Computing Subset Vertex Covers in H-Free Graphs*

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Abstract. We consider a natural generalization of Vertex Cover: the Subset Vertex Cover problem, which is to decide for a graph $G = (V, E)$, a subset $T \subseteq V$ and integer k, if V has a subset S of size at most k, such that S contains at least one end-vertex of every edge incident to a vertex of T. A graph is H-free if it does not contain H as an induced subgraph. We solve two open problems from the literature by proving that Subset Vertex Cover is NP-complete on subcubic (claw,diamond)-free planar graphs and on 2-unipolar graphs, a subclass of $2P_3$-free weakly chordal graphs. Our results show for the first time that Subset Vertex Cover is computationally harder than Vertex Cover (under $\mathsf{P} \neq \mathsf{NP}$). We also prove new polynomial time results. We first give a dichotomy on graphs where $G[T]$ is H-free. Namely, we show that Subset Vertex Cover is polynomial-time solvable on graphs G, for which $G[T]$ is H-free, if $H = sP_1 + tP_2$ and NP-complete otherwise. Moreover, we prove that Subset Vertex Cover is polynomial-time solvable for $(sP_1 + P_2 + P_3)$-free graphs and bounded mim-width graphs. By combining our new results with known results we obtain a partial complexity classification for Subset Vertex Cover on H-free graphs.

1 Introduction

We consider a natural generalization of the classical Vertex Cover problem: the Subset Vertex Cover problem, introduced in [5]. Let $G = (V, E)$ be a graph and T be a subset of V. A set $S \subseteq V$ is a T-*vertex cover* of G if S contains at least one end-vertex of every edge incident to a vertex of T. We note that T itself is a T-vertex cover. However, a graph may have much smaller T-vertex covers. For example, if G is a star whose leaves form T, then the center of G forms a T-vertex cover. We can now define the problem; see also Fig. 1.

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SUBSET VERTEX COVER *Instance:* A graph $G = (V, E)$, a subset $T \subseteq V$, and a positive integer k. *Question:* Does G have a T-vertex cover S_T with $\|S_T\| \leq k$?

If we set $T = V$, then we obtain the VERTEX COVER problem. Hence, as VERTEX COVER is NP-complete, so is SUBSET VERTEX COVER.

To obtain a better understanding of the complexity of an NP-complete graph problem, we may restrict the input to some special graph class. In particular, *hereditary* graph classes, which are the classes closed under vertex deletion, have been studied intensively for this purpose. It is readily seen that a graph class $\mathcal{G}$ is hereditary if and only if $\mathcal{G}$ is characterized by a unique minimal set of forbidden induced subgraphs $\mathcal{F}_G$. Hence, for a systematic study, it is common to first consider the case where $\mathcal{F}_\mathcal{G}$ has size 1. This is also the approach we follow in this paper. So, for a graph H, we set $\mathcal{F}_\mathcal{G} = \{H\}$ for some graph H and consider the class of H-free graphs (graphs that do not contain H as an induced subgraph). We now consider the following research question:

For which graphs H is SUBSET VERTEX COVER*, restricted to H-free graphs, still* NP*-complete and for which graphs H does it become polynomial-time solvable?*

We will also address two open problems posed in [5] (see Section 2 for any undefined terminology):

Q1. What is the complexity of SUBSET VERTEX COVER for claw-free graphs?
Q2. Is SUBSET VERTEX COVER is NP-complete for P_t-free graphs for some t?

The first question is of interest, as VERTEX COVER is polynomial-time solvable even on $rK_{1,3}$-free graphs for every $r \geq 1$ [4], where $rK_{1,3}$ is the disjoint union of r claws (previously this was known for rP_3-free graphs [13] and $2P_3$-free graphs [14]). The second question is of interest due to some recent quasi-polynomial-time results. Namely, Gartland and Lokshtanov [9] proved that for every integer t, VERTEX COVER can be solved in $n^{O(\log^3 n)}$-time for P_t-free

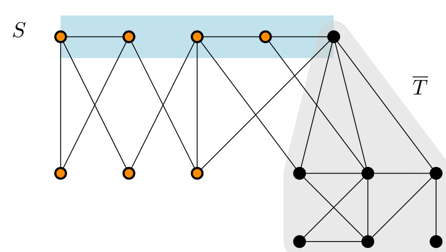

Fig. 1. An instance (G, T, k) of SUBSET VERTEX COVER, where T consists of the orange vertices, together with a solution S (a T-vertex cover of size 5). Note that S consists of four vertices of T and one vertex of $\overline{T} = V \setminus T$.

graphs. Afterwards, Pilipczuk, Pilipczuk and Rzążewski [18] improved the running time to $n^{O(\log^2 n)}$ time. Even more recently, Gartland et al. [10] extended the results of [9,18] from P_t-free graphs to H-free graphs where every connected component of H is a path or a subdivided claw.

Grötschel, Lovász, and Schrijver [11] proved that Vertex Cover can be solved in polynomial time for the class of perfect graphs, which includes well-known graph classes, such as bipartite graphs and (weakly) chordal graphs. Before we present our results, we first briefly discuss the relevant literature.

