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Multi-fractional instantons in $\mathrm{SU}(N)$ Yang-Mills theory on the twisted $\mathbb{T}^4$

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Abstract: We construct analytical self-dual Yang-Mills fractional instanton solutions on a four-torus $\mathbb{T}^4$ with 't Hooft twisted boundary conditions. These instantons possess topological charge $Q = \frac{r}{N}$, where $1 \leq r < N$. To implement the twist, we employ $\mathrm{SU}(N)$ transition functions that satisfy periodicity conditions up to center elements and are embedded into $\mathrm{SU}(k) \times \mathrm{SU}(\ell) \times \mathrm{U}(1) \subset \mathrm{SU}(N)$, where $\ell + k = N$. The self-duality requirement imposes a condition, $kL_1L_2 = r\ell L_3L_4$, on the lengths of the periods of $\mathbb{T}^4$ and yields solutions with abelian field strengths. However, by introducing a detuning parameter $\Delta \equiv (r\ell L_3L_4 - kL_1L_2)/\sqrt{L_1L_2L_3L_4}$, we generate self-dual nonabelian solutions on a general $\mathbb{T}^4$ as an expansion in powers of Δ. We explore the moduli spaces associated with these solutions and find that they exhibit intricate structures. Solutions with topological charges greater than $\frac{1}{N}$ and $k \neq r$ possess non-compact moduli spaces, along which the $\mathcal{O}(\Delta)$ gauge-invariant densities exhibit runaway behavior. On the other hand, solutions with $Q = \frac{r}{N}$ and $k = r$ have compact moduli spaces, whose coordinates correspond to the allowed holonomies in the $\mathrm{SU}(r)$ color space. These solutions can be represented as a sum over r lumps centered around the r distinct holonomies, thus resembling a liquid of instantons. In addition, we show that each lump supports 2 adjoint fermion zero modes.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Anomalies in Field and String Theories

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Contents

1 Introduction, summary, and outlook

Instantons are prominent in studying many nonperturbative phenomena in Yang-Mills theory, including the vacuum structure, condensates, and confinement. One of the least-explored instantons are *'t Hooft fluxes* of $\mathrm{SU}(N)$ gauge theory on the 4-torus $\mathbb{T}^4$ with twisted boundary conditions [1]. Such solutions, found by 't Hooft, carry fractional topological charges and have constant abelian field strength. While the field strength is abelian, for a general number of colors N, the boundary conditions on $\mathbb{T}^4$ are implemented via non-abelian transition functions (i.e. there exists no gauge where all transition functions commute).

Although 't Hooft's solutions have been known since the 1980s, relatively little attention has been devoted to their study since [2]. The notable exception is the work of the Madrid group over many years, reviewed in [3]. The recent development of generalized global symmetries [4] resurrected the interest in this subject. It was shown in [5] that introducing background fields for the 1-form $\mathbb{Z}_N^{(1)}$ center symmetry of Yang-Mills theory can lead to new 't Hooft anomalies, restricting the symmetry realizations and thus the infrared dynamics.

The gauge field of the 1-form symmetry is a 2-form field whose nonvanishing holonomies implement the 't Hooft twist of the boundary conditions on $\mathbb{T}^4$. The fractional 2-form flux is merely an external field that imposes kinematical constraints. On the other hand, finding the field configurations which minimize the action (or energy) in the presence of twists requires dynamical considerations. Recently, the authors questioned the role instantons in the presence of twists could play in determining the dynamics of the theory [6]. In particular, we examined the gaugino condensate in SU(2) super Yang-Mills theory with twists on $\mathbb{T}^4$. The fractional topological charge $Q = \frac{1}{2}$ of the SU(2) solution supports two gaugino zero modes and yields a non-vanishing condensate, which was found to be independent of the torus size. The computations were carried within the limit of the small-torus size, taken to be much smaller than the inverse strong scale, so we remained in the semi-classical domain. Thus, we could perform reliable computations and, thanks to supersymmetry, extract the numerical coefficient of the condensate. However, our computations gave twice the condensate's numerical value on $\mathbb{R}^4$. Thus, our results warrant further examination of the situation for SU(2) and for a general number of colors.

The current work is a continuation of the efforts in this direction. One of the crucial conditions for studying the dynamics is the self-duality of the fractional instantons. A non-self dual solution is not a minimum of the action; it has negative fluctuation modes and hence, is unstable. Insisting on the abelian solutions found by 't Hooft [1], the ratio between the periods of $\mathbb{T}^4$ needs to satisfy a specific condition to respect the self-duality of the solutions. We call such $\mathbb{T}^4$ a self-dual torus. However, in [6], it was found that instantons on the self-dual torus support extra fermion zero modes, more than needed to support the bilinear gaugino condensate.

A way to lift the extra zero modes is to deform the self-dual $\mathbb{T}^4$. The price to pay, insisting on the self-duality of the instantons, is to depart from the simple abelian solutions found by 't Hooft. One is then faced with the fact that a non-abelian analytical solution on a generic $\mathbb{T}^4$ with general 't Hooft twists is not currently known. Furthermore, even a description of the moduli space and of its metric[1] of such self-dual solutions is not available. Fortunately, the authors of [7] developed a systematic approach to obtaining approximate SU(2) nonabelian self-dual solutions as expansion in a small parameter Δ, measuring the deviation from the self-dual torus.[2] The approach in [7] was generalized in [8] to the case of SU(N). Nevertheless, it was only used to obtain solutions with minimal topological charge $Q = \frac{1}{N}$.

In this paper, we carry out a systematic analysis to obtain self-dual solutions with generic topological charge $Q = \frac{r}{N}$, with integer $N > r > 1$, on a non-self dual torus. The main effort of the present work is directed at exploring the structure of the bosonic moduli space of the solutions as well as the fermion zero modes in these backgrounds.

Summary. The main findings of this rather technical paper are described below:

We let L_1, L_2, L_3, L_4 be the lengths of the periods of $\mathbb{T}^4$. Following 't Hooft [1], we embed the SU(N) transition functions and gauge fields in $SU(k) \times SU(\ell) \times U(1) \subset SU(N)$, such that k and ℓ are positive integers and $k + \ell = N$. We choose the transition functions to give rise to 't Hooft twists on $\mathbb{T}^4$ corresponding to topological charge $Q = \frac{r}{N}$ (section 2). Even though the transition functions are fully non-abelian, the original 't Hooft solution with topological charge $Q = \frac{r}{N}$ has only an abelian gauge field A_μ along the U(1) generator.[3] The self-duality of this solution imposes the condition $kL_1L_2 = r\ell L_3L_4$. As already mentioned, a $\mathbb{T}^4$ that satisfies this condition is said to be self-dual.

Next, we define a *detuning parameter* Δ, that measures the deviation from the self-dual $\mathbb{T}^4$, as $\Delta \equiv (r\ell L_3L_4 - kL_1L_2)/\sqrt{L_1L_2L_3L_4}$. Then, the self-dual non-abelian solution is obtained as an expansion in Δ, similar to [7, 8]. The solution now has nontrivial components along the abelian U(1) generator as well as the nonabelian subgroups $SU(k) \times SU(\ell)$. We carry out our analysis to the leading order in Δ, from which we observe the following:

1. To the leading order in Δ, the solution of the self-dual Yang-Mills equations is in one-to-one correspondence with the solution to the Dirac equation of the gaugino zero modes on the self-dual $\mathbb{T}^4$ (section 3). Thus, one can borrow the latter's solutions and show that they satisfy the self-dual Yang-Mills equations to the leading order (section 4).

2. Among all solutions with $Q = \frac{r}{N}$, the ones with $k = r$ stand out. For this case, we find $4r$ arbitrary physical parameters that label the self-dual Yang-Mills solutions, in accordance with the index theorem. We interpret these parameters as the coordinates on the compact moduli space: these are the r $(= k)$ holonomies in the SU(k) color space in each of the 4 spacetime directions (section 5).

3. In addition, we find that gauge-invariant densities for the $k = r$ solutions can be cast into the form of a sum over r identical lumps centered about the values taken by the r $(= k)$ different holonomies. This indicates that a solution with topological charge

[1] These data alone suffice to perform certain instanton computations in supersymmetric theories.

[2] This is the solution used in [6], which, at $\Delta > 0$, supports exactly two zero modes needed to give rise to the bilinear condensate.

[3] See section 3.1: the $Q = \frac{r}{N}$ transition functions are in (3.1) and the abelian solution is in (3.2).

$Q = \frac{r}{N}$ can be thought of as composed of r "elementary," yet strongly overlapping ones—thus, resembling a liquid, rather than a dilute gas [3] (section 6.2.1). See figure 1 for an illustration.

Further support for this interpretation follows from solving the Dirac equation in the background of the full non-abelian solution, showing that 2 fermion zero modes are centered about each of the r holonomies, giving a total of $2r$ fermion zero modes as required by the index theorem (section 6.2.2).

4. We also study the Δ-expansion around the other $Q = \frac{r}{N}$ solutions, the ones with $k \neq r$ (section 5). Here, we find that the moduli space becomes non-compact. To further understand the significance of this finding, we show that gauge-invariant local densities grow without limit in the noncompact moduli directions, clashing with the spirit of the Δ expansion for $k \neq r$ (section 5 and appendix D). This blow-up leads us to conjecture that the only self-dual $Q = \frac{r}{N}$ solutions, obtained via the Δ-expansion, are the ones with $k = r$.

Outlook. There are several directions where this work can be applied to or extended:

The study of the present paper sets the stage for a forthcoming paper to shed light on a few dynamical and kinematical aspects of supersymmetric and non-supersymmetric SU(N) gauge theories. This includes the higher-order condensates, cluster decomposition principle, and exactness/holomorphy of supersymmetric results.

We have yet to achieve a deeper understanding of the apparent failure of the Δ expansion for $k \neq r$ that we observed in the leading order. This may be require better control of the higher orders in the Δ-expansion. Numerical studies of instantons on the twisted torus can also be used to study the convergence of the expansion as well as the approach to various large volume limits.

2 Review of 't Hooft's constant-flux solutions on $\mathbb{T}^4$

This section quickly reviews SU(N) 't Hooft twisted solution on the four-torus $\mathbb{T}^4$. We take the torus to have periods of length L_μ, $\mu = 1, 2, 3, 4$, where μ, ν runs over the spacetime dimensions. The gauge fields A_μ are Hermitian traceless $N \times N$ matrices, and taken to obey the boundary conditions

$$A_\nu(x + L_\mu \hat{e}_\mu) = \Omega_\mu(x) A_\nu(x) \Omega_\mu^{-1}(x) - i\Omega_\mu(x)\partial_\nu \Omega_\mu^{-1}(x)\,, \tag{2.1}$$

upon traversing $\mathbb{T}^4$ in each direction. The transition functions Ω_μ are $N \times N$ unitary matrices, and $\hat{e}_\nu$ are unit vectors in the x_ν direction. The subscript μ in Ω_μ means that the function Ω_μ does not depend on the coordinate x_μ. The boundary condition (2.1) means that the gauge fields A_μ are periodic up to a gauge transformation. Let us for the moment use the short-hand-notation $[\Omega_\mu]A_\nu$ to denote $\Omega_\mu A_\nu \Omega_\mu^{-1} - i\Omega_\mu \partial_\nu \Omega_\mu^{-1}$. Then, the compatibility of (2.1) at the corners of the $x_\mu - x_\nu$ plane of $\mathbb{T}^4$ gives:

$$\begin{aligned} A_\lambda(x + L_\mu \hat{e}_\mu + L_\nu \hat{e}_\nu) &= [\Omega_\mu(x + L_\nu \hat{e}_\nu)][\Omega_\nu(x + L_\mu \hat{e}_\mu)]A_\lambda(x) \\ &= [\Omega_\nu(x + L_\mu \hat{e}_\mu)][\Omega_\mu(x + L_\nu \hat{e}_\nu)]A_\lambda(x)\,, \end{aligned} \tag{2.2}$$

JHEP09(2023)095

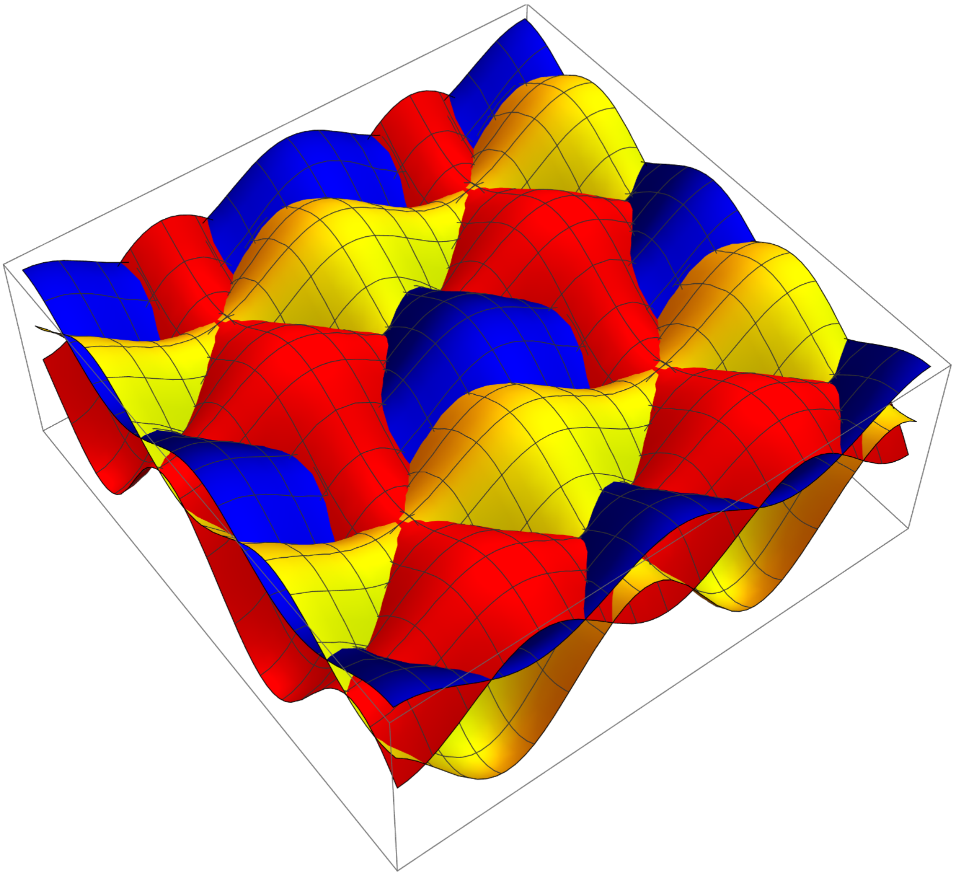

Figure 1. A $3D$ plot of the profile given by eq. (6.11), with $r = 3$, as a function of (x_1, x_2), for fixed (x_3, x_4). For better visualization, we show double the periods in x_1 and x_2. We see three solutions, in red, yellow, and blue, lumped around three distinct centers. These lumps, however, are not well-separated, comprising a liquid rather than a dilute gas. Earlier [9], similar configurations were constructed numerically and used to study confinement, see [3].

JHEP09(2023)095

from which we obtain the periodicity conditions on the transition functions Ω_μ (now giving up the short-hand notation and going back to the original Ω_μ that appears in (2.1))

$$\Omega_\mu(x + \hat{e}_\nu L_\nu)\,\Omega_\nu(x) = e^{i\frac{2\pi n_{\mu\nu}}{N}}\Omega_\nu(x + \hat{e}_\mu L_\mu)\,\Omega_\mu(x)\,. \tag{2.3}$$

Equation (2.3) is the cocycle conditions on the transition functions Ω_μ. The exponent $e^{i\frac{2\pi n_{\mu\nu}}{N}}$, with integers $n_{\mu\nu} = -n_{\nu\mu}$, is the $\mathbb{Z}_N$ center of $\mathrm{SU}(N)$. The freedom to introduce the center stems from the fact that both the transition function and its inverse appear in (2.1).

’t Hooft found a solution to the consistency conditions (2.3) carrying a fractional topological charge by embedding the $\mathrm{SU}(N)$ transition functions $\Omega_\mu(x)$ in $\mathrm{SU}(k)\times\mathrm{SU}(\ell)\times\mathrm{U}(1)\subset\mathrm{SU}(N)$, such that $N = k + \ell$ and writing them in the form

$$\Omega_\mu(x) = P_k^{s_\mu}Q_k^{t_\mu}\otimes P_\ell^{u_\mu}Q_\ell^{v_\mu}\;e^{i\omega\frac{\alpha_{\mu\lambda}x_\lambda}{L_\lambda}}\;. \tag{2.4}$$

Here, $s_\mu, t_\mu, u_\mu, v_\mu$ are integers, a sum over λ is implied in the exponent, and $\alpha_{\mu\lambda}$ is a real matrix with vanishing diagonal components without any (anti-)symmetry properties. The matrices P_k and Q_k (similarly the matrices P_ℓ and Q_ℓ) are the $k \times k$ (similarly $\ell \times \ell$) shift and clock matrices:

$$P_k = \gamma_k \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & & & \\ \dots & & 0 & 1 \\ 1 & 0 \dots & & 0 \end{bmatrix}, \quad Q_k = \gamma_k \, \mathrm{diag}\left[1, e^{\frac{i2\pi}{k}}, e^{2\frac{i2\pi}{k}}, \dots\right], \tag{2.5}$$

which satisfy the relation $P_k Q_k = e^{i\frac{2\pi}{k}} Q_k P_k$. The factor $\gamma_k \equiv e^{\frac{i\pi(1-k)}{k}}$ ensures that $\det Q_k = 1$ and $\det P_k = 1$.

