Global structures from the infrared

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ABSTRACT: Quantum field theories with identical local dynamics can admit different choices of global structure, leading to different partition functions and spectra of extended operators. Such choices can be reformulated in terms of a topological field theory in one dimension higher, the symmetry TFT. In this paper we show that this TFT can be reconstructed from a careful analysis of the infrared Coulomb-like phases. In particular, the TFT matches between the UV and the IR. This provides a purely field theoretical counterpart of several recent results obtained via geometric engineering in various string/M/F theory setups for theories in four and five dimensions that we confirm and extend.

Contents

1 Introduction

Among the most important questions about the dynamics of quantum fields is the task of characterising features that are robust under renormalization flow. One such feature is the anomalous behaviour of global symmetries [1],[1] which are often captured via inflow [2] from an invertible field theory in one dimension higher, known as the *anomaly theory* [3].

This inflow picture can be enriched whenever the QFT has local dynamics compatible with inequivalent global structures (these can be detected by analysing multiple related properties of the theory: the spectra of non-local operators, the spectrum of generalised symmetries, or the partition functions on compact curved spacetimes). When that happens we can split off the choice of global form from the behaviour of the local degrees of freedom, by viewing the local degrees of freedom of the QFT as a theory relative [4] to a non-invertible theory in one dimension higher. A proper QFT, with a fully specified global form, can then be interpreted as the compactification of the bulk non-invertible theory on

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[1]In this paper by anomalies we always mean 't Hooft anomalies for global symmetries.
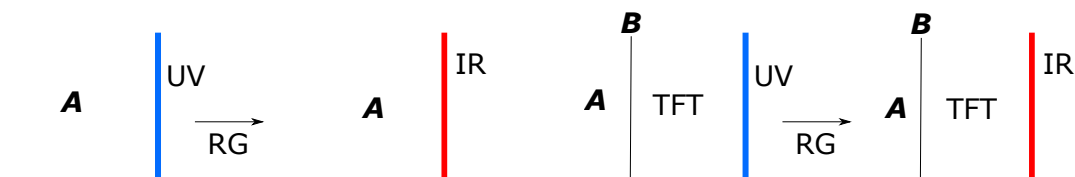
Figure 1. *Left:* inflow picture of anomaly matching for intrinsic QFTs: the anomaly is an invertible TFT A in one dimension higher. *Right:* TFT matching for QFTs with global structure. The choices of global structure are encoded by non-invertible TFT on an interval with a boundary condition interface B to an anomaly theory. This symmetry TFT must match for the theories in the UV and the IR, thus generalizing the anomaly matching procedure.

an interval: on one end of the interval the bulk theory couples to a gapped TFT responsible for choosing the global structure, and on the other end of the interval the bulk theory couples to the local degrees of freedom.[2] This structure is well-known in the context of Lagrangian theories — for an enlightening discussion about the $\mathrm{SU}(N)$ versus $\mathrm{PSU}(N)$ cases we refer our readers to section 6 of [7].

It is generically the case that the resulting QFT after reduction on the interface is anomalous, in which case the picture above needs to be further refined, as elaborated on in [8] (see also [6]): we consider a non-invertible theory on an interval, where a gapped interface on the left connects it to the anomaly theory for the QFT of interest (different global forms have different symmetries and therefore different anomaly theories, which are connected via suitable interfaces to the same non-invertible theory), and the gapless boundary on the right encodes the local degrees of freedom of the QFT. So in this case it is the gapped interface that encodes the choice of global form. As in [8], we refer to the non-invertible theory inside the interval as the *symmetry theory*.

In the presence of anomalies for continuous symmetries the symmetry theory includes sectors of Chern-Simons type, or more generally η-invariants. In this note we focus on the choice of global form, which involves discrete symmetries only, so we do not need to worry about such sectors, and we can restrict ourselves to a part of the symmetry theory which is a proper topological field theory (TFT). By an abuse of language, we will refer to this TFT sector of the symmetry theory as *symmetry TFT*. It is natural to expect this symmetry TFT to be invariant under RG flows triggered by deformations invariant under the symmetries. Equivalently, we expect that the set of choices for the global form of the theory persists at all energy scales — see figure 1.

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[2]It is not always the case that diffeomorphism invariant choices of gapped boundary conditions exist. The 6d $(2,0)$ theories associated to the algebra $\mathfrak{g}$ are examples of consistent local dynamics whose associated bulk theories are expected not to admit such choices for generic $\mathfrak{g}$. (See [5, 6] for an analysis of which $\mathfrak{g}$ are expected to admit diffeomorphism invariant gapped boundary conditions.) This is one way to understand statements in the literature that such theories are not "genuine QFTs". We expect that the bulk theories associated to the 4d $\mathcal{N} = 2$ theories studied in this paper always admit such choices.

We will test this expectation on four dimensional QFTs in four space time dimensions that admit a Coulomb phase, i.e. an infrared regime with an effective description in terms of r independent Maxwell fields.[3] Let us denote such theory $\mathcal{T}$. In this paper we are interested in determining the global structure of $\mathcal{T}$ from the perspective of such infrared regime. All the examples we will consider in this note are $\mathcal{N} = 2$ supersymmetric, but we stress that supersymmetry in itself is not a necessary requirement for the discussion below: the two assumptions we are making on $\mathcal{T}$ are that it has an infrared regime where an effective $\mathrm{U}(1)^r$ gauge theory description is valid, and that the structure of the massive spectrum in this IR regime is sufficiently well understood. The restriction to $\mathcal{N} = 2$ comes from this second assumption, which in the $\mathcal{N} = 2$ context we take to mean the (likely weaker) assumption that the BPS spectrum, which we understand well enough for our needs, is representative of the full spectrum.[4]

The class of $\mathcal{N} = 2$ theories that we study naturally includes geometrically engineered $\mathcal{N} = 2$ SCFTs. In the geometric engineering program [11] one aims to establish a dictionary between the properties of some version of string theory, denoted by $\mathscr{S}$, on a singular background[5] $\mathcal{X}$ and a quantum field theory $\mathcal{T}_{\mathscr{S}/\mathcal{X}}$. From this perspective the symmetry TFT can be recovered via the analysis of the effective theory arising after compactification on $\partial\mathcal{X}$, which captures the anomalies and other features of $\mathcal{T}_{\mathscr{S}/\mathcal{X}}$, including its higher form symmetries in terms of defect groups [6, 8, 12–47]. In particular, if the theory $\mathcal{T}_{\mathscr{S}/\mathcal{X}}$ has non-trivial choices of global structure, in the simplest cases this has been understood [5, 16, 20, 41] in terms of a Heisenberg algebra of non-commuting FMS fluxes on $\partial\mathcal{X}$ [48, 49]. Below we will give an alternative purely field theoretical derivation of this same Heisenberg algebra, showing that it also arises from the infrared perspective on theories with a Coulomb phase. The advantage of this formulation is that it extends field theoretically the results about theories with known Lagrangian formulations to arbitrary SCFTs with mutually non-local massless excitations. In particular, we recover field theoretically the results on global structures of four-dimensional Argyres-Douglas theories that have been obtained via geometric string theory techniques in the literature, and also some results that have no geometric understanding.

Our main result can be derived closely following [50]. In the pure $\mathfrak{u}(1)^r$ gauge theory, before choosing a global form, one can in principle consider Wilson and 't Hooft lines with arbitrary rational dyonic charges. Once we include massive states, two things happen: the set of allowable charges for the lines reduces to those mutually local with respect to the charges of the dynamical states, and some of the line operators get screened. Depending on the structure of the charge lattice of the theory non-trivial lines might remain after

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[3]See e.g. [9, 10] for a similar application in the context of 2d field theories.