Existing Results and Related Work

Whenever Vertex Cover is NP-complete for some graph class $\mathcal{G}$, then so is the more general problem Subset Vertex Cover. Moreover, Subset Vertex Cover can be polynomially reduced to Vertex Cover: given an instance (G, T, k) of the former problem, remove all edges not incident to a vertex of T to obtain an instance (G', k) of the latter problem. Hence, we obtain:

Proposition 1. *The problems* Vertex Cover *and* Subset Vertex Cover *are polynomially equivalent for every graph class closed under edge deletion.*

For example, the class of bipartite graphs is closed under edge deletion and Vertex Cover is polynomial-time solvable on bipartite graphs. Hence, by Proposition 1, Subset Vertex Cover is polynomial-time solvable on bipartite graphs. However, a class of H-free graphs is only closed under edge deletion if H is a complete graph, and Vertex Cover is NP-complete even for triangle-free graphs [19]. This means that there could still exist graphs H such that Vertex Cover and Subset Vertex Cover behave differently if the former problem is (quasi)polynomial-time solvable on H-free graphs. The following well-known result of Alekseev [1] restricts the structure of such graphs H.

Theorem 1 ([1]). *For every graph H that contains a cycle or a connected component with two vertices of degree at least* 3*,* Vertex Cover*, and thus* Subset Vertex Cover*, is* NP*-complete for H-free graphs.*

Due to Theorem 1 and the aforementioned result of Gartland et al. [10], every graph H is now either classified as a quasi-polynomial case or NP-hard case for Vertex Cover. For Subset Vertex Cover the situation is much less clear. So far, only one positive result is known, which is due to Brettell et al. [5].

Theorem 2 ([5]). *For every $s \geq 0$,* Subset Vertex Cover *is polynomial-time solvable on $(sP_1 + P_4)$-free graphs.*

Subset variants of classic graph problems are widely studied, also in the context of H-free graphs. Indeed, Brettell et al. [5] needed Theorem 2 as an auxiliary result in complexity studies for Subset Feedback Vertex Set and Subset Odd Cycle Transversal restricted to H-free graphs. The first problem is to decide for a graph $G = (V, E)$, subset $T \subseteq V$ and integer k, if G has a set S of size at most k such that S contains a vertex of every cycle that intersects T. The second problem is similar but replaces "cycle" by "cycle of odd length". Brettell et al. [5] proved that both these subset transversal problems are polynomial-time solvable on $(sP_1 + P_3)$-free graphs for every $s \geq 0$. They also showed that Odd Cycle Transversal is polynomial-time solvable for P_4-free graphs and NP-complete for split graphs, which form a subclass of $2P_2$-free graphs, whereas NP-completeness for Subset Feedback Vertex Set on split graphs was shown by Fomin et al. [8]. Recently, Paesani et al. [17] extended the result of [5] for Subset Feedback Vertex Set from $(sP_1 + P_3)$-free graphs to $(sP_1 + P_4)$-free graphs for every integer $s \geq 0$. If H contains a cycle or claw, NP-completeness for both subset transversal problems follows from corresponding results for Feedback Vertex Set [16,19] and Odd Cycle Transversal [6].

Combining all the above results leads to the following theorems (see also [5,17]). Here, we write $F \subseteq_i G$ if F is an induced subgraph of G.

Theorem 3. *For a graph H,* Subset Feedback Vertex Set *on H-free graphs is polynomial-time solvable if $H \subseteq_i sP_1 + P_4$ for some $s \geq 0$, and* NP*-complete otherwise.*

Theorem 4. *For a graph $H \neq sP_1 + P_4$ for some $s \geq 1$,* Subset Odd Cycle Transversal *on H-free graphs is polynomial-time solvable if $H = P_4$ or $H \subseteq_i sP_1 + P_3$ for some $s \geq 0$, and* NP*-complete otherwise.*

Our Results

In Section 3 we prove two new hardness results, using the same basis reduction, which may have a wider applicability. We first answer Q1 by proving that Subset Vertex Cover is NP-complete even for subcubic planar line graphs of triangle-free graphs, or equivalently, subcubic planar (claw, diamond)-free graphs.

We then answer Q2 by proving that Subset Vertex Cover is NP-complete even for a 2-unipolar graphs, which are $2P_3$-free (and thus P_7-free).

Our hardness results show a sharp contrast with Vertex Cover, which can be solved in polynomial time for both weakly chordal graphs [11] and $rK_{1,3}$-free graphs for every $r \geq 1$ [4]. Hence, Subset Vertex Cover may be harder than Vertex Cover for a graph class closed under vertex deletion (if $\mathsf{P} \neq \mathsf{NP}$). This is in contrast to graph classes closed under edge deletion (see Proposition 1).

