# Large sample properties of GMM estimators under second-order identification. 

Hugo Kruiniger*<br>Durham University

This version: 24 December 2022

JEL classification: C12, C13, C23.
Keywords: Generalized Method of Moments (GMM) estimation, moment conditions, rank deficiency, rate of convergence, reparametrization, second-order local identification, underidentification.

[^0]
#### Abstract

Dovonon and Hall (Journal of Econometrics, 2018) proposed a limiting distribution theory for GMM estimators for a $p$ - dimensional globally identified parameter vector $\phi$ when local identification conditions fail at first-order but hold at second-order. They assumed that the first-order underidentification is due to the expected Jacobian having rank $p-1$ at the true value $\phi_{0}$, i.e., having a rank deficiency of one. After reparametrizing the model such that the last column of the Jacobian vanishes, they showed that the GMM estimator of the first $p-1$ parameters converges at rate $T^{-1 / 2}$ and the GMM estimator of the remaining parameter, $\widehat{\phi}_{p}$, converges at rate $T^{-1 / 4}$. They also provided a limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ subject to a (non-transparent) condition which they claimed to be not restrictive in general. However, as we show in this paper, their condition is in fact only satisfied when $\phi$ is overidentified and the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$, which is non-standard, depends on whether $\phi$ is exactly identified or overidentified. In particular, the limiting distributions of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ for the cases of exact and overidentification, respectively, are different and are obtained by using expansions of the GMM objective function of different orders. Unsurprisingly, we find that the limiting distribution theories of Dovonon and Hall (2018) for Indirect Inference (II) estimation under two different scenarios with second-order identification where the target function is a GMM estimator of the auxiliary parameter vector, are incomplete for similar reasons. We discuss how our results for GMM estimation can be used to complete both theories and in particular how they can be used to obtain the limiting distributions of the II estimators in the case of exact identification under either scenario.


## 1 Introduction

Global identification is a necessary condition for consistency of an estimator. In models that are linear in the parameters, global identification is equivalent to first-order local identification. However, in models that are nonlinear in the parameters, global identication of the parameter vector may hold even when some of the parameters are not first-order but higher order locally identified although in this case the rate of convergence of the estimators of these parameters is slower than the usual rate.

For the situation where one of the elements of $\phi$, say $\phi_{p}$, is not first-order but only second-order locally identified, Sargan (1983), Rotnitzky et al. (2000) and Kruiniger (2013) developed asymptotic theory for IV estimators, MLEs and Quasi MLEs, respectively. A common finding is that the estimator of the parameter that is only second-order locally identified converges at a quartic root rate, i.e., at rate $T^{-1 / 4}$ and has a non-normal asymptotic distribution, while the estimators of the parameters that are first-order locally identified converge at the usual square root rate, i.e., at rate $T^{-1 / 2}$ and have asymptotic distributions that are mixtures of normal distributions. Furthermore, the limiting distribution of $T^{1 / 2}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ is a mixture of a half-normal distribution and 0 .

Dovonon and Renault (2009) give a formal definition of second-order local identification in the context of GMM estimation. Dovonon and Hall (2018), henceforth DH, present an asymptotic theory for GMM estimators under second-order identification. The limiting distribution they give for the estimator of the second-order locally identified parameter, i.e., $\phi_{p}$, holds if a certain condition is satisfied. However, as we show in this paper, their condition is only satisfied when $\phi$ is overidentified. Furthermore, we show that the limiting distribution of $\widehat{\phi}_{p}$ depends on whether $\phi$ is exactly identified or overidentified. In particular, the limiting distributions of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ for the cases of exact and overidentification, respectively, are different and are obtained by using expansions of the GMM objective function of different orders. The reason for these differences is that in the case of exact identification some terms in the expansion vanish. On the other hand, we find that the formula for the limiting distribution of the GMM estimator of the vector with the other elements of $\phi$ is the same for both cases.

Kruiniger (2018) derived the limiting distributions of two Modified MLEs for a panel ARX(1) model with homoskedastic errors when the autoregressive parameter equals one by viewing them as GMM estimators. In the unit root case the parameter vector is only
second-order locally identified by the objective functions of the Modified MLEs due to the nonlinear terms in the modified score vector. Alvarez and Arellano (2021) found that in the same case (i.e., the case of a unit root and homoskedastic errors) the autoregressive parameter of the panel $\mathrm{AR}(1)$ model is only second-order locally identified by certain nonlinear moment conditions due to Ahn and Schmidt (1995). DH showed that a set of moment conditions that are related to a conditionally heteroskedastic factor model for asset returns has a rank deficient Jacobian matrix and that the vector of parameters in these moment conditions is second-order locally identified. Sargan (1983) discussed IV and FIML estimation of dynamic simultaneous equation models that are linear in the variables and nonlinear in the parameters and where the parameter vector is only second-order locally identified. Finally, Rotnitzky et al. (2000) give additional examples of models where the parameter vector is only second-order locally identified.

The paper is organized as follows. Section 2 briefly reviews GMM estimation under first-order local identification. Section 3 defines second-order identification. Section 4 presents the limiting distribution theory for GMM estimators under second-order local identification and discusses its implications for Indirect Inference (II) estimation under two scenarios with second-order local identification where the target function is a GMM estimator of the auxiliary parameter vector. Section 5 offers some concluding remarks. The appendix contains the proofs.

## 2 GMM under first-order identification

In this section we briefly review the basic GMM framework based on first-order asymptotics, paying special attention to the role of first-order local identification. We first define the GMM estimator and then discuss some first-order asymptotic theory for this estimator. To this end, we introduce the following notation. The model involves the random vector $X$ which is assumed strictly stationary with distribution $P\left(\phi_{0}\right)$ which is indexed by the parameter vector $\phi \in \Phi \subset \mathbb{R}^{p}$. $\phi_{0}$ is the true value of $\phi$.

GMM is a semi-parametric method in the sense that its implementation does not require complete knowledge of $P(\cdot)$ but only population moment conditions implied by this distribution. In view of this, we suppose that the model implies:

$$
\begin{equation*}
E\left[g\left(X, \phi_{0}\right)\right]=0, \tag{1}
\end{equation*}
$$

where $g(\cdot)$ is a $q \times 1$ vector of continuous functions. The GMM estimator of $\phi_{0}$ based on
(1) is defined as:

$$
\begin{equation*}
\widehat{\phi}=\underset{\phi \in \Phi}{\operatorname{argmin}} Q_{T}(\phi), \tag{2}
\end{equation*}
$$

where

$$
Q_{T}(\phi)=m_{T}^{\prime}(\phi) W_{T} m_{T}(\phi) \text { with } m_{T}(\phi)=T^{-1} \sum_{t=1}^{T} g\left(x_{t}, \phi\right),
$$

$W_{T}$ is a positive definite matrix, and $\left\{x_{t}\right\}_{t=1}^{T}$ represents the sample observations on $X$.
We will assume that $q \geq p$ and that $m_{T}(\phi)$ satisfies
Assumption 1 (i) $m_{T}(\phi)=O_{p}(1)$ for all $\phi \in \Phi$; (ii) $T^{1 / 2} m_{T}\left(\phi_{0}\right) \xrightarrow{d} N\left(0, V_{m}\right)$, where $V_{m}$ is a positive definite matrix of finite constants.

To consider the first-order asymptotic properties of GMM estimators, we introduce a number of high level assumptions.

Assumption 2 (i) $W_{T} \xrightarrow{p} W$, a positive definite matrix of constants; (ii) $\Phi$ is a compact set; (iii) $Q_{T}(\phi) \xrightarrow{p} Q(\phi)=m(\phi)^{\prime} W m(\phi)$ uniformly in $\phi$; (iv) $Q(\phi)$ is continuous on $\Phi$; (v) $Q\left(\phi_{0}\right)<Q(\phi) \forall \phi \neq \phi_{0}, \phi \in \Phi$.

Assumption 2(v) serves as a global identification condition. These conditions are sufficient to establish consistency, see, for example, Newey and McFadden (1994).

Proposition 1 If Assumption 2 holds, then $\widehat{\phi} \xrightarrow{p} \phi_{0}$.
Let $M_{T}(\tilde{\phi})=\partial m_{T}(\phi) /\left.\partial \phi^{\prime}\right|_{\phi=\tilde{\phi}}$ and let $N_{\phi, \epsilon}$ be an $\epsilon$-neighbourhood of $\phi_{0}$, that is, $N_{\phi, \epsilon}=\left\{\phi:\left\|\phi-\phi_{0}\right\|<\epsilon\right\}$. We can derive the first-order asymptotic distribution of $\widehat{\phi}$ after adding the following assumption, cf. Newey and McFadden (1994).

Assumption 3 (i) $\phi_{0}$ is an interior point of $\Phi$; (ii) $m_{T}(\phi)$ is continuously differentiable on $N_{\phi, \epsilon} ;($ iii $) M_{T}(\phi) \xrightarrow{p} M(\phi)$ uniformly on $N_{\phi, \epsilon} ;($ iv $) M(\phi)$ is continuous at $\phi_{0} ;(v)$ $M\left(\phi_{0}\right)$ has rank $p$.

Assumption 3(v) is the condition for first-order local identification. It is sufficient but not necessary for local identification of $\phi_{0}$ on $N_{\phi, \epsilon}$, but it is necessary for the development of the standard first-order asymptotic theory.

Proposition 2 If Assumptions 1-3 hold, then $T^{1 / 2}\left(\widehat{\phi}_{M D}-\phi_{0}\right) \xrightarrow{d} N\left(0, V_{\phi}\right)$, where

$$
V_{\phi}=\left[M\left(\phi_{0}\right)^{\prime} W M\left(\phi_{0}\right)\right]^{-1} M\left(\phi_{0}\right)^{\prime} W V_{m} W M\left(\phi_{0}\right)\left[M\left(\phi_{0}\right)^{\prime} W M\left(\phi_{0}\right)\right]^{-1}
$$

Global identification is crucial for consistency; global and first-order local identification are needed for the preceding asymptotic distribution theory.

Given Assumption 2(i), the global identification condition for GMM can be equivalently stated as $E[g(X, \phi)]=0$ has a unique solution at $\phi=\phi_{0}$. The first-order local identification condition can also be stated as $E\left[\partial g(X, \phi) /\left.\partial \phi^{\prime}\right|_{\phi=\phi_{0}}\right]$ has full column rank.