In the rest of this paper, we take primed upper-case Latin letters to denote elements of $k \times k$ matrices: $C', D' = 1, 2, \dots, k$, and the unprimed upper-case Latin letters to denote $\ell \times \ell$ matrices: $C, D = 1, 2, \dots, \ell$. The matrices P_k and Q_k can then be written as $(P_k)_{B'C'} = \delta_{B',C'-1\,(\mathrm{mod}k)}$ and $(Q_k)_{C'B'} = \gamma_k\, e^{i2\pi\frac{C'-1}{k}} \delta_{C'B'}$. The matrix ω is the U(1) generator. It is given by

$$\omega = 2\pi \mathrm{diag}\left[\underbrace{\ell, \ell, \dots, \ell}_{k\,\mathrm{times}}, \underbrace{-k, -k, \dots, -k}_{\ell\,\mathrm{times}}\right], \tag{2.6}$$

and clearly commutes with P_k, P_ℓ, Q_k, Q_ℓ.

Writing the twist matrix $n_{\mu\nu}$ appearing in the cocycle condition (2.3) as $n_{\mu\nu} = n^{(1)}_{\mu\nu} + n^{(2)}_{\mu\nu}$, the antisymmetric part of the coefficients $\alpha_{\mu\nu}$ are taken to be

$$\alpha_{\mu\nu} - \alpha_{\nu\mu} = \frac{n^{(2)}_{\mu\nu}}{N\ell} - \frac{n^{(1)}_{\mu\nu}}{Nk}. \tag{2.7}$$

Recall that $\alpha_{\mu\nu}$ have vanishing diagonal elements; it is convenient, see section 3.1, to choose a particular form for their symmetric part, which amounts to a gauge choice.

A solution of the transition functions (2.4) obeying the cocycle conditions (2.3) with $\alpha_{\mu\nu}$ and $n_{\mu\nu}$ related as in (2.7) can be obtained provided that $s_\mu, t_\mu, u_\mu, v_\mu \in \mathbb{Z}$ can be found such that

$$n^{(1)}_{\mu\nu} = s_\mu t_\nu - s_\nu t_\mu + k A_{\mu\nu}, \quad n^{(2)}_{\mu\nu} = u_\mu v_\nu - v_\nu u_\mu + \ell B_{\mu\nu}, \tag{2.8}$$

where $A_{\mu\nu}$ and $B_{\mu\nu}$ are integers, and

$$n^{(1)}_{\mu\nu} \tilde{n}^{(1)}_{\mu\nu} = 0 \;(\mathrm{mod}\, k), \quad n^{(2)}_{\mu\nu} \tilde{n}^{(2)}_{\mu\nu} = 0 \;(\mathrm{mod}\, \ell), \tag{2.9}$$

and $\tilde{n}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} n_{\alpha\beta}$.

While the details of the derivation are not shown here (see [1]), the data we have given above suffice to check that upon plugging (2.9), (2.8), (2.7) into (2.4) one finds, using (2.6) and (2.5), that the cocycle conditions (2.3) are obeyed, with twist matrices $n_{\mu\nu} = n^{(1)}_{\mu\nu} + n^{(2)}_{\mu\nu}$.

An abelian gauge field configuration along the U(1) generator ω, which obeys the boundary conditions specified by the Ω_μ thus constructed, is given by the expression

$$A_\lambda = -\omega\left(\frac{\alpha_{\mu\lambda}x_\mu}{L_\mu L_\lambda} + \frac{z_\lambda}{L_\lambda}\right). \tag{2.10}$$

The corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is constant everywhere on $\mathbb{T}^4$:

$$F_{\mu\nu} = -\omega\frac{\alpha_{\mu\nu} - \alpha_{\nu\mu}}{L_\mu L_\lambda}. \tag{2.11}$$

The constants z_μ label the holonomies along the U(1) generator, which are translational moduli. This solution carries a fractional topological charge:

$$Q = -\frac{1}{4N}n_{\mu\nu}\tilde{n}_{\mu\nu} = -\frac{n_{12}n_{34} + n_{13}n_{42} + n_{14}n_{23}}{N}. \tag{2.12}$$

Without loss of generality, we can always assume $n_{13} = n_{42} = n_{14} = n_{23} = 0$. Thus, we only consider fluxes in the 1-2 and 3-4 planes. Then, a self-dual solution satisfies the relation $F_{12} = F_{34}$, from which one can find the ratio $\frac{L_1 L_2}{L_3 L_4}$ that defines the self-dual torus. The action of the self-dual solution is

$$S_0 = \frac{1}{2g^2}\int_{\mathbb{T}^4} \mathrm{tr}\,[F_{\mu\nu}F_{\mu\nu}] = \frac{8\pi^2|Q|}{g^2}. \tag{2.13}$$

3 Fermion zero modes in the $Q = \frac{r}{N}$ constant-flux background

In this section, we study the zero modes of the adjoint fermions in the constant-flux abelian background with topological charge $\frac{r}{N}$, described in section 3.1 (see eq. (3.2)). These results are useful when constructing the nonabelian self-dual solution with $Q = \frac{r}{N}$ on the deformed $\mathbb{T}^4$.

We find that there are $2\gcd(k,r)$ dotted (section 3.3) and $2\gcd(k,r)$ undotted (section 3.4.1) *constant* fermion zero modes. We also find $2r$ undotted adjoint fermion zero modes with nontrivial x-dependence (section 3.4.2, see eqs. (3.18)–(3.21) for the explicit solution and appendix A for the rather technical derivation). The latter are the ones determining the bosonic nonabelian self-dual background on the deformed torus in the Δ-expansion.

3.1 The solution with topological charge $Q = \frac{r}{N}$

A solution with topological charge $Q = \frac{r}{N}$ is obtained from section 2 by taking $n_{12}^{(1)} = -r, n_{12}^{(2)} = 0, n_{34}^{(1)} = 0, n_{34}^{(2)} = 1$, and, hence $n_{12} = -r, n_{34} = 1$. We also take $s_1 = -r, t_2 = 1, u_3 = v_4 = 1$ and set $A_{\mu\nu} = B_{\mu\nu} = 0$ and the rest of s_μ, t_μ, u_μ, and v_μ to zero. Thus, without loss of generality, we take $\alpha_{12} = \frac{r}{Nk}\,, \alpha_{21} = 0\,, \alpha_{34} = \frac{1}{N\ell}\,, \alpha_{43} = 0$.

The upshot is that the transition functions (2.4) now read

$$
\begin{aligned}
\Omega_1 &= P_k^{-r} \otimes I_\ell e^{i\omega \frac{r x_2}{N k L_2}} = \begin{bmatrix} P_k^{-r} e^{i 2\pi \ell r \frac{x_2}{N k L_2}} & 0 \\ 0 & e^{-i2\pi r \frac{x_2}{N L_2}} I_\ell \end{bmatrix}, \\
\Omega_2 &= Q_k \otimes I_\ell = \begin{bmatrix} Q_k & 0 \\ 0 & I_\ell \end{bmatrix}, \\
\Omega_3 &= I_k \otimes P_\ell e^{i\omega \frac{x_4}{N \ell L_4}} = \begin{bmatrix} e^{i2\pi \frac{x_4}{N L_4}} I_k & 0 \\ 0 & e^{-i2\pi k \frac{x_4}{N\ell L_4}} P_\ell \end{bmatrix}, \\
\Omega_4 &= I_k \otimes Q_\ell = \begin{bmatrix} I_k & 0 \\ 0 & Q_\ell \end{bmatrix}.
\end{aligned} \tag{3.1}
$$

where we recall that ω is given by (2.6), P and Q in (2.5), and I_k (I_ℓ) denote $k \times k$ ($\ell \times \ell$) unit matrices. Above, we introduced our $k \times \ell$ block-matrix notation, to be used further in this paper.

The reader can use (3.1), recalling that $k + \ell = N$, with k, ℓ being positive integers, and that P and Q are the clock and shift matrices (2.5), to explicitly check that Ω_μ obey the cocycle conditions (2.3) with only $n_{12} = -r$ and $n_{34} = 1$ being nonzero, and that these hold for any $1 \leq r \leq N$. Likewise, it is easy to check that the abelian gauge field and the field strength of the constant flux background

$$
\begin{aligned}
A_1 &= -\omega \frac{z_1}{L_1}, \; A_2 = -\omega \left(\frac{r x_1}{N k L_1 L_2} + \frac{z_2}{L_2} \right), \; A_3 = -\omega \frac{z_3}{L_3}, \; A_4 = -\omega \left(\frac{x_3}{N \ell L_3 L_4} + \frac{z_4}{L_4} \right) \\
F_{12} &= -\omega \frac{r}{N k L_1 L_2}, F_{34} = -\omega \frac{1}{N \ell L_3 L_4}.
\end{aligned} \tag{3.2}
$$

obey the boundary conditions (2.1) with transition functions (3.1).[4]

If we require the self-duality of the solution $F_{12} = F_{34}$, we find that a self-dual torus sides have to obey the relation

$$
\frac{L_1 L_2}{L_3 L_4} = \frac{r \ell}{k}. \tag{3.3}
$$

3.2 Boundary conditions for the adjoint fermions

In the rest of section 3, we solve the Weyl equations $D_\mu \bar{\sigma}_\mu \lambda = 0$ and $D_\mu \sigma_\mu \bar{\lambda} = 0$ for massless adjoint fermions in the background (3.2).[5] This will enable us to understand the fermionic zero modes in the background with topological charge $Q = \frac{r}{N}$ on the self-dual torus. In subsequent sections, the results help the construction of the self-dual bosonic background on the deformed torus in the small-Δ expansion.

Before we begin, let us discuss the moduli of the solution. We first note that the constant holonomies z_μ in the U(1) direction ω, appearing in (3.2), are the most general

[4] If one of k or ℓ is unity, the cocycle conditions with $n_{12} = -r$, $n_{34} = 1$ and the corresponding boundary conditions (2.1) are obeyed with the corresponding P and Q in Ω_μ replaced by unity.

[5] Here, $\sigma_\mu \equiv (i\vec{\sigma}, 1)$, $\bar{\sigma}_\mu \equiv (-i\vec{\sigma}, 1)$, $\vec{\sigma}$ are the Pauli matrices which determine the $\mu = 1, 2, 3$ components of the four-vectors $\sigma_\mu, \bar{\sigma}_\mu$. The Euclidean action for fermions and the matrices σ_μ, $\bar{\sigma}_\mu$, $\sigma_{\mu\nu}$, $\bar{\sigma}_{\mu\nu}$, are as in [10], except that we use hermitean gauge fields, necessitating the replacement $A^{\text{that ref.}} = i A^{\text{this paper}}$.

JHEP09(2023)095

ones commuting with the transition functions (3.1), provided $\gcd(k,r)=1$ (that this is so follows from the discussion below).

However, when $\gcd(k,r)>1$, there are $\gcd(k,r)$ different holonomies permitted for each μ. To work them out for future use, we first note that the holonomies have to be in the Cartan subalgebra, because they have to commute with Q_k and Q_l from (3.1) in order that (2.1) be obeyed. Thus, the additional (to z_μ from (3.2)) holonomies would add, to the background (3.2), $\delta A_\mu = H^{a'}\phi_\mu^{a'} + H^a \phi_\mu^a$, with constant ϕ's, where $H^{a'}$ ($a'=1,\ldots,k-1$) and H^a ($a=1,\ldots l-1$) are the SU(k) and SU(l) Cartan generators, respectively. The generators $H^{a'}$, H^a are extended to have zero entries in their respective complement to SU(N). In addition, $H^{a'}$ and H^a have to commute with the transition functions (3.1), which means that $P_k^{-r}H^{a'}P_k^r = H^{a'}$ and $P_l H^a P_l^{-1} = H^a$. Clearly, there are no nonzero SU(ℓ) generators H^a allowed, thus we set the corresponding holonomies to zero $\phi_\mu^a = 0$. The condition for $H^{a'}$ only allows nonzero $\phi_\mu^{a'}$ if $\gcd(k,r)>1$. If $\gcd(k,r)=k$, any Cartan generator obeys $P_k^{-r}H^{a'}P_k^r = H^{a'}$ and so there are $k-1$ $\phi_\mu^{a'}$'s allowed (for reasons that become clear later, we shall study this case in great detail in what follows). For generic values of $\gcd(k,r)$, $1<\gcd(k,r)\leq k$, there are only $\gcd(k,r)$ holonomies along the SU(k) Cartan generators allowed. Let us now describe them in a manner useful for the future.

For general values of $\gcd(r,k)$, we combine the allowed holonomies in the SU(k) part of SU(N) with the z_μ holonomies (the ones proportional to ω, see (3.2)). We use primed indices $C', B' = 1\ldots k$ to denote the $k\times k$ part of the components of the SU(N) gauge field and unprimed $C,B=1,\ldots \ell$ to denote the SU(ℓ) components. Thus, we describe the general abelian background (3.2) as

$$\hat{A}_\mu = (A_\mu)|_{\text{of eq. (3.2) with } z_\mu=0} + \begin{bmatrix} ||\delta A_{\mu\, C'B'}|| & 0 \\ 0 & ||\delta A_{\mu\, CB}|| \end{bmatrix}, \tag{3.4}$$

using the same block-matrix form as in (3.1), with, e.g. $||\delta A_{\mu\, C'B'}||$ denoting a $k\times k$ matrix with components $\delta A_{\mu\, C'B'}$, etc. All holonomies (including z_μ) are now included in the second term and are given by

$$\begin{aligned} \delta A_{\mu\, CD} &= \delta_{CD}\, 2\pi k \frac{z_\mu}{L_\mu}, \\ \delta A_{\mu\, C'D'} &= \delta_{C'D'}\left(-2\pi\ell\frac{z_\mu}{L_\mu} + \phi_\mu^{C'}\right), \\ \text{where } \phi_\mu^{C'} &= \phi_\mu^{C'-r(\text{mod } k)} \equiv \phi_\mu^{[C'-r]_k} \text{ and } \sum_{C'=1}^{k}\phi_\mu^{C'} = 0. \end{aligned} \tag{3.5}$$

The SU(k) holonomies, denoted by $\phi_\mu^{C'}$, must obey the condition from the last line to ensure commutativity with P_k^r. In (3.5) we also introduced the short-hand notation that we shall often use in this paper:[6]

$$[x]_q \equiv x(\text{mod } q)\,. \tag{3.6}$$

[6]Notice that, to conform to (3.6), in (3.5) and further, since $q(\text{mod}q)=0$, we take the range of the SU(k) index C' to be $0\ldots k-1$ instead of $1\ldots k$. Likewise, we take the range of the unprimed SU(ℓ) indices $0\ldots\ell-1$.

JHEP09(2023)095

We now turn to the adjoint fermions (gauginos), which obey the boundary conditions (2.1) without the inhomogeneous term

$$\lambda(x + L_\mu \hat{e}_\mu) = \Omega_\mu \lambda(x) \Omega_\mu^{-1}\,, \tag{3.7}$$

with Ω_μ from (3.1). Omitting the spinor index, we write the gaugino field, an $N \times N$ traceless matrix, as a block of $k \times k$, $k \times \ell$, $\ell \times k$ and $\ell \times \ell$ matrices (recall $N = k + \ell$):

$$\lambda = \begin{bmatrix} ||\lambda_{C'B'}|| & ||\lambda_{C'B}|| \\ ||\lambda_{CB'}|| & ||\lambda_{CB}|| \end{bmatrix}\,,\; C', B' \in \{0, \ldots k-1\},\; C, B \in \{0, \ldots \ell - 1\}\,, \tag{3.8}$$

obeying the tracelessness condition

$$\sum_{C'=0}^{k-1} \lambda_{C'C'} + \sum_{C=0}^{\ell-1} \lambda_{CC} = 0\,. \tag{3.9}$$

The explicit form of the boundary conditions follows from (3.7) and (3.8). For $\lambda_{C'B'}$, they are

$$\begin{aligned} \lambda_{C'B'}(x + L_1 \hat{e}_1) &= \lambda_{[C'-r]_k\, [B'-r]_k}(x)\,, \\ \lambda_{C'B'}(x + L_2 \hat{e}_2) &= e^{i 2\pi \frac{C'-B'}{k}} \lambda_{C'B'}(x)\,, \\ \lambda_{C'B'}(x + L_3 \hat{e}_3) &= \lambda_{C'B'}(x)\,, \\ \lambda_{C'B'}(x + L_4 \hat{e}_4) &= \lambda_{C'B'}(x)\,, \end{aligned} \tag{3.10}$$

while λ_{CB} obeys

$$\begin{aligned} \lambda_{CB}(x + L_1 \hat{e}_1) &= \lambda_{CB}(x)\,, \\ \lambda_{CB}(x + L_2 \hat{e}_2) &= \lambda_{CB}(x)\,, \\ \lambda_{CB}(x + L_3 \hat{e}_3) &= \lambda_{[C+1]_\ell\, [B+1]_\ell}(x)\,, \\ \lambda_{CB}(x + L_4 \hat{e}_4) &= e^{i 2\pi \frac{C-B}{\ell}} \lambda_{CB}(x)\,, \end{aligned} \tag{3.11}$$

and $\lambda_{C'B}$:

$$\begin{aligned} \lambda_{C'B}(x + L_1 \hat{e}_1) &= \gamma_k^{-r} e^{i 2\pi \frac{r x_2}{k L_2}} \lambda_{[C'-r]_k\, B}(x)\,, \\ \lambda_{C'B}(x + L_2 \hat{e}_2) &= \gamma_k e^{i 2\pi \frac{(C'-1)}{k}} \lambda_{C'B}(x)\,, \\ \lambda_{C'B}(x + L_3 \hat{e}_3) &= \gamma_\ell^{-1} e^{i 2\pi \frac{x_4}{\ell L_4}} \lambda_{C'[B+1]_\ell}(x)\,, \\ \lambda_{C'B}(x + L_4 \hat{e}_4) &= \gamma_\ell^{-1} e^{-i 2\pi \frac{(B-1)}{\ell}} \lambda_{C'B}(x)\,. \end{aligned} \tag{3.12}$$

We also note that $\lambda_{CB'}$ obeys the h.c. conditions to (3.12). In addition, the dotted fermions $\bar{\lambda}$ obey boundary conditions equal to the ones given above, written in terms of a decomposition of $\bar{\lambda}$ in terms of $\bar{\lambda}_{C'B'}$, $\bar{\lambda}_{C'B}$, $\bar{\lambda}_{CB}$ and $\bar{\lambda}_{CB'}$, identical to the one in (3.8).

