquad \text{and} \qquad R^\varphi_{s,t} = \varphi_{s,t} - \varphi_s' S_{s,t},$$

with $\tilde{\varphi}'$ and $R^{\tilde{\varphi}}$ defined similarly. We now aim to estimate each term on the right-hand side of (3.8).

We have

$$|\varphi_0 - \tilde{\varphi}_0| = |f(S_0, A_0) - \tilde{f}(S_0, A_0)| \leq \|f - \tilde{f}\|_\infty \leq \|f - \tilde{f}\|_{C_b^2}, \tag{3.9}$$

and similarly

$$|\varphi_0' - \tilde{\varphi}_0'| = |\mathrm{D}_S f(S_0, A_0) - \mathrm{D}_S \tilde{f}(S_0, A_0)| \leq \|\mathrm{D}_S f - \mathrm{D}_S \tilde{f}\|_\infty \leq \|f - \tilde{f}\|_{C_b^2}. \tag{3.10}$$

Next, for $[s,t] \subseteq [0,T]$, we compute

$$\begin{aligned}(\varphi' - \tilde{\varphi}')_{s,t} &= \mathrm{D}_S f(S_t, A_t) - \mathrm{D}_S f(S_s, A_s) - \mathrm{D}_S \tilde{f}(S_t, A_t) + \mathrm{D}_S \tilde{f}(S_s, A_s) \\ &= \mathrm{D}_S f(S_t, A_s) - \mathrm{D}_S f(S_s, A_s) - \mathrm{D}_S \tilde{f}(S_t, A_s) + \mathrm{D}_S \tilde{f}(S_s, A_s) \\ &\quad + \mathrm{D}_S f(S_t, A_t) - \mathrm{D}_S f(S_t, A_s) - \mathrm{D}_S \tilde{f}(S_t, A_t) + \mathrm{D}_S \tilde{f}(S_t, A_s) \\ &= \int_0^1 \big(\mathrm{D}^2_{SS} f(S_s + \tau S_{s,t}, A_s) - \mathrm{D}^2_{SS} \tilde{f}(S_s + \tau S_{s,t}, A_s)\big) S_{s,t} \,\mathrm{d}\tau \\ &\quad + \int_0^1 \big(\mathrm{D}^2_{SA} f(S_t, A_s + \tau A_{s,t}) - \mathrm{D}^2_{SA} \tilde{f}(S_t, A_s + \tau A_{s,t})\big) A_{s,t} \,\mathrm{d}\tau,\end{aligned}$$

so that

$$\begin{aligned}|(\varphi' - \tilde{\varphi}')_{s,t}| &\leq \|\mathrm{D}^2_{SS} f - \mathrm{D}^2_{SS} \tilde{f}\|_\infty |S_{s,t}| + \|\mathrm{D}^2_{SA} f - \mathrm{D}^2_{SA} \tilde{f}\|_\infty |A_{s,t}| \\ &\leq \|f - \tilde{f}\|_{C^2_b} (|S_{s,t}| + |A_{s,t}|)\end{aligned}$$

and thus

$$\|\varphi' - \tilde{\varphi}'\|_{p,[0,T]} \lesssim \|f - \tilde{f}\|_{C^2_b} (\|S\|_{p,[0,T]} + \|A\|_{1,[0,T]}). \tag{3.11}$$

Finally, we have

$$\begin{aligned}&(R^\varphi - R^{\tilde{\varphi}})_{s,t} \\ &= f(S_t, A_t) - f(S_s, A_s) - \mathrm{D}_S f(S_s, A_s) S_{s,t} - \tilde{f}(S_t, A_t) \\ &\quad + \tilde{f}(S_s, A_s) + \mathrm{D}_S \tilde{f}(S_s, A_s) S_{s,t} \\ &= f(S_t, A_s) - f(S_s, A_s) - \mathrm{D}_S f(S_s, A_s) S_{s,t} - \tilde{f}(S_t, A_s) \\ &\quad + \tilde{f}(S_s, A_s) + \mathrm{D}_S \tilde{f}(S_s, A_s) S_{s,t} \\ &\quad + f(S_t, A_t) - f(S_t, A_s) - \tilde{f}(S_t, A_t) + \tilde{f}(S_t, A_s) \\ &= \int_0^1 \int_0^1 \big(\mathrm{D}^2_{SS} f(S_s + \tau_1 \tau_2 S_{s,t}, A_s) - \mathrm{D}^2_{SS} \tilde{f}(S_s + \tau_1 \tau_2 S_{s,t}, A_s)\big) S^{\otimes 2}_{s,t} \tau_1 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1 \\ &\quad + \int_0^1 \big(\mathrm{D}_A f(S_t, A_s + \tau A_{s,t}) - \mathrm{D}_A \tilde{f}(S_t, A_s + \tau A_{s,t})\big) A_{s,t} \,\mathrm{d}\tau,\end{aligned}$$

so that

$$\begin{aligned}|(R^\varphi - R^{\tilde{\varphi}})_{s,t}| &\leq \|\mathrm{D}^2_{SS} f - \mathrm{D}^2_{SS} \tilde{f}\|_\infty |S_{s,t}|^2 + \|\mathrm{D}_A f - \mathrm{D}_A \tilde{f}\|_\infty |A_{s,t}| \\ &\leq \|f - \tilde{f}\|_{C^2_b} (|S_{s,t}|^2 + |A_{s,t}|)\end{aligned}$$

and hence

$$\|R^\varphi - R^{\tilde{\varphi}}\|_{\frac{p}{2},[0,T]} \lesssim \|f - \tilde{f}\|_{C^2_b} (\|S\|^2_{p,[0,T]} + \|A\|_{1,[0,T]}). \tag{3.12}$$

Substituting (3.9)–(3.12) into (3.8) gives the estimate in (3.5). $\square$

3.3 Path-dependent functionally generated trading strategies

The functionally generated trading strategies considered in Sect. 3.2 could depend on the past prices only through a process of locally finite variation. In some contexts, it is beneficial to work with trading strategies possessing a more general path-dependent structure; see e.g. Schied et al. [45, 46] for more detailed discussions in this direction. A common way to treat path-dependent and non-anticipating trading strategies is the calculus initiated by Dupire [17] and Cont and Fournié [11]. For the sake of brevity, we recall here only the essential definitions and refer to Ananova [3, Sect. 3.1] and [11] for full details.

A functional $F\colon [0,T] \times D([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ is called *non-anticipative* if

$$F(t,S) = F(t,S_{\cdot\wedge t}) \qquad \text{for all } S \in D([0,T];\mathbb{R}^d).$$

As usual, the non-anticipative functionals are defined on the space of stopped paths, defined as the equivalence class in $[0,T] \times D([0,T];\mathbb{R}^d)$ with respect to the equivalence relation

$$(t,S) \sim (t',S') \qquad \Longleftrightarrow \qquad t = t' \text{ and } S_{\cdot\wedge t} = S'_{\cdot\wedge t'}.$$

The resulting space Λ^d_T is equipped with the distance

$$d_\infty\big((t,S),(t',S')\big) := |t-t'| + \sup_{u\in[0,T]} |S_{u\wedge t} - S'_{u\wedge t'}|,$$

and (Λ^d_T, d_∞) is then a complete metric space. We introduce the following spaces:

– $\mathbb{C}^{0,0}_\ell(\Lambda^d_T)$ is the space of left-continuous functionals $F\colon \Lambda^d_T \to \mathbb{R}^d$, i.e., for all $(t,S) \in \Lambda^d_T$ and $\varepsilon > 0$, there exists $\nu > 0$ such that for all $(t',S') \in \Lambda^d_T$,

$$t' < t \text{ and } d_\infty\big((t,S),(t',S')\big) < \nu \qquad \Longrightarrow \qquad |F(t,S) - F(t',S')| < \varepsilon.$$