## 3 Second-order local identification

For our analysis of GMM, we adopt the definition of second-order local identification originally introduced by Dovonon and Renault (2009). To present this definition, we introduce the following notations. Let $m(\phi)=E[g(X, \phi)]$ and

$$
M_{k}^{(2)}\left(\phi_{0}\right)=E\left[\left.\frac{\partial^{2} g_{k}(X, \phi)}{\partial \phi \partial \phi^{\prime}}\right|_{\phi=\phi_{0}}\right], k=1,2, \ldots, q,
$$

where $g_{k}(X, \phi)$ is the $k$ th element of $g(X, \phi)$ and $g(\cdot)$ is defined in (1). Second-order local identification is defined as follows.

Definition 1 The moment condition $m(\phi)=0$ locally identifies $\phi_{0} \in \Phi$ up to the second order if:
(a) $m\left(\phi_{0}\right)=0$.
(b) For all $u$ in the range of $M\left(\phi_{0}\right)^{\prime}$ and all $v$ in the nullspace of $M\left(\phi_{0}\right)$, we have:

$$
\left(M\left(\phi_{0}\right) u+\left(v^{\prime} M_{k}^{(2)}\left(\phi_{0}\right) v\right)_{1 \leq k \leq q}=0\right) \Rightarrow(u=v=0)
$$

The latter condition is derived using a second-order expansion of $m(\phi)$ around $m\left(\phi_{0}\right)$ and can be motivated as follows. For any non-zero $\phi-\phi_{0}$ with $\phi \in N_{\phi, \epsilon}$, we have $\phi-\phi_{0}=c_{1} u+c_{2} v$ where $c_{1}, c_{2}$ are constants such that $c_{1} \neq 0$ and/or $c_{2} \neq 0$. For those directions for which $c_{1}$ is non-zero, the first-order term is non-zero and dominates, and for those directions in which $c_{1}=0$, the second-order term is non-zero. Thus, without requiring the expected Jacobian matrix $M\left(\phi_{0}\right)$ to have full rank, conditions (a) and (b) in Definition 1 guarantee local identification in the sense that there is no sequence of points $\left\{\phi_{n}\right\}$ different from $\phi_{0}$ but converging to $\phi_{0}$ such that $m\left(\phi_{n}\right)=0$ for all $n$. The difference between first-order local identification and second-order local identification (with $M\left(\phi_{0}\right)$ rank deficient) is how sharply $m(\phi)$ moves away from 0 in the neighbourhood of $\phi_{0}$.

Example. Consider the following panel $\mathrm{AR}(1)$ model with individual effects:

$$
\begin{align*}
y_{i, t} & =\rho y_{i, t-1}+w_{i, t}  \tag{3}\\
w_{i, t} & =\eta_{i}+\varepsilon_{i, t}, \text { where } \eta_{i}=(1-\rho) \mu_{i}
\end{align*}
$$

for $i=1, \ldots, N$ and $t=1,2,3$. The number of individuals, $N$, may be large. Note that when $\rho=1$, then $\eta_{i}=0$. Adding the term $x_{i, t}^{\prime} \breve{\beta}$ to (3) with $x_{i, t}$ exogenous and $\breve{\beta}=\beta(1-\rho)$ does not affect the essence of the analysis below except that $p$ and $q$ increase by $\operatorname{dim}(\breve{\beta})$ and $\breve{\beta}$ is first-order locally identified in all cases. We make the following assumption.

Assumption $\mathbf{A}\left\{\eta_{i}, y_{i, 0}, y_{i, 1}, y_{i, 2}, y_{i, 3}\right\}_{i=1}^{N}$ is a random sample from a joint distribution with finite fourth-order moments that satisfies $E\left(\varepsilon_{i, t} \mid \eta_{i}, y_{i, 0}, \ldots, y_{i, t-1}\right)=0$ for $t=1,2,3$.

Let the unconditional variances of the errors be denoted as $E\left(\varepsilon_{i, t}^{2}\right)=\sigma_{t}^{2}$ for $t=1,2,3$. We are interested in GMM estimation of $\rho$. Assumption A implies the following three linear moment conditions for the model in (3), cf. Arellano and Bond (1991):

$$
\begin{equation*}
m_{A B, s, t}(\rho):=E\left[y_{i, t-s}\left(\Delta y_{i, t}-\rho \Delta y_{i, t-1}\right)\right]=0 \text { for } s=2, t \text { and } t=2,3, \tag{4}
\end{equation*}
$$

where $\Delta y_{i, t}=y_{i, t}-y_{i, t-1}$. Assumption A also implies one nonlinear moment condition for the model in (3), cf. Ahn and Schmidt (1995):

$$
\begin{equation*}
m_{A S, 3}(\rho):=E\left[\left(y_{i, 3}-\rho y_{i, 2}\right)\left(\Delta y_{i, 2}-\rho \Delta y_{i, 1}\right)\right]=0 . \tag{5}
\end{equation*}
$$

If $\rho \neq 1$, then $\rho$ is both globally and first-order locally identified by each of the above four moment conditions, while if $\rho=1$, then the first three moment conditions do not help to identify $\rho$ at all, because they are linear in $\rho$ and $d m_{A B, s, t}(\rho) / d \rho=0$ for $s=2, t$ and $t=2,3$. When $\rho=1$ and $\sigma_{1}^{2} \neq \sigma_{2}^{2}$, then $\rho$ is still first-order locally identified by $m_{A S, 3}(\rho)=0$, because $d m_{A S, 3}(\rho) / d \rho=-\sigma_{2}^{2}+\sigma_{1}^{2} \neq 0$, but technically speaking $\rho$ is no longer globally identified because $m_{A S, 3}(\rho)=0$ now has two solutions, i.e., $\rho=1$ and $\rho=\sigma_{2}^{2} / \sigma_{1}^{2}$. However, the fact that $m_{A S, 3}(\rho)=0$ has multiple solutions only in this case (and not when $\rho \neq 1$ ) means that in practice $\rho$ is globally identified in this case (because the occurance of two roots implies that $\rho=1$ ) and that the GMM estimator that exploits $m_{A S, 3}(\rho)=0$ is also consistent in this case, see Kruiniger (2013) for details. If $\rho=1$ and $\sigma_{1}^{2}=\sigma_{2}^{2}$, then $\rho$ is second-order rather than first-order locally identified by $m_{A S, 3}(\rho)=0$, because $d m_{A S, 3}(\rho) / d \rho=0$ and $d^{2} m_{A S, 3}(\rho) / d \rho^{2}=2 \sigma_{1}^{2} \neq 0$, and $\rho$ is also globally identified because the two solutions of $m_{A S, 3}(\rho)=0$ are now both equal to 1 , that is, 1 is a double
root of $m_{A S, 3}(\rho)=0$, cf. Alvarez and Arellano (2021). Finally, as we have just seen, when $\rho=1$, then $\rho$ is only identified by $m_{A S, 3}(\rho)=0$ and not by $m_{A B, s, t}(\rho)=0$ for $s=2, t$ and $t=2,3$, so in this case we really have $q=1$ rather than $q=4$, that is, $\rho$ is exactly identified rather than overidentified because $q=p=1$.

## 4 The limiting distribution of the GMM estimator

In this section we consider the moment condition model (1) and study the asymptotic behaviour of the GMM estimator when $\phi_{0}$ is second-order locally identified because the moment condition exhibits the properties in Definition 1 but the standard local identification condition (Assumption 3(v)) fails.

### 4.1 Main results

We study the asymptotic behaviour of the GMM estimator by restricting ourselves to the case of a rank deficiency of one, i.e., the rank of $M\left(\phi_{0}\right)$ is equal to $p-1$, since this case is relatively easy to analyse compared to the general case. W.l.o.g. we consider the case where the rank deficiency of $M\left(\phi_{0}\right)$ is due to its last column being a null vector ${ }^{1}$ To this end, we partition $\phi$ into $\left(\phi_{1: p-1}^{\prime}, \phi_{p}\right)^{\prime}$ where $\phi_{1: p-1}$ is the vector consisting of the first $p-1$ elements of $\phi$ and $\phi_{p}$ is the $p$-th element of $\phi$. For ease of presentation below, we shorten the subscript and write $\phi_{1}$ for $\phi_{1: p-1}$. Thus $\phi_{0}=\left(\phi_{0,1}^{\prime}, \phi_{0, p}\right)^{\prime}$ where $\phi_{0,1}$ is a $(p-1) \times 1$ vector containing the true value of $\phi_{1: p-1}$ and $\phi_{0, p}$ is the true value of $\phi_{p}$. If $M\left(\phi_{0}\right)$ has rank $p-1$ with $\frac{\partial m}{\partial \phi_{p}}\left(\phi_{0}\right)=0$, then second-order local identification is equivalent to:

$$
\operatorname{Rank}\left(\frac{\partial m}{\partial \phi_{1}^{\prime}}\left(\phi_{0}\right) \frac{\partial^{2} m}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)\right)=p
$$

This is the setting studied by Sargan (1983) for the instrumental variables estimator for a nonlinear in parameters model. We now present the regularity conditions under which we derive the asymptotic distribution of the GMM estimator. Define $D=\frac{\partial m}{\partial \phi_{1}^{\prime}}\left(\phi_{0}\right)$ and $G=$ $\frac{\partial^{2} m}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)$. The next assumption states formally the identification pattern described above.

Assumption 4 (i) $m(\phi)=0 \Leftrightarrow \phi=\phi_{0}$; (ii) $\frac{\partial m}{\partial \phi_{p}}\left(\phi_{0}\right)=0$; (iii) $\operatorname{Rank}(D G)=p$.
We also require the following regularity conditions to hold.