We can now solve the Weyl equations $D_\mu \bar{\sigma}_\mu \lambda = 0$ and $D_\mu \sigma_\mu \bar{\lambda} = 0$ with the above boundary conditions. The covariant derivative is given by $D_\mu = \partial_\mu + i[A_\mu, \,]$ with A_μ already given in (3.4) and (3.5). We solve for the zero modes of the Weyl equation in the abelian background, beginning with the simplest cases.

3.3 Dotted-fermion zero modes

First, we solve the Weyl equation for the dotted fermions, $D_\mu \sigma^\mu \bar\lambda = 0$. Here, we ignore the allowed nonzero holonomies from (3.5), since (as we shall see later) they do not affect the solution in an interesting way. We find, keeping in mind the tracelessness condition (3.9),

$$\begin{aligned} \partial_\mu \sigma^\mu \bar\lambda_{CB\,\dot\alpha} = 0\,, \quad \partial_\mu \sigma^\mu \bar\lambda_{C'B'\,\dot\alpha} = 0, \;\; \text{with } \dot\alpha = \dot 1, \dot 2, \\ \left(\partial_3 - i\partial_4 - \frac{2\pi x_3}{\ell L_3 L_4}\right)\bar\lambda_{C'B\,\dot 1} + \left(\partial_1 - i\partial_2 - \frac{2\pi r x_1}{k L_1 L_2}\right)\bar\lambda_{C'B\,\dot 2} = 0\,, \\ \left(\partial_1 + i\partial_2 + \frac{2\pi r x_1}{k L_1 L_2}\right)\bar\lambda_{C'B\,\dot 1} + \left(-\partial_3 - i\partial_4 - \frac{2\pi x_3}{\ell L_3 L_4}\right)\bar\lambda_{C'B\,\dot 2} = 0\,, \end{aligned} \tag{3.13}$$

and similar equations for $\bar\lambda_{CB'\,\dot\alpha}$. One can convince themselves that there exist no normalizable solutions for $\bar\lambda_{C'B\,\dot\alpha}$ and $\bar\lambda_{CB'\,\dot\alpha}$ obeying the boundary conditions. We shall not repeat the details here but only note that this follows from the analysis of [6] and the realization that normalizability of the solutions on the four torus (after expanding in eigenmodes) ends up requiring normalizability of simple-harmonic oscillator wavefunctions, the solutions of (3.13), in the infinite x_1-x_3 plane (the two oscillators being in the x_1 and x_3 directions).

The only normalizable solution involves the diagonal components $\bar\lambda_{CC\,\dot\alpha}$ and $\bar\lambda_{C'C'\,\dot\alpha}$ and is constant. This is because the boundary conditions (3.11), (3.10) only allow for constant diagonal solutions and also further restrict the solutions as we now discuss. The boundary conditions for the $\ell \times \ell$-components only permit the solution

$$\bar\lambda_{CC\,\dot\alpha} = \bar\vartheta_{\dot\alpha}, \;\; \forall C = 0, \ldots, \ell - 1, \tag{3.14}$$

with equal diagonal entries. Here $\bar\vartheta_{\dot\alpha}$ are two Grassmann variables. The $k \times k$ part of the dotted fermions, $\bar\lambda_{C'C'\,\dot\alpha}$ allows for $\gcd(k,r)$ such solutions (due to the first boundary condition in (3.10)), which can be written as

$$\bar\lambda_{C'C'\,\dot\alpha} = \bar\vartheta_{\dot\alpha}^{[C'-r]_k}, \tag{3.15}$$

for arbitrary Grassmann $\bar\vartheta_{\dot\alpha}^{[C'-r]_k}$. Clearly, for every value of $\dot\alpha$, there are $\gcd(k,r)$ such different $\bar\vartheta_{\dot\alpha}^{[C'-r]_k}$, which one can label $\bar\vartheta^0_{\dot\alpha}$, $\bar\vartheta^1_{\dot\alpha}$ to $\ldots \bar\vartheta_{\dot\alpha}^{\gcd(k,r)-1}$. The tracelessness condition (3.9), however, determines the SU(ℓ) Grassmann variables (3.14) in terms of the SU(k) ones, (3.15).

In conclusion, there are a total of $2\gcd(k,r)$ dotted-fermion zero modes in the constant-flux instanton background.

3.4 Undotted-fermion zero modes

3.4.1 The "diagonal": U(1), SU(ℓ) and SU(k) undotted zero modes

Now, we continue with the undotted fermions λ_{BC} and $\lambda_{B'C'}$, i.e. their componets in the U(1), SU(k) and SU(ℓ) directions. Because the abelian background (3.4), (3.5) commutes with the U(1), SU(k) and SU(ℓ) generators, these "diagonal" components satisfy a free Dirac equation:

$$\begin{aligned} \partial_\mu \bar\sigma_\mu \lambda_{C'B'} &= 0, \\ \partial_\mu \bar\sigma_\mu \lambda_{CB} &= 0, \;\; \text{with } \sum_{C'=0}^{k-1} \lambda_{C'C'} + \sum_{C=0}^{\ell-1} \lambda_{CC} = 0\,, \end{aligned} \tag{3.16}$$

along with the SU(N) tracelessness condition (3.9).

One needs to solve these equations with the boundary conditions (3.10) and (3.11). We now state the results, since the analysis is similar to that in [6, 11]. The first remark is that, following the steps outlined for the dotted zero modes, one finds that there are no normalizable solutions for the components of $\lambda_{C'B'}$ and λ_{CB} with $C' \neq B'$ and $C \neq B$ obeying the boundary conditions.

Next, we note that the only solution for λ_{CC} is the one where $\lambda_{CC\,\alpha} = \eta_\alpha$, with a constant spinor η_α, for all C (this is needed to satisfy (3.11)). The tracelessness condition in (3.16), however, relates this to the $\lambda_{B'B'}$ solutions on which we now focus. The boundary conditions (3.10) are satisfied by the constant solutions

$$\lambda_{B'C'\,\alpha} = \delta_{B'C'} \sum_{j=0}^{\mathrm{gcd(k,r)}-1} \vartheta_\alpha^{(j)} \sum_{n=0}^{\frac{k}{\mathrm{gcd}(k,r)}-1} \delta_{B',[j+nr]_k}, \tag{3.17}$$

with $\mathrm{gcd}(k,r)$ arbitrary constant Grassmann spinors $\vartheta_\alpha^{(j)}$. We conclude that there are $2\mathrm{gcd}(k,r)$ independent zero modes of $\lambda_{B'C'}$ and, from the above remarks, of the all "diagonal" components of the undotted fermions considered in this section.

Note that the number of diagonal undotted zero modes is precisely the same as the number of the dotted fermion zero modes of section 3.3. In particular, the contribution of the zero modes of sections 3.3 and 3.4.1 to the index cancels out.

3.4.2 The "off-diagonal" $k \times \ell$ and $\ell \times k$ undotted zero modes.

The zero modes most worthy of our attention, the ones which determine the nonabelian instanton solution to leading order in Δ, are the ones considered in this section. Finding the off-diagonal undotted zero modes, the ones for $\lambda_{C'B}$ ($k \times \ell$) and $\lambda_{CB'}$ ($\ell \times k$), is the most important and least trivial part of our study. We find that there are r zero modes for $\lambda_{C'B}$ and r zero modes for $\lambda_{CB'}$, in agreement with the index theorem which requires that the number of undotted minus the number of dotted zero modes be $2r$.

The derivation of the results quoted in this section is technically involved and the details are relegated to appendix A. Here, we simply give the explicit formulae for the zero modes for $\lambda_{C'B}$, the $k \times \ell$ ones.[7] We find that in the background (3.4), (3.5), only one spinor component has r normalizable zero modes

$$\begin{aligned} \lambda_{C'B\,1} &= \sum_{p=0}^{\frac{r}{\mathrm{gcd}(k,r)}-1} \eta^{[C'+pk]_r}\, \Phi^{(p)}_{C'B}(x,\hat{\phi}), \\ \lambda_{C'B\,2} &= 0\,. \end{aligned} \tag{3.18}$$

Here, η^j, $j = 0, \ldots, r-1$, are r Grassmann parameters associated with the zero modes (clearly, $[C'+pk]_r$ takes r values). Notice that a given zero mode, proportional to η^j with some $j \in \{0, \ldots, r-1\}$, nontrivially intertwines the gauge indices in (3.18).

Before giving the form of the functions $\Phi^{(p)}$ governing the x-dependence of the zero modes (3.18), we introduce the notation $\hat{\phi}_\mu^{C'}$ to denote the way various gauge field

[7] Noting that the $\ell \times k$ zero modes (which come with their own Grassmann parameters) are obtained by hermitean conjugation of $\Phi^{(p)}$ in (3.18), as per the remark after (3.12).

holonomies appear in the equations governing the off diagonal zero modes. These combine the U(1)-holonomy z_μ with the extra ones allowed when $\gcd(k,r) > 1$, as per the discussion around (3.5):[8]

$$\hat{\phi}_\mu^{C'} \equiv \phi_\mu^{C'} - 2\pi N \frac{z_\mu}{L_\mu}, \text{ with } \hat{\phi}_\mu^{C'} = \hat{\phi}_\mu^{[C'-r]_k}. \tag{3.19}$$

The explicit solution for $\hat{\phi}^{C'}$ obeying the relations above (and from (3.5)) can be written out in a somewhat unwieldy form (which, however, serves to show that there are $\gcd(k,r)$ independent holonomies for each μ)[9]

$$\hat{\phi}_\mu^{C'} = \sum_{j=0}^{\gcd(k,r)-1} \varphi_\mu^j \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1} \delta^{C',[j+nr]_k}\,. \tag{3.20}$$

Here, we use the notation (3.6), taking the range of C' to be $0 \ldots k-1$. The sum over n for each j simply incorporates the fact that the index C' takes values an "orbit" of size $\frac{k}{\gcd(k,r)}$. Each of the $\gcd(k,r)$ "orbits," labelled by j, has the same holonomy ϕ_μ^j and contains values of C' jumping by r units, as required by commutativity of the holonomy with P_k. Although (3.20) explicitly shows that, for each μ, there are $\gcd(k,r)$ independent holonomies φ_μ^j, we prefer to further denote them as $\hat{\phi}_\mu^{C'}$, remembering the relations they obey. However, we make explicit use of (3.20) later on, see section 5.

The zero modes $\lambda_{C'B\,1}$ of (3.18) depend on $(x, \hat{\phi}^{C'}, \eta^j)$. Their x- and $\hat{\phi}$-dependence is through the $\frac{r}{\gcd(k,r)}$ functions $\Phi^{(p)}$, given by (for derivation, see appendix A):

$$\begin{aligned}\Phi^{(p)}_{C'B}(x,\hat{\phi}) = &\sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \;\sum_{n'\in\mathbb{Z}} e^{\frac{i2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{i2\pi x_4}{L_4}\left(n'-\frac{2B-1-\ell}{2\ell}\right)} \\ &\times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} e^{i\frac{\pi(1-\ell)}{\ell}\left(B-\frac{1+\ell(2n'+1)}{2}\right)} \\ &\times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\left(\hat{\phi}_2^{[C']_r}-i\hat{\phi}_1^{[C']_r}\right)-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\ &\times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\left(\hat{\phi}_4^{[C']_r}-i\hat{\phi}_3^{[C']_r}\right)-L_3\left(\ell n'-\frac{2B-1-\ell}{2}\right)\right]^2}.\end{aligned} \tag{3.21}$$

The explicit form of the functions $\Phi^{(p)}$ will be useful later, in our study of the properties of the self-dual fractional instantons on the deformed torus. Eqs. (3.18), (3.19), (3.21) give the general normalizable solution of the massless undotted Weyl equation $D_\mu \bar{\sigma}_\mu \lambda = 0$ for $\lambda_{C'B\,\alpha}$ in the abelian constant-field strength background (3.4), (3.5) of topological charge $Q = \frac{r}{N}$.

In summary of section 3, we found that there is a number of dotted and undotted zero modes in the abelian background of topological charge $\frac{r}{N}$. The total number is consistent with the index theorem. The solutions for the non-constant fermion zero modes will be used to construct the nonabelian self-dual solution of charge $\frac{r}{N}$ on the deformed torus.

[8]The reason that $2\pi N$ (and not $2\pi\ell$) appears here is that $\hat{\phi}^{C'}$ encodes the action of the commutator on the off diagonal components $\lambda_{C'B}$.

[9]We note that this is similar to eq. (3.17) for the undotted diagonal zero modes of the next section.

4 Deforming the self-dual torus: small-Δ expansion for the bosonic background with $Q = \frac{r}{N}$

To remedy the zero modes problem we saw in the previous section, i.e., to lift the dotted zero modes, we now depart from the self-dual torus and search for a self-dual instanton solution with topological charge $Q = \frac{r}{N}$ on a deformed $\mathbb{T}^4$, following the strategy of [7, 8]. We write the general gauge field on the non-self-dual torus in the form

$$A_\mu(x) = \hat{A}_\mu + \mathcal{S}^\omega_\mu(x)\,\omega + \delta_\mu(x)\,. \tag{4.1}$$

Here, ω is the U(1) generator (2.6), $\hat{A}_\mu$ is the abelian gauge field with constant field strength defined previously in (3.4) and $\mathcal{S}^\omega_\mu(x)$ is the nonconstant field component along the U(1) generator. The non-abelian part $\delta_\mu(x)$ is given by the $N\times N$ matrix, which, as earlier in (3.1), (3.4), (3.8), is decomposed in a block form:[10]

$$\delta_\mu = \begin{bmatrix} \mathcal{S}^k_\mu & \mathcal{W}^{k\times\ell}_\mu \\ \mathcal{W}^{\dagger\ell\times k}_\mu & \mathcal{S}^\ell_\mu \end{bmatrix} \quad \left(\equiv \begin{bmatrix} ||\mathcal{S}^k_{\mu\,B'C'}|| & ||\mathcal{W}_{\mu\,B'C}|| \\ ||(\mathcal{W}^\dagger_\mu)_{CB'}|| & ||\mathcal{S}^\ell_{\mu\,BC}|| \end{bmatrix}\right)\,. \tag{4.2}$$

The boundary conditions (2.1) with transition functions (3.1) imply that $\mathcal{S}^\omega_\mu$ satisfy periodic boundary conditions in all directions (because $\hat{A}_\mu$ absorbs the inhomogenous part of (2.1)):

$$\mathcal{S}^\omega_\mu(x + \hat{e}_\nu L_\nu) = \mathcal{S}^\omega_\mu(x)\,. \tag{4.3}$$

On the other hand, $\mathcal{S}^k_\mu$, $\mathcal{S}^\ell_\mu$, $\mathcal{W}^{k\times\ell}_\mu$, and $\mathcal{W}^{\dagger\ell\times k}_\mu$ satisfy exactly the same gaugino-field boundary conditions we discussed in the previous section, and we refrain from repeating (thus, the boundary conditions are given by equations (3.10), (3.11), (3.12), respectively, for $\mathcal{S}^k_\mu$, $\mathcal{S}^\ell_\mu$, $\mathcal{W}^{k\times\ell}_\mu$, recalling (4.2) and Footnote 10).