– $\mathbb{B}(\Lambda^d_T)$ is the space of boundedness-preserving functionals $F\colon \Lambda^d_T \to \mathbb{R}^d$, i.e., for every compact subset $K \subseteq \mathbb{R}^d$ and for every $t_0 \in [0,T]$, there exists $C > 0$ such that for all $t \in [0,t_0]$ and $(t,S) \in \Lambda^d_T$,

$$S([0,t]) \subseteq K \qquad \Longrightarrow \qquad |F(t,S)| < C.$$

– $\mathrm{Lip}(\Lambda^d_T, d_\infty)$ is the space of Lipschitz-continuous functionals $F\colon \Lambda^d_T \to \mathbb{R}^d$, i.e., there exists $C > 0$ such that for all $(t,S),(t',S') \in \Lambda^d_T$,

$$|F(t,S) - F(t',S')| \le C d_\infty\big((t,S),(t',S')\big).$$

We define $\mathbb{C}^{1,1}_b(\Lambda^d_T)$ as the set of non-anticipative functionals $F\colon \Lambda^d_T \to \mathbb{R}$ which are

– horizontally differentiable, i.e., for all $(t,S) \in \Lambda^d_T$,

$$\mathcal{D}F(t,S) = \lim_{h\downarrow 0} \frac{F(t+h,S_{\cdot\wedge t}) - F(t,S_{\cdot\wedge t})}{h}$$

exists, and $\mathcal{D}F$ is continuous at fixed times;

– vertically differentiable, i.e., for all $(t, S) \in \Lambda_T^d$, we have existence of the vertical derivative $\nabla_x F(t, S) = (\partial_i F(t, S))_{i=1,\dots,d}$ with

$$\partial_i F(t, S) = \lim_{h \to 0} \frac{F(t, S_{\cdot\wedge t} + h e_i \mathbf{1}_{[t,T]}) - F(t, S_{\cdot\wedge t})}{h},$$

where $(e_i)_{i=1,\dots,d}$ is the canonical basis of $\mathbb{R}^d$, and $\nabla_x F \in \mathbb{C}_\ell^{0,0}(\Lambda_T^d)$;
– such that $\mathcal{D}F, \nabla_x F \in \mathbb{B}(\Lambda_T^d)$.

Corollary 3.7 *If $F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$ with F and $\nabla_x F$ in $\mathrm{Lip}(\Lambda_T^d, d_\infty)$, and if $S \in \Omega_p$, then the path-dependent functionally generated trading strategy $F(\cdot, S)$ is an admissible strategy in the sense of Definition 3.2.*

Proof That $(F(\cdot, S), \nabla_x F(\cdot, S))$ is a controlled path with respect to S immediately follows from Ananova [2, Lemma 5.12]; see also Ananova [3, Lemma 3.7]. The admissibility condition regarding the jump times of $F(\cdot, S)$ is ensured by the Lipschitz-continuity of F. $\square$

Remark 3.8 The standard examples of sufficiently regular path-dependent functionals are functionals which depend on the running maximum or on a notion of the average of the underlying path; see e.g. Ananova [3, Example 4.4]. Further examples include

(i) $F(t, S_{\cdot\wedge t}) := \int_0^t \psi(s, S_{\cdot\wedge s-}) \, \mathrm{d}[S]_s$,
(ii) $F(t, S_{\cdot\wedge t}) := \int_0^t \nabla_x f(s, S_{\cdot\wedge s}) \, \mathrm{d}S_s$,
(iii) $F(t, S_{\cdot\wedge t}) := \sum_{i=1}^d (\int_0^t (S_t^i - S_s^i) f_i(S_s^i) \, \mathrm{d}S_s^i - \int_0^t f_i(S_s^i) \, \mathrm{d}[S^i]_s)$

for a twice differentiable function $f = (f_1, \dots, f_d) \colon \mathbb{R}^d \to \mathbb{R}^d$ and a left-continuous and locally bounded function $\psi \colon \Lambda_T^d \to \mathbb{R}^{d\times d}$; see Chiu and Cont [10, Example 4.18].

3.4 Cover's universal portfolio

While functionally generated trading strategies are most prominent in the literature regarding hedging and control problems in mathematical finance, various other trading strategies with desirable properties have been considered. One example coming from portfolio theory is Cover's [12] universal portfolio. The basic idea is to invest not according to one specific trading strategy, but according to a mixture of all admissible strategies. Following Cuchiero et al. [13], we introduce here a model-free analogue of Cover's universal portfolio.

Let $\mathcal{Z}$ be a Borel-measurable subset of $C_b^2(\mathbb{R}^d; \mathbb{R}^d)$ and suppose that ν is a probability measure on $\mathcal{Z}$. A model-free version of Cover's universal portfolio φ^ν is then given by

$$\varphi_t^\nu := \int_{\mathcal{Z}} \varphi_t^f \, \mathrm{d}\nu(f), \qquad t \in [0, \infty), \tag{3.13}$$

where $\varphi^f = f(S)$ is the portfolio generated by f for some fixed $S \in \Omega_p$.

Lemma 3.9 *Let $S \in \Omega_p$ and let ν be a probability measure on $\mathcal{Z}$ as above. If*

$$\int_{\mathcal{Z}} \|f\|_{C_b^2} \,\mathrm{d}\nu(f) < \infty,$$

then Cover's universal portfolio φ^ν as defined in (3.13) is an admissible strategy in the sense of Definition 3.2.

Proof Let $T > 0$. We know from Proposition 3.6 that for each $f \in \mathcal{Z}$, the corresponding functionally generated portfolio $\varphi = \varphi^f$ is an admissible strategy. Let φ' and R^φ denote the corresponding Gubinelli derivative and remainder term. It follows from the inequalities in (3.6) and (3.7) that

$$\begin{aligned} |\varphi_0| + \|\varphi, \varphi'\|_{\mathcal{V}_S^{p,p/2}} &= |\varphi_0| + |\varphi'_0| + \|\varphi'\|_{p,[0,T]} + \|R^\varphi\|_{\frac{p}{2},[0,T]} \\ &\lesssim \|f\|_{C_b^2}(1 + \|S\|_{p,[0,T]}^2), \end{aligned}$$

and hence that

$$\int_{\mathcal{Z}} (|\varphi_0| + \|\varphi, \varphi'\|_{\mathcal{V}_S^{p,p/2}}) \,\mathrm{d}\nu \lesssim (1 + \|S\|_{p,[0,T]}^2) \int_{\mathcal{Z}} \|f\|_{C_b^2} \,\mathrm{d}\nu < \infty. \tag{3.14}$$

Recall that the map $(\varphi, \varphi') \mapsto |\varphi_0| + \|\varphi, \varphi'\|_{\mathcal{V}_S^{p,p/2}}$ is a norm on the Banach space $\mathcal{V}_S^{p,\frac{p}{2}}$ of controlled paths. Note also that the subset of controlled paths $(\varphi, \varphi') \in \mathcal{V}_S^{p,\frac{p}{2}}$ satisfying $J_\varphi \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$ is a closed linear subspace of $\mathcal{V}_S^{p,\frac{p}{2}}$ and thus itself a Banach space. It follows from the integrability condition in (3.14) that the integral in (3.13) exists as a well-defined Bochner integral, and defines a controlled path $(\varphi^\nu, (\varphi^\nu)') \in \mathcal{V}_S^{p,\frac{p}{2}}$ satisfying $J_{\varphi^\nu} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$. □

3.5 Functionally generated portfolios from stochastic portfolio theory

In this section, we briefly discuss admissible strategies appearing in stochastic portfolio theory as initiated by Fernholz [18]; see also e.g. Strong [48] and Karatzas and Ruf [33]. Following the model-free framework for stochastic portfolio theory as introduced in Schied et al. [46] and Cuchiero et al. [13], we consider a d-dimensional càdlàg path $S = (S^1, \dots, S^d)$ such that $S_t^i > 0$ for all $t \geq 0$ and $i = 1, \dots, d$. The total capitalisation Σ is defined by $\Sigma_t := S_t^1 + \cdots + S_t^d$, and the relative market weight process μ is given by

$$\mu_t^i := \frac{S_t^i}{\Sigma_t} = \frac{S_t^i}{S_t^1 + \cdots + S_t^d}, \qquad i = 1, \dots, d,$$

which takes values in the open simplex

$$\Delta_+^d := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1, x_i > 0 \text{ for all } i \right\}.$$

We impose that the market weight process $\mu = (\mu^1, \ldots, \mu^d)$ satisfies property (RIE) with respect to some $p \in (2, 3)$ and a given sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of nested partitions. More precisely, we assume that μ is a price path in the sense of Definition 3.1.