[^1]Assumption 5 (i) $m_{T}(\phi)$ has partial derivatives up to order 3 if $q>p$ and up to order 5 if $q=p$ in a neighbourhood $N_{\phi, \epsilon}$ of $\phi_{0}$ and the derivatives of $m_{T}(\phi)$ converge in probability uniformly on $N_{\phi, \epsilon}$ to those of $m(\phi)$.
(ii) $\sqrt{T}\binom{m_{T}\left(\phi_{0}\right)}{\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)} \xrightarrow{d}\binom{\mathbb{Z}_{0}}{\mathbb{Z}_{1}}$.
(iii) $W_{T}-W=o_{p}\left(T^{-1 / 4}\right), \frac{\partial m_{T}}{\partial \phi_{1}^{1}}\left(\phi_{0}\right)-D=O_{p}\left(T^{-1 / 2}\right), \frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)-G=O_{p}\left(T^{-1 / 2}\right)$, $\frac{\partial^{2} m_{T}}{\partial \phi_{1}^{\prime} \partial \phi_{p}}\left(\phi_{0}\right)-G_{1 p}=o_{p}(1)$ and $\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{3}}\left(\phi_{0}\right)-L=o_{p}(1)$, and if $q=p, \frac{\partial^{3} m_{T}}{\partial \phi_{1}^{\prime} \partial \phi_{p}^{2}}\left(\phi_{0}\right)-G_{1 p p}=o_{p}(1)$, $\frac{\partial^{4} m_{T}}{\partial \phi_{1}^{\prime} \partial \phi_{p}^{3}}\left(\phi_{0}\right)-G_{1 p p p}=o_{p}(1), \frac{\partial^{4} m_{T}}{\partial \phi_{p}^{4}}\left(\phi_{0}\right)-F=o_{p}(1)$ and $\frac{\partial^{2} m_{k, T}}{\partial \phi_{1} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)-K_{k}=o_{p}(1)$ for $k=$ $1,2, \ldots, q$, with $G_{1 p}=\frac{\partial^{2} m}{\partial \phi_{1}^{\prime} \partial \phi_{p}}\left(\phi_{0}\right), L=\frac{\partial^{3} m}{\partial \phi_{p}^{3}}\left(\phi_{0}\right), G_{1 p p}=\frac{\partial^{3} m}{\partial \phi_{1}^{2} \partial \phi_{p}^{2}}\left(\phi_{0}\right), G_{1 p p p}=\frac{\partial^{4} m}{\partial \phi_{1}^{\prime} \partial \phi_{p}^{3}}\left(\phi_{0}\right)$, $F=\frac{\partial^{4} m}{\partial \phi_{p}^{4}}\left(\phi_{0}\right)$ and $K_{k}=\frac{\partial^{2} m_{k}}{\partial \phi_{1} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)$ for $k=1,2, \ldots, q$, where $m_{k, T}(\phi)\left(m_{k}(\phi)\right)$ is the kth element of $m_{T}(\phi)($ of $m(\phi))$.

These conditions are stronger than those imposed in the standard first-order asymptotic analysis. The derivation of the asymptotic distribution of the GMM estimator requires an expansion of $Q_{T}(\phi)$ involving derivatives of $m_{T}(\phi)$ up to the third order when $q>p$ and up to the fifth order when $q=p$, and the uniform convergence guaranteed by Assumption 5(i) is useful to control the remainder of our expansions. The orders of our expansions of $Q_{T}(\phi)$ for the cases $q>p$ and $q=p$ are different because in the case of exact identification some terms in the expansion vanish. Assumption 5(ii) states that $\sqrt{T}\left(m_{T}\left(\phi_{0}\right)^{\prime}, \partial m_{T}\left(\phi_{0}\right)^{\prime} / \partial \phi_{p}\right)^{\prime}$ converges in distribution. Under Assumption 4 and additional mild conditions on $g\left(X, \phi_{0}\right)$ and $\frac{\partial g}{\partial \phi_{p}}\left(X, \phi_{0}\right)$, the central limit theorem guarantees that $\left(\mathbb{Z}_{0}, \mathbb{Z}_{1}\right)^{\prime} \sim N(0, v)$, with $v=\lim _{T \rightarrow \infty} \operatorname{Var}\left[\sqrt{T}\left(m_{T}\left(\phi_{0}\right)^{\prime}, \partial m_{T}\left(\phi_{0}\right)^{\prime} / \partial \phi_{p}\right)^{\prime}\right]$. Assumption 5 (iii) imposes the asymptotic order of magnitude on the differences between some sample dependent quantities and their probability limits. These orders of magnitude are enough to make these differences negligible in the expansions. Assumption 5(iii) is not particularly restrictive since most of the orders of magnitude imposed are guaranteed by the central limit theorem.

To facilitate the presentation of our main result in this section, we introduce the following definitions. Let $M_{d}$ be the matrix of the orthogonal projection on the orthogonal complement of $W^{1 / 2} D$ :

$$
M_{d}=I_{q}-W^{1 / 2} D\left(D^{\prime} W D\right)^{-1} D^{\prime} W^{1 / 2}
$$

where $I_{q}$ is the identity matrix of size $q$, let $P_{g}$ be the matrix of the orthogonal projection
on $M_{d} W^{1 / 2} G$ :

$$
P_{g}=M_{d} W^{1 / 2} G\left(G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\right)^{-1} G^{\prime} W^{1 / 2} M_{d},
$$

and let $M_{d g}$ be the matrix of the orthogonal projection on the orthogonal complement of $\left(W^{1 / 2} D \quad W^{1 / 2} G\right):$

$$
M_{d g}=M_{d}-P_{g} .
$$

Let

$$
\begin{align*}
\mathbb{R}_{1}= & \left(\mathbb{Z}_{0}^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} G^{\prime}-G^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} \mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \times \\
& \left(\frac{1}{3} L+G_{1 p} H G\right) / \sigma_{G}+\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2}\left(\mathbb{Z}_{1}+G_{1 p} H \mathbb{Z}_{0}\right), \tag{6}
\end{align*}
$$

with $\sigma_{G}=G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G$ and $H=-\left(D^{\prime} W D\right)^{-1} D^{\prime} W$. In addition, let $V=-2 \mathbb{Z} \mathbf{1}(\mathbb{Z}<$ 0) $/ \sigma_{G}$, where $\mathbb{Z}=G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0}$ and $\mathbf{1}(\cdot)$ is the usual indicator function.

The following lemma, which is based on Theorem 1 in DH , and theorem give the asymptotic properties of the GMM estimator $\widehat{\phi}$ under Assumptions 2, 4 and 5.

Lemma 1 (Dovonon and Hall (2018)) Under Assumptions 2, 4 and 5, we have:
(a) $\widehat{\phi}_{1}-\phi_{0,1}=O_{p}\left(T^{-1 / 2}\right)$ and $\widehat{\phi}_{p}-\phi_{0, p}=O_{p}\left(T^{-1 / 4}\right)$;
(b) if in addition $\phi_{0} \in \operatorname{interior}(\Phi)$, then

$$
\binom{\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)}{\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}} \xrightarrow{d}\binom{H \mathbb{Z}_{0}+H G V / 2}{V} .
$$

Theorem 1 Under Assumptions 2, 4 and 5, and if $\phi_{0} \in \operatorname{interior}(\Phi)$, we have:
(a) if in addition $q>p$, then

$$
T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \xrightarrow{d}(-1)^{B_{1}} \sqrt{V},
$$

with $B_{1}=\mathbf{1}\left(\mathbb{R}_{1} \geq 0\right)$;
(b) if in addition $q=p$ and $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$, where $\mathbb{R}_{2}$ is defined in the proof below equation (19), then

$$
T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \xrightarrow{d}(-1)^{B_{2}} \sqrt{V},
$$

with $B_{2}=1\left(\mathbb{R}_{2} \geq 0\right)$.
Parts (a) and (b) of Lemma 1 are the same as parts (a) and (b) of Theorem 1 in DH. In the Appendix we provide an alternative, self-contained proof for part (b) of Lemma 1. There we also provide a proof for our Theorem 1.

### 4.2 Discussion

Our theorem is different from part (c) of Theorem 1 of DH . The latter only states that $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ converges in distribution to the limiting distribution given in part (a) of our Theorem 1 under the condition that $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right)=0$. DH claim in their Remark 1 that this condition "is not expected to be restrictive in general ...", although they also add the following caveat: "However, when $q=p=1$ (one moment restriction with one non first-order locally identified parameter), we can see that $\mathbb{R}_{1}=0$." However, as we show in the proof of Lemma 2 in the Appendix, the condition $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right)=0$ is actually only satisfied when $\phi$ is overidentified, i.e., when $q>p$; when $\phi$ is exactly identified, i.e., when $q=p$, then $\operatorname{Pr}\left(\mathbb{R}_{1}=0 \mid \mathbb{Z}<0\right)=1$ and hence $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right) \neq 0$. In other words, the theory of DH only provides the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ for the case where $\phi$ is overidentified.

In part (b) of Theorem 1 we provide the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ for the case where $\phi$ is exactly identified and $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$. The difference between the limiting distributions of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ given in parts (a) and (b) is related to the difference between the distributions of the Bernoulli r.v.'s $B_{1}$ and $B_{2}$ that determine the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ when $\mathbb{Z}<0$. The condition $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$ holds if $F+3!G_{1 p p} H G+\frac{4!}{2} \tilde{\lambda}_{3} \neq$ 0 , where $\tilde{\lambda}_{3}=\left(\lambda_{3,1} \ldots \lambda_{3, q}\right)^{\prime}$ with $\lambda_{3, k}=\frac{1}{4} G^{\prime} H^{\prime} K_{k} H G$ for $k=1,2, \ldots, q$, cf. Lemma 2(b). When $q=p>1$ and $F+3!G_{1 p p} H G+\frac{4!}{2} \tilde{\lambda}_{3}=0$, then the condition $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$ may still hold, but $\mathbb{R}_{2}=0$ when $F=0, G_{1 p p}=0$ and $K_{k}=0$ for $k=1,2, \ldots, q$. When $q=p=1$, then $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$ if only if $F \neq 0$. If $q=p$ and $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)>0$, then one may still be able to describe the limiting distribution (of the sign) of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$, see later in this subsection.

Part (a) of Lemma 1 gives the rates of convergence of $\widehat{\phi}_{1}$ and $\widehat{\phi}_{p}$. Because $\phi_{1}$ is firstorder identified and $\phi_{p}$ is second-order identified, $\widehat{\phi}_{1}-\phi_{0,1}$ converges at the usual rate $T^{-1 / 2}$ while $\widehat{\phi}_{p}-\phi_{0, p}$ converges at the slower rate $T^{-1 / 4}$.

Part (b) of Lemma 1 gives the limiting distribution of $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right), \sqrt{T}\left(\widehat{\phi}_{p}-\right.\right.$ $\left.\phi_{0, p}\right)^{2}$ ). This result is obtained by minimizing the sum of the leading $O_{p}\left(T^{-1}\right)$ terms of an expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ around $\phi_{0}$ which are collected into $\underline{K}_{T}\left(\phi_{0}\right)$ as given by (10) in the Appendix. As $\underline{K}_{T}\left(\phi_{0}\right)$ is a quadratic function of $\left(\widehat{\phi}_{1}-\phi_{0,1}\right)$ and $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ only, it only allows one to obtain the limiting distribution of $T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right|$. To obtain the limiting distribution of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ one needs to employ a higher order expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ which includes an odd power of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$.