The field strength of (4.1), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, is given by

$$\begin{aligned} F_{\mu\nu} &= \hat{F}_{\mu\nu} + F^s_{\mu\nu}\omega + \hat{D}_\mu\delta_\nu - \hat{D}_\nu\delta_\mu + i\left[\mathcal{S}^\omega_\mu\omega, \delta_\nu\right] + i\left[\delta_\mu, \mathcal{S}^\omega_\nu\omega\right] + i[\delta_\mu, \delta_\nu]\,, \\ &\equiv \hat{F}_{\mu\nu} + F^s_{\mu\nu}\omega + \begin{bmatrix} F^k_{\mu\nu} & \mathcal{F}^{k\times\ell}_{\mu\nu} \\ \mathcal{F}^{\dagger\ell\times k}_{\mu\nu} & F^\ell_{\mu\nu} \end{bmatrix}\,, \end{aligned} \tag{4.4}$$

where $\hat{D}_\mu = \partial_\mu + i[\hat{A}_\mu, \;]$ is the covariant derivative w.r.t. the gauge field $\hat{A}_\mu$. Using (4.1), (4.2), we obtain:

$$\begin{aligned} F^s_{\mu\nu} &= \partial_\mu\mathcal{S}^\omega_\nu - \partial_\nu\mathcal{S}^\omega_\mu\,, \\ F^k_{\mu\nu} &= \partial_\mu\mathcal{S}^k_\nu - \partial_\nu\mathcal{S}^k_\mu + i[\mathcal{S}^k_\mu, \mathcal{S}^k_\nu] + i\mathcal{W}^{k\times\ell}_\mu\mathcal{W}^{\dagger\ell\times k}_\nu - i\mathcal{W}^{k\times\ell}_\nu\mathcal{W}^{\dagger\ell\times k}_\mu\,, \\ F^\ell_{\mu\nu} &= \partial_\mu\mathcal{S}^\ell_\nu - \partial_\nu\mathcal{S}^\ell_\mu + i[\mathcal{S}^\ell_\mu, \mathcal{S}^\ell_\nu] + i\mathcal{W}^{\dagger\ell\times k}_\mu\mathcal{W}^{k\times\ell}_\nu - i\mathcal{W}^{\dagger\ell\times k}_\nu\mathcal{W}^{k\times\ell}_\mu\,, \\ \mathcal{F}^{k\times\ell}_{\mu\nu} &= \hat{D}_\mu\mathcal{W}^{k\times\ell}_\nu - \hat{D}_\nu\mathcal{W}^{k\times\ell}_\mu + i\mathcal{S}^k_\mu\mathcal{W}^{k\times\ell}_\nu - i\mathcal{S}^k_\nu\mathcal{W}^{k\times\ell}_\mu + i\mathcal{W}^{k\times\ell}_\mu\mathcal{S}^\ell_\nu - i\mathcal{W}^{k\times\ell}_\nu\mathcal{S}^\ell_\mu \\ &\quad + i2\pi N\left(\mathcal{S}^\omega_\mu\mathcal{W}^{k\times\ell}_\nu - \mathcal{S}^\omega_\nu\mathcal{W}^{k\times\ell}_\mu\right)\,, \end{aligned} \tag{4.5}$$

[10] Here $\mathcal{S}^k_\mu$ and $\mathcal{S}^l_\mu$ are traceless $su(k)$- and $su(l)$-algebra elements, respectively, while $\mathcal{W}^{k\times\ell}_\mu$ is a complex $k\times\ell$ matrix with $\mathcal{W}^{\dagger\ell\times k}_\mu$ its hermitean conjugate. In the second (bracketed) term in (4.2) we have indicated the index notation used earlier in describing the zero modes of the adjoint fermions, recall (3.18). Here, we find it convenient to use the block matrix notation $S^k, S^\ell, W^{k\times\ell}, W^{\dagger\,\ell\times k}$ and will revert to using indices $B'C', B'C$, etc., when needed.

where $\hat{D}_\mu \mathcal{W}_\nu^{k\times\ell}$ is understood as

$$\hat{D}_\mu \mathcal{W}_\nu^{k\times\ell} = \left[\partial_\mu + i2\pi N \hat{A}_\mu^\omega\right] \mathcal{W}_\nu^{k\times\ell}\,, \tag{4.6}$$

and we have written $\hat{A}_\mu = \hat{A}_\mu^\omega \omega$, for $\hat{A}_\mu$ from (3.4).[11] Similarly,

$$\hat{D}_\mu \mathcal{W}_\nu^{\dagger\ell\times k} = \left[\partial_\mu - i2\pi N \hat{A}_\mu^\omega\right] \mathcal{W}_\nu^{\dagger\ell\times k}\,. \tag{4.7}$$

Next, we impose self duality on the background (4.1) on the deformed $\mathbb{T}^4$. Imposing self-duality is equivalent (see e.g. [10]) to imposing the constraint on the field strength

$$\bar{\sigma}_{\mu\nu} F_{\mu\nu} = 0\,. \tag{4.8}$$

where[12] $\bar{\sigma}_{\mu\nu} = \frac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)$. Now, we recall $\hat{F}_{\mu\nu} = \hat{F}^\omega_{\mu\nu}\omega$, and use (3.2) to find $\hat{F}^\omega_{12} = -\frac{r}{NkL_1L_2}$ and $\hat{F}^\omega_{34} = -\frac{1}{N\ell L_3 L_4}$. Recalling the properties of the self-dual $\mathbb{T}^4$, eq. (3.3), we also define the parameter Δ, which parametrizes the deviation from the self-dual torus:

$$\Delta \equiv \frac{r\ell L_3 L_4 - kL_1L_2}{\sqrt{V}}\,. \tag{4.9}$$

We assume, without loss of generality, $\Delta \geq 0$. Thus, we find that

$$\hat{F}^\omega_{\mu\nu}\bar{\sigma}_{\mu\nu} = -\frac{2i\Delta}{Nk\ell\sqrt{V}}\sigma_3\,. \tag{4.10}$$

To continue, for every four-vector $\mathcal{V}_\mu$, we define the quaternions $\mathcal{V} \equiv \sigma_\mu \mathcal{V}_\mu$ and $\bar{\mathcal{V}} \equiv \bar{\sigma}_\mu \mathcal{V}_\mu$. Then, using (4.5) and (4.10), we find that self-duality (4.8) implies that

$$\frac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu} = \left(-\frac{i\Delta}{Nk\ell\sqrt{V}}\sigma_3 + \bar{\partial}\mathcal{S}^\omega - \partial_\mu \mathcal{S}^\omega_\mu\right)\omega + \begin{bmatrix} \mathcal{A}^k & \mathcal{A}^{k\times\ell} \\ \mathcal{A}^{\dagger\ell\times k} & \mathcal{A}^\ell \end{bmatrix} = 0\,, \tag{4.11}$$

where[13]

$$\begin{aligned}
\mathcal{A}^k &= \bar{\partial}\mathcal{S}^k - \partial_\mu \mathcal{S}^k_\mu - i\bar{\mathcal{S}}^k\mathcal{S}^k + i\mathcal{S}^k_\mu\mathcal{S}^k_\mu + i\bar{\mathcal{W}}^{k\times\ell}\mathcal{W}^{\dagger\ell\times k} - i\mathcal{W}^{k\times\ell}_\mu\mathcal{W}^{\dagger\ell\times k}_\mu\,,\\
\mathcal{A}^{k\times\ell} &= \bar{\hat{D}}\mathcal{W}^{k\times\ell} - \hat{D}_\mu\mathcal{W}^{k\times\ell}_\mu + i\bar{\mathcal{S}}^k\mathcal{W}^{k\times\ell} - i\mathcal{S}^k_\mu\mathcal{W}^{k\times\ell}_\mu + i\bar{\mathcal{W}}^{k\times\ell}\mathcal{S}^\ell - i\mathcal{W}^{k\times\ell}_\mu\mathcal{S}^\ell_\mu\\
&\quad + i2\pi N\left(\bar{\mathcal{S}}^\omega\mathcal{W}^{k\times\ell} - \mathcal{S}^\omega_\mu\mathcal{W}^{k\times\ell}_\mu\right)\,,\\
\mathcal{A}^\ell &= \bar{\partial}\mathcal{S}^\ell - \partial_\mu\mathcal{S}^\ell_\mu - i\bar{\mathcal{S}}^\ell\mathcal{S}^\ell + i\mathcal{S}^\ell_\mu\mathcal{S}^\ell_\mu + i\bar{\mathcal{W}}^{\dagger\ell\times k}\mathcal{W}^{k\times\ell} - i\mathcal{W}^{\dagger\ell\times k}_\mu\mathcal{W}^{k\times\ell}_\mu\,.
\end{aligned} \tag{4.12}$$

In order to remove gauge redundancies, we impose the background gauge condition with respect to the field $\hat{A}_\mu$:

$$\hat{D}_\mu A_\mu = 0 \tag{4.13}$$

[11]For brevity, the nontrivial holonomies' (allowed when $\gcd(k,r) > 1$) are not explicitly shown here. They should, however, be included in the covariant derivatives in (4.6), (4.7) and our final solution (4.21) does take these into account.

[12]Recall that the matrices σ_μ, $\bar{\sigma}_\mu$ were defined in Footnote 5.

[13]Here and below, the terms that have sums over μ should be multiplied by unit quaternion σ_4, which we have omitted for brevity. Thus, temporarily not denoting explicitly that these are $k\times\ell$ matrices, we warn the reader to keep in mind the difference between the quaternions, $\mathcal{W} \equiv \mathcal{W}_\mu\sigma_\mu$, $\bar{\mathcal{W}} = \bar{\sigma}_\mu W_\mu$, and the four-vector $\mathcal{W}_\mu$ and, furthermore, note that $\mathcal{W}^\dagger = \sigma_\mu W^\dagger_\mu$ and $\bar{\mathcal{W}}^\dagger = \bar{\sigma}_\mu W^\dagger_\mu$.

which in components reads:

$$\partial_\mu \mathcal{S}^\omega_\mu = 0\,, \partial_\mu \mathcal{S}^k_\mu = 0\,, \partial_\mu \mathcal{S}^\ell_\mu = 0\,, \hat{D}_\mu \mathcal{W}^{k\times\ell}_\mu = 0\,, \hat{D}_\mu \mathcal{W}^{\dagger \ell\times k}_\mu = 0\,. \tag{4.14}$$

Using (4.14) in (4.11), we find the set of equations imposing the self-duality condition on the background (4.1):

$$\begin{aligned}
\left(-\frac{i2\pi\Delta}{Nk\sqrt{V}}\sigma_3 + 2\pi\ell\bar{\partial}\mathcal{S}^\omega\right) I_k + \bar{\partial}\mathcal{S}^k - i\bar{\mathcal{S}}^k\mathcal{S}^k + i\mathcal{S}^k_\mu\mathcal{S}^k_\mu + i\bar{\mathcal{W}}^{k\times\ell}\mathcal{W}^{\dagger\ell\times k} - i\mathcal{W}^{k\times\ell}_\mu\mathcal{W}^{\dagger\ell\times k}_\mu &= 0\,,\\
\left(\frac{i2\pi\Delta}{N\ell\sqrt{V}}\sigma_3 - 2\pi k\bar{\partial}\mathcal{S}^\omega\right) I_\ell + \bar{\partial}\mathcal{S}^\ell - i\bar{\mathcal{S}}^\ell\mathcal{S}^\ell + i\mathcal{S}^\ell_\mu\mathcal{S}^\ell_\mu + i\bar{\mathcal{W}}^{\dagger\ell\times k}\mathcal{W}^{k\times\ell} - i\mathcal{W}^{\dagger\ell\times k}_\mu\mathcal{W}^{k\times\ell}_\mu &= 0\,,\\
\bar{\hat{D}}\mathcal{W}^{k\times\ell} + i\bar{\mathcal{S}}^k\mathcal{W}^{k\times\ell} - i\mathcal{S}^k_\mu\mathcal{W}^{k\times\ell}_\mu + i\bar{\mathcal{W}}^{k\times\ell}\mathcal{S}^\ell - i\mathcal{W}^{k\times\ell}_\mu\mathcal{S}^\ell_\mu + i2\pi N\left(\bar{\mathcal{S}}^\omega\mathcal{W}^{k\times\ell} - \mathcal{S}^\omega_\mu\mathcal{W}^{k\times\ell}_\mu\right) &= 0\,.
\end{aligned} \tag{4.15}$$

We note that here $\bar{\hat{D}} \equiv \bar{\sigma}_\mu \hat{D}_\mu$, precisely the Weyl operator for the undotted fermions, whose zero modes were studied in section 3.4.

The idea of the method introduced in [7] is that a solution of the self-duality conditions (4.15) can be obtained via series expansions in the deformation parameter Δ of (4.9). The approximate solution of the self-duality equations thus obtained is then also an approximation to the minimal action solution of the equations of motion, i.e. a fractional instanton with $Q = \frac{r}{N}$. Comparing the Δ scaling of the various terms in (4.15), the Δ-expansion is found to have the following form

$$\begin{aligned}
\mathcal{W}^{k\times\ell} &= \sqrt{\Delta}\sum_{a=0}^{\infty}\Delta^a\mathcal{W}^{(a)k\times\ell}\,,\\
\mathcal{S} &= \Delta\sum_{a=0}^{\infty}\Delta^a\mathcal{S}^{(a)}\,,
\end{aligned} \tag{4.16}$$

where $\mathcal{S}$ accounts for $\mathcal{S}^\omega$, $\mathcal{S}^k$, and $\mathcal{S}^\ell$.

We proceed to leading order[14] in Δ by considering solutions of $\mathcal{W}^{k\times\ell}$ to order $\sqrt{\Delta}$ and $\mathcal{S}$ to order Δ, thus keeping only the terms $\mathcal{S}^{(0)}$ and $\mathcal{W}^{(0)}$ in (4.16). Then, to this order, (4.15) gives

$$\begin{aligned}
\left(-\frac{i2\pi}{Nk\sqrt{V}}\sigma_3 + 2\pi\ell\bar{\partial}\mathcal{S}^{(0)\omega}\right) I_k + \bar{\partial}\mathcal{S}^{(0)k} + i\bar{\mathcal{W}}^{(0)k\times\ell}\mathcal{W}^{\dagger(0)\ell\times k} - i\mathcal{W}^{(0)k\times\ell}_\mu\mathcal{W}^{\dagger(0)\ell\times k}_\mu &= 0\,,\\
\left(\frac{i2\pi}{N\ell\sqrt{V}}\sigma_3 - 2\pi k\bar{\partial}\mathcal{S}^{(0)\omega}\right) I_\ell + \bar{\partial}\mathcal{S}^{(0)\ell} + i\bar{\mathcal{W}}^{\dagger(0)\ell\times k}\mathcal{W}^{(0)k\times\ell} - i\mathcal{W}^{\dagger(0)\ell\times k}_\mu\mathcal{W}^{(0)k\times\ell}_\mu &= 0\,,
\end{aligned} \tag{4.17}$$

and

$$\bar{\hat{D}}\mathcal{W}^{(0)k\times\ell} = 0\,. \tag{4.18}$$

The strategy of solving the leading-order equations (4.17), (4.18) is as follows:

[14] The Δ expansion was tested to high orders, and found to converge (even to the infinite volume limit) in the two dimensional abelian Higgs model in [12]. Convergence is not well understood for the general case of SU(N) in four dimensions. For SU(2), the comparisons with the exact numerical solution (obtained by minimizing the lattice Yang-Mills action) of [7] give evidence for the convergence of the expansion for small Δ. It should be possible to analytically study the properties of higher orders in the expansion (4.16) of the solutions of (4.15); however, this rather formidable task is left for the future.

1. Solve (4.18) for the quaternions $\mathcal{W}^{(0)k\times\ell}$. This equation has the form of two copies of the undotted fermion zero-mode equation, whose general normalizable solutions were already found in section 3.4.2, recall (3.18).

2. Next, plug the general solution of (4.18) into (4.17). The result is a set of first-order differential equations for the quaternions $\mathcal{S}^{(0)}$, with periodic boundary conditions for $\mathcal{S}^{(0)\omega}$ and with $\mathcal{S}^{(0)k}, \mathcal{S}^{(0)\ell}$, obeying (3.10), (3.11), respectively. The resulting equations for $\mathcal{S}^{(0)}$ have nonvanishing source terms, comprised of a constant piece (the one proportional to σ_3 in (4.17)) and of terms quadratic in the just-found general solution of (4.18), $\mathcal{W}^{(0)k\times\ell}$. Consistency of these equations requires that the source term be orthogonal to the zero modes of the differential operator acting on the various components of $\mathcal{S}^{(0)}$.

3. One then needs to determine the zero modes of $\bar{\partial}$, the operator acting on $\mathcal{S}^{(\prime)}$, obeying the appropriate boundary conditions. This task was already accomplished in section 3.4.1, since $\bar{\partial}$ is simply the undotted diagonal Weyl operator. We then require orthogonality of these zero modes to the source terms in (4.17). On one hand, this will be shown to provide restrictions on the arbitrary coefficients appearing in the general solution of (4.18), $\mathcal{W}^{(0)k\times\ell}$. The coefficients left arbitrary determine the moduli space of the multi-fractional instanton. On the other hand, imposing consistency of (4.17) allows one to determine $\mathcal{S}^{(0)}$ by expanding both sides in a chosen basis of functions and equating the coefficients on both sides.

The procedure outlined above can be, in principle, iterated to higher orders. The way this procedure works to higher orders was, in principle, studied in [12]. However, implementing it to determine the higher-order solution becomes technically challenging. Here, we shall only study the leading-order and determine the constraints of the arbitrary coefficients in $\mathcal{W}^{(0)\,k\times\ell}$, which restrict the moduli space of the multi-fractional instantons.

To begin implementing the above steps, we start with (4.18), written explicitly as

$$\bar{\sigma}_\mu \hat{D}_\mu \begin{bmatrix} \mathcal{W}_4^{(0)k\times\ell} + i\mathcal{W}_3^{(0)k\times\ell} & \mathcal{W}_2^{(0)k\times\ell} + i\mathcal{W}_1^{(0)k\times\ell} \\ -\mathcal{W}_2^{(0)k\times\ell} + i\mathcal{W}_1^{(0)k\times\ell} & \mathcal{W}_4^{(0)k\times\ell} - i\mathcal{W}_3^{(0)k\times\ell} \end{bmatrix} = 0\,, \tag{4.19}$$

where $\hat{D}_\mu = \partial_\mu + i\,[\hat{A}_\mu,]$ is the covariant derivative in the background (3.4). As already stated, (4.19) represent two copies of the undotted gaugino zero mode equations in the $\Delta = 0$ background A^ω, one for each column of the $\mathcal{W}$-quaternion given above. Further, as for the gauginos, one can show that normalizability on $\mathbb{T}^4$ requires normalizability in the infinite x_1, x_3 plane of the simple harmonic oscillator wave functions, the solutions of (4.19). Thus, we borrow the solutions for the gauginos from section 3.4.2, we find that equations (4.19) have normalizable solutions if and only if

$$\mathcal{W}_4^{(0)k\times\ell} = i\mathcal{W}_3^{(0)k\times\ell}\,, \quad \mathcal{W}_2^{(0)k\times\ell} = i\mathcal{W}_1^{(0)k\times\ell} \tag{4.20}$$

noting that these are nothing but the conditions of vanishing of $\lambda_{C'B\,2}$, recall (3.18). The solutions for $\mathcal{W}_4^{(0)k\times\ell}, \mathcal{W}_2^{(0)k\times\ell}$ are then borrowed from (3.18):[15]

$$\left(\mathcal{W}_2^{(0)k\times\ell}\right)_{C'C} = V^{-1/4} \sum_{p=0}^{\frac{r}{\gcd(k,r)}-1} \mathcal{C}_2^{[C'+pk]_r} \Phi^{(p)}_{C'C}(x,\hat{\phi}) =: W_{2\,C'C}\,,$$
$$\left(\mathcal{W}_4^{(0)k\times\ell}\right)_{C'C} = V^{-1/4} \sum_{p=0}^{\frac{r}{\gcd(k,r)}-1} \mathcal{C}_4^{[C'+pk]_r} \Phi^{(p)}_{C'C}(x,\hat{\phi}) =: W_{4\,C'C}\,, \tag{4.21}$$

where $\Phi^{(p)}_{C'C}(x,\hat{\phi})$ are given by (3.21) and the volume factor is included for future convenience. Thus, there are $2r$ arbitrary coefficients $\mathcal{C}_2^{[C'+pk]_r}$ and $\mathcal{C}_4^{[C'+pk]_r}$, which are now complex bosonic variables. In the following, we shall discuss the physical significance of $\mathcal{C}_{2,4}$.