[4]We emphasize that this as an assumption. A known example where the non-BPS spectrum includes states that would invalidate an analysis based on the BPS spectrum only is the type I string, with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$. In this case the spinor representation appears as a non-BPS brane with discrete K-theory charge; the BPS spectrum appears in vector representations only. In all examples without torsional charges we know the BPS spectrum is representative. It is tempting to conjecture that this is a general fact.

[5]Here we let $\mathscr{S}$ denote also M-theory or F-theory, and by $\mathcal{X}$ we denote schematically the whole data needed to prescribe a BPS background for $\mathscr{S}$ of the form $\mathbb{R}^d \times \mathcal{X}$ giving rise to a QFT in d spacetime dimensions.

screening. Generically not all remaining lines will be mutually local, so an specification of a global form will consist, as in [50], on a specification of a maximal subset of lines that will be genuine line operators in our theory. The rest of the lines should then be viewed as open surface operators.

We emphasise that from this point of view the usual choice of having line operators with arbitrary integer dyonic charge is a possible choice for the maximally commuting set of line operators, but as shown below this is not the only possible choice given any fixed lattice of charged states. (In making this statement we assume that we have normalised our charges so that the lattice of charged states is a sublattice of $\mathbb{Z}^{2r}$, which will be the case throughout the paper.)

The argument in terms of screening of lines given above can be recast as the derivation of a symmetry TFT for the theory of the form[6]

$$S_{5d} = 2\pi i \sum_{j=1}^{r} n_j \int_{\mathcal{M}_5} b_{e,j}^{(2)} \wedge db_{m,j}^{(2)} \tag{1.1}$$

where the n_j are positive integers. Below we will give an explicit prescription for how to extract the integers n_j from the Coulomb phase of the theory. When some of these integers differ from 1 we recover the Heisenberg algebra of non-commuting fluxes we found from geometric engineering in terms of this bulk TFT.

All conventional 4d $\mathcal{N} = 2$ gauge theories can be analysed with our methods: giving generic vevs in the Cartan of the gauge group to the adjoint scalars in the $\mathcal{N} = 2$ vector multiplets, breaks the gauge group to $\mathrm{U}(1)^r$, giving us a plethora of consistency checks. Moreover, this feature is also shared by all non-conventional 4d $\mathcal{N} = 2$ SQFTs that have a Coulomb branch of complex dimension r. In this case the integer r is known as the 'rank' of the corresponding non-conventional SQFT. From this perspective our results extend and generalize results that have appeared previously in the literature by giving a common ground for many computations of defect groups in various dimensions.

We conclude this paper by extending our formulation to other theories with IR phases under control, namely 6d SCFTs with a tensor branch, and 5d SCFT with a Coulomb branch. Also for those systems we find a symmetry TFT that has the structure of a BF theory in 7 ad 6 dimensions respectively, which is responsible for the global structure of the theories in question.

The structure of this paper is as follows. In section 2 we review some well-known features of Maxwell type theories and their higher form symmetries to set up our notation and conventions. We proceed by revisiting the field theoretical origin of global structures for 4d theories from an infrared perspective. In section 3 we summarise the main features of BF theories we will need, and derive our main result. Results in these two sections only assume the theory under scrutiny has a Coulomb phase, and are independent from supersymmetry. In section 4 however, we apply this result in the context of $\mathcal{N} = 2$ theories where the charge lattice of the theories can be explicitly calculated, from this we recover the well-known center symmetries and global structures of conventional gauge theories,

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[6]Throughout the paper we will use conventions where the b fields are periodic with period 1, instead of 2π.
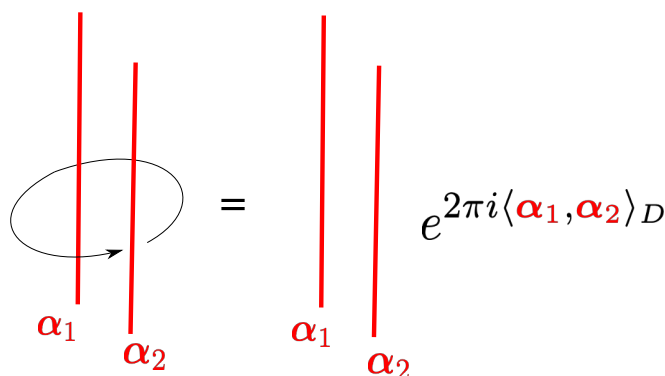
Figure 2. Aharonov-Bohm effect for lines: monodromies can give rise to phases proportional to the corresponding Dirac pairings. Only lines which have charges satisfying the Dirac quantization condition can be simultaneously genuine. The presence of non-genuine lines is the hallmark of a relative QFT — see e.g. section 2 of [41] for a nice review.

and we also reproduce and extend the results obtained about non-conventional SCFTs via geometric engineering methods. In section 5 we give a generalization of our findings to theories in various dimensions. In section 6 we present our conclusions and comment on future directions and applications.

2 Genuine and non-genuine lines from the infrared

### 2.1 Line operators from the Coulomb phase

In what follows we explore the constraints on the line operators of Maxwell theory that arise from the presence of charge dyonic BPS states in the spectrum. Many important aspects of the analysis below can be found in section 4.1 of [7], see also appendix C in [51].

The dynamics of a theory in a Coulomb phase can be described in terms of r copies of the $\mathfrak{u}(1)$ Maxwell theory coupled to massive states. (There can be non-generic points where some of these massive states can become massless.) Ignoring the massive states for a moment, at a generic point we have a higher 1-form global symmetry of the form

$$\left(\mathrm{U}(1)_e^{(1)} \times \mathrm{U}(1)_m^{(1)}\right)^r \tag{2.1}$$

where $\mathrm{U}(1)_e^{(1)} \times \mathrm{U}(1)_m^{(1)}$ are the 1-form global symmetries of Maxwell theory. The operators charged under these symmetries are dyonic lines. We denote the electric and magnetic charges of one such line by a $2r$-component vector $\boldsymbol{\alpha}$. As in [50, 52], two lines with charges $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ can only be genuine operators in the Maxwell theory (as opposed to open Gukov-Witten surface operators [53, 54]) if their Dirac pairing $\left\langle \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \right\rangle_D := \sum_{i=1}^r (\boldsymbol{\alpha}_1^{2i-1} \boldsymbol{\alpha}_2^{2i} - \boldsymbol{\alpha}_1^{2i} \boldsymbol{\alpha}_2^{2i-1})$ is an integer.[7] (See figure 2.) This is often accomplished by requiring $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{Z}^{2r}$, but in the absence of a dynamical spectrum of point particles this is not necessary for consistency of the theory.

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[7]Relatedly, we can determine the commutation relations of the line operators in a spatial slice using the exponentiated form of the commutators found in [55], which leads to a commutator $\exp(2\pi i \left\langle \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \right\rangle_D)$.
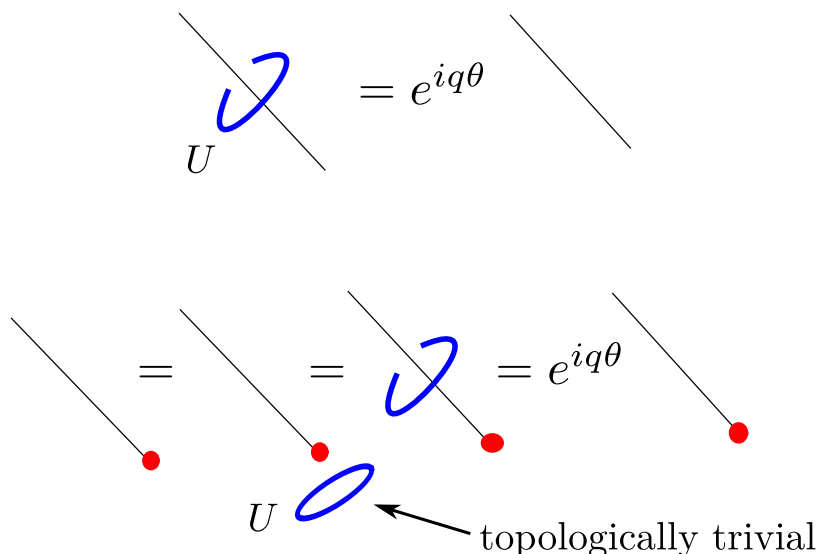
Figure 3. *Top:* action of a U(1)$^{(1)}$-form symmetry on a line operator. *Bottom:* constraint on the higher form symmetry in presence of charged particles (red dot at the end of the line). The presence of a particle of charge $q \in \mathbb{Z}$ enforces $e^{i\theta q} = 1$ which in turns entail that only rotations with phase $\theta = 2\pi k/q$ are allowed, thus breaking U(1)$^{(1)}$ down to $\mathbb{Z}_q^{(1)}$.