In Section 3 we also prove that Subset Vertex Cover is NP-complete for inputs (G, T, k) if the subgraph $G[T]$ of G induced by T is P_3-free. On the other hand, our first positive result, shown in Section 4, shows that the problem is polynomial-time solvable if $G[T]$ is sP_2-free for any $s \geq 2$. In Section 4 we also prove that Subset Vertex Cover can be solved in polynomial time for $(sP_1 + P_2 + P_3)$-free graphs for every $s \geq 1$. Our positive results generalize known results for Vertex Cover. The first result also implies that Subset Vertex Cover is polynomial-time solvable for split graphs, contrasting our NP-completeness result for 2-unipolar graphs, which are generalized split, $2P_3$-free, and weakly chordal. Combining our new results with Theorem 2 gives us a partial classification and a dichotomy, both of which are proven in Section 5.

Theorem 5. *For a graph $H \neq rP_1+sP_2+P_3$ for any $r \geq 0$, $s \geq 2$; $rP_1+sP_2+P_4$ for any $r \geq 0$, $s \geq 1$; or $rP_1+sP_2+P_t$ for any $r \geq 0$, $s \geq 0$, $t \in \{5,6\}$,* Subset Vertex Cover *on H-free graphs is polynomial-time solvable if $H \subseteq_i sP_1+P_2+P_3$, sP_2, or sP_1+P_4 for some $s \geq 1$, and* NP*-complete otherwise.*

Theorem 6. *For a graph H,* Subset Vertex Cover *on instances (G,T,k), where $G[T]$ is H-free, is polynomial-time solvable if $H \subseteq_i sP_2$ for some $s \geq 1$, and* NP*-complete otherwise.*

Theorems 3–6 show that Subset Vertex Cover on H-free graphs can be solved in polynomial time for infinitely more graphs H than Subset Feedback Vertex Set and Subset Odd Cycle Transversal. This is in line with the behaviour of the corresponding original (non-subset) problems.

In Section 6 we discuss some directions for future work, which naturally originate from the above results and our final new result, which is proven in the full version of our paper[4], and which states that Subset Vertex Cover is polynomial-time solvable on every graph class of bounded mim-width, such as the class of circular-arc graphs.

2 Preliminaries

Let $G=(V,E)$ be a graph. The *degree* of a vertex $u \in V$ is the size of its *neighbourhood* $N(u)=\{v \mid uv \in E\}$. We say that G is *subcubic* if every vertex of G has degree at most 3. An independent set I in G is *maximal* if there exists no independent set I' in G with $I \subsetneq I'$. Similarly, a vertex cover S of G is *minimal* if there no vertex cover S' in G with $S' \subsetneq S$. For a graph H we write $H \subseteq_i G$ if H is an *induced* subgraph of G, that is, G can be modified into H by a sequence of vertex deletions. If G does not contain H as an induced subgraph, G is *H-free*. For a set of graphs $\mathcal{H}$, G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. If $\mathcal{H}=\{H_1,\ldots,H_p\}$ for some $p \geq 1$, we also write that G is *$(H_1,\ldots,H_p)$-free.*

The *line graph* of a graph $G=(V,E)$ is the graph $L(G)$ that has vertex set E and an edge between two vertices e and f if and only if e and f share a common end-vertex in G. The *complement* $\overline{G}$ of a graph $G=(V,E)$ has vertex set V and an edge between two vertices u and v if and only if $uv \notin E$.

For two vertex-disjoint graphs F and G, the *disjoint union* $F+G$ is the graph $(V(F) \cup V(G), E(F) \cup E(G))$. We denote the disjoint union of s copies of the same graph G by sG. A *linear forest* is a disjoint union of one or more paths.

Let C_s be the cycle on s vertices; P_t the path on t vertices; K_r the complete graph on r vertices; and $K_{1,r}$ the star on $(r+1)$ vertices. The graph $C_3=K_3$ is the *triangle*; the graph $K_{1,3}$ the *claw*, and the graph $\overline{2P_1+P_2}$ is the *diamond* (so the diamond is obtained from the K_4 after deleting one edge). The *subdivision* of an edge uv replaces uv with a new vertex w and edges uw, wv. A *subdivided claw* is obtained from the claw by subdividing each of its edges zero or more times.

[4] The full version is available on arXiv, see `https://arxiv.org/abs/2307.05701`.

A graph is *chordal* if it has no induced C_s for any $s \geq 4$. A graph is *weakly chordal* if it has no induced C_s and no induced $\overline{C_s}$ for any $s \geq 5$. A cycle C_s or an anti-cycle $\overline{C_s}$ is *odd* if it has an odd number of vertices. By the Strong Perfect Graph Theorem [7], a graph is *perfect* if it has no odd induced C_s and no odd induced $\overline{C_s}$ for any $s \geq 5$. Every chordal graph is weakly chordal, and every weakly chordal graph is perfect. A graph $G = (V, E)$ is *unipolar* if V can be partitioned into two sets V_1 and V_2, where $G[V_1]$ is a complete graph and $G[V_2]$ is a disjoint union of complete graphs. If every connected component of $G[V_2]$ has size at most 2, then G is 2-*unipolar*. Unipolar graphs form a subclass of *generalized split graphs*, which are the graphs that are unipolar or their complement is unipolar. It can also be readily checked that every 2-unipolar graph is weakly chordal (but not necessarily chordal, as evidenced by $G = C_4$).