We now continue with the next step: imposing orthogonality to the various zero modes of $\bar{\partial} = \bar{\sigma}_\mu \partial_\mu$, the solutions of the equation $\bar{\partial}\mathcal{S}^{(0)} = 0$. Notice that $\bar{\partial}$ is precisely the Weyl operator for the diagonal undotted fermions discussed in section 3.4.1 and that we shall borrow our results from that section shortly. To continue, however, it is convenient to rewrite (4.17) using the index notation, recalling eq. (4.2) and Footnote 10. This necessitates using (4.20) and the definition of the quaternions, in order to express everything through the general solutions of (4.18), denoted by $W_{4\,(\text{or}2)\,C'C}$ of (4.21). This produces, from the first equation of (4.17), an equation determining $\mathcal{S}_{C'B'}$ (which includes the component $\mathcal{S}^\omega\omega$ from (4.1)):

$$\begin{aligned}&\bar{\partial}\mathcal{S}_{C'B'} \\ &= i\begin{pmatrix} \frac{2\pi}{Nk\sqrt{V}}\delta_{C'B'} - 2\,(W_2W_2^* - W_4W_4^*)_{C'B'} & 4\,(W_2W_4^*)_{C'B'} \\ 4\,(W_2W_4^*)_{C'B'} & -\frac{2\pi}{Nk\sqrt{V}}\delta_{C'B'} + 2\,(W_2W_2^* - W_4W_4^*)_{C'B'}\end{pmatrix},\end{aligned} \tag{4.22}$$

where we introduced the shorthand notation, $(W_2W_4^*)_{C'B'} \equiv W_{2\,C'D}W^*_{4\,B'D}$, with a sum over D implied, and similar for the other contractions. Likewise, the equation for $\mathcal{S}_{CB}$ obtained from the second of eqs. (4.17) reads:

$$\begin{aligned}&\bar{\partial}\mathcal{S}_{CB} \\ &= i\begin{pmatrix} -\frac{2\pi}{N\ell\sqrt{V}}\delta_{CB} + 2(W_2^*W_2 - W_4^*W_4)_{CB} & -4(W_4^*W_2)_{CB} \\ -4(W_2^*W_4)_{CB} & \frac{2\pi}{N\ell\sqrt{V}}\delta_{CB} - 2(W_2^*W_2 - W_4^*W_4)_{CB}\end{pmatrix},\end{aligned} \tag{4.23}$$

using a similar shorthand (e.g. $(W_2^*W_2)_{CB} \equiv W^*_{2\,D'C}W_{2\,D'B}$ with a sum over D').

Next, we recall that the operator $\bar{\partial}$ is the Weyl operator for the diagonal undotted fermions, whose zero modes were determined in section 3.4.1. We also recall that $\mathcal{S}$ is a quaternion, hence (similar to (4.18)), we can think of $\mathcal{S}$ as of two sets of Weyl fermions, one for each column of the quaternion matrix. We can thus borrow the results for the zero modes, recalling (3.16) and (3.17), and then impose their orthogonality of the r.h.s. of (4.22), (4.23). As shown there, undotted fermions have $2\gcd(k,r)$ constant zero modes.

[15] For further use, in (4.21), we also introduced the short-hand notation $W_{2\,C'C}$ and $W_{4\,C'C}$ for the general solutions of (4.18).

JHEP09(2023)095

This implies that there are $4\gcd(k,r)$ zero modes of $\mathcal{S}$, which we label by an arbitrary *quaternionic* coefficient $\epsilon^{(j)}$, $j = 0, \ldots, \gcd(k,r)-1$. The corresponding wave functions, which we denote $s_{B'C'}$ and s_{BC}, have only diagonal entries

$$\begin{aligned} s_{B'C'} &= \delta_{B'C'} \sum_{j=0}^{\gcd(k,r)-1} \epsilon^{(j)} \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1} \delta_{B',[j+nr]_k}\,, \\ s_{BC} &= -\frac{\delta_{BC}}{\ell} \sum_{B'=0}^{k-1} s_{B'B'}\,,\ \forall B = 0, \ldots, \ell-1\,. \end{aligned} \tag{4.24}$$

The simplest condition is the orthogonality of s_{BC} (which is simply a constant quaternionic mode) to the source term in the equation for $\mathcal{S}_{CB}$. Multiplying the source term by the s_{BC} zero mode, taking the trace, and integrating over $\mathbb{T}^4$, we find that orthogonality implies that the integral of the trace of the r.h.s. over $\mathbb{T}^4$ should vanish for every entry in the quaternion source on the r.h.s. of (4.23). Explicitly, this gives the conditions

$$\begin{aligned} \int_{\mathbb{T}^4} (W^*_{2\,B'C} W_{2\,B'C} - W^*_{4\,B'C} W_{4\,B'C}) &= \frac{\pi}{N}\sqrt{V}\,, \\ \int_{\mathbb{T}^4} W^*_{4\,B'C} W_{2\,B'C} &= 0\,, \end{aligned} \tag{4.25}$$

with a sum over the full range of repeated indices implied.

However, the conditions imposed by orthogonality to the $4\gcd(k,r)$ zero modes $s_{B'B'}$ labelled by $\epsilon^{(j)}$ are more detailed than (4.25). Proceeding similar to the above, we find the $\gcd(k,r)$ conditions:

$$\begin{aligned} &\sum_{B=0}^{\ell-1} \sum_{C'=0}^{k-1} \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1} \delta_{C',[j+nr]_k} \int_{\mathbb{T}^4} (W_{2\,C'B} W^*_{2\,C'B} - W_{4\,C'B} W^*_{4\,C'B}) = \frac{\pi}{N\gcd(k,r)}\sqrt{V} \\ &\sum_{B=0}^{\ell-1} \sum_{C'=0}^{k-1} \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1} \delta_{C',[j+nr]_k} \int_{\mathbb{T}^4} W^*_{4\,C'B} W_{2\,C'B} = 0, \quad j = 0, \ldots, \gcd(k,r)-1\,. \end{aligned} \tag{4.26}$$

That the above $\gcd(k,r)$ conditions are more general than (4.25) follows by observing that summing up the $\gcd(k,r)$ conditions in each line of (4.26) (i.e., summing over j) we obtain (4.25).

The importance of the conditions (4.26) is that they restrict the $2r$ complex coefficients $\mathcal{C}_2$ and $\mathcal{C}_4$, and thus determine the moduli space of the multifractional instanton. Studying this is the subject of the next section.

5 The moduli of the $Q = \frac{r}{N}$ bosonic solution: compact vs. noncompact

To study the constraints (4.25), (4.26) with W_2 and W_4 from (4.21), we now define, for each $j = 0, \ldots, \gcd(k,r)-1$ and $a, b \in \{2,4\}$:

$$I^{ab}_j = \sum_{C'=0}^{k-1} \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1} \delta_{C',[j+nr]_k} \sum_{p,p'=0}^{\frac{r}{\gcd(k,r)}-1} \frac{\mathcal{C}_a^{[C'+pk]_r}\, \mathcal{C}_b^{*\,[C'+p'k]_r}}{\sqrt{V}} \int_{\mathbb{T}^4} \sum_{B=0}^{\ell-1} \Phi^{(p)}_{C'B} \Phi^{(p')\,*}_{C'B}\,. \tag{5.1}$$

In terms of I_j^{ab}, the constraints (4.25), (4.26) take the form:

$$
\begin{aligned}
I_j^{22} - I_j^{44} &= \frac{\pi\sqrt{V}}{\gcd(k,r)N}, \\
I_j^{42} &= 0, \quad \text{where} \ \ j = 0, \dots, \gcd(k,r) - 1\,.
\end{aligned}
\tag{5.2}
$$

The expressions (5.1) are evaluated in appendix B. Substituting I_j^{ab} from (B.6) in, we find the constraints (4.25), (4.26) expressed in terms of the undetermined complex coefficients $\mathcal{C}_2^A$ and $\mathcal{C}_4^A$ from the solution of the equations for $\mathcal{W}_\mu$ (4.21):[16]

$$
\begin{aligned}
\sum_{A_j \in S_j} \mathcal{C}_2^{A_j}\, \mathcal{C}_2^{*\,A_j} - \mathcal{C}_4^{A_j}\, \mathcal{C}_4^{*\,A_j} &= \frac{2\pi}{\gcd(k,r)N} \sqrt{\frac{rL_1L_3}{\ell k L_2 L_4}}\; e^{-\frac{L_1L_2k}{2\pi r}(\varphi_1^j)^2}\, e^{-\frac{L_3L_4\ell}{2\pi}(\varphi_3^j)^2}, \\
\sum_{A_j \in S_j} \mathcal{C}_2^{A_j}\, \mathcal{C}_4^{*\,A_j} &= 0\,.
\end{aligned}
\tag{5.3}
$$

Here, S_j are $\gcd(k,r)$ sets of integers ($\in \{0,\dots,r-1\}$), defined in (B.4) and repeated here for convenience:

$$
S_j = \left\{ [[j+nr]_k + pk]_r, \text{for } n = 0, \dots \frac{k}{\gcd(k,r)} - 1, \text{and } p = 0, \dots, \frac{r}{\gcd(k,r)} - 1 \right\}.
$$

Repeated entries in S_j are identified so that each set has $\frac{r}{\gcd(k,r)}$ elements. The union of all sets S_j is the set $\{0,\dots,r-1\}$.

As we shall shortly see, the structure of the "moduli space" of $\mathcal{C}_{2,4}^A$ defined by (5.3) is quite rich. Let us, however, first count the number of moduli for general k and $r > 1$, taking into account the constraints (5.3). First, there are $4\gcd(k,r)$ Wilson lines φ_μ^j, as per (3.20). Then, there are $2r$ real components of $\mathcal{C}_2^A$ and $2r$ real components of $\mathcal{C}_4^A$. Thus the total number of real moduli is $4r + 4\gcd(k,r)$. These are subject to the constraints of eq. (5.3): the $\gcd(k,r)$ real constraints on the first line and $2\gcd(k,r)$ real constraints on the second line. Thus, it would appear that the number of moduli minus the number of constraints is $4r + \gcd(k,r)$. We notice, however, that the gauge conditions (4.14) are invariant under constant gauge transformations in the $\gcd(k,r)$ Cartan directions, the ones along the allowed holonomies (3.20) (i.e. ones that commute with the transition functions).[17] Thus, the total number of bosonic moduli for $k \neq r > 1$ is $4r$, as required by the index theorem for a selfdual solution.

We now consider the various cases in detail:

1. **The case $\mathbf{k = r}$.** This case is singled out by the fact that there are k complex coefficients $\mathcal{C}_2^A$ (and k $\mathcal{C}_4^A$). In addition, the r sets S_j are such that each contains a

[16] We also note that the origin of the $(\varphi_{1,3}^j)^2$-terms on the r.h.s. is in the imaginary $\hat{\phi}_1, \hat{\phi}_3$-terms appearing in the last two lines in $\Phi^{(p)}$ from (3.21). One can show that they can be absorbed in the definition of the coefficients $\mathcal{C}^j$ (or η^j).

[17] In the next section, we shall explicitly see that no gauge invariant characterizing the instanton depends on these phases.

single element, one of the r allowed values of A. Thus the $r(=k)$ constraints become, with c a real number, determined by the r.h.s. of (5.3):

$$\begin{aligned} \mathcal{C}_2^A\, \mathcal{C}_2^{*\,A} - \mathcal{C}_4^A\, \mathcal{C}_4^{*\,A} &= c^2 \text{ (no sum over } A), \\ \mathcal{C}_2^A\, \mathcal{C}_4^{*\,A} = 0 \qquad &\Longrightarrow \mathcal{C}_4^A = 0,\ \mathcal{C}_2^A = e^{i\alpha_A} c,\ \forall\ A \in \{0, \ldots, r-1\}. \end{aligned} \tag{5.4}$$

Thus, all "moduli" $\mathcal{C}_{2,4}^A$ are fixed up to r undetermined phases α_A. These phases are unphysical and correspond to the already mentioned ability to perform r (=gcd(k,r)) constant gauge transformations preserving the gauge conditions (4.14). Thus, the only moduli left are the r phases φ_μ^j, $j = 0, \ldots, r$, recall (3.20).

Thus the multifractional instanton obtained for $k = r$, with $Q = \frac{r}{N}$, has $4r$ compact moduli, as expected from the index theorem. Further studies of the instantons for $k = r$ and the interpretation of these moduli will be discussed in the next section.

2. **The case $\mathbf{k \neq r, r > 1}$.**[18] This case is quite different. Here the r sets S_j contain more than a single number each. Thus, the second equation in (5.4) does not set any modulus to zero (recall that it required that all $\mathcal{C}_4^A$ vanish for $k = r$). Instead, as we argue below, the constraints permit the moduli $\mathcal{C}_{2,4}$ to grow without bound, thus making the "moduli" space noncompact.

   To illustrate the noncompactness for $k \neq r > 1$, we abandon generality and focus on a simple example $r = 2, k = 3$, a case with gcd$(k,r) = 1$ (we shall further use this example in the following). Here, there is only a single set S_j, $S_0 = \{0, 1\}$ and after the following relabeling, with all x's and y's real,[19]

$$\mathcal{C}_2^0 \to x_1 + iy_1\,, \quad \mathcal{C}_4^0 \to x_2 + iy_2\,, \quad \mathcal{C}_2^1 \to x_3 + iy_3\,, \quad \mathcal{C}_4^1 \to x_4 + iy_4\,, \tag{5.5}$$

   we obtain for eqs. (5.3):

$$\begin{aligned} x_1^2 + y_1^2 + x_3^2 + y_3^2 - x_2^2 - y_2^2 - x_4^2 - y_4^2 &= 1\,, \\ x_1 x_2 + y_1 y_2 + x_3 x_4 + y_3 y_4 &= 0\,, \\ x_2 y_1 - x_1 y_2 + y_3 x_4 - x_3 y_4 &= 0\,. \end{aligned} \tag{5.6}$$

   Conditions (5.6) eliminate 3 out of 8 real parameters, leaving 4 physical parameters that parameterize the moduli space in addition to the single arbitrary unphysical phase mentioned above (recall that here gcd(k,r)=1).

   The moduli space spanned by the hypersurface given by the constraints (5.6) is non-compact. To see this, we set for simplicity $x_2 = y_1 = y_3 = x_4 = 0$. Then, the constraints become

$$x_1 y_2 = -x_3 y_4\,, \quad x_1^2 - y_2^2 + x_3^2 - y_4^2 = 1\,. \tag{5.7}$$

[18] We do not consider $r = 1$ here, as it was studied in detail before [8]. As is also easy to see from our formulae, for $r = 1$, the moduli $\mathcal{C}_{2,4}$ are also fixed.

[19] A trivial rescaling setting the r.h.s. of the first equation in (5.3) to unity is not explicitly shown.

For every $x_3 = y_4 \in (-\infty, \infty)$ we find

$$x_1^2 = \frac{x_3^4}{x_1^2} + 1\,, \tag{5.8}$$

which has at least two real solutions of x_1. We also find that $x_1 \to \infty$ as $x_3 = y_4 \to \infty$. We conclude that the moduli space is non-compact. For a later convenience, we parametrize the asymptotic region ($u \to \infty$) of this noncompact direction of the moduli space as

$$\mathcal{C}_2^0 \sim \pm u\,, \quad \mathcal{C}_2^1 \sim u\,, \quad \mathcal{C}_4^0 \sim \mp i u\,, \quad \mathcal{C}_4^1 \sim i u\,. \tag{5.9}$$

It is easy to see, even without following the derivation, that (5.9) obey (5.3) with vanishing r.h.s., i.e. at $u \to \infty$

The presence of noncompact moduli for the $k \neq r$ instantons is difficult to interpret in a $\mathbb{T}^4$ geometry. In the later sections, we shall see that on this noncompact moduli space, $\mathcal{O}(\Delta)$ gauge invariants characterizing the multifractional instanton grow without bounds — see the end of section 6.1 for a brief discussion of the blowup and appendix D for details of its derivation. This blow up clashes with the spirit of the Δ expansion. As we mentioned in the Introduction, it would be nice to achieve a deeper understanding of this finding.

6 Local gauge invariants of the $Q = \frac{r}{N}$ solution and its "dissociation"

In this section, we give expressions for local gauge invariant densities characterizing the multifractional instanton to order Δ. These expressions are evaluated in the appendices. We use the results to, first, show that $\mathcal{O}(\Delta)$ local gauge invariants grow without bound along the noncompact moduli directions found for $k \neq r$, and, second, to argue for the fractionalization of the $k = r$ multifractional instanton into r identical lumps located at positions on $\mathbb{T}^4$ determined by the r distinct holonomies/moduli.