This structure gets simplified once we couple the pure Maxwell theory to charged dynamical states. We will refer to this theory as the "UV theory", in contrast to the "IR theory" by which we mean pure Maxwell with no dynamical states. The electromagnetic charges of the dynamical states live on a lattice $\Gamma \subseteq \mathbb{Z}^{2r}$. In practice, it is convenient to choose a basis γ_i of generators for Γ, and this gives rise to an explicit expression for the Dirac pairing in terms of $\mathcal{Q}_{ij} = \langle \gamma_i, \gamma_j \rangle_D \in \mathbb{Z}$.

The first effect of coupling the theory to charged dynamical states is that the spectrum of admissible lines is reduced. Given a line with charges $\boldsymbol{\alpha}$, it is only admissible if

$$\langle \boldsymbol{\alpha}, \gamma \rangle_D \in \mathbb{Z} \quad \text{for all } \gamma \in \Gamma \,. \tag{2.2}$$

In other words, $\boldsymbol{\alpha} \in \Gamma^*$, the dual lattice. The lines in Γ^* do not need to commute — see figure 2. Integrally charged line operators always commute, as pointed out in [49], but Γ^* is not necessarily an integral lattice. This is the Coulomb branch counterpart of the effect discussed at length by [50, 52]: the same local dynamics can be compatible with inequivalent global structures, and different global forms of the theory can be detected from the spectrum of line defects in four-dimensions.

Moreover, the spectrum of allowed charges for genuine line defects is also constrained as a sublattice of $\Gamma_L \subset \Gamma^*$ consisting of a maximally mutually local collection of defect charges satisfying the Dirac quantization condition $\langle \boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}} \rangle_D \in \mathbb{Z}$. Inequivalent global forms of the theory correspond to inequivalent choices of sublattices Γ_L after screening.

Indeed, a second effect is that the higher form symmetry group (2.1) is explicitly broken to a subgroup via screening. This can be understood as follows: the line defects can have endpoints corresponding to charged operators, and this constrains the electric

and magnetic symmetry operators in the UV theory to be those under which lines which can end are neutral. For an example see figure 3 where we illustrate the breaking of a $U(1)^{(1)}$ higher symmetry to $\mathbb{Z}_q^{(1)}$ in presence of a particle of charge $q \in \mathbb{Z}$. From this perspective, the symmetry (2.1) is an emergent symmetry in the IR, which is broken by massive states that at sufficiently high energies become dynamical. For this reason the UV theory has a much smaller 1-form global symmetry, which oftentimes is completely trivial (corresponding to the cases when the spectrum of the theory is complete, meaning that all defect charges can be screened).[8] A related effect is that the potential background fluxes that the theory admits get reduced: a background for $U(1)_e$ can be understood as a modification for the quantisation condition for the magnetic flux (see e.g. section 5 of [57] for a discussion), so in the presence of electrically charged states only those backgrounds that result in quantisation conditions compatible with the charges of the dynamical matter are allowed. For instance, if all of our electrically charged dynamical particles have even charge, we can only have background magnetic fluxes with holonomies 0 and $\frac{1}{2}$. In general, the holonomies of the electric and magnetic fluxes must be such that their Dirac pairing with all dynamical states is integral.

The screening of Γ^* with respect to Γ gives rise to 1-form factor of the defect group, $\mathbb{D}^{(1)} = \Gamma^*/\Gamma$ [12]. The actual higher form symmetry for a given theory corresponding to the sublattice Γ_L is Γ_L/Γ [50, 52].

2.2 An example: the $\mathcal{N} = 2$ $\mathfrak{su}(2)$ theory on the Coulomb branch

As a simple illustration of the previous discussion, here we briefly review the results of [52] about the global form of the $\mathfrak{su}(2)$ gauge theories. From the Seiberg-Witten solution it is known that the BPS spectrum of the $\mathfrak{su}(2)$ theory can be generated as bound states of two mutually non-local excitations with charges γ_1 and γ_2 such that $\langle \gamma_1, \gamma_2 \rangle_D = -2$. In the choice of electromagnetic duality frame by Seiberg and Witten one can view γ_1 as a monopole of charge $(0, 1)$ and γ_2 as a dyon of charge $(2, -1)$. To determine the dual lattice Γ^* we can proceed as follows

$$\langle \boldsymbol{\alpha}, \gamma_1 \rangle_D = \alpha_1 \qquad \langle \boldsymbol{\alpha}, \gamma_2 \rangle_D = -\alpha_1 - 2\alpha_2 \qquad (2.3)$$

Then the first equation implies that $\alpha_1 \in \mathbb{Z}$, while the second implies that $\alpha_2 \in \frac{1}{2}\mathbb{Z}$. Let us consider the possible mutually local sublattices of Γ^* from this perspective, with respect to the corresponding screenings. There are three minimal defect charges which get nontrivial monodromies with respect to one another, corresponding to non-integer quantized Dirac pairings, namely

$$(1, 0) \qquad \left(0, \frac{1}{2}\right) \qquad \text{and} \qquad \left(1, \frac{1}{2}\right). \qquad (2.4)$$

Let us first consider the sublattice $\Gamma^*_{(1,0)} \subset \Gamma^*$ that contains the line defect with charge $(1, 0)$. The requirement of maximal mutual locality then implies that

$$\langle (1, 0), \boldsymbol{\alpha} \rangle_D = \alpha_2 \in \mathbb{Z} \qquad (2.5)$$

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[8]For a recent discussion about this point and applications beyond the scope of the present work, we refer our readers to the nice work [56].

Hence the charges of the line defects in $\Gamma^*_{(1,0)}$ have the form (n, m) where n, m are both integers. Considering the screening by the charges γ_1 and γ_2 corresponds to identifying

$$(n, m) \sim (n', m') + k_1 \gamma_1 + k_2 \gamma_2 \tag{2.6}$$

where $k_i \in \mathbb{Z}$. We see that we are left only with two equivalence classes in $\Gamma^*_{(1,0)}$, namely $[(1,0)]$ and $[(0,0)]$, hence we obtain an electric 1-form symmetry $(\mathbb{Z}_2)^{(1)}_e$ corresponding (with a natural choice of duality frame) to the gauge group SU(2). Now consider the lattice $\Gamma^*_{(0,\frac{1}{2})}$ which contains the line with charge $(0, \frac{1}{2})$. The requirement of maximal mutual locality for this class of charges is

$$\langle \left(0, \frac{1}{2}\right), \boldsymbol{\alpha} \rangle_D = -\frac{\alpha_1}{2} \in \mathbb{Z} \Rightarrow \alpha_1 \in 2\mathbb{Z} \tag{2.7}$$

Hence the charges of the line defects in $\Gamma^*_{(0,\frac{1}{2})}$ have the form $(2n, m/2)$ where n, m are both integers. The screening equivalence relation is again

$$(2n, m/2) \sim (2n', m'/2) + k_1 \gamma_1 + k_2 \gamma_2. \tag{2.8}$$

We see that we are left with only two equivalence classes again $[(0,0)]$ and $[(0, \frac{1}{2})]$ corresponding to a magnetic 1-form symmetry $(\mathbb{Z}_2)^{(1)}_m$ which gives a gauge group SO(3)$_+$. Now, consider $\Gamma^*_{(1,\frac{1}{2})}$: procceeding analogously, we obtain that

$$
\begin{aligned}
\langle \left(1, \frac{1}{2}\right), \boldsymbol{\alpha} \rangle_D &= \alpha_2 - \frac{\alpha_1}{2} \in \mathbb{Z} \\
&\Rightarrow (\alpha_1, \alpha_2) \in \{((2n+1), (2m+1)/2)\} \text{ or } \{(2n, m)\} \qquad n, m \in \mathbb{Z}
\end{aligned}
\tag{2.9}
$$

Again by screening we see that we obtain only two equivalence classes: $[(0,0)]$ and $[(1, \frac{1}{2})]$, corresponding to a $(\mathbb{Z}_2)^{(1)}_{\text{diag}}$ 1-form symmetry, which corresponds to the gauge group SO(3)$_-$.