For an integer r, a graph G' is an r-*subdivision* of a graph G if G' can be obtained from G by subdividing every edge of G r times, that is, by replacing each edge $uv \in E(G)$ with a path from u to v of length $r + 1$.

3 NP-Hardness Results

In this section we prove our hardness results for Subset Vertex Cover, using the following notation. Let G be a graph with an independent set I. We say that we *augment* G by adding a (possibly empty) set F of edges between some pairs of vertices of I. We call the resulting graph an I-*augmentation* of G.

The following lemma forms the basis for our hardness gadgets.

Lemma 1. *Every vertex cover of a graph $G = (V, E)$ with an independent set I is a $(V \setminus I)$-vertex cover of every I-augmentation of G, and vice versa.*

Proof. Let G' be an I-augmentation of G. Consider a vertex cover S of G. For a contradiction, assume that S is not a $(V \setminus I)$-vertex cover of G'. Then $G' - S$ must contain an edge uv with at least one of u, v belonging to $V \setminus I$. As $G - S$ is an independent set, uv belongs to $E(G') \setminus E(G)$ implying that both u and v belong to I, a contradiction.

Now consider a $(V \setminus I)$-vertex cover S' of G'. For a contradiction, assume that S' is not a vertex cover of G. Then $G - S'$ must contain an edge uv (so $uv \in E$). As G' is a supergraph of G, we find that $G' - S'$ also contains the edge uv. As S' is a $(V \setminus I)$-vertex cover of G', both u and v must belong to I. As $uv \in E$, this contradicts the fact that I is an independent set. □

To use Lemma 1 we need one other lemma, which follows directly from an observation due to Poljak [19].

Lemma 2 ([19]). *For an integer r, a graph G with m edges has an independent set of size k if and only if the $2r$-subdivision of G has an independent set of size $k + rm$.*

We are now ready to prove our first two hardness results. Recall that a graph is (claw, diamond)-free if and only if it is a line graph of a triangle-free graph.

Hence, the result in particular implies NP-hardness of Subset Vertex Cover for line graphs. Recall also that we denote the claw and diamond by $K_{1,3}$ and $\overline{2P_1 + P_2}$, respectively.

Theorem 7. Subset Vertex Cover *is* NP*-complete for* $(K_{1,3}, \overline{2P_1 + P_2})$*-free subcubic planar graphs.*

Proof. We reduce from Vertex Cover, which is NP-complete even for cubic planar graphs [15]. As an n-vertex graph has a vertex cover of size at most k if and only if it has an independent set of size at least $n-k$, we find that Vertex Cover is NP-complete even for subcubic planar graphs that are 4-subdivisions due to an application of Lemma 2 with $r = 2$ (note that subdividing an edge preserves both maximum degree and planarity). So, let (G, k) be an instance of Vertex Cover, where $G = (V, E)$ is a subcubic planar graph that is a 4-subdivision of some cubic planar graph G^*, and k is an integer.

In G, we let $U = V(G^*)$ and W be the subset of $V(G) \setminus U$ that consists of all neighbours of vertices of U. Note that W is an independent set in G. We construct a W-augmentation G' as follows.

For every vertex $u \in U$ of degree 3 in G, we pick two arbitrary neighbours of u (which both belong to W) and add an edge between them. It is readily seen that G' is $(K_{1,3}, \overline{2P_1 + P_2})$-free, planar and subcubic. By Lemma 1, it holds that G has a vertex cover of size at most k if and only if G' has a $(V \setminus W)$-vertex cover of size at most k. □

See the full version of our paper for the proof of our second hardness result. It can be readily checked that 2-unipolar graphs are $(2C_3, C_5, C_6, C_3 + P_3, 2P_3, \overline{P_6}, \overline{C_6})$-free graphs, and thus are $2P_3$-free weakly chordal.

Theorem 8. Subset Vertex Cover *is* NP*-complete for instances* (G, T, k)*, for which* G *is* 2*-unipolar and* $G[T]$ *is a disjoint union of edges.*

4 Polynomial-Time Results

In this section, we prove our polynomial-time results. We start with the case where $H = sP_2$ for some $s \geq 1$. For this case we need the following two well-known results. The *delay* of an enumeration algorithm is the maximum of the time taken before the first output and that between any pair of consecutive outputs.

Theorem 9 ([2]). *For every constant* $s \geq 1$*, the number of maximal independent sets of an* sP_2*-free graph on* n *vertices is at most* $n^{2s} + 1$*.*

Theorem 10 ([20]). *For every constant* $s \geq 1$*, it is possible to enumerate all maximal independent sets of an* sP_2*-free graph* G *on* n *vertices and* m *edges with a delay of* $O(nm)$*.*

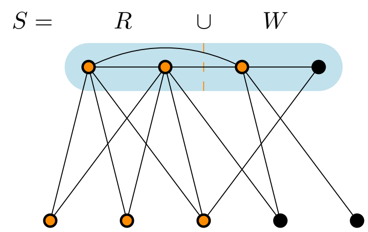

Fig. 2. An example of the $2P_2$-free graph G' of the proof of Theorem 11. Here, T consists of the orange vertices. A solution S can be split up into a minimal vertex cover R of $G'[T]$ and a vertex cover W of $G[V \setminus R]$.