6.1 Gauge-invariant local densities to order Δ and their blow up for $k \neq r$

The gauge-invariant local density of the lowest scaling dimension is

$$\mathrm{tr}\left[F_{\mu_1\nu_1} F_{\mu_2\nu_2}\right]\,, \tag{6.1}$$

where

$$F_{\mu\nu} = \left(F^\omega_{\mu\nu} + F^s_{\mu\nu}\right)\omega + \begin{bmatrix} F^k_{\mu\nu} & \mathcal{F}_{\mu\nu} \\ \mathcal{F}^\dagger_{\mu\nu} & F^\ell_{\mu\nu} \end{bmatrix}\,, \tag{6.2}$$

and we recall that the components of (6.2) were already defined in (4.5).[20]

In appendix C, we compute the various field strength components appearing in (6.2) to order Δ (shown in eq. (C.13)) as well as the action density and action. Then, following

[20] For brevity, we have omitted the $k \times \ell$ and $\ell \times k$ superscripts in writing (6.2).

the same steps used in deriving the action density there, we obtain for eq. (6.1) to order Δ

$$
\begin{aligned}
&\mathrm{tr}\left[F_{\mu_1\nu_1}F_{\mu_2\nu_2}\right]\\
&= \mathrm{tr}[\omega^2]\left\{\hat{F}^\omega_{\mu_1\nu_1}\hat{F}^\omega_{\mu_2\nu_2} + \Delta\hat{F}^\omega_{\mu_1\nu_1}\left(\partial_{\mu_2}\mathcal{S}^{(0)\omega}_{\nu_2} - \partial_{\nu_2}\mathcal{S}^{(0)\omega}_{\mu_2}\right) + \Delta\hat{F}^\omega_{\mu_2\nu_2}\left(\partial_{\mu_1}\mathcal{S}^{(0)\omega}_{\nu_1} - \partial_{\nu_1}\mathcal{S}^{(0)\omega}_{\mu_1}\right)\right\}\\
&\quad +2\pi\ell\Delta\hat{F}^\omega_{\mu_1\nu_1}\mathrm{tr}_k\left[\partial_{\mu_2}\mathcal{S}^{(0)k}_{\nu_2} - \partial_{\nu_2}\mathcal{S}^{(0)k}_{\mu_2}\right] + 2\pi\ell\Delta\hat{F}^\omega_{\mu_2\nu_2}\mathrm{tr}_k\left[\partial_{\mu_1}\mathcal{S}^{(0)k}_{\nu_1} - \partial_{\nu_1}\mathcal{S}^{(0)k}_{\mu_1}\right]\\
&\quad -2\pi k\Delta\hat{F}^\omega_{\mu_1\nu_1}\mathrm{tr}_\ell\left[\partial_{\mu_2}\mathcal{S}^{(0)\ell}_{\nu_2} - \partial_{\nu_2}\mathcal{S}^{(0)\ell}_{\mu_2}\right] - 2\pi k\Delta\hat{F}^\omega_{\mu_2\nu_2}\mathrm{tr}_\ell\left[\partial_{\mu_1}\mathcal{S}^{(0)\ell}_{\nu_1} - \partial_{\nu_1}\mathcal{S}^{(0)\ell}_{\mu_1}\right]\\
&\quad +i2\pi N\Delta\hat{F}^\omega_{\mu_1\nu_1}\mathrm{tr}_k\left[\mathcal{W}_{\mu_2}\mathcal{W}^\dagger_{\nu_2} - \mathcal{W}_{\nu_2}\mathcal{W}^\dagger_{\mu_2}\right] + i2\pi N\Delta\hat{F}^\omega_{\mu_2\nu_2}\mathrm{tr}_k\left[\mathcal{W}_{\mu_1}\mathcal{W}^\dagger_{\nu_1} - \mathcal{W}_{\nu_1}\mathcal{W}^\dagger_{\mu_1}\right]\\
&\quad +\Delta\mathrm{tr}_k\left(\mathcal{F}_{\mu_1\nu_1}\mathcal{F}^\dagger_{\mu_2\nu_2}\right) + \Delta\mathrm{tr}_\ell\left(\mathcal{F}^\dagger_{\mu_1\nu_1}\mathcal{F}_{\mu_2\nu_2}\right). \qquad (6.3)
\end{aligned}
$$

Using $\mathrm{tr}_\ell\mathcal{S}^{(0\ell)}_\mu = \mathrm{tr}_k\mathcal{S}^{(0k)}_\mu = 0$, we obtain

$$
\begin{aligned}
&\mathrm{tr}\left[F_{\mu_1\nu_1}F_{\mu_2\nu_2}\right]\\
&= \mathrm{tr}[\omega^2]\left\{\hat{F}^\omega_{\mu_1\nu_1}\hat{F}^\omega_{\mu_2\nu_2} + \Delta\hat{F}^\omega_{\mu_1\nu_1}\left(\partial_{\mu_2}\mathcal{S}^{(0)\omega}_{\nu_2} - \partial_{\nu_2}\mathcal{S}^{(0)\omega}_{\mu_2}\right) + \Delta\hat{F}^\omega_{\mu_2\nu_2}\left(\partial_{\mu_1}\mathcal{S}^{(0)\omega}_{\nu_1} - \partial_{\nu_1}\mathcal{S}^{(0)\omega}_{\mu_1}\right)\right\}\\
&\quad +i2\pi N\Delta\hat{F}^\omega_{\mu_1\nu_1}\mathrm{tr}_k\left[\mathcal{W}_{\mu_2}\mathcal{W}^\dagger_{\nu_2} - \mathcal{W}_{\nu_2}\mathcal{W}^\dagger_{\mu_2}\right] + i2\pi N\Delta\hat{F}^\omega_{\mu_2\nu_2}\mathrm{tr}_k\left[\mathcal{W}_{\mu_1}\mathcal{W}^\dagger_{\nu_1} - \mathcal{W}_{\nu_1}\mathcal{W}^\dagger_{\mu_1}\right]\\
&\quad +\Delta\mathrm{tr}_k\left(\mathcal{F}_{\mu_1\nu_1}\mathcal{F}^\dagger_{\mu_2\nu_2}\right) + \Delta\mathrm{tr}_\ell\left(\mathcal{F}^\dagger_{\mu_1\nu_1}\mathcal{F}_{\mu_2\nu_2}\right). \qquad (6.4)
\end{aligned}
$$

In appendix D, we compute (for definiteness) the gauge invariant density $\mathrm{tr}\left[F_{34}F_{34}\right]$ for the $k \neq r$ solution and show that it grows without bounds along the noncompact moduli direction of (5.9). This local gauge invariant, from (6.4), is given by

$$
\begin{aligned}
&\mathrm{tr}\left[F_{34}F_{34}\right]\\
&= \mathrm{tr}[\omega^2]\left\{\hat{F}^\omega_{34}\hat{F}^\omega_{34} + 2\Delta\hat{F}^\omega_{34}\left(\partial_3\mathcal{S}^{(0)\omega}_4 - \partial_4\mathcal{S}^{(0)\omega}_3\right)\right\} + i4\pi N\Delta\hat{F}^\omega_{34}\mathrm{tr}_k\left[\mathcal{W}_3\mathcal{W}^\dagger_4 - \mathcal{W}_4\mathcal{W}^\dagger_3\right]\\
&= \mathrm{tr}[\omega^2]\left\{\hat{F}^\omega_{34}\hat{F}^\omega_{34} + 2\Delta\hat{F}^\omega_{34}\left(\partial_3\mathcal{S}^{(0)\omega}_4 - \partial_4\mathcal{S}^{(0)\omega}_3\right)\right\} + 8\pi N\Delta\hat{F}^\omega_{34}\mathrm{tr}_k\left[\mathcal{W}_4\mathcal{W}^\dagger_4\right], \qquad (6.5)
\end{aligned}
$$

and we used $\mathcal{W}_3 = -i\mathcal{W}_4$.

To show the blow up, we use the example $r = 2$, $k = 3$ studied in section 5. In appendix D, we show that in the noncompact direction (5.9) the $\mathcal{O}(\Delta)$ gauge invariant blows up as $u \to \infty$. This runaway behaviour of local gauge invariant densities along the noncompact moduli space runs counter the spirit of the Δ-expansion. Thus, in what follows, we concentrate on the properties of the $k = r$ solutions with compact moduli space.

6.2 Fractionalization of solutions with topological charges $r > 1$

6.2.1 Bosonic gauge invariant densities

In this section, we use the results for the local gauge invariants to argue that instantons with topological charges $r > 1$ dissociate into r identical components. It is clear from the discussion in the previous section that unless one takes $k = r$, one faces the undesired runaway behavior of the gauge-invariant densities. Thus, we limit our discussion to the case $k = r$, where we show that the gauge-invariant densities take the form of a sum over r independent lumps centered around r distinct holonomies.

To this end, consider (6.4) taking $\mu_1 = \mu_3 = 1, \mu_2 = \mu_4 = 2$. Thus, one obtains

$$\mathrm{tr}\,[F_{12}F_{12}] = \mathrm{tr}[\omega^2]\left\{\hat{F}^{\omega}_{12}\hat{F}^{\omega}_{12} + 2\Delta\hat{F}^{\omega}_{12}\left(\partial_1\mathcal{S}_2^{(0)\omega} - \partial_2\mathcal{S}_1^{(0)\omega}\right)\right\} + 8\pi N\Delta\hat{F}^{\omega}_{12}\mathrm{tr}_k\left[\mathcal{W}_2\mathcal{W}_2^{\dagger}\right], \tag{6.6}$$

where, using (D.3), we find

$$\left(\partial_1\mathcal{S}_2^{(0)\omega} - \partial_2\mathcal{S}_1^{(0)\omega}\right) = -\left(\pi\ell k\Box\right)^{-1}\left(\partial_1^2 + \partial_2^2\right)\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_2^{\dagger(0)}\right]. \tag{6.7}$$

Here,

$$\mathcal{W}^{(0)}_{2\,C',C}(x) = V^{-1/4}\mathcal{C}_2^{C'}\Phi^{(0)}_{C',C}(x,\hat{\phi}), \quad C' = 1,2,\ldots,k=r, \quad C = 1,2,\ldots,\ell. \tag{6.8}$$

It is more convenient to express $\Phi^{(0)}_{C',C}(x,\hat{\phi})$ in the form given in (A.54)

$$\begin{aligned}\Phi^{(0)}_{C',C}(x,\hat{\phi}) = {} & e^{\frac{kL_1L_2}{2\pi r}\hat{\phi}_1^{C'}\left(i\hat{\phi}_2^{C'}+\hat{\phi}_1^{C'}/2\right)}e^{\frac{\ell L_3L_4}{2\pi}\hat{\phi}_3^{C'}\left(i\hat{\phi}_4^{C'}+\hat{\phi}_3^{C'}/2\right)}e^{-i\hat{\phi}_1^{C'}x_1}e^{-i\hat{\phi}_3^{C'}x_3}\\
&\times\sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}}e^{i\left(\frac{2\pi x_2}{L_2}+L_1\hat{\phi}_1^{C'}\right)\left(m'+\frac{2C'-1-k}{2k}\right)}e^{i\left(\frac{2\pi x_4}{L_4}+\ell L_3\hat{\phi}_3^{C'}\right)\left(n'-\frac{2C-1-\ell}{2\ell}\right)}\\
&\times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)}e^{i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)}\\
&\times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{L_1L_2}{2\pi}\hat{\phi}_2^{C'}-\frac{L_1}{k}\left(km'+\frac{2C'-1-k}{2}\right)\right]^2}\\
&\times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\hat{\phi}_4^{C'}-L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right]^2}.\end{aligned} \tag{6.9}$$

The above eqs. (6.7), (6.6) imply that the computation of the gauge-invariant density $\mathrm{tr}\,[F_{12}F_{12}]$ requires finding the quantity

$$\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_2^{\dagger(0)}\right] = \sum_{C'=1}^{r}\left(\sum_{C=1}^{\ell}|\mathcal{C}_2^{C'}|^2|\Phi^{(0)}_{C',C}(x,\hat{\phi})|^2\right). \tag{6.10}$$

To further study (6.10), we need to take into account the fact that the r coefficients $\mathcal{C}_2$ are determined by the top equation in (5.3), as described in (5.4). It is important that $\mathcal{C}_2$ do depend on the holonomies, which were absorbed into the coefficient c in (5.4). Taking this into account,[21] we find, after some rearrangement, that the expression (6.10), which determines $\mathrm{tr}\,[F_{12}F_{12}]$ to order Δ has the following form:[22]

$$\begin{aligned}&\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_2^{\dagger(0)}\right] \\
&\sim\sum_{C'=1}^{r}\left|\sum_{m'\in\mathbb{Z}}e^{i\left(\frac{2\pi x_2}{L_2}+L_1\hat{\phi}_1^{C'}\right)m'-\frac{\pi}{L_1L_2}\left[x_1-\frac{L_1L_2}{2\pi}\hat{\phi}_2^{C'}-\frac{L_1C'}{r}-L_1\left(m'-\frac{1+r}{2r}\right)\right]^2}\right|^2\\
&\quad\times\left|\sum_{n'\in\mathbb{Z}}e^{i\left(\frac{2\pi x_4}{\ell L_4}+L_3\hat{\phi}_3^{C'}\right)n'-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\hat{\phi}_4^{C'}-L_3\left(\ell n'+\frac{1+\ell}{2}\right)\right]^2}\right|^2\\
&=:\sum_{C'=1}^{r}F\left(x_1-\frac{L_1L_2}{2\pi}\hat{\phi}_2^{C'}-\frac{L_1C'}{r},\ x_2+\frac{L_1L_2}{2\pi}\hat{\phi}_1^{C'},\ x_3-\frac{\ell L_3L_4}{2\pi}\hat{\phi}_4^{C'},\ x_4+\frac{\ell L_3L_4}{2\pi}\hat{\phi}_3^{C'}\right).\end{aligned} \tag{6.11}$$

[21] The $\hat{\phi}_{1,3}$-dependence of $\mathcal{C}_2$ cancels the $(\hat{\phi}_1)^2$ and $(\hat{\phi}_3)^2$ terms in the exponent on the first line of (6.9). This ensures that gauge invariant quantities have periodic dependence on the holonomies.

[22] Up to an inessential L_μ, r, ℓ, N-dependent constant which can be easily determined.

As indicated on the last line above, for every $C' = 1, 2, \ldots, r$, the summand is given by the same function $F(x_1, x_2, x_3, x_4)$, implicitly defined above, but centered at a different point x_μ on $\mathbb{T}^4$. The position of each lump is determined by the moduli $\hat{\phi}_\mu^{C'}$, $\mu = 1, 2, 3, 4$, $C' = 1, \ldots, r$. The size of the lumps is, of course, set by the size of $\mathbb{T}^4$, the only scale of the problem. Thus, the "lumps" we find are not well isolated, but strongly overlapping, rather like a liquid than a dilute gas (see figure 1 for an illustration).

6.2.2 Fermionic zero modes and their localization

The conclusion of the above analysis is that the local gauge invariant density of the multi-fractional instanton, tr $[F_{12}F_{12}]$, takes the form of a sum of r identical lumps, each centered at r distinct holonomies. Thus, the solution of topological charge r/N can be thought of as composed of r distinct lumps. Each lump is expected to contribute $1/N$-th of the total topological charge.

This expectation is strengthened by considering the fermion zero modes in the $Q = \frac{r}{N}$ self-dual solution. In appendix E, we show that there are $2r$ zero modes, labeled by a 2-spinor $\bar{\eta}_\alpha^{C'}$, with $C' = 1, \ldots r$. To order $\mathcal{O}(\sqrt{\Delta})$, the x-dependence of the zero modes appears in the off-diagonal components:

$$\begin{aligned} \lambda_{1\, C'D} &\sim \bar{\eta}_2^{C'} (\partial_3 + i\hat{\phi}_3^{C'}) \Phi^{(0)}_{C',C}(x, \hat{\phi})) \equiv \bar{\eta}_2^{C'} \mathcal{G}^{(0)}_{3\, C'D}(x, \hat{\phi}^{C'}), \\ \lambda_{2\, C'D} &= 0\,. \end{aligned} \tag{6.12}$$

with the expression for $\mathcal{G}^{(0)}_{3\, C'D}(x, \hat{\phi}^{C'})$ given in appendix C, see (C.9). Likewise, the zero mode wave function in the other off-diagonal component is

$$\begin{aligned} \lambda_{1\, DC'} &= 0, \\ \lambda_{2\, DC'} &\sim \bar{\eta}_1^{C'} \mathcal{G}^{*\,(0)}_{3\, C'D}(x, \hat{\phi}^{C'}). \end{aligned} \tag{6.13}$$

Even without consulting the explicit expression, it is clear that the C'-th zero mode only depends on $\hat{\phi}_\mu^{C'}$, which, therefore, governs its x_μ-dependence, similar to (6.11) above.