We stress here that the above result is independent of the choice of electromagnetic frame: we can choose to work with any different basis as long as we are preserving the Dirac paring. For instance, one could do the analysis working with $\gamma_1 = (1, 1)$ and $\gamma_2 = (1, -1)$. In this case the charges in Γ^* have the form

$$\langle \boldsymbol{\alpha}, \gamma_1 \rangle_D = \alpha_1 - \alpha_2 \qquad \langle \boldsymbol{\alpha}, \gamma_2 \rangle_D = -\alpha_1 - \alpha_2 \tag{2.10}$$

And therefore one obtains

$$(\alpha_1, \alpha_2) \in \left\{ \left(\frac{2m+1}{2}, \frac{2n+1}{2}\right) \right\} \text{ or } \{(m, n)\} \qquad m, n \in \mathbb{Z}. \tag{2.11}$$

With this choice of duality frame we have the following defect charges that would violate Dirac quantization

$$\left(\frac{1}{2}, \frac{1}{2}\right) \qquad \left(\frac{1}{2}, -\frac{1}{2}\right) \qquad \text{and} \qquad (0, 1). \tag{2.12}$$

These correspond to the three choices of lattices above, if we identify the direction $(1, 1)$ with the magnetic charge and the direction $(0, 1)$ with the electric one.

3 Symmetry TFT and global structure

We have just argued that whenever the lattice Γ of charges particles in Maxwell theory is not unimodular we have the possibility of having choices of global structure, encoded as choices of maximal sets Γ_L of commuting elements in Γ^*/Γ. This is perhaps a surprising statement, as we typically don't think of Maxwell theory as admitting different global forms. The key difference between our analysis and the usual analysis is that we do not impose a priori integral quantization for the electric and magnetic fluxes in the theory, but rather accept as valid any flux quantisation structure compatible with the dynamical matter content. For instance, if all electrically charged states have even charge we include half-integrally quantised fluxes in the path integral for Maxwell theory. If we are considering theories with only electrically charged states (in some duality frame) then this is purely a matter of convention, and the half-integrality can be rescaled away. In contrast, the theories of interest to us are richer, and include dyonic states, which lead to genuinely different prescriptions for which fluxes to sum over.

Our task is therefore classifying all the possibilities for flux quantisation conditions compatible with the local dynamics. As mentioned above, the choice of quantisation for the electric and magnetic fields can be understood as a choice of background fields for the electric and magnetic 1-form symmetries. So our problem may be recast as the determination of which choices of background fluxes are allowed in a given quantum theory, given the dynamical matter content. This kind of problem has a familiar solution (see for instance [58]): the possible flux choices can be understood as states in the Hilbert space of a (generically non-invertible) BF theory in one dimension higher.

3.1 A quick review of BF theory in $D+1$ dimensions

In this section we review some basic details of the BF theory, following the discussion in [59]. These details are well-known, and can be skipped by *cognoscenti*. A BF theory is a TFT in $D+1$ dimensions with action

$$S = 2\pi i n \int_{\mathcal{M}^{D+1}} b^{(q+1)} \wedge db^{(D-q-1)} \tag{3.1}$$

where $b^{(q+1)}$ and $b^{(D-q-1)}$ are a $(q+1)$-form and a $(D-q-1)$-form. We stress that the coefficient n which multiplies the action must be an integer for $\exp(-S)$ to be well defined and compatible with the local U(1) gauge transformations

$$b^{(q+1)} \to b^{(q+1)} + d\lambda^{(q)} \qquad b^{(D-q-1)} \to b^{(D-q-1)} + d\lambda^{(D-q-2)} \,. \tag{3.2}$$

Here we are being naive and focusing only on the local structure of these higher gauge transformations. More refined experts might look into the full fledged gerby behavior of these Deligne-Beilinson cocylces. For our purposes in this paper the above description will suffice. Notice that upon a gauge variations we obtain

$$S \to S + 2\pi i n \int_{\partial\mathcal{M}^{D+1}} \lambda^{(q)} \wedge db^{(D-q-1)} \,. \tag{3.3}$$

This boundary term is crucial for applications to inflow and generalizations that we are after in this paper.

The theory is topological for a simple reason: the equation of motion for this theory are

$$f^{(q+2)} = db^{(q+1)} = 0 \qquad\qquad f^{(D-q)} = db^{(D-q-1)} = 0 \tag{3.4}$$

and forbid any local propagating degree of freedom. The theory has nevertheless interesting non-local gauge invariant operators corresponding to closed $(q+1)$-dimensional and $(D-q-1)$-dimensional hypersurfaces of $\mathcal{M}^{D+1}$, that generalize the familiar Wilson lines:

$$\begin{aligned}
\mathcal{W}_{\Sigma^{(q+1)}} &= \exp 2\pi i \int_{\Sigma^{(q+1)}} b^{(q+1)} \\
\mathcal{W}_{\Sigma^{(D-q-1)}} &= \exp 2\pi i \int_{\Sigma^{(D-q-1)}} b^{(D-q-1)}
\end{aligned} \tag{3.5}$$

To determine the algebra of these operators notice that the insertion of $\mathcal{W}_{\Sigma^{(q+1)}}$ in a correlator can be absorbed in the action introducing a source term in the equations of motion (3.4)

$$n f^{(D-q)} = \delta_{\Sigma^{q+1}} \tag{3.6}$$

where $\delta_{\Sigma^{(q+1)}}$ is the Poincaré dual to the cycle $\Sigma^{(q+1)}$ in $\mathcal{M}^{D+1}$. As a result we obtain that

$$\langle \mathcal{W}_{\Sigma^{(q+1)}} \mathcal{W}_{\Sigma^{(D-q-1)}} \rangle = \exp\left(\frac{2\pi i}{n} \ell(\Sigma^{(q+1)}, \Sigma^{(D-q-1)}) \right) \tag{3.7}$$

where $\ell(\Sigma^{(q+1)}, \Sigma^{(D-q-1)})$ is the linking number of $\Sigma^{(q+1)}$ and $\Sigma^{(D-q-1)}$ in $\mathcal{M}^{D+1}$. Equivalently, restricting everything along a spatial slice and considering a Hamiltonian quantization, these generalized Wilson lines form a Heisenberg algebra

$$\mathcal{W}_{\Sigma^{(q+1)}} \mathcal{W}_{\Sigma^{(D-q-1)}} = \exp \frac{2\pi i}{n} \left(\Sigma^{(q+1)} \cdot \Sigma^{(D-q-1)} \right) \mathcal{W}_{\Sigma^{(D-q-1)}} \mathcal{W}_{\Sigma^{(q+1)}} \tag{3.8}$$

where $\cdot$ is the intersection pairing along the spatial slice for the Hamiltonian quantization. Since this Heisenberg algebra is nontrivial the Hilbert space associated to a generic codimension one submanifold $\mathcal{M}^D$ will have dimension greater than one.