Given a controlled path $\theta \in \mathcal{V}_\mu^{q,r}$, we define its associated value evolution V^θ by $V_t^\theta := \sum_{i=1}^d \theta_t^i \mu_t^i$ for $t \geq 0$. We also denote $Q_t^\theta := V_t^\theta - V_0^\theta - \int_0^t \theta_s \,\mathrm{d}\mu_s$, where $\int_0^t \theta_s \,\mathrm{d}\mu_s$ is interpreted as a rough integral as in Remark 2.5. Similarly to classical mathematical finance (see e.g. Karatzas and Ruf [33, Proposition 2.3]), we can show that every controlled path $\theta \in \mathcal{V}_\mu^{q,r}$ induces a self-financing trading strategy φ, and that every such strategy φ is itself a controlled path in $\mathcal{V}_\mu^{q,r}$.

Proposition 3.10 *Given $\theta \in \mathcal{V}_\mu^{q,r}$ and a constant $C \in \mathbb{R}$, we introduce*

$$\varphi_t^i := \theta_t^i - Q_t^\theta - C, \qquad t \geq 0, i = 1, \ldots, d.$$

Then the resulting path $\varphi = (\varphi^1, \ldots, \varphi^d)$ is a controlled path in $\mathcal{V}_\mu^{q,r}$, and φ is self-financing in the sense that

$$V_t^\varphi - V_0^\varphi = \int_0^t \varphi_s \,\mathrm{d}\mu_s, \qquad t \geq 0,$$

where $V_t^\varphi = \sum_{i=1}^d \varphi_t^i \mu_t^i$. Moreover, if θ is an admissible strategy in the sense of Definition 3.2, then so is φ.

Proof Since $\theta \in \mathcal{V}_\mu^{q,r}$ is a controlled path, by (the trivial extension to càdlàg paths of) Allan et al. [1, Lemma A.1], the path $V^\theta = \sum_{i=1}^d \theta^i \mu^i$ is also a controlled path. Recalling Remark 2.6, we also have that the rough integral $\int_0^\cdot \theta \,\mathrm{d}\mu$ is itself a controlled path (with Gubinelli derivative equal to θ). It is then clear that $Q^\theta = V^\theta - V_0^\theta - \int_0^\cdot \theta_s \,\mathrm{d}\mu_s$ is a controlled path, and hence that $\varphi^i = \theta^i - Q^\theta - C$ is as well for every $i = 1, \ldots, d$, so that $\varphi \in \mathcal{V}_\mu^{q,r}$. The self-financing property of φ can be verified by following the proof of [33, Proposition 2.3].

We know from Proposition 2.14 and the fact that μ satisfies property (RIE) that the jump times J_μ of μ satisfy $J_\mu \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$. It is also clear that the jump times of the integral $\int_0^\cdot \theta \,\mathrm{d}\mu$ form a subset of J_μ. Thus under the assumption that θ is an admissible strategy, so that $J_\theta \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$, we also deduce that $J_\varphi \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$ so that φ is itself an admissible strategy. □

The following definition introduces notions which can be seen as the rough path counterparts of the functionally generated strategies induced by regular and Lyapunov functions from stochastic portfolio theory; see [33, Definitions 3.1 and 3.3].

Definition 3.11 We say that a continuous function $G \colon \Delta_+^d \to \mathbb{R}$ is *regular* for the market weight process μ if

(i) there exists a measurable function $\mathrm{D}G = (\mathrm{D}_1 G, \ldots, \mathrm{D}_d G)$ on Δ_+^d such that the path $\theta = (\theta^1, \ldots, \theta^d)$ given by $\theta^i = \mathrm{D}_i G(\mu)$ for $i = 1, \ldots, d$ is an admissible strategy (in the sense of Definition 3.2), and

(ii) the (càdlàg) path Γ^G given by

$$\Gamma_t^G := G(\mu_0) - G(\mu_t) + \int_0^t \theta_s \, \mathrm{d}\mu_s, \qquad t \geq 0,$$

has locally bounded variation.

We say that a regular function $G \colon \Delta_+^d \to \mathbb{R}$ is a *Lyapunov* function for the market weight process μ if the path Γ^G is also nondecreasing.

Example 3.12 Suppose that $G \colon \Delta_+^d \to \mathbb{R}$ is a C^3 function. It is then straightforward to see that $\theta = \mathrm{D}G(\mu)$ defines a controlled path, with $\mathrm{D}G$ here defined as the classical gradient of G. It also follows from the Itô formula for rough paths (recall Remark 2.20) that

$$\Gamma_t^G = -\frac{1}{2} \int_0^t \mathrm{D}^2 G(\mu_s) \, \mathrm{d}[\mu]_s^c - \sum_{s \leq t} \big(G(\mu_s) - G(\mu_{s-}) - \mathrm{D}G(\mu_{s-}) \Delta \mu_s \big), \quad (3.15)$$

where $[\mu]^c$ is the continuous part of the quadratic variation of μ and $\Delta\mu_s = \mu_s - \mu_{s-}$ denotes the jump of μ at time s.

If we also assume that G is concave, then we infer from (3.15) that Γ^G is nondecreasing. In this case, G is a Lyapunov function in the sense of Definition 3.11. Two important examples of such functions are the Gibbs entropy function defined by $H(x) := \sum_{i=1}^d x_i \log \frac{1}{x_i}$, and the quadratic function where $Q^{(c)}(x) := c - \sum_{i=1}^d x_i^2$ for some $c \in \mathbb{R}$; see [33, Sect. 5.1].

Remark 3.13 If μ is realised by a semimartingale model, then any continuous concave (but not necessarily C^3) function G can be a candidate for a Lyapunov function in the sense of [33, Definition 3.3]. This is essentially because in stochastic portfolio theory, one only needs the "supergradients" $\mathrm{D}G$ to be measurable, so that $\mathrm{D}G(\mu)$ is integrable with respect to the semimartingale μ. In contrast, in our purely pathwise setup, measurability of $\mathrm{D}G$ alone is not enough to ensure that $\theta = \mathrm{D}G(\mu)$ is controlled by μ; so we require more regularity of the generating function G. This illustrates a difference between stochastic integration in a probabilistic setting and rough integration when only a single deterministic path is considered; for more detailed discussions on this theme, we refer to Allan et al. [1]. On the other hand, as shown in the previous example, many important Lyapunov functions from stochastic portfolio theory (such as the Gibbs entropy function) are actually smooth; so they do induce controlled paths (and indeed admissible strategies in the sense of Definition 3.2) for almost all trajectories of μ, and the stability results established in Sect. 3 remain valid for these functionally generated strategies. It would be interesting to explore further Lyapunov functions for rough paths, but this is beyond the scope of the present paper.

Based on the previous observations, we can also extend the following notions from classical stochastic portfolio theory (e.g. [33]) to our rough path setting. The proof is straightforward and therefore omitted for brevity.