We show a slightly stronger result than proving that Subset Vertex Cover is polynomial-time solvable for sP_2-free graphs. The idea behind the algorithm is to remove any edges between vertices in $V \setminus T$, as these edges are irrelevant. As a consequence, we may leave the graph class, but this is not necessarily an obstacle. For example, if $G[T]$ is a complete graph, or T is an independent set, we can easily solve the problem. Both cases are generalized by the result below.

Theorem 11. *For every $s \geq 1$,* Subset Vertex Cover *can be solved in polynomial time for instances (G, T, k) for which $G[T]$ is sP_2-free.*

Proof. Let $s \geq 1$, and let (G, T, k) be an instance of Subset Vertex Cover where $G = (V, E)$ is a graph such that $G[T]$ is sP_2-free. Let $G' = (V, E')$ be the graph obtained from G after removing every edge between two vertices of $V \setminus T$, so $G'[V \setminus T]$ is edgeless. We observe that G has a T-vertex cover of size at most k if and only if G' has a T-vertex cover of size at most k. Moreover, $G'[T]$ is sP_2-free, and we can obtain G' in $O(|E(G)|)$ time. Hence, from now on, we consider the instance (G', T, k).

We first prove the following two claims, see Figure 2 for an illustration.

Claim 1. A subset $S \subseteq V(G')$ is a T-vertex cover of G' if and only if $S = R \cup W$ for a minimal vertex cover R of $G'[T]$ and a vertex cover W of $G'[V \setminus R]$.

We prove Claim 1 as follows. Let $S \subseteq V(G')$. First assume that S is a T-vertex cover of G'. Let $I = V \setminus S$. As S is a T-vertex cover, $T \cap I$ is an independent set. Hence, S contains a minimal vertex cover R of $G'[T]$. As $G'[V \setminus T]$ is edgeless, S is a vertex cover of G, or in other words, I is an independent set. In particular, this means that $W \setminus R$ is a vertex cover of $G'[V \setminus R]$.

Now assume that $S = R \cup W$ for a minimal vertex cover R of $G'[T]$ and a vertex cover W of $G'[V \setminus R]$. For a contradiction, suppose that S is not a T-vertex cover of G'. Then $G' - S$ contains an edge $uv \in E'$, where at least one of u, v belongs to T. First suppose that both u and v belong to T. As R is a vertex cover of $G'[T]$, at least one of u, v belongs to $R \subseteq S$, a contradiction. Hence, exactly one of u, v belongs to T, say $u \in T$ and $v \in V \setminus T$, so in particular, $v \notin R$. As $R \subseteq S$, we find that $u \notin R$. Hence, both u and v belong to $V \setminus R$. As W is a vertex cover of $V \setminus R$, this means that at least one of u, v belongs to $W \subseteq S$, a contradiction. This proves the claim. $\diamond$

Claim 2. For every minimal vertex cover R of $G'[T]$, the graph $G'[V \setminus R]$ is bipartite.

We prove Claim 2 as follows. As R is a vertex cover of $G'[T]$, we find that $T \setminus R$ is an independent set. As $G'[V \setminus T]$ is edgeless by construction of G', this means that $G'[V \setminus R]$ is bipartite with partition classes $T \setminus R$ and $V \setminus T$. $\diamond$

We are now ready to give our algorithm. We enumerate the minimal vertex covers of $G'[T]$. For every minimal vertex cover R, we compute a minimum vertex cover W of $G'[V \setminus R]$. In the end, we return the smallest $S = R \cup W$ that we found.

The correctness of our algorithm follows from Claim 1. It remains to analyse the running time. As $G'[T]$ is sP_2-free, we can enumerate all maximal independent sets I of $G'[T]$ and thus all minimal vertex covers $R = T \setminus I$ of $G'[T]$ in $(n^{2s}+1) \cdot O(nm)$ time due to Theorems 9 and 10. For a minimal vertex cover R, the graph $G'[V \setminus R]$ is bipartite by Claim 2. Hence, we can compute a minimum vertex cover W of $G'[V \setminus R]$ in polynomial time by applying König's Theorem. We conclude that the total running time is polynomial. $\square$

For our next result (Theorem 12) we need two known results as lemmas.

Lemma 3 ([5]). *If* Subset Vertex Cover *is polynomial-time solvable on H-free graphs for some H, then it is so on $(H + P_1)$-free graphs.*

Lemma 4 ([4]). *For every $r \geq 1$,* Vertex Cover *is polynomial-time solvable on $rK_{1,3}$-free graphs.*

We are now ready to prove our second polynomial-time result.