3.2 Global structures from the infrared

We now specialize the discussion in the previous section to the case of interest for this paper in which $D = 4$ and $q = 1$. Our main claim is that the introduction of the dynamical states on Maxwell theory leads to a 4d theory relative to the 5d theory[9]

$$\mathcal{S} = \frac{2\pi i}{2} \int_{\mathcal{M}^5} \mathcal{Q}^{\alpha\beta} b_\alpha^{(2)} \wedge db_\beta^{(2)} = \frac{2\pi i}{2} \int_{\mathcal{M}^5} \langle b^{(2)}, db^{(2)} \rangle_D \tag{3.9}$$

where the skew-symmetric matrix $\mathcal{Q}^{\alpha\beta}$ is the $2r \times 2r$ Dirac pairing for the BPS states in 4d, and $b^{(2)}$ is a $2r$ component 2-form (or more precisely, a $2r$-dimensional vector of degree

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[9]We stress that this is not the whole action of the symmetry TFT, just the part from which the choice of global structures will follow.

3 differential characters, but as mentioned above we will not need to worry about such topological subtleties here).

This action can be justified as follows. Note first that since $\mathcal{Q}$ is an antisymmetric matrix we can do an invertible integral change of basis to bring it into a block diagonal form

$$
\widetilde{\mathcal{Q}} = \left(
\begin{array}{ccccccc|c}
0 & n_1 & & & \cdots & & & \\
-n_1 & 0 & & & & & & \\
& & \ddots & & & & & \vdots \\
\vdots & & & 0 & n_k & & & \\
& & & -n_k & 0 & & & \\
& & & & & \ddots & & \\
& & & & & & 0 & n_r \\
& \cdots & & & & -n_r & 0 & \\
\hline
& & & & & & & \mathbf{0}_{f \times f}
\end{array}
\right) \tag{3.10}
$$

with the n_i integral and non-negative (see theorem IV.1 of [60] as well as the discussion in [61]). In practice the n_i can be obtained easily by going to the Smith normal form of $\mathcal{Q}$. If we name the components of $b^{(2)} = (b_{e,1}^{(2)}, b_{m,1}^{(2)}, b_{e,2}^2, \ldots)$ we see that our 5d theory becomes (up to boundary counterterms that we are neglecting)

$$
\mathcal{S}_5 = 2\pi i \sum_{i=1}^{r} n_i \int_{\mathcal{M}^5} b_{e,i}^{(2)} \wedge db_{m,i}^{(2)}. \tag{3.11}
$$

So our 5d theory decomposes into a sum of decoupled sectors.

In order to justify (3.11), consider the effect of going to the basis giving (3.10) on the boundary field theory. The generators γ_i in this basis satisfy $\left\langle \gamma_{2i-1}, \gamma_{2i} \right\rangle_D = -\left\langle \gamma_{2i}, \gamma_{2i-1} \right\rangle_D = n_i$ and zero otherwise. So the problem reduces to a situation similar to what we found in the $\mathfrak{su}(2)$ case in section 2.2. For simplicity we will henceforth focus on the first block.

By an SL$(2, \mathbb{Z})$ transformation we can further choose $\gamma_1 = (0, 1)$, $\gamma_2 = (n_1, k)$, with $k \in \mathbb{Z}$. We see that all states in this basis have electric charge divisible by n_1, so we can consistently introduce 't Hooft lines of charge $1/n_1$. Wilson lines, on the other hand, necessarily have integral charge. More generally, the spectrum of allowed lines has charges $(p, q/n_1)$, with $p, q \in \mathbb{Z}$. This implies, in turn, that the electric background fields have periodicity 1, but magnetic background fields have periodicity n_1. This means that the allowed flux operators on the 5d BF theory are of the form $\mathcal{W}_e^p \mathcal{W}_m^{q/n_1}$, with $p, q \in \mathbb{Z}$ (but n_1 not necessarily dividing q). Equivalently, we can think of the line operators as being the boundary of Gukov-Witten open surface operators, which when pulled to the 5d bulk become the operators in the BF theory.

A BF theory with action

$$\mathcal{S}_5 = 2\pi i \int_{\mathcal{M}^5} b_e^{(2)} \wedge db_m^{(2)} \tag{3.12}$$

where $b_m^{(2)}$ has periodicity n_1 (so $\mathcal{W}_m^{1/n_1}$ is allowed) and $b_e^{(2)}$ periodicity 1 is the same as a BF theory with action

$$\mathcal{S}_5 = 2\pi i \, n_1 \int_{\mathcal{M}^5} b_e^{(2)} \wedge d\tilde{b}_m^{(2)} \tag{3.13}$$

with both $b_e^{(2)}$ and $b_m^{(2)}$ of periodicity 1 (via $b_m^{(2)} := n_1 \tilde{b}_m^{(2)}$). Unwinding the choices of basis, this proves that (3.9) is indeed the bulk theory for Maxwell theory with pairing $\mathcal{Q}$.

From our review in section 3.1 it follows that this TFT has an algebra of generalized Wilson lines of the form

$$\mathcal{W}_{\Sigma^2}^{e,i} = \exp 2\pi i \int_{\Sigma_2} b_{e,i}^{(2)} \qquad \mathcal{W}_{\Sigma^2}^{m,i} = \exp 2\pi i \int_{\Sigma_2} b_{m,i}^{(2)} \tag{3.14}$$

which along a spatial slice satisfy an Heisenberg algebra

$$\mathcal{W}_{\Sigma^2}^{e,i} \mathcal{W}_{\widehat{\Sigma}^2}^{m,i} = \exp\left(\frac{2\pi i}{n_i} (\Sigma^2 \cdot \widehat{\Sigma}^2) \right) \mathcal{W}_{\widehat{\Sigma}^2}^{m,i} \mathcal{W}_{\Sigma^2}^{e,i} \,. \tag{3.15}$$

This entails that whenever one of the n_i's is different from one, we obtain a Hilbert space of dimension greater than one for the theory on the boundary. From the point of view of the four dimensional theory, these generalised Wilson lines are the operators measuring background flux for the 1-form symmetries, so the fact that they do not commute implies that we cannot choose Dirichlet boundary conditions for all fluxes simultaneously, as argued originally in the holographic context in [58]. In the specific case of geometrically engineered four dimensional theories our field theory discussion will reproduce the results obtained from geometry previously in [19, 20, 30].

4 Examples and consistency checks from $\mathcal{N} = 2$ theories

In order to compute the Dirac pairing $\mathcal{Q}$ we need to know the charge lattice of the theory, meaning the electromagnetic charges of the particles in the spectrum of the theory. In this paper, for concreteness, we focus on examples arising in the context of 4d $\mathcal{N} = 2$ theories where we can easily extract this information exploiting BPS quivers [62]. Here we are assuming that the BPS spectrum faithfully reproduces the charge lattice, meaning that we are assuming that in any charge sector populated by states we can always find a BPS representative. By definition of BPS quiver the Dirac pairing is captured by the quiver exchange matrix

$$\mathcal{Q}_{ij} = \#(\text{ arrows } i \to j) - \#(\text{ arrows } j \to i) \tag{4.1}$$

where one works with an extended charge lattice with a number of generators that equals the number of electric, magnetic and flavour charges (where the latter are taken in the Cartan of the flavour symmetry F of the theory). The generators of the charge lattice are in one to one correspondence with the nodes of the BPS quiver and all other stable BPS states

$\mathfrak{g}$	$\mathbb{D}^{(1)}$	
A_n	$\mathbb{Z}_{n+1} \oplus \mathbb{Z}_{n+1}$	
D_n	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	if $2 \mid n$
	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	if $2 \nmid n$
E_6	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	
E_7, B_n, C_n	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	
E_8, G_2, F_4	0	

Table 1. Defect groups for the pure $\mathcal{N} = 2$ SYM theories with algebra $\mathfrak{g}$.

have charges that can be expressed as linear combinations of the form $\gamma = \sum_{i=1}^{2r+f} M_i \gamma_i$ where $M_i \in \mathbb{Z}_{\geq 0}$ for all i (particles) or $M_i \in \mathbb{Z}_{\leq 0}$ for all i (antiparticles). In particular, we are granted that the γ_i are in the spectrum and hence our screening argument is faithfully representing the 1-form symmetry.