Corollary 3.14 *Suppose that the market weight process $\mu = (\mu^1, \dots, \mu^d)$ is a price path in the sense of Definition 3.1, and let $\theta = (\theta^1, \dots, \theta^d)$ be an admissible strategy for μ in the sense of Definition 3.2. Let G be a regular function for μ in the sense of Definition 3.11. Then the following paths are also admissible strategies for μ:*

(i) *the additively generated strategy $\varphi_t^i = \theta_t^i - Q_t^\theta - C_0$, $i = 1, \dots, d$, where we set $Q_t^\theta = V_t^\theta - V_0^\theta - \int_0^t \theta_s \, \mathrm{d}\mu_s$ and $C_0 = \sum_{j=1}^d \mu_0^i \mathrm{D}_j G(\mu_0) - G(\mu_0)$;*

(ii) *the portfolio weights associated to φ, given by $\pi_t^i = \frac{\mu_t^i \varphi_t^i}{\sum_{j=1}^d \mu_t^j \varphi_t^j}$, $i = 1, \dots, d$;*

(iii) *the multiplicatively generated strategy $\Phi_t^i = \eta_t^i - Q_t^\eta - C_0$, where we set $\eta_t^i = \theta_t^i \exp(\int_0^t \frac{\mathrm{d}\Gamma_s^G}{G(\mu_s)})$, $i = 1, \dots, d$, and $Q_t^\eta = V_t^\eta - V_0^\eta - \int_0^t \eta_s \, \mathrm{d}\mu_s$, and the function G is also assumed to be positive and bounded away from zero;*

(iv) *the portfolio weights associated to Φ, given by*

$$\Pi_t^i = \mu_t^i \bigg(1 + \frac{1}{G(\mu_t)} \Big(\mathrm{D}_i G(\mu_t) - \sum_{j=1}^d \mathrm{D}_j G(\mu_t) \mu_t^j\Big)\bigg), \qquad i = 1, \dots, d.$$

4 Semimartingales and typical price paths satisfy property (RIE)

In this section, we show that many stochastic processes commonly used to model the price evolutions on financial markets satisfy property (RIE) along a suitable sequence of partitions. In particular, we verify that property (RIE) holds for semimartingales and for typical price paths in the sense of Vovk [51].

4.1 Semimartingales

Semimartingales, such as geometric Brownian motion and Markov jump-diffusion processes, serve as the most frequently used stochastic processes to model price evolutions on financial markets. For more details on semimartingales and Itô integration, we refer to the standard textbook by Protter [43, Chap. II].

Usually the considered class of semimartingales is restricted to those satisfying the condition "no free lunch with vanishing risk" in classical mathematical finance, see e.g. Delbaen and Schachermayer [15], or the condition "no unbounded profit with bounded risk" (NUPBR) in stochastic portfolio theory, see e.g. Karatzas and Kardaras [31]. Such a restriction is not required here. Property (RIE) is fulfilled by general càdlàg semimartingales with respect to any $p \in (2, 3)$ and a suitable (random) sequence of partitions.

Let us fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ satisfies the usual conditions of completeness and right-continuity. On this, we consider a d-dimensional càdlàg semimartingale $X = (X_t)_{t \in [0,\infty)}$.

For each $n \in \mathbb{N}$, we introduce stopping times $(\tau_k^n)_{k \in \mathbb{N} \cup \{0\}}$ by $\tau_0^n = 0$ and

$$\tau_k^n := \inf\{t > \tau_{k-1}^n : |X_t - X_{\tau_{k-1}^n}| \geq 2^{-n}\}, \qquad k \in \mathbb{N}.$$

We then define a sequence $(\mathcal{P}_X^n)_{n \in \mathbb{N}}$ of partitions by

$$\mathcal{P}_X^n := \{\tau_k^m : m \leq n, k \in \mathbb{N} \cup \{0\}\}.$$

Note that $(\mathcal{P}^n_X)_{n\in\mathbb{N}}$ is a nested sequence of adapted partitions. However, this sequence of partitions will not have vanishing mesh size if the path X has an interval of constancy. To amend this, we proceed as follows. For each $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned}\sigma^n_k &:= \tau^n_{k+1} \wedge \inf\{t > \tau^n_k : \exists\, \delta > 0 \text{ such that } X_s = X_t \text{ for all } s \in [t, t+\delta]\},\\ \varsigma^n_k &:= \tau^n_{k+1} \wedge \inf\{t > \sigma^n_k : X_t \neq X_{\sigma^n_k}\},\end{aligned}$$

that is, the beginning and end points of the first interval of constancy of X within the interval $[\tau^n_k, \tau^n_{k+1}]$. For each $i \in \mathbb{N}$, we then define $\rho^n_{k,i} := (\sigma^n_k + i2^{-n}) \wedge \varsigma^n_k$. Clearly, for each n, k, we have $\rho^n_{k,i} = \varsigma^n_k$ for all but finitely many $i \in \mathbb{N}$ (provided that $\varsigma^n_k < \infty$). Finally, we define

$$\mathcal{Q}^n_X := \mathcal{P}^n_X \cup \left\{\sigma^m_k : m \leq n, k \in \mathbb{N} \cup \{0\}\right\} \cup \left\{\rho^m_{k,i} : m \leq n, k \in \mathbb{N} \cup \{0\}, i \in \mathbb{N}\right\}.$$

The sequence $(\mathcal{Q}^n_X)_{n\in\mathbb{N}}$ of partitions is still nested and moreover has vanishing mesh size on every compact time interval. Moreover, it is straightforward to see that all the points τ^n_k, σ^n_k and $\rho^n_{k,i}$ appearing in the partitions $(\mathcal{Q}^n_X)_{n\in\mathbb{N}}$ are stopping times with respect to $\mathbb{F}$. In the next proposition, we show that X satisfies property (RIE) with respect to any $p \in (2, 3)$ and the sequence $(\mathcal{Q}^n_X)_{n\in\mathbb{N}}$ of partitions.

Proposition 4.1 *Let $p \in (2, 3)$, and let X be a d-dimensional càdlàg semimartingale. Then almost every sample path of X is a price path in the sense of Definition 3.1.*

Proof *Step 1.* Fix a $T > 0$. Let $\int_0^\cdot X_{u-} \otimes \mathrm{d}X_u$ denote the Itô integral of X with respect to itself. For each $n \in \mathbb{N}$, we define the discretised process $X^n = (X^n_t)_{t\in[0,T]}$ by

$$X^n_t := \sum_{[u,v]\in\mathcal{Q}^n_X([0,T])} X_u \mathbf{1}_{[u,v)}(t), \qquad t \in [0, T], \tag{4.1}$$

so that in particular $\int_0^t X^n_{u-} \otimes \mathrm{d}X_u = \sum_{[u,v]\in\mathcal{Q}^n_X([0,T])} X_u \otimes X_{u\wedge t, v\wedge t}$ for $t \in [0, T]$. By the definition of the partition $\mathcal{Q}^n_X$, we have

$$\|X^n_{\cdot-} - X_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n} \qquad \text{for all } n \geq 1.$$

An application of the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma as in the proof of Liu and Prömel [35, Proposition 3.4] then yields the existence of a measurable set $\Omega' \subseteq \Omega$ with full measure such that for every $\omega \in \Omega'$ and every $\varepsilon \in (0, 1)$, there exists a constant $C = C(\varepsilon, \omega)$ such that

$$\left\|\left(\int_0^\cdot X^n_{u-} \otimes \mathrm{d}X_u - \int_0^\cdot X_{u-} \otimes \mathrm{d}X_u\right)(\omega)\right\|_{\infty,[0,T]} \leq C2^{-n(1-\varepsilon)}, \qquad \forall n \geq 1. \tag{4.2}$$

Thus we have $X^n(\omega) \to X(\omega)$ and $\int_0^\cdot X^n_{u-} \otimes \mathrm{d}X_u(\omega) \to \int_0^\cdot X_{u-} \otimes \mathrm{d}X_u(\omega)$ uniformly as $n \to \infty$, for every $\omega \in \Omega'$.