When $q>p$, the limiting distribution of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ can be obtained from the $O_{p}\left(T^{-5 / 4}\right)$ terms in the expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$. In that case we have:

$$
m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})=\underline{K}_{T}\left(\phi_{0, p}\right)+\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \times 2 R_{1 T}+o_{p}\left(T^{-5 / 4}\right),
$$

where $\underline{K}_{T}\left(\phi_{0, p}\right)$ and $R_{1 T}$ are quadratic functions of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$. We show in the Appendix that $T R_{1 T} \xrightarrow{d} \mathbb{R}_{1}$ and that $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right)=0$ if $q>p$. When $Z_{T} \equiv G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} m_{T}\left(\phi_{0}\right)<$ $0, q>p$ and $T$ is large, the minimum of $m_{T}^{\prime}(\phi) W_{T} m_{T}(\phi)$ is reached when $\left(\phi_{p}-\phi_{0, p}\right)$ has the opposite sign to $R_{1 T}$. This suggests that when $q>p$, then the limiting distribution of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ can be described by $(-1)^{B_{1}}$ with $B_{1}=\mathbf{1}\left(\mathbb{R}_{1} \geq 0\right)$.

When $q=p$, then $\operatorname{Pr}\left(\mathbb{R}_{1}=0 \mid \mathbb{Z}<0\right)=1$. In fact, when $q=p$, then $\mathbb{R}_{1}=0$, see the end of the proof of Lemma 2(a1). However, if $q=p$ and $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$, then the limiting distribution of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ can be obtained from the $O_{p}\left(T^{-7 / 4}\right)$ terms in the expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$. In that case we have:
$m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})=\underline{K}_{T}\left(\phi_{0, p}\right)+\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \times 2 R_{1 T}+O_{p}\left(T^{-6 / 4}\right)+\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \times 2 R_{2 T}+o_{p}\left(T^{-7 / 4}\right)$,
where the $O_{p}\left(T^{-6 / 4}\right)$ term and $R_{2 T}$ are cubic functions of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$. We show in the Appendix that $T^{6 / 4} R_{2 T} \xrightarrow{d} \mathbb{R}_{2}$ if $q=p$. When $Z_{T}<0, q=p, \operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$ and $T$ is large, then the minimum of $m_{T}^{\prime}(\phi) W_{T} m_{T}(\phi)$ is reached when $\left(\phi_{p}-\phi_{0, p}\right)$ has the opposite sign to $R_{2 T}$. This suggests that if $q=p$ and $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$, then the limiting distribution of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ can be described by $(-1)^{B_{2}}$ with $B_{2}=\mathbf{1}\left(\mathbb{R}_{2} \geq 0\right)$. If $q=p$ and $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)>0$, then we may still be able to characterize the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ by using an expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ of an even higher order. However, if, for instance, $q=p=1$ and $\frac{\partial^{k} m}{\partial \phi_{p}^{k}}\left(\phi_{0}\right)=0$ for $k \geq 4$, then we cannot characterize the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ and the latter does not have a proper limiting distribution, whereas $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ has one, which is given in Lemma 1 (b).

When $\phi$ is exactly identified and $\operatorname{dim}(\phi)>1$, i.e., when $q=p>1$, then the limiting distribution of $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right), T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)\right)$ still depends on the choice of the weight matrix $W_{T}$ unlike the limiting distribution of a GMM estimator of a parameter vector that is wholly first-order identified. The reason for this dependence is that $\widehat{\phi}_{1}$ and $\widehat{\phi}_{p}$ converge at different rates and as a consequence the sum of the leading terms in the expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$, i.e., $\underline{K}_{T}\left(\phi_{0}\right)$ only depends on even powers of $\left(\phi_{p}-\phi_{0, p}\right)$, which cannot be negative. Therefore minimization of $\underline{K}_{T}\left(\phi_{0}\right)$ is constrained rather than unconstrained and that is why $W_{T}$ also matters for the limiting distribution when $q=p$.

When $Z_{T}<0, \underline{K}_{T}\left(\phi_{0}\right)$ is minimized at $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}=-2 \sqrt{T} Z_{T} / \sigma_{G}$, whereas when $Z_{T} \geq 0, \underline{K}_{T}\left(\phi_{0}\right)$ is minimized at $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}=0$. Given the value of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$, $\underline{K}_{T}\left(\phi_{0}\right)$ is minimized at $\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)=H\left(\sqrt{T} m_{T}\left(\phi_{0}\right)+\frac{1}{2} G \sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)$. This explains why the limiting distribution of $\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)$ shown in Lemma 1 is a mixture of two normal distributions, namely the limiting distribution of $H\left(\sqrt{T} m_{T}\left(\phi_{0}\right)-G \sqrt{T} Z_{T} / \sigma_{G}\right)$ given $\sqrt{T} Z_{T}<0$ and the limiting distribution of $H\left(\sqrt{T} m_{T}\left(\phi_{0}\right)\right)$ given $\sqrt{T} Z_{T} \geq 0$ with mixing probabilities $\operatorname{Pr}(\mathbb{Z}<0)=\frac{1}{2}$ and $\operatorname{Pr}(\mathbb{Z} \geq 0)=\frac{1}{2}$. Furthermore, note that when $q>p$, then $\operatorname{Pr}\left(B_{1}=1\right) \neq \frac{1}{2}$, and when $q=p$, then $\operatorname{Pr}\left(B_{2}=1\right) \neq \frac{1}{2}$. Hence the limiting distributions of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ given in Theorem 1 are generally asymmetric around 0 .

### 4.3 Examples with exact identification

Kruiniger (2018) derived the limiting distributions of two Modified MLEs for the panel ARX(1) model with homoskedastic errors when the autoregressive parameter equals one by viewing them as GMM estimators. In the unit root case the autoregressive parameter is only second-order locally identified by the objective functions of both Modified MLEs due to the nonlinear terms in the modified score vector. Furthermore, the parameter vector is obviously exactly identified and the condition $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$ holds because the condition $F+3!G_{1 p p} H G+\frac{4!}{2} \tilde{\lambda}_{3} \neq 0$ is satisfied. It is therefore unsurprising that the limiting distributions obtained by Kruiniger (2018) for both Modified MLEs of the autoregressive parameter are in agreement with Theorem 1(b) above.

We now return to the example given in section 3. Recall that when $\rho=1$ and $\sigma_{1}^{2}=\sigma_{2}^{2}$, then $\rho$ is only second-order locally identified by the four moment conditions mentioned in that example. However, in this example $q=p=1$ and $F=0$ because there is only one nonlinear moment condition, i.e., $m_{A S, 3}(\rho)=0$, which is quadratic in $\rho$. Therefore in this case the limiting distribution of $T^{1 / 4}(\hat{\rho}-1)$ cannot be obtained from Theorem 1 (b) above. In fact, it does not exist at all. To see this, note that when $\rho=1$, then $m_{A S, 3}(r)=E\left[\left(\varepsilon_{i, 3}+(1-r) y_{i, 2}\right)\left(\Delta \varepsilon_{i, 2}+(1-r) \varepsilon_{i, 1}\right)\right]$. Hence, when $N$ is large, the GMM estimator for $\rho$ is approximately equal to the solution of $N^{-1 / 2} \sum_{i=1}^{N}\left[(\hat{\rho}-1)^{2} y_{i, 2} \varepsilon_{i, 1}-(\widehat{\rho}-1) \times\right.$ $\left.\left(\varepsilon_{i, 1} \varepsilon_{i, 3}+y_{i, 2} \Delta \varepsilon_{i, 2}\right)+\varepsilon_{i, 3} \Delta \varepsilon_{i, 2}\right]=0$. However, the linear term $-(\widehat{\rho}-1) N^{-1 / 2} \sum_{i=1}^{N}\left(\varepsilon_{i, 1} \varepsilon_{i, 3}+\right.$ $\left.y_{i, 2} \Delta \varepsilon_{i, 2}\right)=o_{p}(1)$ because $N^{1 / 4}(\widehat{\rho}-1)=O_{p}(1)$ and $N^{-1 / 2} \sum_{i=1}^{N}\left(\varepsilon_{i, 1} \varepsilon_{i, 3}+y_{i, 2} \Delta \varepsilon_{i, 2}\right)=$ $O_{p}(1)$. Thus when $N$ tends to infinity, the sample counterpart of $m_{A S, 3}(\rho)=0$ only determines the behaviour of $N^{1 / 2}(\widehat{\rho}-1)^{2}$, i.e., $N^{1 / 2}(\widehat{\rho}-1)^{2} \xrightarrow{d}-\widetilde{\mathbb{Z}}_{0} \mathbf{1}\left(\widetilde{\mathbb{Z}}_{0}<0\right) / \sigma^{2}$ with $N^{-1 / 2} \sum_{i=1}^{N}\left(\varepsilon_{i, 3} \Delta \varepsilon_{i, 2}\right) \xrightarrow{d} \widetilde{\mathbb{Z}}_{0}$, but can no longer determine the sign of $N^{1 / 4}(\widehat{\rho}-1)$.

### 4.4 GMM-based inference under second-order identification

The limiting distributions in Theorem 1 and part (b) of Lemma 1 are non-standard but easy to simulate. Approximations to these limiting distributions can be obtained by drawing randomly copies of $\left(\mathbb{Z}_{0}^{\prime}, \mathbb{Z}_{1}^{\prime}\right)^{\prime}$ from $N(0, \widehat{v})$, where $\widehat{v}$ is a consistent estimator of $v$, and using consistent estimators of $W, D, G, L, G_{1 p}, G_{1 p p}, G_{1 p p p}, F$ and $K_{k}$ for $k=1,2, \ldots, q$ as required. One can use the quantiles of the simulated distributions to construct confidence sets for the elements of $\phi_{0}$.

Dovonon, Hall and Kleibergen (2020) studied and compared the local power properties of various test-statistics for conducting inference in moment conditions models that locally identify the parameters only to second order. The tests considered include tests for $H_{0}: \phi_{0}=a$, where $a$ is a known vector, such as the conventional Wald and LM tests, the Generalized Anderson-Rubin (GAR) test (Anderson and Rubin, 1949; Staiger and Stock, 1997; Stock and Wright, 2000), the KLM test (Kleibergen 2002; 2005) and the GMM extension of Moreira's (2003) Conditional LR test, also known as the GMM-M test (Kleibergen, 2005), and tests for $H_{0}: m\left(\phi_{0}\right)=0$, such as the identification-robust J test of Kleibergen (2005) and the GAR test. Under the null hypothesis the conventional LM and Wald test-statistics have non-standard limiting distributions, although the LM test-statistic converges to a $\chi^{2}$ r.v. in a special case; the distribution of the Wald teststatistic depends on $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ only through $T^{1 / 2}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$, which has a limiting distribution that is a mixture of a half-normal distribution and 0 . All the other teststatistics are robust to weak and second-order local identification and have the same limiting distribution under the null hypothesis as they would have under first-order local identification. Apart from the Wald test, all the tests can also be used when the Jacobian is rank deficient by more than one. Dovonon, Hall and Kleibergen (2020) found that in a particular panel $\operatorname{AR}(1)$ model, the Wald test of the unit root hypothesis has better power than the GAR, KLM, LM and GMM-M tests.