4.1 Pure $\mathcal{N} = 2$ SYM theories with algebra $\mathfrak{g}$

We begin considering the center symmetries of pure SYM theories with gauge algebra $\mathfrak{g}$. The BPS quivers for all the ADE cases are described in [62, 63] and those for the non-simple laced cases where given in [64]. We emphasize that in the latter case we have no IIB geometric engineering construction for the BPS quiver, so the agreement that we will find between the field theory expectation and the result from the analysis in terms of the BPS quiver can be taken as evidence that our discussion above remains valid for those cases for which the string theory analysis is not available.

We find that for all these semi-simple Lie algebras, the Dirac pairing $\mathcal{Q}$ can be written in the block diagonal form and the corresponding defect groups are given in 1. This is clear from the structure of the Dirac pairing for such models, which is given by

$$\mathcal{Q}_{\mathfrak{g}} = \left(\begin{array}{c|c} C_{\mathfrak{g}} - C_{\mathfrak{g}}^t & C_{\mathfrak{g}}^t \\ \hline -C_{\mathfrak{g}} & 0 \end{array} \right) \tag{4.2}$$

where $C_{\mathfrak{g}}$ is the $r \times r$ Cartan matrix for the Lie algebra $\mathfrak{g}$. For example, the BPS quivers for gauge groups $\mathrm{SU}(N+1)$ and $\mathrm{USp}(2N)$ are

$$\tag{4.3}$$
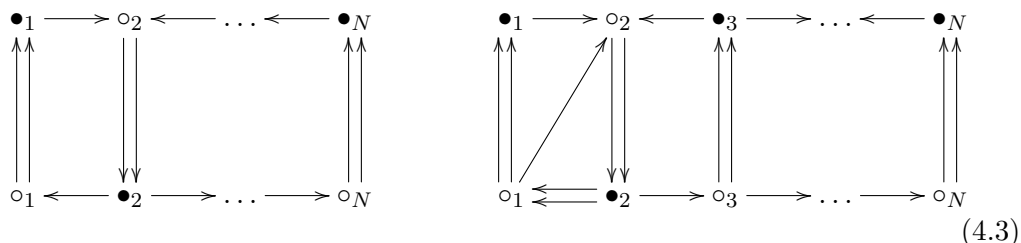
respectively. From these we obtain the block diagonal forms

$$\widetilde{\mathcal{Q}}_{A_N} = \begin{pmatrix} 0 & N+1 & & & \\ -(N+1) & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad \widetilde{\mathcal{Q}}_{C_N} = \begin{pmatrix} 0 & 2 & & & \\ -2 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}, \qquad (4.4)$$

respectively. Thus, it is clear that,

$$\mathbb{D}_{A_N}^{(1)} = (\mathbb{Z}_{N+1})_e^{(1)} \oplus (\mathbb{Z}_{N+1})_m^{(1)}, \quad \mathbb{D}_{C_N}^{(1)} = (\mathbb{Z}_2)_e^{(1)} \oplus (\mathbb{Z}_2)_m^{(1)}. \qquad (4.5)$$

Moreover, the resulting Heisenberg algebra precisely reproduces the expected global forms for the corresponding Lie groups [5]. The check for all other groups is carried in a similar fashion and we consistently recover all results in table 1.

4.2 Example of $\mathcal{N} = 2^*$ theories with algebra $\mathfrak{g}$

In order to obtain the $\mathcal{N} = 2^*$ quivers for a simple gauge theory with gauge algebra of ADE type, one can start from the affine quiver $A(1,1)\square\widehat{G}$ and replace one of the Kronecker subquivers that correspond to a node with Dynkin weight 1 in the affine $\widehat{G}$ diagram with a single node $*$, giving a 'specialization' of the corresponding quiver in the language of [64]. Graphically such operation corresponds to
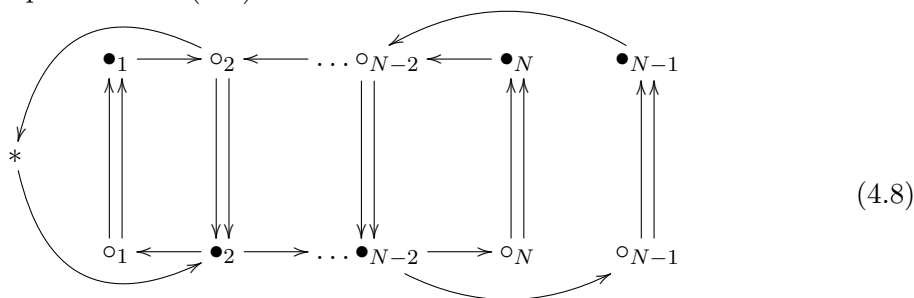

$$(4.6)$$
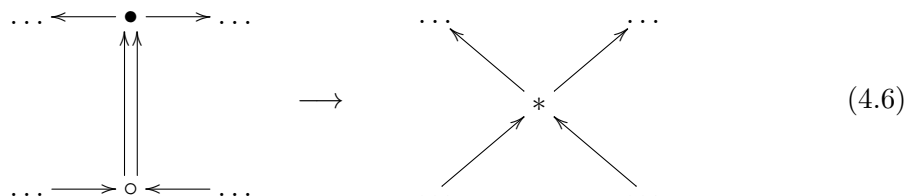
Using this trick, for instance, the BPS quiver for $\text{SU}(N+1)$ $\mathcal{N} = 2^*$ is

$$(4.7)$$
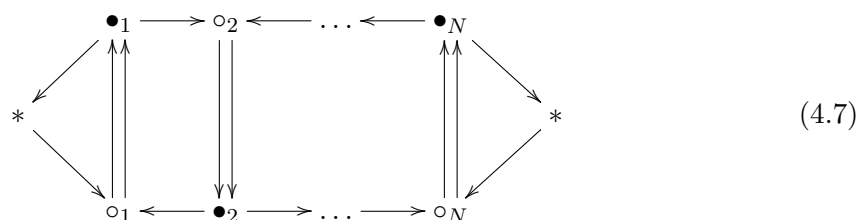
where the nodes $*$ appearing on the left and on the right of the above equation need to be identified. The resulting $\mathcal{Q}_{ij}$ can be read off straightforwardly from equation (4.1).

Similarly, the BPS quiver for $SO(2N)$ $\mathcal{N} = 2^*$ is

$$\text{(4.8)}$$

Also in these examples we obtain perfect agreement with the defect groups we expect from gauge theory. For instance starting from the quiver of $SU(N)$ $\mathcal{N} = 2^*$ we obtain

$$\widetilde{\mathcal{Q}}_{SU(N)\,\mathcal{N}=2^*} = \begin{pmatrix} 0 & N & & & \\ -N & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad \text{(4.9)}$$

as expected.