Theorem 12. *For every integer s,* Subset Vertex Cover *is polynomial-time solvable on $(sP_1 + P_2 + P_3)$-free graphs.*

Proof. Due to Lemma 3, we can take $s = 0$, so we only need to give a polynomial-time algorithm for $(P_2 + P_3)$-free graphs. Hence, let (G, T, k) be an instance of Subset Vertex Cover, where $G = (V, E)$ is a $(P_2 + P_3)$-free graph.

First compute a minimum vertex cover of G. As G is (P_2+P_3)-free, and thus $2K_{1,3}$-free, this takes polynomial time by Lemma 4. Remember the solution S_{VC}.

We now compute a minimum T-vertex cover S of G that is not a vertex cover of G. Then $G - S$ must contain an edge between two vertices in $G - T$. We branch by considering all $O(n^2)$ options of choosing this edge. For each chosen edge uv we do as follows. As both u and v will belong to $G - S$ for the T-vertex cover S of G that we are trying to construct, we first add every neighbour of u or v that belongs to T to S.

Let T' consist of all vertices of T that are neither adjacent to u nor to v. As G is $(P_2 + P_3)$-free and $uv \in E$, we find that $G[T']$ is P_3-free and thus a disjoint union of complete graphs. We call a connected component of $G[T']$ *large* if it has at least two vertices; else we call it *small* (so every small component of $G[T']$ is an isolated vertex). See also Figure 3 for an illustration.

Case 1. The graph $G[T']$ has at most two large connected components.
Let D_1 and D_2 be the large connected components of $G[T']$ (if they exist). As $V(D_1)$ and $V(D_2)$ are cliques in $G[T]$, at most one vertex of D_1 and at most one vertex of D_2 can belong to $G-S$. We branch by considering all $O(n^2)$ options of choosing at most one vertex of D_1 and at most one vertex of D_2 to be these vertices. For each choice of vertices we do as follows. We add all other vertices of D_1 and D_2 to S. Let T^* be the set of vertices of T that we have not added to S. Then T^* is an independent set.

We delete every edge between any two vertices in $G-T$. Now the graph G^* induced by the vertices of $T^* \cup (V \setminus T)$ is bipartite (with partition classes T^* and $V \setminus T$). It remains to compute a minimum vertex cover S^* of G^*. This can be done in polynomial time by applying König's Theorem. We let S consist of S^* together with all vertices of T that we had added in S already.

For each branch, we remember the output, and in the end we take a smallest set S found and compare its size with the size of S_{VC}, again taking a smallest set as the final solution.

Case 2. The graph $G[T']$ has at least three large connected components.
Let $D_1, \ldots, D_p$, for some $p \geq 3$, be the large connected components of $G[T']$. Let A consists of all the vertices of the small connected components of $G[T']$.

We first consider the case where $G-S$ will contain a vertex $w \in V \setminus T$ with one of the following properties:

1. for some i, w has a neighbour and a non-neighbour in D_i; or
2. for some i, j with $i \neq j$, w has a neighbour in D_i and a neighbour in D_j; or
3. for some i, w has a neighbour in D_i and a neighbour in A.

We say that a vertex w in $G-S$ is *semi-complete* to some D_i if w is adjacent to all vertices of D_i except at most one. We show the following claim that holds if the solution S that we are trying to construct contains a vertex $w \in V \setminus (S \cup T)$ that satisfies one of the three properties above. See Figure 3 for an illustration.

Claim. Every vertex $w \in V \setminus (S \cup T)$ that satisfies one of the properties 1-3 is semi-complete to every $V(D_j)$.

We prove the Claim as follows. Let $w \in V \setminus (S \cup T)$. First assume w satisfies Property 1. Let x and y be vertices of some D_i, say D_1, such that $wx \in E$ and $wy \notin E$. For a contradiction, assume w is not semi-complete to some D_j. Hence, D_j contains vertices y' and y'', such that $wy' \notin E$ and $wy'' \notin E$. If $j \geq 2$, then $\{y', y'', w, x, y\}$ induces a $P_2 + P_3$ (as D_1 and D_j are complete graphs). This contradicts that G is $(P_2 + P_3)$-free. Hence, w is semi-complete to every $V(D_j)$ with $j \geq 2$. Now suppose $j = 1$. As $p \geq 3$, the graphs D_2 and D_3 exist. As w is semi-complete to every $V(D_j)$ for $j \geq 2$ and every D_j is large, there exist vertices $x' \in V(D_2)$ and $x'' \in V(D_3)$ such that $wx' \in E$ and $wx'' \in E$. However, now $\{y', y'', x', w, x''\}$ induces a $P_2 + P_3$, a contradiction.