Kruiniger (2018) discusses a Quasi LM test for $H_{0}: \phi_{0}=a$, when $\phi_{0, p}$ is possibly only second-order locally identified. Specifically, Kruiniger's (2018) Quasi LM test-statistic generalizes the LM test-statistic $W_{n}^{(2)}(\theta)$ in Bottai (2003), who studied the asymptotic behaviour of several tests and confidence regions in identifiable one-dimensional parametric models with a smooth likelihood function and Fisher information equal to zero at some point in the parameter space, in two ways, namely by allowing for several parameters in
the model and by relaxing Bottai's ML setup to a Quasi ML setup. Under $H_{0}$ both LM test-statistics have a $\chi^{2}$-distribution, also when one of the parameters that appears in the null hypothesis is only second-order locally identified. In the latter case, the score that corresponds to that parameter and appears in the LM test-statistic under first-order local identification will be replaced by its first-derivative. Kruiniger (2018) shows that his Quasi LM test and the confidence region that is based on inverting his test-statistic have correct asymptotic size in a uniform sense.

Finally, Lee and Liao (2018) pointed out that when (a part of) $\phi_{0}$ is only secondorder locally identified by the original set of moment conditions, then Jacobian-based moment conditions can be used to obtain GMM estimators and overidentification-teststatistics with standard asymptotic properties. They then noted that the asymptotic normal distributions of such GMM estimators can be used to conduct standard inference on $\phi_{0}$. However, their tests and confidence intervals are only valid when (a part of) $\phi_{0}$ is only second-order locally identified by the original set of moment conditions and hence they obviously do not have correct asymptotic size in a uniform sense.

### 4.5 Monte Carlo results

Using simulations, Kruiniger (2018) and DH studied the finite sample properties of specific GMM estimators under second-order identification in the cases of exact and overidentification, respectively. Both papers found that the GMM estimator of $\phi_{0, p}$ is biased. These findings are related to the asymmetry of the limiting distributions of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in these cases, which are given in Theorem 1 above $\sqrt[2]{ }$ DH also studied the coverage rates of two kinds of confidence intervals for $\phi_{0, p}$ in a model with $p=1$ that are based on a GMM estimator that exploits $q>p$ moment conditions and use analytic quantiles of its limiting distribution given in Lemma 1(b) above and simulated quantiles of its limiting distribution given in Theorem 1(1a) above, respectively, when $\phi_{0, p}$ is locally identified at the second-order and they concluded that these limiting distributions give a reasonable approximation to the behaviour of the GMM estimator for $\phi_{0, p}$.

[^2]
### 4.6 Indirect Inference

Dovonon and Hall (2018) also considered the limiting distribution of an Indirect Inference (II) estimator under second-order local identification. Specifically, DH focused on an II estimator for the parameter vector $\theta_{0} \in \Omega \subset \mathbb{R}^{p}$ which is defined by the following set-up: the auxiliary model consists of a set of $q$ population moment conditions indexed by a vector of auxiliary parameters $h \in \mathcal{H} \subset \mathbb{R}^{l}$ and the target function for the II estimation is a GMM estimator of the auxiliary parameter vector. Within this framework, there are two types of identification conditions: one set involving the binding function, and the other involving the auxiliary parameters. The standard first-order asymptotic theory is premised on the assumption that the binding function satisfies global and first-order local identification conditions and the auxiliary parameters are globally and first-order locally identified within the auxiliary model. DH presents the limiting distribution of the II estimator under the following two scenarios: (i) the binding function satisfies the global and first-order local identification conditions and the auxiliary parameters are globally identified but only locally identified at second order; (ii) the binding function satisfies the global identification condition but only satisfies the local identification condition at second order, and the auxiliary parameters are globally and first-order locally identified.

Unsurprisingly, the limiting distribution theories of DH for II estimation under these two different scenarios with second-order identification, i.e., their Theorems 2 and 3(b) are incomplete for similar reasons as their limiting distribution theory for GMM estimation is: their limiting distributions for scenario (i) and scenario (ii) are only valid in the case of overidentification, that is, when $q>l$ and when $l>p$, respectively, or in terms of their conditions, when $\operatorname{Pr}\left(\mathbb{R}_{1}^{(a)}=0\right)=0$ and $\operatorname{Pr}\left(\mathbb{R}_{1}^{(b)}(s)=0\right)=0$, respectively, where $\mathbb{R}_{1}^{(a)}$ and $\mathbb{R}_{1}^{(b)}(s)$ are defined similarly as $\mathbb{R}_{1}$, see DH . Our results and derivations for GMM estimation under second-order local identification can be used to complete both theories and in particular can be used to straightforwardly derive the limiting distributions of the II estimators in the case of exact identification under either scenario. The representations of these distributions are obtained by replacing $\mathbb{R}_{1}^{(a)}$ and $\mathbb{R}_{1}^{(b)}(s)$ (implicit) in Theorems 2 and $3(\mathrm{~b})$ in DH by $\mathbb{R}_{2}^{(a)}$ and $\mathbb{R}_{2}^{(b)}(s)$, respectively, which are defined similarly as $\mathbb{R}_{2}$ above just like $\mathbb{R}_{1}^{(a)}$ and $\mathbb{R}_{1}^{(b)}(s)$ are defined similarly as $\mathbb{R}_{1}$.

## 5 Concluding remarks

The limiting distribution theory of DH (2018) for GMM estimators under second-order local identification depends on a non-transparent condition, namely that $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right)=0$ where $\mathbb{R}_{1}$ is defined in (6). We have shown that this condition is only satisfied when $\phi$ is overidentified and derived the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ for the case where $\phi$ is exactly identified. This distribution is different from that of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ given in DH for the case where $\phi$ is overidentified. In particular, the limiting distributions of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ for the cases of exact and overidentification, respectively, are different and are obtained by using expansions of the GMM objective function of different orders. We have also pointed out that the limiting distribution theories of DH for Indirect Inference (II) estimation under two different scenarios with second-order identification where the target function is a GMM estimator of the auxiliary parameter vector, are incomplete for similar reasons and we have discussed how they can be completed.

The asymptotic theory for GMM estimators that has been discussed in this paper can be generalized in two directions: (i) one can consider cases where the (expected) Jacobian matrix has a rank deficiency that is higher than one, and/or (ii) local identification of an order that is higher than two. In the case of second-order identification where the Jacobian has a rank deficiency of $r d$ (with $r d \in \mathbb{N} \backslash\{0,1\}$ ), one can reparametrize the model in such a way that the last $r d$ columns of the Jacobian are zero, and we expect that the asymptotic theory is similar to the theory for the case of a rank deficiency of one apart from the fact that in the current case there are now $r d$ GMM estimators that converge at rate $T^{-1 / 4}$. In the case of local identification of order $s$ (with $s>1$ ), we expect that the GMM estimator(s) of the higher-order identified parameter(s) converge(s) at rate $T^{-1 /(2 s)}$. Like Rotnitzky et al. (2000), one also needs to distinguish between cases where $s$ is even and cases where $s$ is odd: when $s$ is even, we expect that the limiting distribution(s) of the GMM estimator(s) of the higher-order identified parameter(s) is/are a mixture of a spike at the true value and a non-standard distribution, while when $s$ is odd, we expect that their limiting distribution(s) is/are equal to the distribution of the $s-t h$ root of a normal random variable. Furthermore, when $s$ is even, the GMM estimators of the remaining parameters converge at the usual rate $T^{-1 / 2}$ and have a limiting distribution that is a mixture of two normal distributions, whereas when $s$ is odd, they converge at the usual rate $T^{-1 / 2}$ and have a normal limiting distribution.

## 6 Appendix. Proofs

## Proof of Lemma 1.

(a) For a proof for this part we refer to the proof of part (a) of Theorem 1 in DH.
(b) Here we provide an alternative to the proof of DH. Our proof is self-contained, unlike their proof, and it is also more straightforward than their proof.

Using $\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)=O_{p}\left(T^{-1 / 2}\right)$ and $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)=o_{p}(1)$ (from Proposition 1), DH show in the proof for part (a) of their Theorem 1 that

$$
\begin{equation*}
m_{T}(\widehat{\phi})=m_{T}\left(\phi_{0}\right)+\frac{\partial m_{T}}{\partial \phi_{1}^{\prime}}\left(\bar{\phi}_{1}, \widehat{\phi}_{p}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+\frac{1}{2} \frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0,1}, \bar{\phi}_{p}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+o_{p}\left(T^{-1 / 2}\right) . \tag{7}
\end{equation*}
$$

where $\bar{\phi}_{1} \in\left(\phi_{0,1}, \widehat{\phi}_{1}\right)$ and may differ from row to row, and where $\bar{\phi}_{p} \in\left(\phi_{0, p}, \widehat{\phi}_{p}\right)$ and may differ from row to row.