### 4.3 Adding matter in various representations

The BPS quiver for a gauge theory with gauge group $\mathfrak{g}$ and matter in an principal representation R is easily obtained [62]. There is a one-to-one correspondence between principal representations and nodes of the Dynkin diagram of $\mathfrak{g}$. We can schematically summarize it as follows: the nodes of the Dynkin diagram are in one-to-one correspondence with the basis elements of the weight lattice, and principal representations are such that their highest weight $w(R_i)$ is $w(R_i) = \delta_{ij}\omega_j$, where ω_j is the weight basis. Graphically, the i-th node in the Dynkin diagram

$$\circ \text{------} \circ \text{------} \cdots \text{------} \bullet_i \text{------} \cdots \text{------} \circ \text{------} \cdots \qquad \text{(4.10)}$$

correspond to the principal representation R_i. For the pure SYM BPS quiver of type $\mathfrak{g}$ there is a one-to-one correspondence between the nodes of the Dynkin diagram and full Kronecker subquivers. To obtain the quiver for SYM coupled to the i-th principal representation one adds a node to the BPS quiver connected to the i-th Kronecker subquiver as follows

$$\text{(4.11)}$$
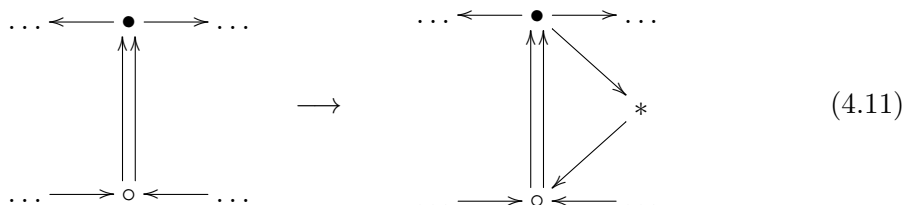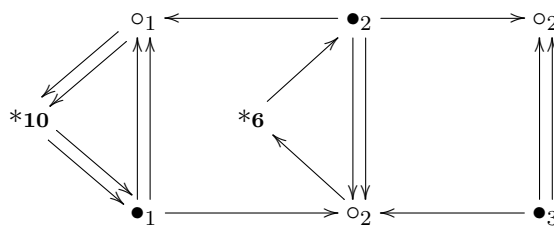
Figure 4. The BPS quiver for SU(4) with matter in the $\mathbf{6} \oplus \mathbf{10}$.

Notice that this prescription is compatible with the prescription discussed in the previous section about $\mathcal{N} = 2^*$ theories whenever the adjoint is a principal representation (e.g. this is the case for SO($2N$) above).

In general, if the extra matter corresponds to a tensor product of principal representations $R_i \otimes R_j$ the corresponding BPS quiver is obtained by connecting the extra node to the rest of the quiver with an oriented triangle for each of the corresponding Kroneckers. An example for this prescription is the BPS quiver for the adjoint of $\mathfrak{su}(N)$ in the previous section. The quiver in equation (4.7) is the BPS quiver for the representation

$$\mathbf{N} \otimes \overline{\mathbf{N}} = 1 \oplus \text{Adj}. \tag{4.12}$$

and it corresponds to the tensor product of the fundamental $\bullet_1 \text{------} \cdots$ and the antifundamental $\cdots \text{------} \bullet_N$ representations of SU(N).

It is amusing to check explicitly that the breaking of the center symmetry by the N-ality of the corresponding representation is respected, thus giving further consistency checks to our general result. As an explicit example of how this works in practice let us discuss here the case of the BPS quiver for a Lagrangian theory with gauge group SU(4) and matter in the direct sum of a symmetric two-index representation of SU(4) and an antisymmetric two-index representation. Since both the $\mathbf{6}$ and the $\mathbf{10}$ of SU(4) have quadrality 2, we expect the center symmetry of SU(4) to be broken down to $\mathbb{Z}_2$ in this example. The resulting quiver is in figure 4. We indeed find a defect group $\mathbb{D}^{(1)}$ given by two copies of $\mathbb{Z}_2^{(1)}$ and a single non-trivial BF coupling $n_1 = 2$, compatible with the field theory expectations.

4.4 Non-Lagrangian theories

The results in this paper confirm field theoretically all the results we obtained from studying the 4d $\mathcal{N} = 2$ theories arising from IIB on isolated hypersurface singularities. This follows from a simple remark: in the IIB geometric engineering of both Lagrangian and non-Lagrangian theories the Dirac pairing is captured by the intersection form among the special Lagrangian vanishing 3-cycles of the corresponding CY (see e.g. [65]). In that context the charge lattice of the theory is a sublattice of $H_3(\mathcal{X}, \mathbb{Z})$ given by stable collections of wrapped D3 branes. The quiver captures precisely this information, and the nodes of the quiver give rise to a collection of 3-cycles that are always stable, in regions of the

moduli space that are compatible with that quiver descriptions. The intersection pairing is therefore identified with the Dirac pairing, and that is precisely the quantity which enters in all the computations we carried out in our previous paper on the subject and that determines the structure of the Heisenberg algebra of FMS fluxes. For all these examples, therefore, our field theoretical results and the ones obtained from geometric engineering agree by construction. In particular, this confirms previous results [19, 20, 30, 34, 36, 41, 66] about the Argyres-Douglas theories of type (G, G') constructed by Cecotti, Neitzke, and Vafa [63], the various Arnol'd SCFTs [67, 68] and other theories originating from singularity theory [69], as well as the SCFTs of type $D_p(G)$ [70, 71].

We stress that our methods extend straightforwardly to all other theories with a known BPS quiver.

As a further example of an application, we present in the rest of this section an analysis of the rank one theories with known BPS quivers, namely theories in the $\mathcal{I}_1$ series and in the $\mathcal{I}_4$ series in table 1 of [72]. Therefore we obtain examples in all possible characteristic dimensions [73]. The BPS quivers for these theories have been obtained in [74] — see section 4 of [75] for a review (see also [62, 71, 76] for previous results on the topic as well as the nice works [77, 78] for a more recent take on the subject). The resulting quivers have the form

$$ \text{(4.13)} $$

where:

- f is the rank of the flavor symmetry of the rank 1 theory of interest;

- $q_{\mathcal{K}}$ is a positive integer denoting the multiplicity of the arrows $\circ \to \bullet$ determined as follows:

$$ q_{\mathcal{K}} = \begin{cases} 3 & \text{for } \mathcal{K} = II^*, III^*, IV^* \\ 2 & \text{for } \mathcal{K} = I_0^* \\ 1 & \text{for } \mathcal{K} = II, III, IV \end{cases} \tag{4.14} $$

- $a_1, a_2, \ldots, a_f$ are positive integers denoting the multiplicities of the arrows $\bullet \to *_i \to \circ$, determined from the decomposition of the Kodaira fiber

$$ \mathcal{K} \to I_1, I_1, I_{(a_1)^2}, I_{(a_2)^2}, \ldots, I_{(a_f)^2}. \tag{4.15} $$

As an example for the E_8 Minhan-Nemeshansky theory [79] we have $\mathcal{K} = II^*$ and $II^* \to (I_1)^{10}$, hence $q_{\mathcal{K}} = 3$, $f = 8$ and $a_i = 1$ for all $i = 1, \ldots, 8$. As another example, for the Argyres-Wittig theory with a flavor symmetry with Lie algebra C_5 we have $II^* \to (I_1)^6, I_4$, hence $q_{\mathcal{K}} = 3$, $f = 5$ and $a_1 = 2$ while $a_{2,3,4,5} = 1$.

Exploiting the data of table 1 of [72] it is straightforward to read off the corresponding BPS quivers for these theories, then by (4.1) the resulting BF levels follow by our method.

The result we obtain is that for all these theories $n_1 = 1$, but for the case $\mathcal{K} = I_0^*$ with decomposition $I_0^* \to I_1{}^2, I_4$ which corresponds to SU(2) $\mathcal{N} = 2^*$. In this case we obtain $n_1 = 2$, as we already discussed above.