Explicitly, one can construct $\mathcal{O}(\Delta)$ gauge invariants formed from the zero modes, to find, as for the bosonic invariants, that they are governed by a "lumpy" structure, with each of the r lumps supporting 2 zero modes located at a position governed by the moduli $\hat{\phi}_\mu^{C'}$. Explicitly, we find that the order-Δ gauge invariants built from the fermion zero modes contain terms like

$$\begin{aligned} &\sum_{C',D} \lambda_{1C'D} \lambda_{2DC'} \\ &\sim \sum_{C'} \bar{\eta}_1^{C'} \bar{\eta}_2^{C'} \left| \sum_m e^{i\frac{2\pi m}{L_2}(x_2 + \frac{L_1 L_2}{2\pi}\hat{\phi}_1^{C'}) - \frac{\pi}{L_1 L_2}\left[x_1 - \frac{L_1 L_2}{2\pi}\hat{\phi}_2^{C'} - \frac{L_1 C'}{r} + L_1 \frac{1+r}{2r} - L_1 m\right]^2} \right|^2 \\ &\times \left| \sum_n \left(x_3 - \frac{\ell L_3 L_4}{2\pi}\hat{\phi}_4^{C'} - L_3 \ell n - L_3 \frac{1+\ell}{2} \right) e^{i\frac{2\pi n}{\ell L_4}(x_4 + \frac{\ell L_3 L_4}{2\pi}\hat{\phi}_3^{C'}) - \frac{\pi}{\ell L_3 L_4}\left[x_3 - \frac{\ell L_3 L_4}{2\pi}\hat{\phi}_4^{C'} - L_3\left(\ell n + \frac{1+\ell}{2}\right)\right]^2} \right|^2. \end{aligned} \tag{6.14}$$

This expression shows the same "localization" properties (determined by the holonomies $\hat{\phi}^{C'}$) of the fermion zeromodes that were made evident for the bosonic solution in (6.11). It is also clear that the C'^{th} fermion zero mode vanishes at the position determined by the C'^{th} holonomy.

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A Derivation of the off-diagonal fermion zero modes

A.1 The zero modes at zero holonomy

Within this appendix, we present the derivation of one of the main results in the main text, denoted as eq. (3.18). Our objective revolves around solving the off-diagonal fermion zero modes of the Dirac equation $D_\mu \bar{\sigma}^\mu \lambda = 0$. This equation pertains to the 't Hooft flux background, wherein the covariant derivative takes the form $D_\mu = \partial_\mu + i[A_\mu, \,]$. To streamline our approach, we commence by deactivating the holonomies. Subsequently, we can reintroduce them once we have obtained a general solution.

Using (3.2) and writing $A_\mu \equiv A_\mu^\omega \omega$, we find the commutator

$$[A_\mu, \lambda] = 2\pi A_\mu^\omega \begin{bmatrix} 0 & N||\lambda_{C'C}|| \\ -N||\lambda_{CC'}|| & 0 \end{bmatrix}, \tag{A.1}$$

In this appendix we take the range of C and C' to be $C = 1, 2, \ldots, \ell$ and $C' = 1, 2, \ldots, k$. Substituting the above result into the Dirac equation, $D_\mu \bar{\sigma}^\mu \lambda = 0$, we obtain for $\lambda_{C'C}$ (and similarly for $\lambda_{CC'}$ after replacing $+i2\pi N \to -i2\pi N$):

$$\bar{\sigma}^\mu \left[\partial_\mu \lambda_{C'C} + i2\pi N A_\mu^\omega \lambda_{C'C}\right] = 0\,. \tag{A.2}$$

Writing $\lambda_{C'C}$ in terms of its two spinor components $\lambda_{C'C\,1}$ and $\lambda_{C'C\,2}$, the Dirac equation reads:

$$\begin{aligned} \left(\partial_1 - i\partial_2 - \frac{2\pi r x_1}{kL_1L_2}\right)\lambda_{C'C\,2} + \left(\partial_3 + i\partial_4 + \frac{2\pi x_3}{\ell L_3 L_4}\right)\lambda_{C'C\,1} &= 0\,, \\ \left(\partial_1 + i\partial_2 + \frac{2\pi r x_1}{kL_1L_2}\right)\lambda_{C'C\,1} + \left(-\partial_3 + i\partial_4 + \frac{2\pi x_3}{\ell L_3 L_4}\right)\lambda^\beta_{C'C\,2} &= 0\,. \end{aligned} \tag{A.3}$$

A normalizable solution to the above equations can be found provided that we set $\lambda_{C'C\,2} = 0$; an assertion that will be revisited below in the light of the most general normalizable solution on $\mathbb{T}^4$ we shall construct.

We proceed further by defining the functions $U_{C'C}$ via:

$$\lambda_{C'C\,1} \equiv e^{-\frac{\pi r x_1^2}{kL_1L_2}} e^{-\frac{\pi x_3^2}{\ell L_3 L_4}} U_{C'C}\,, \tag{A.4}$$

which reduces (A.3) to the two simple equations

$$(\partial_1 + i\partial_2)\, U_{C'C} = 0\,, \quad (\partial_3 + i\partial_4)\, U_{C'C} = 0\,. \tag{A.5}$$

By defnining the complex variables $w_1 \equiv x_1 + ix_2$ and $w_2 \equiv x_3 + ix_4$, we can cast (A.5) in the form

$$\frac{\partial U_{C'C}}{\partial \bar{w}_1} = 0\,, \quad \frac{\partial U_{C'C}}{\partial \bar{w}_2} = 0\,, \tag{A.6}$$

and, thus, we conclude that $U_{C'C}$ is an analytic function of w_1 and w_2:

$$U_{C'C} = U_{C'C}(w_1, w_2)\,. \tag{A.7}$$

We can also write the boundary conditions (3.12) as (the cyclic nature of the matrix elements, i.e., $U_{C'C} \equiv U_{C'+k;C+\ell}$ will be imposed below):

$$\begin{aligned}
U_{C'C}(w_1 + L_1, w_2) &= \gamma_k^{-r} e^{\frac{\pi r L_1}{k L_2} + \frac{2\pi r w_1}{k L_2}} U_{C'-r\,C}(w_1, w_2)\,,\\
U_{C'C}(w_1 + iL_2, w_2) &= \gamma_k e^{i\frac{2\pi(C'-1)}{k}} U_{C'C}(w_1, w_2)\,,\\
U_{C'C}(w_1, w_2 + L_3) &= \gamma_\ell^{-1} e^{\frac{\pi L_3}{\ell L_4} + \frac{2\pi w_2}{\ell L_4}} U_{C'\,C+1}(w_1, w_2)\,,\\
U_{C'C}(w_1, w_2 + iL_4) &= \gamma_\ell^{-1} e^{-i\frac{2\pi(C-1)}{\ell}} U_{C'C}(w_1, w_2)\,.
\end{aligned} \tag{A.8}$$

We notice that the transformation properties under imaginary shifts of w_1 by iL_2 and w_2 by iL_4 are satisfied by writing $U_{C'C}(w_1, w_2)$ as the phase factor

$$e^{\frac{w_1}{L_2}\frac{\pi}{k}(2C'-1-k) - \frac{w_2}{L_4}\frac{\pi}{\ell}(2C-\ell-1)} \tag{A.9}$$

times an analytic function which is periodic w.r.t. these imaginary shifts, i.e., is a linear combination of functions $e^{2\pi n\frac{w_1}{L_2} + 2\pi m\frac{w_2}{L_4}}$ where $n, m \in \mathbb{Z}$.[23] Thus, the expression for $U_{C'C}$ has the form

$$U_{C'C}(w_1, w_2) = e^{\frac{\pi w_1(2C'-1-k)}{k L_2}} e^{-\frac{\pi w_2(2C-1-\ell)}{\ell L_4}} \sum_{m,n \in \mathbb{Z}} d_{C',C,m,n} e^{2\pi m\frac{w_1}{L_2} + 2\pi n\frac{w_2}{L_4}}\,. \tag{A.10}$$

Our next task is determining the coefficients $d_{C',C,\,m,n}$. Using the first and third BCs in (A.8), we obtain the recurrence relations

$$d_{C',C,m,n} = e^{-i\frac{\pi r(1-k)}{k}} e^{-\frac{\pi L_1(2C'-1-k)}{k L_2} - \frac{2\pi m L_1}{L_2} + \frac{\pi r L_1}{k L_2}} d_{C'-r,C,m,n}\,, \tag{A.11}$$

and

$$d_{C',C+1,m,n} = e^{i\frac{\pi(1-\ell)}{\ell}} e^{\frac{\pi(-2C+(2n+1)\ell)L_3}{\ell L_4}} d_{C',C,m,n}\,. \tag{A.12}$$

These recurrence relations must be supplemented with boundary conditions that need to be satisfied to guarantee the cyclic nature of the solution, i.e., $U_{C'C}(w_1, w_2) = U_{C'+k\,C}(w_1, w_2) = U_{C'\,C+\ell}(w_1, w_2)$. First, using $U_{C'1}(w_1, w_2) = U_{C'\,1+\ell}(w_1, w_2)$ along with the third equation in (A.8), we obtain the following relationship between the elements $C = 1$ and $C = \ell$ in SU(ℓ):

$$U_{C'\,C=\ell}(w_1, w_2 + L_3) = \gamma_\ell^{-1} e^{\frac{\pi L_3}{\ell L_4}} e^{\frac{2\pi w_2}{\ell L_4}} U_{C'\,C=1}(w_1, w_2)\,, \tag{A.13}$$

[23]The periodicity in imaginary shifts requires the exponential dependence, while the rest follows by holomorphy. The functions $e^{2\pi n\frac{w_2}{L_4}}$ are normalizable on $\mathbb{T}^2$, and the ones with different n's are orthogonal, as enforced by the imaginary part of integrals over x_2.

which yields via (A.10):

$$d_{C',\ell,m,n} = e^{-i\frac{\pi(\ell-1)}{\ell}} e^{\frac{\pi(1-2n)L_3}{L_4}} d_{C',1,m,n-1}\,. \tag{A.14}$$

We can generalize (A.12) and (A.14) to

$$\begin{aligned} d_{C',C,m,n} &= e^{-i\frac{\pi(1-\ell)}{\ell}} e^{\frac{-\pi(-2C+(2n+1)\ell)L_3}{\ell L_4}} d_{C',C+1,m,n}\,, \quad \text{if } C+1<\ell \\ d_{C',C,m,n} &= e^{-i\frac{\pi(1-\ell)}{\ell}} e^{\frac{-\pi(-2C+(2n+1)\ell)L_3}{\ell L_4}} d_{C',C_{\text{new}},m,n-1}\,, \quad C_{\text{new}} = C+1-\ell \quad \text{if } C+1>\ell\,. \end{aligned} \tag{A.15}$$

We must also find the boundary condition for the recurrence relation (A.11). Using $U_{1C}(w_1,w_2) = U_{1+k\,C}(w_1,w_2)$ along with the first equation in (A.8), we obtain the following relationship between the elements $C'=1$ and $C'=k-(r-1)$ in SU(k):

$$U_{C'=1\,C}(w_1+L_1,w_2) = \gamma_k^{-r} e^{\frac{\pi r L_1}{kL_2}} e^{\frac{2\pi r}{kL_2}w_1} U_{C'=k-(r-1)\,C}(w_1,w_2)\,, \tag{A.16}$$

which yields via (A.10):

$$d_{1,C,m,n} = e^{-i\frac{\pi r(1-k)}{k}} e^{\frac{\pi(r-1+k-2mk)}{k}\frac{L_1}{L_2}} d_{k-(r-1),C,m-1,n}\,. \tag{A.17}$$

Notice that we had to shift m by one unit since, according to the first equation in (A.8), a shift in the L_1 direction relates the element $C'=1$ to the element $C'=1-r$. However, since $1-r\leq 0$, we needed to replace $C'=1-r$ by a new $C'_{\text{new}} = k-(r-1)$. This replacement forces us to shift $m\to m-1$ to obey the boundary condition (A.8) in the L_1 direction. This shift in m always happens whenever the matrix elements have $C'-r\leq 0$. We may generalize (A.17) for any C' whenever the first condition (A.11) yields $d_{C'=C-r,C,m,n}$ with $C'<0$. Demanding the cyclicity $U_{C'+k\,C}(x) = U_{C'\,C}(x)$, we replace (A.11) and (A.17) with

$$\begin{aligned} d_{C',C,m,n} &= e^{-i\frac{\pi r(1-k)}{k}} e^{-\frac{\pi L_1(2C'-1-k)}{kL_2} - \frac{2\pi m L_1}{L_2} + \frac{\pi r L_1}{kL_2}} d_{C'-r,C,m,n}\,, \quad \text{if } C'-r>0\,, \\ d_{C',C,m,n} &= e^{-i\frac{\pi r(1-k)}{k}} e^{-\frac{\pi L_1(2C'-1-k)}{kL_2} - \frac{2\pi m L_1}{L_2} + \frac{\pi r L_1}{kL_2}} d_{C'_{\text{new}},C,m-1,n}\,, \\ & \qquad C'_{\text{new}} = C'-r+k\,, \quad \text{if } C'-r\leq 0\,. \end{aligned} \tag{A.18}$$

Now we come to the solution of the difference equation (A.15). This is a first-order difference equation, and thus, it yields a single solution. To this end, we substitute the following form

$$d_{C',C,m,n} = F(C',m) e^{-\frac{\pi L_3}{\ell L_4}[C+S(n)]^2} \tag{A.19}$$

into the first equation in (A.15), to obtain

$$S(n) = -\frac{1+(2n+1)\ell}{2}\,. \tag{A.20}$$

Thus,

$$d_{C',C,m,n} = F(C',m) e^{-\frac{\pi L_3}{\ell L_4}\left(C - i\frac{L_4(1-\ell)}{2L_3} - \frac{1+\ell(2n+1)}{2}\right)^2}\,. \tag{A.21}$$

It is straightforward to check that the solution (A.21) obeys (A.15).

On the other hand, the recurrence relation (A.18) is a difference equation of order r, and thus, it should yield r independent solutions. To solve it, we parameterize it as

$$d_{C',C,m,n} = e^{-\frac{\pi L_1}{krL_2}\left(C'+i\frac{L_2 r(1-k)}{2L_1}+S'(m)\right)^2}\,, \tag{A.22}$$

and, inserting into the first equation in (A.18), we find

$$S'(m) = -\frac{1+k(1-2m)}{2}\,. \tag{A.23}$$

We can check that (A.22), (A.23) satisfy (A.18). Combining (A.21) and (A.22), we obtain the final answer

$$d_{C',C,m,n} = e^{-\frac{\pi L_3}{\ell L_4}\left(C-i\frac{(1-\ell)L_4}{2L_3}-\frac{1+\ell(2n+1)}{2}\right)^2} e^{-\frac{\pi L_1}{krL_2}\left(C'+i\frac{r(1-k)L_2}{2L_1}-\frac{1+k(1-2m)}{2}\right)^2}\,. \tag{A.24}$$

Notice that $d_{C',C,m,n} \to e^{-\frac{\pi L_3}{L_4}\ell n^2} e^{-\frac{\pi L_1}{rL_2}km^2}$ as $n, m \to \infty$, and thus, the series (A.10) rapidly converges. Had we not set $\lambda_{C'C\,2} = 0$ in (A.3), we would have obtained a divergent series in m, n, and thus, no normalizable zero modes could be found.

What is not immediately clear from (A.24) is that there are r independent solutions of $U_{C'C}$; this should be expected since (A.18) is a difference equation of order r. The r independent solutions of $U_{C'C}$ follow from two distinct cases.

1. If $\gcd(r,k) = r$, we can show that there are r independent coefficients

$$d_{C'=1,C,m,n},\ d_{C'=2,C,m,n},\ \ldots, d_{C'=r,C,m,n}\,, \tag{A.25}$$

   and the sums over m, n in (A.10) are over all integers. The r independent coefficients yield r independent solutions.

2. If $\gcd(r,k) = 1$ and $r > 1$, then the set of integers m in (A.10) bifurcates into r sets such that the sum over $m \in \mathbb{Z}$ in (A.10) is divided into $m_j = n_j r + n$, $n_j \in \mathbb{Z}$, $n = 0, 1, \ldots, r-1$. These form r independent orbits that correspond to r independent solutions.

The general situation, $1 < \gcd(r,k) < r$, is a combination of both cases.

To ease our discussion, we consider a few examples to understand the essence of each case. First, consider case 1 above, and take as an example $k = 6, r = 2$, where $\gcd(6,2) = 2$. Using (A.18), we see that the coefficients $d_{C',C,m,n}$ are related via (here we ignore C and n since they do not play a role. Also the arrow indicates the relations between the coefficients as we traverse the L_1 direction, without caring about the pre-coefficients):

$$\begin{aligned} &d_{1,m} \to d_{5,m-1} \to d_{3,m-1} \to d_{1,m-1} \to d_{5,m-2} \to \ldots\,, \\ &d_{2,m} \to d_{6,m-1} \to d_{4,m-1} \to d_{2,m-1} \to d_{6,m-2} \to \ldots\,. \end{aligned} \tag{A.26}$$

Each line depicts a set of coefficients, and the coefficients of line 1 and line 2 are independent in that a coefficient in line 1 cannot be reached via a coefficient in line 2 and vice versa.

Notice also, for example, as we start from $d_{1,m}$ and traverse the L_1 direction 3 times, we obtain the shifted $d_{1,m-1}$ by one unit. Thus, we need to sum over all integers m in every line. This gives the two independent solutions.

Next, consider case 2. For example, take $k = 6, r = 5$, where $\gcd(k, r) = 1$. Applying (A.18) we find

$$d_{1,m} \to d_{2,m-1} \to d_{3,m-2} \to d_{4,m-3} \to d_{5,m-4} \to d_{6,m-5} \to d_{1,m-5} \to d_{2,m-6} \cdots . \tag{A.27}$$

Thus, the fact that $d_{1,m}$ shifts to $d_{1,m-5}$ and $d_{2,m-1}$ to $d_{2,m-6}$, etc. means that the set of integers m bifurcates into 5 sets: $m = 5m'+p$, $p = 0, 1, 2, 3, 4$ and $m' \in \mathbb{Z}$. Thus, we obtain 5 independent orbits corresponding to 5 independent solutions.