Step 2. We choose $q_0 \in (2,3)$ close enough to 2 and $\varepsilon \in (0,1)$ small enough such that

$$\frac{p}{2} > \max\left\{\frac{q_0}{2}, q_0 - 1\right\} \qquad \text{and} \qquad \frac{p}{2} \geq \frac{q_0 - 1 - \varepsilon}{1 - \varepsilon}. \tag{4.3}$$

We also fix a control function w_{X,q_0} such that

$$|X_{s,t}|^{q_0} \leq w_{X,q_0}(s,t) \qquad \text{for all } (s,t) \in \Delta_{[0,T]}. \tag{4.4}$$

This is always possible as X has almost surely finite q-variation for every $q > 2$ (see e.g. Lépingle [34, Théorème 1]), and so without loss of generality, we may assume that $X(\omega)$ has finite q_0-variation for every $\omega \in \Omega'$. Note that by (4.3), we have $p > q_0$ and consequently (by increasing the values of w_{X,q_0} by a multiplicative constant if necessary) we can also assume that

$$\sup_{(s,t)\in\Delta_{[0,T]}} \frac{|X_{s,t}|^p}{w_{X,q_0}(s,t)} \leq 1. \tag{4.5}$$

Let $0 \leq s < t \leq T$ be such that $s = \tau^n_{k_0}$ and $t = \tau^n_{k_0+N}$ for some $n \in \mathbb{N}$, $k_0 \in \mathbb{N} \cup \{0\}$ and $N \geq 1$. By the superadditivity of the control function w_{X,q_0}, there must exist an $\ell \in \{1, 2, \ldots, N-1\}$ such that

$$w_{X,q_0}(\tau^n_{k_0+\ell-1}, \tau^n_{k_0+\ell+1}) \leq \frac{2}{N-1} w_{X,q_0}(s,t).$$

Thus by (4.4),

$$\begin{aligned}
&|X_{\tau^n_{k_0+\ell-1}} \otimes X_{\tau^n_{k_0+\ell-1},\tau^n_{k_0+\ell}} + X_{\tau^n_{k_0+\ell}} \otimes X_{\tau^n_{k_0+\ell},\tau^n_{k_0+\ell+1}} - X_{\tau^n_{k_0+\ell-1}} \otimes X_{\tau^n_{k_0+\ell-1},\tau^n_{k_0+\ell+1}}| \\
&= |X_{\tau^n_{k_0+\ell-1},\tau^n_{k_0+\ell}} \otimes X_{\tau^n_{k_0+\ell},\tau^n_{k_0+\ell+1}}| \\
&\leq w_{X,q_0}(\tau^n_{k_0+\ell-1}, \tau^n_{k_0+\ell+1})^{2/q_0} \\
&\leq \left(\frac{2}{N-1} w_{X,q_0}(s,t)\right)^{2/q_0}.
\end{aligned}$$

By successively removing in this manner all the intermediate points from the partition $\{s = \tau^n_{k_0}, \tau^n_{k_0+1}, \ldots, \tau^n_{k_0+N} = t\}$, we obtain the estimate

$$\begin{aligned}
\left|\sum_{i=0}^{N-1} X_{\tau^n_{k_0+i}} \otimes X_{\tau^n_{k_0+i},\tau^n_{k_0+i+1}} - X_s \otimes X_{s,t}\right| &\leq \sum_{j=2}^{N} \left(\frac{2}{j-1} w_{X,q_0}(s,t)\right)^{2/q_0} \\
&\lesssim N^{1-2/q_0} w_{X,q_0}(s,t)^{2/q_0}.
\end{aligned}$$

Since

$$w_{X,q_0}(s,t) \geq \sum_{i=0}^{N-1} w_{X,q_0}(\tau^n_{k_0+i}, \tau^n_{k_0+i+1}) \geq \sum_{i=0}^{N-1} |X_{\tau^n_{k_0+i},\tau^n_{k_0+i+1}}|^{q_0} \geq N 2^{-nq_0},$$

we obtain $N \leq 2^{nq_0} w_{X,q_0}(s,t)$. Substituting this into the above, we have

$$\left| \int_s^t X_{u-}^n \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right| \lesssim 2^{n(q_0-2)} w_{X,q_0}(s,t), \tag{4.6}$$

where the discretised process X^n is defined relative to the partition $\{\tau_0^n, \tau_1^n, \tau_2^n, \dots\}$.

If more generally $0 \leq s < t \leq T$ are such that $s, t \in \mathcal{P}_X^n$, then $s = \tau_{k_1}^{m_1}$ and $t = \tau_{k_2}^{m_2}$ for some $m_1, m_2 \leq n$ and $k_1, k_2 \in \mathbb{N} \cup \{0\}$. In this case, the number of partition points N above satisfies $N \leq \sum_{m=1}^n 2^{mq_0} w_{X,q_0}(s,t) \lesssim 2^{nq_0} w_{X,q_0}(s,t)$, and we thus still obtain the same bound in (4.6). If we further allow the pair of times s, t to include the times σ_k^m for $m \leq n$ and $k \in \mathbb{N} \cup \{0\}$, this at most doubles the total number of partition points N so that we can again obtain the same bound. Since the points $\rho_{k,i}^m$ lie inside the interval $[\sigma_k^m, \varsigma_k^m]$ on which X is constant, it is clear for instance that $X_{\sigma_k^m, \rho_{k,i}^m} = 0$ and $\int_{\sigma_k^m}^{\rho_{k,i}^m} X_{u-}^n \otimes \mathrm{d}X_u = 0$ for every $i \in \mathbb{N}$, and it follows that these terms do not contribute anything to the bound above.

Thus the bound in (4.6) actually holds for all $0 \leq s < t \leq T$ with $s, t \in \mathcal{Q}_X^n$, with X^n defined relative to the partition $\mathcal{Q}_X^n$ as in (4.1).

Step 3. We first consider the case that $w_{X,q_0}(s,t)^{\frac{2}{p(1-\varepsilon)}} \leq 2^{-n}$. Then it follows from (4.3) and (4.6) that there exists a constant $C_1 = C_1(\omega, q_0, \varepsilon)$ such that

$$\left| \int_s^t X_{u-}^n \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right|^{\frac{p}{2}} \leq C_1 w_{X,q_0}(s,t). \tag{4.7}$$

Now we consider the case that $w_{X,q_0}(s,t)^{\frac{2}{p(1-\varepsilon)}} \geq 2^{-n}$. Let $\mathbb{X}_{s,t} := \int_s^t X_{s,u-} \otimes \mathrm{d}X_u$ be the second level component of the Itô lift of X. By Liu and Prömel [35, Proposition 3.4], we know that $\mathbb{X}$ possesses finite $\frac{p}{2}$-variation; that is, there exists a control function $w_{\mathbb{X},\frac{p}{2}}$ such that

$$\sup_{(s,t) \in \Delta_{[0,T]}} \frac{|\mathbb{X}_{s,t}|^{\frac{p}{2}}}{w_{\mathbb{X},\frac{p}{2}}(s,t)} \leq 1.$$