The result we obtain for the global forms of these theories can be also recovered exploiting the fact that these models arise as fixed points of supersymmetry enhancing RG flows, starting from $\mathcal{N} = 1$ Lagrangians [80]. Another interesting class of susy enhancing RGs are those of Maruyoshi-Song type which give results for the theories in the $\mathcal{I}_1$ series [81–83]. For the theory in the $\mathcal{I}_4$ series with global symmetry USp(4) × U(1) as well as for the E_6 MN theory, UV $\mathcal{N} = 1$ theories have been obtained by brane bending and deconfinement [84]. In all these examples we reproduce easily the fact that the UV theory has a trivial defect group, thus confirming our findings.

5 Generalization to other dimensions

In this section we quickly comment about the generalization of the argument above to some other theories in higher dimensions. In general we expect the $D + 1$ TFT action will contain terms of the following form (with an additional factor of 2 in the self-dual case)

$$\mathcal{S}_{D+1}^{TFT} \supseteq 2\pi i \sum_{q=0}^{D} \mathcal{Q}_q^{\alpha,\beta} \int_{D+1} b_\alpha^{(q+1)} \wedge db_\beta^{(D-q-1)} \, . \tag{5.1}$$

The latter are relevant to probe more general global forms of QFTs in D-dimensions with different kinds of higher q-form symmetries corresponding to defects with non-trivial generalized Dirac strings. We stress that other couplings can be allowed, which in this paper we are omitting. In these cases the symmetry properties of the matrices $\mathcal{Q}_q^{\alpha,\beta}$ depend crucially on D and on q. For instance for $D = 6$ and $q = 2$, we have a symmetric pairing that was explored in the context of defect groups and global structures of 6d (2,0) and (1,0) theories [12, 15]. In what follows we quickly address the IR origin of the global structures for the cases of 6d (2,0) SCFTs and of 5d SCFTs.

5.1 The case of 6d (2,0) theories

In the case of 6d (2,0) theories we have an analogue of the Coulomb branch, where the nonabelian string dynamics reduces to an abelian one, the so-called *tensor branch*. Along the tensor branch we have a $(\mathrm{U}(1)_e^{(2)})^r$ higher 2-form symmetry, which has 3-form currents $J^{(3)} = h_i^{(3)}$ corresponding to the anti-self-dual 3-form curvatures $h_i^{(3)} = db_i^{(2)}$ where $b_i^{(2)}$ are the 2-form fields in the 6d tensormultiples. We can couple the latter to background fluxes $B_i^{(3)}$, which have 3-form background gauge transformations analogous to the discussion we had for the Maxwell theory. Also in this case, when we excite the (2,0) BPS strings, the current conservation equation $d * J^{(3)} = 0$ is broken by the presence of sources for the $h_i^{(3)}$ fluxes, represented by the string charges.

The effect of such breaking is again detected by the inflow mechanism which associates to the tensor branch a 7d BF like theory of the form

$$\mathcal{S}_7 \supseteq \frac{2\pi i}{2} C_{\mathfrak{g}}^{\alpha\beta} \int_7 b_\alpha^{(3)} \wedge db_\beta^{(3)} \tag{5.2}$$

where $C_{\mathfrak{g}}^{\alpha\beta}$ is the Cartan matrix of the Lie algebra of type $\mathfrak{g}$, which gives the BPS string Dirac pairing for the (2,0) theory of type $\mathfrak{g}$ in 6d, again accounting for the 't Hooft screening [12].

There is a crucial difference between six dimensions and four dimensions: in six dimensions the Dirac pairing in 6d is a symmetric matrix, which is compatible with the symmetry properties of the 7-form in the action 5.2. For this reason, 6d strings can be non-mutually local with respect to themselves. As examples one can consider e.g. the rank one non Higgsable cluster theories, 6d SCFTs with tensor branch of the form $\overset{\mathfrak{g}}{n}$. For these models we obtain

$$\mathcal{S}_7 \supseteq \frac{2\pi i n}{2} \int_7 b^{(3)} \wedge db^{(3)} \tag{5.3}$$

This physical distinction comes together with a very important mathematical distinction: while the theory of skew-symmetric integral bilinear forms relevant to the four dimensional case is very simple (and in particular implies that a change of basis to the simple block diagonal form (3.10) always exists), the theory of symmetric bilinear forms over the integers, relevant in the six dimensional case, is significantly more complicated. For instance, one can show that the Cartan matrix arising in the $\mathfrak{su}(2)$ case cannot be taken to diagonal form via an integral congruence [85].

Interestingly, it is the diagonal form that appears in the holographic result arising from the reduction of the Chern-Simons couplings of M-theory [58, 86]

$$\mathcal{S}_7(A_{N-1}) \supseteq \frac{2\pi i}{2} N \int_7 c^{(3)} \wedge dc^{(3)} + \dots \tag{5.4}$$

The equivalence of this BF theory, up to an invertible sector, with the theory (5.2) is shown in appendix F of [6].

5.2 The case of 5d theories

Another class of theories with interesting Coulomb phases is provided by the 5d SCFTs with non-trivial ranks. Along a 5d Coulomb branch we have an emergent $(\mathrm{U}(1)_e^{(1)} \times \mathrm{U}(1)_m^{(2)})^r$ higher form symmetry. The latter is similarly broken to subgroups by the spectrum and, *mutatis mutandis*, the same logic applies. The resulting BF-theory in this case has the form

$$\mathcal{S}_6 \supseteq 2\pi i \mathcal{Q}^{\alpha\beta} \int_6 b_{e,\alpha}^{(2)} \wedge db_{m,\beta}^{(3)} \tag{5.5}$$

where $\mathcal{Q}^{\alpha\beta}$ is the Dirac pairing among the BPS electric states and the 5d BPS monopole strings. This latter quantity determines the structure of the Heisenberg algebras of non-commuting fluxes via a field redefinition to its Smith normal form, thus reproducing all the results obtained via M-theory in this context (see e.g. [16, 17, 22, 46, 47, 87]).[10] For this class of theories, it is known there might be further terms in the bulk TFT [8]: it should be possible to recover the latter in terms of the infrared as well, however this would go beyond the modest scope of this short note.

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[10]The Dirac pairing $\mathcal{Q}^{\alpha\beta}$ coincides with the intersection form between 2-cycles and 4-cycles relevant for the computation of the global forms from M-theory geometric engineering.

As a consistency check of the above formula, one can consider the 4d KK theory of the corresponding SCFT, obtained from circle reduction. These systems have 5d BPS quivers [88]. The latter can be exploited in a way analogous to the one we discussed above to capture the global structure of the 4d KK theory so obtained. See reference [87] for an application of this idea in the context of 5d orbifold SCFTs discussed e.g. in [89].

6   Conclusions

In this paper we have begun exploring a mechanism to recover the global structure of a given QFT from an infrared phase which is under perturbative control. Our main result is to recover, from Coulomb-like phases, the Heisenberg algebra of non-commuting fluxes that was found in the geometric engineering analysis in purely field theoretical terms.

An interesting question that we leave for future analysis is to characterize the full structure of the symmetry TFT from an IR perspective. We expect this to be possible by a suitable extension of the 't Hooft anomaly matching argument: while the theory on the boundary flows, the symmetry TFT in the bulk must match along the flow. In this short note, we recovered the term responsible for the possible choices of global structures, but we stress that, for instance, we expect mixing terms between the various higher form fields in the symmetry TFT. The latter are not captured by the argument presented here, but are known to arise from a geometric engineering perspective [8, 44].

In this context a direction that we find particularly interesting is the question of recovering higher group structures or more general non-invertible symmetries from the IR. For two-groups, evidence that this is indeed possible in some cases was given in the context of little string theories in the papers [25, 51, 90]. There it was shown that higher group structure constants are related to specific terms in the anomaly polynomial of the corresponding little strings. Based on that analysis we conjecture the anomaly theories on the worldvolumes of the various BPS degrees of freedom of the boundary QFTs of interest must know about these finer details of the symmetry TFT. A similar effect was recently exploited to unravel certain non-invertible symmetries in [91, 92].

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