From (a) and (7), we have

$$
m_{T}(\widehat{\phi})=m_{T}\left(\phi_{0}\right)+D\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+o_{p}\left(T^{-1 / 2}\right) .
$$

The first-order condition for an interior solution is given by:

$$
\frac{\partial m_{T}^{\prime}}{\partial \phi}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})=0
$$

In the direction of $\phi_{1}$, this amounts to

$$
\left(D^{\prime}+o_{p}(1)\right) W\left(\sqrt{T} m_{T}\left(\phi_{0}\right)+D \sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+\frac{1}{2} G \sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+o_{p}(1)\right)=0 .
$$

This gives:

$$
\begin{equation*}
\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)=-\left(D^{\prime} W D\right)^{-1} D^{\prime} W\left(\sqrt{T} m_{T}\left(\phi_{0}\right)+\frac{1}{2} G \sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)+o_{p}(1) . \tag{8}
\end{equation*}
$$

A Taylor expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ around $\phi_{0}$ up to second-order gives:

$$
\begin{equation*}
Q_{T}(\widehat{\phi})=m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})=m_{T}^{\prime}(\widehat{\phi}) W m_{T}(\widehat{\phi})+o_{p}\left(T^{-1}\right)=\underline{K}_{T}\left(\phi_{0}\right)+o_{p}\left(T^{-1}\right) \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
\underline{K}_{T}\left(\phi_{0}\right)= & m_{T}^{\prime}\left(\phi_{0}\right) W m_{T}\left(\phi_{0}\right)+2 m_{T}^{\prime}\left(\phi_{0}\right) W\left(D\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right) \\
& +\left(D\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)^{\prime} W\left(D\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right) \tag{10}
\end{align*}
$$

Defining $Z_{0 T}=m_{T}\left(\phi_{0}\right)$ and replacing $\left(\widehat{\phi}_{1}-\phi_{0,1}\right)$ in (10) by its expression from (8), the leading $O_{p}\left(T^{-1}\right)$ term in the expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ is obtained as $\underline{K}_{T}\left(\phi_{0, p}\right)$ with

$$
\begin{equation*}
\underline{K}_{T}\left(\phi_{0, p}\right)=Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} Z_{0 T}+Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+\frac{1}{4} G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4} \tag{11}
\end{equation*}
$$

Let $Z_{T}=Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G$. If $Z_{T}<0$, then $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ is minimized at

$$
\begin{equation*}
\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}=-2 \frac{Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G}{G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G}+o_{p}\left(T^{-1 / 2}\right) \tag{12}
\end{equation*}
$$

If $Z_{T} \geq 0$, then $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ is minimized at $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}=o_{p}\left(T^{-1 / 2}\right)$.
Since $\sqrt{T} m_{T}\left(\phi_{0}\right)$ and $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ are $O_{p}(1)$, the pair is jointly $O_{p}(1)$ and by Prohorov's theorem, any subsequence of them has a further subsequence that jointly converges in distribution towards, say, $\left(\mathbb{Z}_{0}, V\right)$. Hence,
$T m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi}) \xrightarrow{d} \underline{K}\left(\phi_{0}\right) \equiv \mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0}+\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G V+\frac{1}{4} G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G V^{2}$.
Let $\mathbb{Z}=\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G$. If $\mathbb{Z}<0$, then $\underline{K}\left(\phi_{0}\right)$ is minimized at

$$
V=-2 \frac{\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G}{G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G}=-2 \frac{\mathbb{Z}}{\sigma_{G}}
$$

If $\mathbb{Z} \geq 0$, then $\underline{K}\left(\phi_{0}\right)$ is minimized at $V=0$.
Either way, we conclude that $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2} \xrightarrow{d} V=-2 \mathbb{Z} \mathbf{1}(\mathbb{Z}<0) / \sigma_{G}$.
Finally, using (8) we obtain that $\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right) \xrightarrow{d} H\left(\mathbb{Z}_{0}+\frac{1}{2} G V\right)$, where $H=$ $-\left(D^{\prime} W D\right)^{-1} D^{\prime} W$.

## Proof of Theorem 1.

The limiting distribution for $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ that is specified in part (c) of Theorem 1 of DH (2018) is only valid in the case of overidentification, i.e., when $q>p$, and DH's proof of this claim is only valid when $q>p$. Here we provide a more straightforward and more complete proof of this claim for the case $q>p$, i.e., of part (a) of our Theorem 1, and also derive the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in the case of exact identification, i.e., when $q=p$, which is given in part (b) of our Theorem 1.

Proof of (a): In our proof for part (b) of Lemma 1 we have derived the limiting distribution of $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$. To get the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$, it remains to characterize its sign when $\mathbb{Z}<0$.

The powers of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in an expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ around $\phi_{0}$ up to $O_{p}\left(T^{-1}\right)$ are even, cf. $\underline{K}_{T}\left(\phi_{0, p}\right)$ in (11), and therefore we cannot characterize the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\right.$ $\phi_{0, p}$ ) by using such an expansion. However, following the approach of Rotnitzky et al. (2000) for the ML estimator, we can do this when $q>p$ by expanding $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ up to $o_{p}\left(T^{-5 / 4}\right)$. Specifically, the $O_{p}\left(T^{-5 / 4}\right)$ terms in a Taylor expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ up to $o_{p}\left(T^{-5 / 4}\right)$ will provide the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in case $q>p$ as we will now show.
$Q_{T}(\widehat{\phi})=m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})=m_{T}^{\prime}(\widehat{\phi}) W m_{T}(\widehat{\phi})+o_{p}\left(T^{-5 / 4}\right)=\underline{K}_{T}\left(\phi_{0, p}\right)+\underline{R}_{1 T}\left(\phi_{0}\right)+o_{p}\left(T^{-5 / 4}\right)$
with $\underline{R}_{1 T}\left(\phi_{0}\right)=\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \times 2 R_{1 T}$ where

$$
\begin{align*}
R_{1 T}= & \left(m_{T}\left(\phi_{0}\right)+\frac{\partial m_{T}}{\partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)+\frac{\partial^{2} m_{T}}{\partial \phi_{p} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)+ \\
& \frac{1}{3!}\left(m_{T}\left(\phi_{0}\right)+\frac{\partial m_{T}}{\partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{3}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!}\left(\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)+\frac{\partial^{2} m_{T}}{\partial \phi_{p} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{3!2!}\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{3}}\left(\phi_{0}\right)\right)^{\prime} W\left(\frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4} . \tag{13}
\end{align*}
$$

Defining $Z_{1 T}=\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)$ and replacing $\left(\widehat{\phi}_{1}-\phi_{0,1}\right)$ in (13) by its expression from (8), we obtain

$$
\begin{aligned}
R_{1 T}= & \left(Z_{0 T}+D H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)^{\prime} W\left(Z_{1 T}+G_{1 p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)+ \\
& \frac{1}{3!}\left(Z_{0 T}+D H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)^{\prime} W L\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+
\end{aligned}
$$

$$
\begin{align*}
& \frac{1}{2!}\left(Z_{1 T}+G_{1 p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)^{\prime} W G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{3!2!} L^{\prime} W G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+o_{p}\left(T^{-1}\right) \\
= & Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} Z_{1 T}+Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G_{1 p} H Z_{0 T}+ \\
& \left(\frac{1}{3} Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} L+Z_{1 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G+\right. \\
& \left.G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G_{1 p} H Z_{0 T}+Z_{0 T}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G_{1 p} H G\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \left(\frac{1}{6} G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} L+\frac{1}{2} G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G_{1 p} H G\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+o_{p}\left(T^{-1}\right) . \tag{14}
\end{align*}
$$

At the minimum of $Q_{T}(\phi)$, we expect $\underline{R}_{1 T}\left(\phi_{0}\right)$ to be negative, i.e., $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ and $R_{1 T}$ have opposite sign. Hence,

$$
T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)=(-1)^{B_{1 T}} T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right|
$$

with $B_{1 T}=\mathbf{1}\left(T R_{1 T} \geq 0\right)$. After replacing $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ in (14) by its expression from (12) and scaling (14) by $T$, we can see, using the continuous mapping theorem, that $T R_{1 T}$ converges in distribution towards $\mathbb{R}_{1}$ :

$$
\begin{align*}
\mathbb{R}_{1}= & \left(\mathbb{Z}_{0}^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} G^{\prime}-G^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} \mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2}\left(\frac{1}{3} L+G_{1 p} H G\right) / \sigma_{G}+ \\
& \mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2}\left(\mathbb{Z}_{1}+G_{1 p} H \mathbb{Z}_{0}\right) \tag{15}
\end{align*}
$$

We actually have that $\left(\sqrt{T} Z_{0 T}, \sqrt{T} Z_{1 T}, T R_{1 T}\right)$ converges in distribution towards $\left(\mathbb{Z}_{0}, \mathbb{Z}_{1}, \mathbb{R}_{1}\right)$. According to our Lemma 2 , when $q>p, \mathbb{R}_{1}$ does not have an atom of probability at 0 .

Applying a version of Lemma 1 of DH (2018), we have $\left(\sqrt{T} Z_{0 T}, \sqrt{T} Z_{1 T},(-1)^{B_{1 T}}\right) \xrightarrow{d}$ $\left(\mathbb{Z}_{0}, \mathbb{Z}_{1},(-1)^{B_{1}}\right)$, where $B_{1}=\mathbf{1}\left(\mathbb{R}_{1} \geq 0\right)$. Since $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right), T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right|,(-1)^{B_{1 T}}\right)=$ $O_{p}(1)$, any subsequence of the left hand side has a further subsequence that converges in distribution. Using part (b) of Lemma 1 , such a subsequence obeys $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right.$, $\left.T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right|,(-1)^{B_{1 T}}\right) \xrightarrow{d}\left(H \mathbb{Z}_{0}+H G V / 2, \sqrt{V},(-1)^{B_{1}}\right)$. Since the limit distribution does not depend on the subsequence, the whole sequence converges towards that limit. By the continuous mapping theorem, we deduce that: $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right), T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)\right) \xrightarrow{d}$ $\left(H \mathbb{Z}_{0}+H G V / 2,(-1)^{B_{1}} \sqrt{V}\right)$.

Proof of (b): Part (b) of Lemma 1 gives the limiting distribution of $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$. To get the limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$, it remains to characterize its sign when $\mathbb{Z}<0$. According to our Lemma 2, when $q=p, \operatorname{Pr}\left(\mathbb{R}_{1}=0 \mid \mathbb{Z}<0\right)=1$. Hence
when $q=p,(-1)^{B_{1}}$ with $B_{1}=\mathbf{1}\left(\mathbb{R}_{1} \geq 0\right)$ will not correctly describe the behaviour of the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in its limiting distribution. However, when $q=p$, we can characterize the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ by expanding $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ around $\phi_{0}$ up to $o_{p}\left(T^{-7 / 4}\right)$. Specifically, the $O_{p}\left(T^{-7 / 4}\right)$ terms in a Taylor expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ up to $o_{p}\left(T^{-7 / 4}\right)$ will provide the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in case $q=p$ as we will now show.