Then in view of (4.2), we obtain

$$\begin{aligned}
\left| \int_s^t X_{u-}^n \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right| &\leq 2 \left\| \int_0^\cdot X_{u-}^n \otimes \mathrm{d}X_u - \int_0^\cdot X_{u-} \otimes \mathrm{d}X_u \right\|_{\infty,[0,T]} \\
&\quad + |\mathbb{X}_{s,t}| \\
&\leq C_2 \big(2^{-n(1-\varepsilon)} + w_{\mathbb{X},\frac{p}{2}}(s,t)^{\frac{2}{p}} \big) \\
&\leq C_2 \big(w_{X,q_0}(s,t)^{\frac{2}{p}} + w_{\mathbb{X},\frac{p}{2}}(s,t)^{\frac{2}{p}} \big)
\end{aligned}$$

for some constant $C_2 = C_2(\varepsilon, \omega)$, and hence

$$\left| \int_s^t X_{u-}^n \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right|^{\frac{p}{2}} \leq C_3 \big(w_{X,q_0}(s,t) + w_{\mathbb{X},\frac{p}{2}}(s,t) \big), \tag{4.8}$$

where $C_3 = (2C_2)^{\frac{p}{2}}$. Letting $\tilde{w}_{X,p}(s,t) := 2(1+C_1+C_3)(w_{X,q_0}(s,t)+w_{\mathbb{X},\frac{p}{2}}(s,t))$ and combining (4.5), (4.7) and (4.8), we conclude that for every $\omega \in \Omega'$,

$$\sup_{(s,t)\in\Delta_{[0,T]}} \frac{|X_{s,t}|^p}{\tilde{w}_{X,p}(s,t)} + \sup_{n\in\mathbb{N}} \sup_{\substack{(s,t)\in\Delta_{[0,T]} \\ s,t\in\mathcal{Q}_X^n}} \frac{|\int_s^t X_{u-}^n \otimes \mathrm{d}X_u - X_s \otimes X_{s,t}|^{\frac{p}{2}}}{\tilde{w}_{X,p}(s,t)} \leq 1,$$

from which property (RIE) follows. □

Remark 4.2 The result of Proposition 4.1 implies that almost every sample path of a semimartingale X satisfies (RIE) with respect to a suitable sequence of partitions (which depends on the choice of sample path), and may therefore be canonically lifted to a rough path. We also saw in Step 1 of the proof of Proposition 4.1 that the Riemann sums $\int_0^\cdot X_{u-}^n \otimes \mathrm{d}X_u$ converge almost surely to the Itô integral $\int_0^\cdot X_{u-} \otimes \mathrm{d}X_u$, from which it follows that the rough path lift constructed via property (RIE) is nothing but the standard Itô-rough path lift of X, i.e., the function $(s,t) \mapsto \int_s^t X_{s,u} \otimes \mathrm{d}X_u$ with the integral defined in the sense of Itô. Thus the rough path lift itself does not actually depend on the choice of sequence of partitions. For (almost) every sample path, the sequence of partitions specified in property (RIE) is merely a choice of partitions along which the value of the Itô integral may be well approximated by left-point Riemann sums in a pathwise fashion.

This is analogous to Föllmer integration in which the quadratic variation (as opposed to the 2-variation) of a semimartingale is well defined as a stochastic process; but to make sense of the quadratic variation in a pathwise sense requires a suitable choice of sequence of partitions, which depends on the sample path being considered. Indeed, given any continuous path, it is possible to find a sequence of partitions along which the quadratic variation of the path is equal to zero (see Freedman [23, (70) Proposition]). Similarly, it is natural to allow the sequence of partitions specified in property (RIE) to depend on the path being considered; but in practice, there typically exists a natural choice for this sequence which results in the desired rough path.

Remark 4.3 Proposition 4.1 holds true even if the semimartingale X takes values in an (infinite dimensional) Hilbert space E, as long as the norm on $E \otimes E$ is admissible in the sense of Lyons et al. [40, Definition 1.25]. In particular, an extension of Proposition 4.1 to so-called piecewise semimartingales, which were introduced in Strong [47, Definition 2.2] as generalised semimartingales with an image dimension evolving randomly in time, appears to be straightforward to implement. As discussed in Karatzas and Kim [32, Remark 6.2 and Sect. 7], piecewise semimartingales provide a realistic framework to model so-called open markets, which are financial markets with an evolving number of traded assets.

4.2 Generalised semimartingales

It is a well observed fact in the empirical literature, see e.g. Lo [36], that price processes appear regularly in financial markets which are not semimartingales. Motivated by this fact, many researchers have proposed and investigated financial models based on fractional Brownian motions; see for instance Jarrow et al.

[29], Cheridito [8] or Bender [6]. One example of such models are the so-called mixed Black–Scholes models. In these models, the (one-dimensional) price process $S = (S_t)_{t\in[0,\infty)}$ is usually given by

$$S_t := s_0 \exp(\sigma W_t + \eta Y_t + \nu t + \mu t^{2H}), \qquad t \in [0, \infty), \tag{4.9}$$

for constants $s_0, \sigma, \eta > 0$ and $\nu, \mu \in \mathbb{R}$, where $W = (W_t)_{t\in[0,\infty)}$ is a standard Brownian motion and $Y = (Y_t)_{t\in[0,\infty)}$ is a fractional Brownian motion with some Hurst index $H \in (0, 1)$. Multi-dimensional versions of the mixed Black–Scholes model (4.9) can be obtained by standard modifications. Notice that while the price process S as defined in (4.9) is not a semimartingale if $H \neq 1/2$, the mixed Black–Scholes model (4.9) is still arbitrage-free when restricting the admissible trading strategies to classes of trading strategies which, roughly speaking, exclude continuous rebalancing of the positions in the underlying market; cf. [29, 8, 6]. In particular, the mixed Black–Scholes model (4.9) is free of simple arbitrage opportunities if $H > 1/2$, as proved in [6, Sect. 4.1].

In order to demonstrate that property (RIE) is satisfied by various financial models based on fractional Brownian motion, we consider the following class of generalised semimartingales. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a complete right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$, let Z be a d-dimensional process admitting the decomposition

$$Z_t = X_t + Y_t, \qquad t \in [0, \infty),$$

where X is a semimartingale and Y is a càdlàg adapted process with finite q-variation for some $q \in [1, 2)$. Processes Z of this form are sometimes called *Young semimartingales* and belong to the class of càdlàg Dirichlet processes in the sense of Föllmer [20].

We introduce stopping times (τ_k^n) such that for each $n \in \mathbb{N}$, $\tau_0^n = 0$ and

$$\tau_k^n := \inf\{t > \tau_{k-1}^n : |X_t - X_{\tau_{k-1}^n}| \geq 2^{-n} \text{ or } |Y_t - Y_{\tau_{k-1}^n}| \geq 2^{-n}\}$$

for $k \in \mathbb{N}$, and set

$$\mathcal{P}_Z^n := \left\{\tau_k^m : m \leq n, k \in \mathbb{N} \cup \{0\}\right\}.$$

As in the previous section, since we insist that the sequence of partitions in property (RIE) has vanishing mesh size, we also define

$$\mathcal{Q}_Z^n := \mathcal{P}_Z^n \cup \left\{\sigma_k^m : m \leq n, k \in \mathbb{N} \cup \{0\}\right\} \cup \left\{\rho_{k,i}^m : m \leq n, k \in \mathbb{N} \cup \{0\}, i \in \mathbb{N}\right\},$$

where the times σ_k^m and $\rho_{k,i}^m$ are defined analogously as in Sect. 4.1.

Proposition 4.4 *Let $Z = X + Y$ be a d-dimensional process such that X is a càdlàg semimartingale and Y a càdlàg process with finite q-variation for some $q \in [1, 2)$. Then for any $p \in (2, 3)$ such that $1/p + 1/q > 1$, almost every sample path of Z is a price path in the sense of Definition 3.1.*

Proof *Step 1.* It is enough to prove that almost all sample paths of Z satisfy property (RIE) along $(\mathcal{Q}^n_Z([0,T]))_{n\in\mathbb{N}}$ for any $T > 0$. To this end, for each $n \in \mathbb{N}$, we set

$$Z^n_t := \sum_{[u,v]\in\mathcal{Q}^n_Z([0,T])} Z_u \mathbf{1}_{[u,v)}(t), \qquad t \in [0,T],$$

and define X^n and Y^n in the same way with respect to the partition $\mathcal{Q}^n_Z([0,T])$.