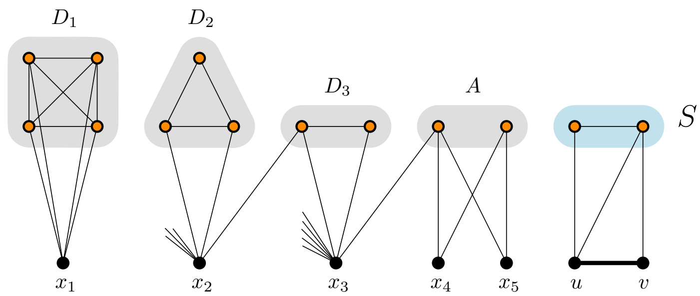

Fig. 3. An illustration of the graph G in the proof of Theorem 12, where T consists of the orange vertices, and $p = 3$. Edges in $G[V \setminus T]$ are not drawn, and for x_2 and x_3 some edges are partially drawn. None of x_1, x_4, x_5 satisfy a property; x_2 satisfies Property 1 for D_2 and Property 2 for D_2 and D_3; and x_3 satisfies Property 3 for D_3.

Now assume w satisfies Property 2, say w is adjacent to $x_1 \in V(D_1)$ and to $x_2 \in V(D_2)$. Suppose w is not semi-complete to some $V(D_j)$. If $j \geq 3$, then the two non-neighbours of w in D_j, together with x_1, w, x_2, form an induced $P_2 + P_3$, a contradiction. Hence, w is semi-complete to every $V(D_j)$ for $j \geq 3$. If $j \in \{1, 2\}$, say $j = 1$, then let y, y' be two non-neighbours of w in D_1 and let x_3 be a neighbour of w in D_3. Now, $\{y, y', x_2, w, x_3\}$ induces a $P_2 + P_3$, a contradiction. Hence, w is semi-complete to $V(D_1)$ and $V(D_2)$ as well.

Finally, assume w satisfies Property 3, say w is adjacent to $z \in A$ and $x_1 \in V(D_1)$. If w not semi-complete to $V(D_j)$ for some $j \geq 2$, then two non-neighbours of w in D_j, with z, w, x_1, form an induced $P_2 + P_3$, a contradiction. Hence, w is semi-complete to every $V(D_j)$ with $j \geq 2$. As before, by using a neighbour of w in D_2 and one in D_3, we find that w is also semi-complete to $V(D_1)$. $\diamond$

We now branch by considering all $O(n)$ options for choosing a vertex $w \in V \setminus (S \cup T)$ that satisfies one of the properties 1-3. For each chosen vertex w, we do as follows. We remove all its neighbours in T, and add them to S. By the above Claim, the remaining vertices in T form an independent set. We delete any edge between two vertices from $V \setminus T$, so $V \setminus T$ becomes an independent set as well. It remains to compute, in polynomial time by König's Theorem, a minimum vertex cover in the resulting bipartite graph and add this vertex cover to S. For each branch, we store S. After processing all of the $O(n)$ branches, we keep a smallest S, which we denote by S^*.

We are left to compute a smallest T-vertex cover S of G over all T-vertex covers that contain every vertex from $V \setminus T$ that satisfy one of the properties 1–3. We do this as follows. First, we put all vertices from $V \setminus T$ that satisfy one of the three properties 1–3 to the solution S that we are trying to construct. Let G^* be the remaining graph. We do not need to put any vertex from any connected component of G^* that contains no vertex from T in S.

Now consider the connected component D_1' of G^* that contains the vertices from D_1. As D_1' contains no vertices from $V \setminus T$ satisfying properties 2 or 3, we find that D_1' contains no vertices from A or from any D_j with $j \geq 2$, so

$V(D_1') \cap T = V(D_1)$. Suppose there exists a vertex v in $V(D_1') \setminus V(D_1)$, which we may assume has a neighbour in D_1 (as D_1' is connected). Then, v is complete to D_1 as it does not satisfy Property 1. Then, we must put at least $|V(D_1)|$ vertices from D_1' in S, so we might just as well put every vertex of D_1 in S. As $V(D_1') \cap T = V(D_1)$, this suffices. If $D_1' = D_1$, then we put all vertices of D_1 except for one arbitrary vertex of D_1 in S.

We do the same as we did for D_1 for the connected components $D_2', \ldots, D_p'$ of G^* that contain $V(D_2), \ldots V(D_p)$, respectively.

Now, it remains to consider the induced subgraph F of G^* that consists of connected components containing the vertices of A. Recall that A is an independent set. We delete every edge between two vertices in $V \setminus T$, resulting in another independent set. This changes F into a bipartite graph and we can compute a minimum vertex cover S_F of F in polynomial time due to König's Theorem. We put S_F to S and compare the size of S with the size of S^* and S_{VC}, and pick the one with smallest size as our solution.

The correctness of our algorithm follows from the above description. The number of branches is $O(n^4)$ in Case 1 and $O(n^3)$ in Case 2. As each branch takes polynomial time to process, this means that the total running time of our algorithm is polynomial. This completes our proof. □

5 The Proof of Theorems 5 and 6

We first prove Theorem 5, which we restate below.