First we note that the powers of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ in the $O_{p}\left(T^{-6 / 4}\right)$ terms in an expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ up to $o_{p}\left(T^{-7 / 4}\right)$ are all even and therefore these terms do not provide information on the sign of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$. Next, we consider the $O_{p}\left(T^{-7 / 4}\right)$ terms in the Taylor expansion of $m_{T}^{\prime}(\widehat{\phi}) W_{T} m_{T}(\widehat{\phi})$ around $\phi_{0}$ :

$$
\underline{R}_{2, T}\left(\phi_{0}\right)=\left(\widehat{\phi}_{p}-\phi_{0, p}\right) \times 2 R_{2, T} \text { where }
$$

$$
\begin{align*}
R_{2, T}= & \frac{1}{2!}\left(\kappa_{1, T} \ldots \kappa_{q, T}\right) W\left(\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)+\frac{\partial^{2} m_{T}}{\partial \phi_{p} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)+ \\
& \frac{1}{2!3!}\left(\kappa_{1, T} \ldots \kappa_{q, T}\right) W\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{3}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{3!}\left(m_{T}\left(\phi_{0}\right)+\frac{\partial m_{T}}{\partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial^{4} m_{T}}{\partial \phi_{p}^{3} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!}\left(\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)+\frac{\partial^{2} m_{T}}{\partial \phi_{p} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{2} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!3!}\left(\frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)\right)^{\prime} W\left(\frac{\partial^{4} m_{T}}{\partial \phi_{p}^{3} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{3!2!}\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{3}}\left(\phi_{0}\right)\right)^{\prime} W\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{2} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{5!}\left(m_{T}\left(\phi_{0}\right)+\frac{\partial m_{T}}{\partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial^{5} m_{T}}{\partial \phi_{p}^{5}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{4!}\left(\frac{\partial m_{T}}{\partial \phi_{p}}\left(\phi_{0}\right)+\frac{\partial^{2} m_{T}}{\partial \phi_{p} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right)^{\prime} W\left(\frac{\partial^{4} m_{T}}{\partial \phi_{p}^{4}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{2!5!}\left(\frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)\right)^{\prime} W\left(\frac{\partial^{5} m_{T}}{\partial \phi_{p}^{5}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+ \\
& \frac{1}{3!4!}\left(\frac{\partial^{3} m_{T}}{\partial \phi_{p}^{3}}\left(\phi_{0}\right)\right)^{\prime} W\left(\frac{\partial^{4} m_{T}}{\partial \phi_{p}^{4}}\left(\phi_{0}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6} . \tag{16}
\end{align*}
$$

with $\kappa_{k, T}=\left(\widehat{\phi}_{1}-\phi_{0,1}\right)^{\prime} K_{k, T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)$ and $K_{k, T}=\frac{\partial^{2} m_{k, T}}{\partial \phi_{1} \partial \phi_{1}^{\prime}}\left(\phi_{0}\right)$ for $k=1,2, \ldots, q$, where $m_{k, T}(\phi)$ is the kth element of $m_{T}(\phi)$.

Recalling that when $q=p$, then $m(\widehat{\phi})=0$ and hence $m_{T}\left(\phi_{0}\right)+\frac{\partial m_{T}}{\partial \phi_{1}^{1}}\left(\phi_{0}\right)\left(\widehat{\phi}_{1}-\phi_{0,1}\right)+$ $\frac{1}{2} \frac{\partial^{2} m_{T}}{\partial \phi_{p}^{2}}\left(\phi_{0}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}=o_{p}\left(T^{-1 / 2}\right)$ (cf. (7)), and replacing $\left(\widehat{\phi}_{1}-\phi_{0,1}\right)$ in (16) by its
expression from (8), we obtain

$$
\begin{align*}
R_{2, T}= & \frac{1}{2!}\left(\lambda_{1, T} \ldots \lambda_{q, T}\right) W\left(Z_{1 T}+G_{1 p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)+ \\
& \frac{1}{2!3!}\left(\lambda_{1, T} \ldots \lambda_{q, T}\right) W L\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!}\left(Z_{1 T}+G_{1 p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)^{\prime} W\left(G_{1 p p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{3!2!} L^{\prime} W\left(G_{1 p p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{4!}\left(Z_{1 T}+G_{1 p} H\left(Z_{0 T}+\frac{1}{2} G\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)\right)^{\prime} W F\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{3!4!} L^{\prime} W F\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+o_{p}\left(T^{-6 / 4}\right) . \tag{17}
\end{align*}
$$

with $\lambda_{k, T}=\lambda_{1, k, T}+\lambda_{2, k, T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+\lambda_{3, k}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}$, where $\lambda_{1, k, T}=Z_{0 T}^{\prime} H^{\prime} K_{k} H Z_{0 T}$, $\lambda_{2, k, T}=G^{\prime} H^{\prime} K_{k} H Z_{0 T}$ and $\lambda_{3, k}=\frac{1}{4} G^{\prime} H^{\prime} K_{k} H G$ for $k=1,2, \ldots, q$. Let $\tilde{\lambda}_{i, T}=\left(\lambda_{i, 1, T} \ldots \lambda_{i, q, T}\right)^{\prime}$ for $i=1,2$ and recall that $\tilde{\lambda}_{3}=\left(\lambda_{3,1} \ldots \lambda_{3, q}\right)^{\prime}$. Combining powers in (17) we get

$$
\begin{align*}
R_{2, T}= & \frac{1}{2!} \tilde{\lambda}_{1, T}^{\prime} W\left(Z_{1 T}+G_{1 p} H Z_{0 T}\right)+ \\
& \frac{1}{2!} \tilde{\lambda}_{1, T}^{\prime} W\left(\frac{1}{2} G_{1 p} H G+\frac{1}{3!} L\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!} \tilde{\lambda}_{2, T}^{\prime} W\left(Z_{1 T}+G_{1 p} H Z_{0 T}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!} \tilde{\lambda}_{2, T}^{\prime} W\left(\frac{1}{2} G_{1 p} H G+\frac{1}{3!} L\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{2!} \tilde{\lambda}_{3}^{\prime} W\left(Z_{1 T}+G_{1 p} H Z_{0 T}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{2!} \tilde{\lambda}_{3}^{\prime} W\left(\frac{1}{2} G_{1 p} H G+\frac{1}{3!} L\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+ \\
& \frac{1}{2!}\left(Z_{1 T}+G_{1 p} H Z_{0 T}\right)^{\prime} W\left(G_{1 p p} H Z_{0 T}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}+ \\
& \frac{1}{2!}\left(Z_{1 T}+G_{1 p} H Z_{0 T}\right)^{\prime} W\left(\frac{1}{2} G_{1 p p} H G\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{2!}\left(\frac{1}{2} G_{1 p} H G\right)^{\prime} W\left(G_{1 p p} H Z_{0 T}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+ \\
& \frac{1}{2!}\left(\frac{1}{2} G_{1 p} H G\right)^{\prime} W\left(\frac{1}{2} G_{1 p p} H G\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+ \\
& \frac{1}{3!2!} L^{\prime} W\left(G_{1 p p} H Z_{0 T}\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+\frac{1}{3!2!} L^{\prime} W\left(\frac{1}{2} G_{1 p p} H G\right)\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+ \\
& \frac{1}{4!}\left(Z_{1 T}+G_{1 p} H Z_{0 T}\right)^{\prime} W F\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{4}+\frac{1}{4!}\left(\frac{1}{2} G_{1 p} H G\right)^{\prime} W F\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+ \\
& \frac{1}{3!4!} L^{\prime} W F\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{6}+o_{p}\left(T^{-6 / 4}\right) . \tag{18}
\end{align*}
$$

At the minimum of $Q_{T}(\phi)$, we expect $\underline{R}_{2, T}\left(\phi_{0}\right)$ to be negative, i.e., $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ and $R_{2, T}$ have opposite sign. Hence,

$$
\begin{equation*}
T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)=(-1)^{B_{2, T}} T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right| \tag{19}
\end{equation*}
$$

with $B_{2, T}=\mathbf{1}\left(T^{6 / 4} R_{2, T} \geq 0\right)$. After replacing $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ in (18) by its expression from (12) and scaling (18) by $T^{6 / 4}$, we can see, using the continuous mapping theorem, that $T^{6 / 4} R_{2, T}$ converges in distribution towards, say, $\mathbb{R}_{2}$. The formula for $\mathbb{R}_{2}$ is given by (18) with $Z_{0 T}, Z_{1 T}$ and powers of $\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}$ replaced by $\mathbb{Z}_{0}, \mathbb{Z}_{1}$ and powers of $\left(-2 \mathbb{Z} \mathbf{1}(\mathbb{Z}<0) / \sigma_{G}\right)$, and with the unspecified $o_{p}\left(T^{-6 / 4}\right)$ term at the very end of (18) omitted.

We actually have that $\left(\sqrt{T} Z_{0 T}, \sqrt{T} Z_{1 T}, T^{6 / 4} R_{2, T}\right)$ converges in distribution towards $\left(\mathbb{Z}_{0}, \mathbb{Z}_{1}, \mathbb{R}_{2}\right)$. Furthermore, recall that we have assumed that $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$.

Applying a version of Lemma 1 of DH (2018), we have $\left(\sqrt{T} Z_{0 T}, \sqrt{T} Z_{1 T},(-1)^{B_{2, T}}\right) \xrightarrow{d}$ $\left(\mathbb{Z}_{0}, \mathbb{Z}_{1},(-1)^{B_{2}}\right)$, where $B_{2}=\mathbf{1}\left(\mathbb{R}_{2} \geq 0\right)$. Since $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right), T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right|,(-1)^{B_{2, T}}\right)=$ $O_{p}(1)$, any subsequence of the left hand side has a further subsequence that converges in distribution. Using part (b) of Lemma 1 , such a subsequence obeys $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right)\right.$, $\left.T^{1 / 4}\left|\widehat{\phi}_{p}-\phi_{0, p}\right|,(-1)^{B_{2, T}}\right) \xrightarrow{d}\left(H \mathbb{Z}_{0}+H G V / 2, \sqrt{V},(-1)^{B_{2}}\right)$. Since the limit distribution does not depend on the subsequence, the whole sequence converges towards that limit. By the continuous mapping theorem, we deduce that: $\left(\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right), T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)\right) \xrightarrow{d}$ $\left(H \mathbb{Z}_{0}+H G V / 2,(-1)^{B_{2}} \sqrt{V}\right)$.

Lemma 2 (a1) When $q=p$, then $\operatorname{Pr}\left(\mathbb{R}_{1}=0 \mid \mathbb{Z}<0\right)=1$.
(a2) When $q>p$, then $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right)=0$.
(b) When $q=p$ and $F+3!G_{1 p p} H G+\frac{4!}{2} \tilde{\lambda}_{3} \neq 0$, then $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$.

Proof of Lemma 2. $\mathbb{R}_{1}$ is the sum of two terms with the first term given in the first line of (15) and the second term given in the second line of (15).
(a1) We will first show that

$$
\left(\mathbb{Z}_{0}^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} G^{\prime}-G^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} \mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)=0
$$

so that the first term of $\mathbb{R}_{1}$ in (15) equals 0 when $\mathbb{Z}<0$.