Let $X = M + A$ be a decomposition of the semimartingale X such that M is a locally square-integrable martingale and A is of bounded variation. By then setting $B := A + Y$, we can write $Z = M + B$, where B has finite q-variation. We define the Itô integral of Z with respect to itself by

$$\int_0^t Z_{u-} \otimes \mathrm{d}Z_u := \int_0^t Z_{u-} \otimes \mathrm{d}M_u + \int_0^t Z_{u-} \otimes \mathrm{d}B_u,$$

where the first integral on the right-hand side is an Itô integral and the second is interpreted as a Young integral, which exists because Z has finite p-variation and we have $1/p + 1/q > 1$; see e.g. Friz and Zhang [26]. Then since $\|Z^n_{\cdot-} - Z_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n}$, the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma imply that for almost all ω and for every $\varepsilon \in (0,1)$, there exists a constant $C = C(\omega,\varepsilon)$ such that for all $n \in \mathbb{N}$,

$$\left\| \int_0^\cdot Z^n_{u-} \otimes \mathrm{d}M_u - \int_0^\cdot Z_{u-} \otimes \mathrm{d}M_u \right\|_{\infty,[0,T]} \leq C 2^{-n(1-\varepsilon)}; \tag{4.10}$$

cf. the proof of Liu and Prömel [35, Proposition 3.4]. By a standard estimate for Young integrals (e.g. [26, Proposition 2.4]), for every $t \in [0,T]$, we also have (noting that $Z^n_0 = Z_0$ for all n) that

$$\left| \int_0^t (Z^n_{u-} - Z_{u-}) \otimes \mathrm{d}B_u \right| \leq C \|Z^n_{\cdot-} - Z_{\cdot-}\|_{p,[0,t]} \|B\|_{q,[0,T]}$$

for some constant $C = C(p,q)$. Since $\|Z^n_{\cdot-}\|_{p_0,[0,t]} \leq \|Z\|_{p_0,[0,T]}$ for every n and every $p_0 \in (2,p)$, a routine interpolation argument shows that for each $n \in \mathbb{N}$,

$$\|Z^n_{\cdot-} - Z_{\cdot-}\|_{p,[0,T]} \leq C \|Z^n_{\cdot-} - Z_{\cdot-}\|_{\infty,[0,T]}^{1-\frac{p_0}{p}} \|Z\|_{p_0,[0,T]}^{\frac{p_0}{p}}$$

for some constant $C = C(p,p_0)$. Hence, since $\|Z^n_{\cdot-} - Z_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n}$ for all n, we have

$$\lim_{n\to\infty} \left\| \int_0^\cdot Z^n_{u-} \otimes \mathrm{d}B_u - \int_0^\cdot Z_{u-} \otimes \mathrm{d}B_u \right\|_{\infty,[0,T]} = 0. \tag{4.11}$$

Combining the bound in (4.10) with (4.11), we conclude that almost surely, the integral $\int_0^\cdot Z^n_{u-} \otimes \mathrm{d}Z_u$ converges uniformly to $\int_0^\cdot Z_{u-} \otimes \mathrm{d}Z_u$.

Step 2. For every $n \in \mathbb{N}$, we set

$$\begin{aligned}\mathbb{Z}^n_{s,t} &:= \int_s^t Z^n_{s,u-} \otimes \mathrm{d}Z_u \\ &= \int_s^t X^n_{s,u-} \otimes \mathrm{d}X_u + \int_s^t Y^n_{s,u-} \otimes \mathrm{d}Y_u + \int_s^t X^n_{s,u-} \otimes \mathrm{d}Y_u + \int_s^t Y^n_{s,u-} \otimes \mathrm{d}X_u\end{aligned}$$

for $(s,t) \in \Delta_{[0,T]}$. Moreover, we define

$$\mathbb{X}^n_{s,t} := \int_s^t X^n_{s,u-} \otimes \mathrm{d}X_u, \qquad \mathbb{Y}^n_{s,t} := \int_s^t Y^n_{s,u-} \otimes \mathrm{d}Y_u,$$
$$\mathbb{XY}^n_{s,t} := \int_s^t X^n_{s,u-} \otimes \mathrm{d}Y_u, \qquad \mathbb{YX}^n_{s,t} := \int_s^t Y^n_{s,u-} \otimes \mathrm{d}X_u$$

for $(s,t) \in \Delta_{[0,T]}$. We seek a control function c such that

$$\sup_{n\in\mathbb{N}} \sup_{\substack{(s,t)\in\Delta_{[0,T]} \\ s,t\in\mathcal{Q}^n_Z([0,T])}} \frac{|\mathbb{Z}^n_{s,t}|^{\frac{p}{2}}}{c(s,t)} \leq 1.$$

Towards this end, we first construct a control function $c_{\mathbb{X}}$ such that the above bound holds for $\mathbb{X}^n$. Since $\|X^n_{\cdot-} - X_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n}$, we still have the bound in (4.2). We also choose $q_0 \in (2,3)$ and $\varepsilon \in (0,1)$ as in (4.3), and let w_{X,q_0} and $w_{Y,q}$ be control functions dominating the q_0-variation and q-variation for X and Y, respectively. From the definition of the stopping times $\tau^m_k \in \mathcal{P}^n_Z$ and the fact that $q < 2 < q_0$, it is easy to check that for all $s < t$ with $s,t \in \mathcal{P}^n_Z$, the number N of partition points in $\mathcal{P}^n_Z$ between s and t can be bounded by

$$N \leq \sum_{m=1}^n \big(2^{mq_0} w_{X,q_0}(s,t) + 2^{mq} w_{Y,q}(s,t)\big) \lesssim 2^{nq_0} w_{q_0,q}(s,t),$$

where $w_{q_0,q}(s,t) := w_{X,q_0}(s,t) + w_{Y,q}(s,t)$. By the same argument as in Step 2 of the proof of Proposition 4.1, we deduce that for all $n \in N$ and all $s < t$ with $s,t \in \mathcal{Q}^n_Z$,

$$\left| \int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right| \lesssim 2^{n(q_0-2)} w_{q_0,q}(s,t),$$

which as in Step 3 of the proof of Proposition 4.1 allows us to deduce the existence of a control function $c_{\mathbb{X}}$ such that

$$\sup_{n\in\mathbb{N}} \sup_{\substack{(s,t)\in\Delta_{[0,T]} \\ s,t\in\mathcal{Q}^n_Z([0,T])}} \frac{|\mathbb{X}^n_{s,t}|^{\frac{p}{2}}}{c_{\mathbb{X}}(s,t)} \leq 1.$$

Next we use the local estimates of Young integration to show that there exists a control $c_{\mathbb{XY}}$ such that the above bound holds for $\mathbb{XY}^n$ and $c_{\mathbb{XY}}$. Indeed, by [26, Proposition 2.4], we have for all $s < t$ in $[0, T]$ that

$$\begin{aligned}\left|\int_s^t X^n_{s,u-} \otimes \mathrm{d}Y_u\right|^{p/2} &\le C_{p,q}\|Y\|^{p/2}_{q,[s,t]|}\|X^n\|^{p/2}_{p,[s,t]}\\ &\le C_{p,q}\|Y\|^{p/2}_{q,[s,t]}\|X\|^{p/2}_{p,[s,t]}\\ &\le C_{p,q} w_{Y,q}(s,t)^{p/2q} w_{X,p}(s,t)^{1/2}\end{aligned}$$

for some constant $C_{p,q}$ depending only on p and q. Since $p \in (2, 3)$ and $q \in [1, 2)$, we have $p/2q > 1/2$ and thus $p/2q + 1/2 > 1$, which implies that the function $(s,t) \mapsto w_{Y,q}(s,t)^{p/2q} w_{X,p}(s,t)^{1/2}$ is superadditive and hence itself a control function. Thus the control function $c_{\mathbb{X}\mathbb{Y}} := C_{p,q} w_{Y,q}(s,t)^{p/2q} w_{X,p}(s,t)^{1/2}$ gives the desired bound. Similarly, we can find control functions $c_{\mathbb{Y}\mathbb{X}}$ and $c_{\mathbb{Y}}$ for $\mathbb{Y}\mathbb{X}^n$ and $\mathbb{Y}^n$, respectively. Hence our claim follows by noting that

$$|\mathbb{Z}^n| \le |\mathbb{X}^n| + |\mathbb{Y}^n| + |\mathbb{X}\mathbb{Y}^n| + |\mathbb{Y}\mathbb{X}^n|.$$

We thus deduce that the sample paths of Z almost surely satisfy property (RIE) along $(\mathcal{Q}^n_Z([0,T]))_{n\in\mathbb{N}}$. □

4.3 Typical price paths

The notion of "typical price paths" was introduced by Vovk, who introduced a model-free hedging-based approach to mathematical finance allowing to investigate the sample path properties of such "typical price paths" based on arbitrage considerations; see for instance Vovk [49, 51] or Perkowski and Prömel [42]. Let us briefly recall the basic setting and definitions of Vovk's approach.