Theorem 5 (restated). *For a graph $H \neq rP_1 + sP_2 + P_3$ for any $r \geq 0$, $s \geq 2$; $rP_1 + sP_2 + P_4$ for any $r \geq 0$, $s \geq 1$; or $rP_1 + sP_2 + P_t$ for any $r \geq 0$, $s \geq 0$, $t \in \{5, 6\}$,* Subset Vertex Cover *on H-free graphs is polynomial-time solvable if $H \subseteq_i sP_1 + P_2 + P_3$, sP_2, or $sP_1 + P_4$ for some $s \geq 1$, and* NP*-complete otherwise.*

Proof. Let H be a graph not equal to $rP_1 + sP_2 + P_3$ for any $r \geq 0$, $s \geq 2$; $rP_1 + sP_2 + P_4$ for any $r \geq 0$, $s \geq 1$; or $rP_1 + sP_2 + P_t$ for any $r \geq 0$, $s \geq 0$, $t \in \{5, 6\}$. If H has a cycle, then we apply Theorem 1. Else, H is a forest. If H has a vertex of degree at least 3, then the class of H-free graphs contains all $K_{1,3}$-free graphs, and we apply Theorem 7. Else, H is a linear forest. If H contains an induced $2P_3$, then we apply Theorem 8. If not, then $H \subseteq_i sP_1 + P_2 + P_3$, sP_2, or $sP_1 + P_4$ for some $s \geq 1$. In the first case, apply Theorem 12; in the second case Theorem 11; and in the third case Theorem 2. □

We now prove Theorem 6, which we restate below.

Theorem 6 (restated). *For a graph H,* Subset Vertex Cover *on instances (G, T, k), where $G[T]$ is H-free, is polynomial-time solvable if $H \subseteq_i sP_2$ for some $s \geq 1$, and* NP*-complete otherwise.*

Proof. First suppose $P_3 \subseteq_i H$. As a graph that is a disjoint union of edges is P_3-free, we can apply Theorem 8. Now suppose H is P_3-free. Then $H \subseteq_i sP_2$ for some $s \geq 1$, and we apply Theorem 11. □

6 Conclusions

Apart from giving a dichotomy for Subset Vertex Cover restricted to instances (G, T, k) where $G[T]$ is H-free (Theorem 6), we gave a partial classification of Subset Vertex Cover for H-free graphs (Theorem 5). Our partial classification resolved two open problems from the literature and showed that for some hereditary graph classes, Subset Vertex Cover is computationally harder than Vertex Cover (if $\mathsf{P} \neq \mathsf{NP}$). This is in contrast to the situation for graph classes closed under edge deletion. Hence, Subset Vertex Cover is worth studying on its own, instead of only as an auxiliary problem (as in [5]).

Our results raise the question whether there exist other hereditary graph classes on which Subset Vertex Cover is computationally harder than Vertex Cover. Recall that Vertex Cover is polynomial-time solvable for perfect graphs [11], and thus for weakly chordal graphs and chordal graphs. On the other hand, we showed that Subset Vertex Cover is NP-complete for 2-unipolar graphs, a subclass of $2P_3$-free weakly chordal graphs. Hence, as the first candidate graph class to answer this question, we propose the class of chordal graphs. A standard approach for Vertex Cover on chordal graphs is dynamic programming over the clique tree of a chordal graph. However, a naive dynamic programming algorithm over the clique tree does not work for Subset Vertex Cover, as we may need to remember an exponential number of subsets of a bag (clique) and the bags can have arbitrarily large size. In the full version of our paper, we show that Subset Vertex Cover can be solved in polynomial time on graphs of bounded mim-width. Using known results, this immediately implies the following:

Corollary 1. Subset Vertex Cover *can be solved in polynomial time on interval and circular-arc graphs.*

Corollary 1 makes the open question of the complexity of Subset Vertex Cover on chordal graphs, a superclass of the class of interval graphs, even more pressing. Recall that Subset Feedback Vertex Set, which is also solvable in polynomial time for graphs of bounded mim-width [3], is NP-complete for split graphs and thus for chordal graphs [8].

We note that our polynomial algorithms for Subset Vertex Cover for sP_2-free graphs and $(P_2 + P_3)$-free graphs can easily be adapted for Weighted Subset Vertex Cover for sP_2-free graphs and $(P_2 + P_3)$-free graphs. every $s \geq 1$ [5] (see also Theorem 4).

Finally, to complete the classification of Subset Vertex Cover for H-free graphs we need to solve the open cases where $H = sP_2 + P_3$ for $s \geq 2$; or $H = sP_2 + P_4$ for $s \geq 1$; or $H = sP_2 + P_t$ for $s \geq 0$ and $t \in \{5, 6\}$. Brettell et al. [5] asked what the complexity of Subset Vertex Cover is for P_5-free graphs. In contrast, Vertex Cover is polynomial-time solvable even for P_6-free graphs [12]. However, the open cases where $H = sP_2 + P_t$ ($s \geq 1$ and $t \in \{4, 5, 6\}$) are even open for Vertex Cover on H-free graphs (though a quasi-polynomial time algorithm is known [9,18]). So for those cases we may want to first restrict ourselves to Vertex Cover instead of Subset Vertex Cover.

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