Recall that

$$
P_{g}=M_{d} W^{1 / 2} G\left(G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\right)^{-1} G^{\prime} W^{1 / 2} M_{d}
$$

where $M_{d}=I-W^{1 / 2} D\left(D^{\prime} W D\right)^{-1} D^{\prime} W^{1 / 2}$.
Define

$$
Z_{1}=-2 \mathbb{Z} / \sigma_{G}=-2\left(G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\right)^{-1} G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0}
$$

Note that $\mathbf{1}(\mathbb{Z}<0)=\mathbf{1}\left(Z_{1}>0\right)$ and $\sqrt{T}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2} \rightarrow^{d} V=Z_{1} \mathbf{1}\left(Z_{1}>0\right)$.
Now

$$
\begin{gathered}
\left(\mathbb{Z}_{0}^{\prime} W^{1 / 2} P_{g} W^{1 / 2} \mathbb{Z}_{0} G^{\prime}-G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0} \mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)= \\
\left(\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\left(G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\right)^{-1} G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0} G^{\prime}-\right. \\
\left.G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0} \mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)= \\
\left(-\frac{1}{2} \mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G Z_{1} G^{\prime}-G^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbb{Z}_{0} \mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)= \\
-\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d} W^{1 / 2} G\left(\frac{1}{2} Z_{1} G^{\prime}+\mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)=0
\end{gathered}
$$

because $\left(\mathbb{Z}_{0}+\frac{1}{2} Z_{1} G\right)^{\prime} W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}\left(Z_{1}>0\right)=0$. The latter can be shown as follows. If $q=p$ and $Z_{T}<0$, then upon replacing $\left(\widehat{\phi}_{1}-\phi_{0,1},\left(\widehat{\phi}_{p}-\phi_{0, p}\right)^{2}\right)^{\prime}$ in (10) by $-\left(D \frac{1}{2} G\right)^{-1} m_{T}\left(\phi_{0}\right)+o_{p}\left(T^{-1 / 2}\right)$, we obtain $\underline{K}_{T}\left(\phi_{0}\right)=o_{p}\left(T^{-1}\right)$. Hence it follows from (9) that if $q=p$ and $Z_{T}<0$, then $m_{T}^{\prime}(\widehat{\phi}) W m_{T}(\widehat{\phi}) \leq o_{p}\left(T^{-1}\right)$ and hence $m_{T}(\widehat{\phi})=o_{p}\left(T^{-1 / 2}\right)$. Using this result, that $\sqrt{T} Z_{T} \rightarrow^{d} \mathbb{Z}$ and that $\sqrt{T}\left(\widehat{\phi}_{1}-\phi_{0,1}\right) \rightarrow^{d} \mathbb{Z}_{2} \equiv H \mathbb{Z}_{0}+H G V / 2$, it follows from (7) that $\left(\mathbb{Z}_{0}+D \mathbb{Z}_{2}+\frac{1}{2} Z_{1} G\right) \mathbf{1}\left(Z_{1}>0\right)=0.3$ Hence

$$
\begin{aligned}
W^{-1 / 2} M_{d} W^{1 / 2}\left(\mathbb{Z}_{0}+\frac{1}{2} Z_{1} G\right) \mathbf{1}\left(Z_{1}\right. & >0)= \\
\left(I-D\left(D^{\prime} W D\right)^{-1} D^{\prime} W\right)\left(\mathbb{Z}_{0}+D \mathbb{Z}_{2}+\frac{1}{2} Z_{1} G\right) \mathbf{1}\left(Z_{1}\right. & >0)
\end{aligned}
$$

Next, we will show that $\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)=0$ so that the second term of $\mathbb{R}_{1}$ in (15)) equals 0 when $\mathbb{Z}<0$. Using $M_{d g}=M_{d}-P_{g}$ and the definition of $Z_{1}$, we obtain

$$
\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2} \mathbf{1}(\mathbb{Z}<0)=\left(\mathbb{Z}_{0}^{\prime}+\frac{1}{2} Z_{1} G^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2} \mathbf{1}\left(Z_{1}>0\right)=0
$$

again because $W^{-1 / 2} M_{d} W^{1 / 2}\left(\mathbb{Z}_{0}+\frac{1}{2} Z_{1} G\right) \mathbf{1}\left(Z_{1}>0\right)=0$. We conclude that if $q=p$, then $\mathbb{R}_{1} \mathbf{1}(\mathbb{Z}<0)=0$, which means that $\operatorname{Pr}\left(\mathbb{R}_{1}=0 \mid \mathbb{Z}<0\right)=1$.

[^3]We can actually show that $\mathbb{R}_{1}=0$ when $q=p$. It is easily verified that the matrix $M_{d g}$ projects a vector on the orthogonal complement of $\left(W^{1 / 2} D W^{1 / 2} G\right)$. When $q=p$, $\operatorname{Rank}(D G)=p=q$. Moreover, $W$ has full rank. Hence when $q=p$, then $M_{d g}=\mathbf{0}$ and $\left(\frac{1}{2} Z_{1} G^{\prime}+\mathbb{Z}_{0}^{\prime}\right) W^{1 / 2} M_{d} W^{1 / 2}=\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2}=0$ so that both terms of $\mathbb{R}_{1}$ equal 0.
(a2) Note that only the second term of $\mathbb{R}_{1}$, that is, $\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2}\left(\mathbb{Z}_{1}+G_{1 p} H \mathbb{Z}_{0}\right)$ depends on $\mathbb{Z}_{1}$. It therefore suffices to show that $\operatorname{Pr}\left(\mathbb{Z}_{0}^{\prime} W^{1 / 2} M_{d g} W^{1 / 2} \mathbb{Z}_{1}=0\right)=0$. The latter follows from $M_{d g}^{2}=M_{d g}, \operatorname{Rank}\left(M_{d g}\right)=q-p>0$, and the fact that $\left(\left(W^{1 / 2} \mathbb{Z}_{0}\right)^{\prime}\left(W^{1 / 2} \mathbb{Z}_{1}\right)^{\prime}\right)^{\prime}$ has a continuous multivariate (normal) distribution with mean zero and a covariance matrix of full rank. We conclude that when $q>p, \operatorname{Pr}\left(\mathbb{R}_{1}=0\right)=0$.
(b) There is no combination of terms in the formula for $\mathbb{R}_{2}$ that includes $\frac{1}{4!} \mathbb{Z}_{1}^{\prime} W(F+$ $\left.3!G_{1 p p} H G+\frac{4!}{2} \tilde{\lambda}_{3}\right) \times\left(-2 \mathbb{Z} / \sigma_{G}\right)^{2}$ and equals zero. We conclude that when $q=p$ and $F+3!G_{1 p p} H G+\frac{4!}{2} \tilde{\lambda}_{3} \neq 0$, then $\operatorname{Pr}\left(\mathbb{R}_{2}=0\right)=0$.

## References

[1] Ahn, S.C., and P. Schmidt, 1995, Efficient estimation of models for dynamic panel data, Journal of Econometrics 68, 5-28.
[2] Alvarez, J., and M. Arellano, 2021, Robust likelihood estimation of dynamic panel data models, forthcoming in Journal of Econometrics.
[3] Anderson, T. W., and H. Rubin, 1949, Estimation of the parameters of a single equation in a complete system of stochastic equations. The Annals of Mathematical Statistics 20, 46-63.
[4] Arellano, M., and S. Bond, 1991, Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations, Review of Economic Studies LVIII, 277-297.
[5] Bottai, M., 2003, Confidence regions when the Fisher information is zero, Biometrika 90, 73-84.
[6] Dovonon, P., and A. Hall, 2018, The asymptotic properties of GMM and Indirect Inference under second-order identification. Journal of Econometrics 205, 76-111.
[7] Dovonon, P., Hall, A. R., and F. Kleibergen, 2020, Inference in second-order identified models. Journal of Econometrics 218, 346-372.
[8] Dovonon, P., and E. Renault, 2009, GMM overidentification test with first order underidentification. Working paper, University of North Carolina at Chapel Hill. A version of this WP is also available at SSRN: http://dx.doi.org/10.2139/ssrn. 1500013
[9] Kleibergen, F., 2002, Pivotal statistics for testing structural parameters in instrumental variables regression. Econometrica 70, 1781-1803.
[10] Kleibergen, F., 2005, Testing parameters in GMM without assuming that they are identified. Econometrica 73, 1103-1123.
[11] Kruiniger, H., 2013, Quasi ML estimation of the panel AR(1) model with arbitrary initial conditions. Journal of Econometrics 173, 175-188.
[12] Kruiniger, H., 2018, A further look at Modied ML estimation of the panel AR(1) model with fixed effects and arbitrary initial conditions. MPRA Working paper 88623, https://mpra.ub.uni-muenchen.de/88623/ (revised on 26 August 2022)
[13] Lee, J. H., and Z. Liao, 2018, On standard inference for GMM with local identification failure of known forms. Econometric Theory 34, 790-814.
[14] Moreira, M. J., 2003, A conditional likelihood ratio test for structural models. Econometrica 71, 1027-1048.
[15] Newey, W.K., and D. McFadden, 1994, Large sample estimation and hypothesis testing. In: Engle, R.F., and D. McFadden (eds.), Handbook of econometrics, vol. 4, North-Holland, Elsevier, Amsterdam, pp. 2111-2245.
[16] Rotnitzky, A., D.R. Cox, M. Bottai, and J. Robins, 2000, Likelihood-based inference with singular information matrix. Bernoulli 6, 243-284.
[17] Sargan, J. D., 1983, Identification and lack of identification. Econometrica 51, 16051633.
[18] Staiger, D., and J. H. Stock, 1997, Instrumental Variables Regression with Weak Instruments. Econometrica 65, 557-586.
[19] Stock, J. H., and J. H. Wright, 2000, GMM with weak identification. Econometrica 68, 1055-1096.


[^0]:    *Address: hugo.kruiniger@durham.ac.uk; Department of Economics, 1 Mill Hill Lane, Durham DH1 3LB, England.

[^1]:    ${ }^{1}$ As mentioned by Sargan (1983) and shown by DH, any model with an expected Jacobian that has a rank deficiency of one can be brought into this configuration by reparametrizing the model as needed.

[^2]:    ${ }^{2}$ Recall that in the case of exact identification Theorem 1 of DH does not provide a limiting distribution of $T^{1 / 4}\left(\widehat{\phi}_{p}-\phi_{0, p}\right)$ because $\operatorname{Pr}\left(\mathbb{R}_{1}=0\right) \neq 0$ and hence its sign is not well characterized in the limit in this case.

[^3]:    ${ }^{3}$ I thank P. Dovonon and A. Hall for pointing out a flaw in an earlier version of this argument.