We denote by $\Omega_+ := D([0,\infty); \mathbb{R}^d_+)$ the space of all nonnegative càdlàg functions $\omega\colon [0,\infty) \to \mathbb{R}^d_+$. For each $t \in [0,\infty)$, $\mathcal{F}^\circ_t$ is defined as the smallest σ-algebra on Ω_+ that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable, and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}^\circ_t$. Stopping times $\tau\colon \Omega_+ \to [0,\infty) \cup \{\infty\}$ with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual. The coordinate process on Ω_+ is denoted by S, i.e., $S_t(\omega) := \omega(t)$ for $t \in [0,\infty)$.

A process $H\colon \Omega_+ \times [0,\infty) \to \mathbb{R}^d$ is a *simple (trading) strategy* if there exist a sequence of stopping times $0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots$ such that for every $\omega \in \Omega_+$, there exists an $N(\omega) \in \mathbb{N}$ such that $\sigma_n(\omega) = \sigma_{n+1}(\omega)$ for all $n \ge N(\omega)$, and a sequence of $\mathcal{F}_{\sigma_n}$-measurable bounded functions $h_n\colon \Omega_+ \to \mathbb{R}^d$ such that we have $H_t(\omega) = \sum_{n=0}^\infty h_n(\omega)\mathbf{1}_{(\sigma_n(\omega),\sigma_{n+1}(\omega)]}(t)$ for $t \in [0,\infty)$. For a simple strategy H, the corresponding integral process

$$(H \cdot S)_t(\omega) := \sum_{n=0}^\infty h_n(\omega) S_{\sigma_n\wedge t,\sigma_{n+1}\wedge t}(\omega)$$

is well defined for all $(t,\omega) \in [0,\infty) \times \Omega_+$. For $\lambda > 0$, we write $\mathcal{H}_\lambda$ for the set of all simple strategies H such that $(H \cdot S)_t(\omega) \ge -\lambda$ for all $(t,\omega) \in [0,\infty) \times \Omega_+$.

Definition 4.5 *Vovk's outer measure* $\overline{P}$ of a set $A \subseteq \Omega_+$ is defined as the minimal super-hedging price for $\mathbf{1}_A$, that is,

$$\overline{P}(A) := \inf \Big\{ \lambda > 0 : \exists\, (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ such that} \\ \liminf_{n\to\infty} \big(\lambda + (H^n \cdot S)_T(\omega) \big) \geq \mathbf{1}_A(\omega), \forall \omega \in \Omega_+ \Big\}.$$

A set $A \subseteq \Omega_+$ is called a *nullset* if $\overline{P}(A) = 0$. A property (P) holds for *typical price paths* if the set A where (P) is violated is a nullset.

Remark 4.6 Loosely speaking, the outer measure $\overline{P}$ corresponds to the (model-free) notion of "no unbounded profit with bounded risk"; see Perkowski and Prömel [42, Sect. 2.2] for a more detailed discussion in this direction. Furthermore, the outer measure $\overline{P}$ dominates all local martingale measures on the space Ω_+; see Łochowski et al. [37, Lemma 2.3 and Proposition 2.5]. As a consequence, all results proved for typical price paths hold simultaneously under all martingale measures (or in other words quasi-surely with respect to all martingale measures).

Let us recall that typical price paths are of finite p-variation for every $p > 2$ (see Vovk [50, Theorem 1]), and that Vovk's model-free framework allows setting up a model-free Itô integration; see e.g. Łochowski et al. [37].

Lemma 4.7 *Typical price paths are price paths in the sense of Definition 3.1, with any $p \in (2, 3)$.*

The proof of Lemma 4.7 works verbatim as that of Proposition 4.1, keeping in mind Liu and Prömel [35, Proposition 3.10] and [37, Corollary 4.9], and is therefore omitted for brevity.

4.4 Consistency of rough and stochastic integration

In a probabilistic framework when the underlying process is a semimartingale, one can employ either rough or stochastic Itô integration. In this subsection, we briefly demonstrate that under property (RIE), these two integrals actually coincide almost surely whenever both are defined.

Let us again fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ satisfies the usual conditions.

Proposition 4.8 *Let X be a d-dimensional càdlàg semimartingale and Y an adapted càdlàg process such that for almost every $\omega \in \Omega$, the path $Y(\omega) \in \mathcal{A}_{X(\omega)}$ is an admissible strategy (in the sense of Definition 3.2). Then the rough and Itô integrals of Y against X coincide almost surely. That is,*

$$\int_0^t Y_s(\omega)\, \mathrm{d}\mathbf{X}_s(\omega) = \bigg(\int_0^t Y_{s-}\, \mathrm{d}X_s \bigg)(\omega) \qquad \textit{for all } t \in [0, \infty)$$

holds for almost every $\omega \in \Omega$, where $\mathbf{X}(\omega)$ is the canonical rough path lift of $X(\omega)$ as defined in Lemma 2.13.

Proof Fix $T > 0$. By Proposition 4.1, we know that for any $p \in (2, 3)$, almost every sample path of X is a price path and satisfies property (RIE) along a nested sequence of adapted partitions $\mathcal{P}_X^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, of the interval $[0, T]$ with vanishing mesh size. Since $Y(\omega) \in \mathcal{A}_{X(\omega)}$, there exists a càdlàg process Y' such that $(Y(\omega), Y'(\omega)) \in \mathcal{V}_{X(\omega)}^{q,r}$ is a controlled path on $[0, T]$ for almost every $\omega \in \Omega$ and some suitable numbers q, r.

By Protter [43, Theorem II.21], we have that

$$\sum_{k=0}^{N_n-1} Y_{t_k^n} X_{t_k^n \wedge t, t_{k+1}^n \wedge t} \longrightarrow \int_0^t Y_{s-} \,\mathrm{d}X_s \qquad \text{as } n \to \infty, \tag{4.12}$$

where the convergence holds uniformly (in $t \in [0, T]$) in probability. By taking a subsequence if necessary, we can then assume that the (uniform) convergence in (4.12) holds almost surely. On the other hand, by Theorem 2.15, we know that for almost every $\omega \in \Omega$,

$$\sum_{k=0}^{N_n-1} Y_{t_k^n}(\omega) X_{t_k^n \wedge t, t_{k+1}^n \wedge t}(\omega) \longrightarrow \int_0^t Y_s(\omega) \,\mathrm{d}\mathbf{X}_s(\omega) \qquad \text{as } n \to \infty \tag{4.13}$$

uniformly for $t \in [0, T]$. Combining (4.12) and (4.13), we deduce that almost surely, $\int_0^t Y_s \,\mathrm{d}\mathbf{X}_s = \int_0^t Y_{s-} \,\mathrm{d}X_s$ for all $t \in [0, T]$. Since $T > 0$ was arbitrary, the result follows. □

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Declarations

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