Finally, consider the general case $1 < \gcd(r, k) < r$, and take, for example, $k = 6, r = 4$, where $\gcd(6, 4) = 2$. Here, we find

$$\begin{aligned} &d_{1,m} \to d_{3,m-1} \to d_{5,m-2} \to d_{1,m-2} \to \ldots , \\ &d_{2,m} \to d_{4,m-1} \to d_{6,m-2} \to d_{2,m-2} \to \ldots . \end{aligned} \tag{A.28}$$

The two lines depict independent solutions. However, we also find that there are independent orbits within each line. For example, $d_{1,m}$ shifts to $d_{1,m-2}$, etc. Thus, the integers are divided into two sets, odd and even. We conclude that there are two orbits in each line, and in total, we have 4 independent solutions, as expected. In this general case, we find that a simple relation gives the r solutions:

$$r = \underbrace{\gcd(k, r)}_{\text{number of vertical lines, case (1)}} \times \underbrace{\frac{r}{\gcd(k, r)}}_{\text{independent orbits, case (2)}} . \tag{A.29}$$

It is best to cast the above findings in a more effective compact notation. To this end, we define the functions:

$$\begin{aligned} \tilde{\Phi}^{(p)}_{C'C}(x) \equiv\ & e^{-\frac{\pi r x_1^2}{kL_1L_2}} e^{-\frac{\pi x_3^2}{\ell L_3 L_4}} \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \ \sum_{n'\in\mathbb{Z}} \\ &\times e^{-\frac{\pi L_3}{\ell L_4}\left(C-i\frac{(1-\ell)L_4}{2L_3}-\frac{1+\ell(2n'+1)}{2}\right)^2} e^{-\frac{\pi L_1}{krL_2}\left(C'+i\frac{r(1-k)L_2}{2L_1}-\frac{1+k(1-2m)}{2}\right)^2} \\ &\times e^{\frac{2\pi w_1}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{2\pi w_2}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} , \end{aligned} \tag{A.30}$$

for $p = 0, 1, \ldots, \frac{r}{\gcd(k,r)} - 1$. Thus, there are $\frac{r}{\gcd(k,r)}$ independent solutions correspond to $\frac{r}{\gcd(k,r)}$ independent orbits. We can also rewrite $\tilde{\Phi}^{(p)}_{C',C}$ conveniently as

$$\begin{aligned} \tilde{\Phi}^{(p)}_{C'C}(x) =& \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \ \sum_{n'\in\mathbb{Z}} \Bigg\{ e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{i\frac{2\pi x_4}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \\ &\times e^{\frac{\pi r(1-k)^2 L_2}{4kL_1}-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} \times e^{\frac{\pi(1-\ell)^2 L_3}{4\ell L_4}+i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)} \\ &\times e^{-\frac{\pi r}{kL_1L_2}\left(x_1-\frac{L_1(2mk+2C'-1-k)}{2r}\right)^2} e^{-\frac{\pi}{\ell L_3L_4}\left(x_3-\frac{L_3((2n'+1)\ell-(2C-1))}{2}\right)^2} \Bigg\} . \end{aligned} \tag{A.31}$$

JHEP09(2023)095

Since the terms $e^{\frac{\pi r(1-k)^2 L_2}{4kL_1}}$ and $e^{\frac{\pi(1-\ell)^2 L_3}{4\ell L_4}}$ are independent of m, n, C, C', we may drop them and define the function $\Phi^{(p)}_{C'C}(x)$ as:

$$\Phi^{(p)}_{C'C}(x) \equiv \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \sum_{n'\in\mathbb{Z}} \left\{ e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{i\frac{2\pi x_4}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \right.$$
$$\times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} \times e^{i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)}$$
$$\left. \times e^{-\frac{\pi r}{kL_1L_2}\left(x_1-\frac{L_1(2mk+2C'-1-k)}{2r}\right)^2} e^{-\frac{\pi}{\ell L_3 L_4}\left(x_3-\frac{L_3((2n'+1)\ell-(2C-1))}{2}\right)^2} \right\}. \quad \text{(A.32)}$$

The functions $\Phi^{(p)}_{C'C}(x)$ solve the equation

$$\bar{\sigma}^\mu \left[\partial_\mu \Phi^{(p)}_{C'C} + i2\pi N A^\omega_\mu \Phi^{(p)}_{C'C}\right] = 0\,, \quad \text{(A.33)}$$

and satisfy the boundary conditions

$$\begin{aligned}
\Phi^{(p)}_{C'C}(x+\hat{e}_1L_1) &= e^{-i\frac{\pi r(1-k)}{k}} e^{i\frac{2\pi r x_2}{kL_2}} \Phi^{(p)}_{[C'-r]_k\, C}(x)\,,\\
\Phi^{(p)}_{C'C}(x+\hat{e}_2L_2) &= e^{i\frac{2\pi(2C'-1-k)}{2k}} \Phi^{(p)}_{C'C}(x)\,,\\
\Phi^{(p)}_{C'C}(x+\hat{e}_3L_3) &= e^{-i\frac{\pi(1-\ell)}{\ell}} e^{i\frac{2\pi x_4}{\ell L_4}} \Phi^{(p)}_{C'\,[C+1]_\ell}(x)\,,\\
\Phi^{(p)}_{C'C}(x+\hat{e}_4L_4) &= e^{-i\frac{2\pi(2C-1-\ell)}{2\ell}} \Phi^{(p)}_{C'C}(x)\,,
\end{aligned} \quad \text{(A.34)}$$

which are the exact same boundary conditions (3.12). The entries with $C' = j, j+\gcd(k,r), j+2\gcd(k,r), \ldots, j+k-\gcd(k,r)$, for every $j = 1, 2, \ldots, \gcd(k,r)$, are shuffled to each other as we traverse the L_1 direction. Thus, the rows with $C' = 1, 2, \ldots, \gcd(k,r)$ are independent. In total, there are $\gcd(k,r) \times \frac{r}{\gcd(k,r)} = r$ independent solutions. In addition, $\Phi^{(p)}_{C'C}$ satisfy the cyclic properties:

$$\begin{aligned}
\Phi^{(p)}_{C'+k\, C}(x) &= \Phi^{(p+1)}_{C'C}(x)\,,\\
\Phi^{(p)}_{C'C}(x) &= \Phi^{\left(p+\frac{r}{\gcd(k,r)}\right)}_{C'C}(x)\,.
\end{aligned} \quad \text{(A.35)}$$

Notice the intertwining between the shift in p by 1 and C' by k.

We can use (A.35), noticing the intertwining between the shift in p and C', to write the r independent zero modes of the Dirac equation as

$$\lambda_{C'C}(x) = \sum_{p=0}^{\frac{r}{\gcd(k,r)}-1} \begin{bmatrix} \eta^{[C'+pk]_r} \\ 0 \end{bmatrix} \Phi^{(p)}_{C'C}(x)\,, \quad \text{(A.36)}$$

where $[x]_r \equiv x \bmod r$, and it is obvious that $\eta^{[C'+pk]_r}$ yields r independent coefficients. This is the desired equation (3.18) without holonomies.

A.2 Turning on holonomies

Next, we turn on the SU(k) space holonomies. In particular, the gauge field is now given by

$$A_\mu = -\left[\hat{A}^\omega_\mu + \phi_\mu\right]\omega + H^{a'}\phi^{a'}_\mu\,, \tag{A.37}$$

where $\phi_\mu = z_\mu/L_\mu$ are the abelian holonomies, $H^{a'}$, $a' = 1, 2, \ldots, k-1$ are the $k-1$ Cartan generators of the $su(k)$ algebra, and $\phi^{a'}_\mu$ are $k-1$ holonomies in every direction $\mu = 1, 2, 3, 4$. Next, we need to compute the commutator:

$$[H^{a'}\phi^{a'}_\mu, ||\lambda||_{C'C}] = \left(H^{a'}\phi^{a'}_\mu\right)_{C'C'}\lambda_{C'C} \equiv \phi^{C'}_\mu\lambda_{C'C}\,. \tag{A.38}$$

Recalling (A.1), we find it convenient to define

$$\hat{\phi}^{C'}_\mu = \phi^{C'}_\mu - 2\pi N\phi_\mu\,. \tag{A.39}$$

Noticing that A_μ has to commute with the transition functions, then out of k holonomies, there are at most gcd(k, r) holonomies in every spacetime direction. Thus, we find that $\hat{\phi}^{C'}_\mu = \hat{\phi}^{C'+r}_\mu$, or we can express this fact as

$$\hat{\phi}^{C'}_\mu = \hat{\phi}^{[C']_r}_\mu\,. \tag{A.40}$$

Using the above information in the Dirac equation $\bar{\sigma}^\mu D_\mu\lambda = 0$, we find (compare with (A.3))

$$\begin{aligned}\left(\partial_3 + i\hat{\phi}^{C'}_3 + i\partial_4 - \hat{\phi}^{[C']_r}_4 + \frac{2\pi x_3}{\ell L_3 L_4}\right)\lambda_{1C',C} = 0\,,\\ \left(\partial_1 + i\hat{\phi}^{C'}_1 + i\partial_2 - \hat{\phi}^{[C']_r}_2 + \frac{2\pi r x_1}{k L_1 L_2}\right)\lambda_{1C',C} = 0\,.\end{aligned} \tag{A.41}$$

and we have set $\lambda_{C'C\,2} = 0$, as in (A.3).

Next, we use the field redefinition

$$\lambda_{C'C\,1} = e^{-\frac{\pi r x_1^2}{k L_1 L_2}}e^{-\frac{\pi x_3^2}{L_3 L_4}}e^{-i x_\mu \hat{\phi}^{[C']_r}_\mu}U_{C'C} \tag{A.42}$$

in (A.41) to find that $U_{C'C}$ obeys the equations

$$(\partial_1 + i\partial_2)\,U_{C'C} = 0\,, \quad (\partial_3 + i\partial_4)\,U_{C'C} = 0\,. \tag{A.43}$$

These equations, as before, imply that $U_{C'C}$ is an analytic function of $w_1 \equiv x_1 + ix_2$ and $w_2 \equiv x_3 + ix_4$.

The BCS (3.12) can be rewritten in terms of the functions $U_{C'C}$:

$$\begin{aligned}U_{C'C}(w_1 + L_1, w_2) &= \gamma_k^{-r}e^{\frac{\pi r L_1}{k L_2} + \frac{2\pi r}{k L_2}w_1 + iL_1\hat{\phi}^{[C']_r}_1}\,U_{C'-r\,C}(w_1, w_2)\,,\\ U_{C'C}(w_1 + iL_2, w_2) &= e^{i\frac{\pi}{k}(2C'-1-k) + iL_2\hat{\phi}^{[C']_r}_2}\,U_{C'C}(w_1, w_2)\,,\\ U_{C'C}(w_1, w_2 + L_3) &= \gamma_\ell^{-1}e^{\frac{\pi L_3}{\ell L_4} + \frac{2\pi}{\ell L_4}w_2 + iL_3\hat{\phi}^{[C']_r}_3}\,U_{C'\,C+1}(w_1, w_2)\,,\\ U_{C'C}(w_1, w_2 + iL_4) &= e^{-i\frac{\pi}{\ell}(2C-\ell-1) + iL_4\hat{\phi}^{[C']_r}_4}\,U_{C'C}(w_1, w_2)\,.\end{aligned} \tag{A.44}$$

Similar to (A.10), the transformation properties under imaginary shifts of w_1 by iL_2 and w_2 by iL_4 are satisfied by writing $U_{C'C}(w_1, w_2)$ as the phase factor

$$e^{\frac{w_1}{L_2}\frac{\pi}{k}(2C'-1-k)+w_1\hat{\phi}_2^{[C']_r}-\frac{w_2}{L_4}\frac{\pi}{\ell}(2C-\ell-1)+w_2\hat{\phi}_4^{[C']_r}} \tag{A.45}$$

times an analytic function which is periodic w.r.t. these imaginary shifts. Thus, the expression for $U_{C'C}$ takes the form

$$\bar{U}_{C',C}(w_1, w_2) = e^{w_1\hat{\phi}_2^{[C']_r}+w_2\hat{\phi}_4^{[C']_r}+\frac{\pi w_1}{kL_2}(2C'-1-k)-\frac{\pi w_2}{\ell L_4}(2C-1-\ell)} \sum_{m,n\in\mathbb{Z}} d_{C',C,m,n}\, e^{2\pi m\frac{w_1}{L_2}+2\pi n\frac{w_2}{L_4}}\,, \tag{A.46}$$

which differs from (A.10) by the prefactor $e^{w_1\hat{\phi}_2^{[C']_r}+w_2\hat{\phi}_4^{[C']_r}}$.

As we proceed in the absence of holonomies, our next step involves determining the coefficients $d_{C',C,m,n}$ by utilizing the first and third boundary conditions in (A.44). These conditions lead to the following recurrence relations:

$$d_{C',C,m,n} = e^{-i\frac{\pi r(1-k)}{k}} e^{-\frac{\pi L_1}{kL_2}(2C'-r-1+(2m-1)k)} e^{iL_1(\hat{\phi}_1^{[C']_r}+i\hat{\phi}_2^{[C']_r})}\, d_{C'-r,C,m,n}\,, \tag{A.47}$$

and

$$d_{C',C,m,n} = e^{-i\frac{\pi(1-\ell)}{\ell}} e^{\frac{\pi L_3}{\ell L_4}(2C-(2n+1)\ell)} e^{iL_3(\hat{\phi}_3^{[C']_r}+i\hat{\phi}_4^{[C']_r})}\, d_{C',C+1,m,n}\,. \tag{A.48}$$

We observe that (A.47) and (A.48) become identical to (A.11) and (A.12) respectively, when we replace:

$$\begin{aligned} m &\longrightarrow m - \frac{iL_2}{2\pi}\left(\hat{\phi}_1^{[C']_r} + i\hat{\phi}_2^{[C']_r}\right)\,, \\ n &\longrightarrow n - \frac{iL_4}{2\pi}\left(\hat{\phi}_3^{[C']_r} + i\hat{\phi}_4^{[C']_r}\right)\,, \end{aligned} \tag{A.49}$$

in (A.11) and (A.12). Consequently, the solution to (A.47) and (A.48) is identical to (A.24) after making the replacement (A.49):

$$\begin{aligned} d_{C',C,m,n} = {} & e^{-\frac{\pi L_3}{\ell L_4}\left[C-i\frac{(1-\ell)L_4}{2L_3}-\frac{1+\ell(2n+1)}{2}+i\frac{\ell L_4}{2\pi}\left(\hat{\phi}_3^{[C']_r}+i\hat{\phi}_4^{[C']_r}\right)\right]^2} \\ & \times e^{-\frac{\pi L_1}{krL_2}\left[C'+i\frac{r(1-k)L_2}{2L_1}-\frac{1+k(1-2m)}{2}-i\frac{kL_2}{2\pi}\left(\hat{\phi}_1^{[C']_r}+i\hat{\phi}_2^{[C']_r}\right)\right]^2}\,, \end{aligned} \tag{A.50}$$

and we used the fact that $\phi_\mu^{[C']_r} = \phi_\mu^{[C'-r]_r}$.

Then, all the analyses in the absence of holonomies repeat precisely, with $\tilde{\Phi}^{(p)}_{C'C}(x, \hat{\phi})$ now given by the expression

$$\begin{aligned} \tilde{\Phi}^{(p)}_{C'C}(x, \hat{\phi}) \equiv {} & e^{-ix_\mu\hat{\phi}_\mu^{[C']_r}} e^{w_1\hat{\phi}_2^{[C'}]_r} e^{w_2\hat{\phi}_4^{[C']_r}} e^{-\frac{\pi r x_1^2}{kL_1L_2}} e^{-\frac{\pi x_3^2}{\ell L_3 L_4}} \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \; \sum_{n'\in\mathbb{Z}} \\ & \times e^{\frac{2\pi w_1}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{2\pi w_2}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \\ & \times e^{-\frac{\pi L_3}{\ell L_4}\left[C-i\frac{(1-\ell)L_4}{2L_3}-\frac{1+\ell(2n'+1)}{2}+i\frac{\ell L_4}{2\pi}\left(\hat{\phi}_3^{[C']_r}+i\hat{\phi}_4^{[C']_r}\right)\right]^2} \\ & \times e^{-\frac{\pi L_1}{krL_2}\left[C'+i\frac{r(1-k)L_2}{2L_1}-\frac{1+k(1-2m)}{2}-i\frac{kL_2}{2\pi}\left(\hat{\phi}_1^{[C']_r}+i\hat{\phi}_2^{[C']_r}\right)\right]^2}\,, \end{aligned} \tag{A.51}$$

JHEP09(2023)095

where the tilde service as a reminder that these are not precisely the functions we define in the bulk of the paper. The latter will be defined momentarily. Manipulating, we can rewrite $\tilde{\Phi}^{(p)}_{C'C}(x,\hat{\phi})$ in the more convenient form (easier form for taking derivatives)

$$
\begin{aligned}
\tilde{\Phi}^{(p)}_{C'C}(x,\hat{\phi}) = & \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \; \sum_{n'\in\mathbb{Z}} e^{\frac{i2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{i2\pi x_4}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \\
& \times e^{\frac{\pi r(1-k)^2 L_2}{4kL_1} - i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2} - i\frac{kL_2}{2\pi}\left(\hat{\phi}_1^{[C']_r}+i\hat{\phi}_2^{[C']_r}\right)\right)} \\
& \times e^{\frac{\pi(1-\ell)^2 L_3}{4\ell L_4} + i\frac{\pi(1-\ell)}{\ell}\left(B-\frac{1+\ell(2n'+1)}{2} + i\frac{\ell L_4}{2\pi}\left(\hat{\phi}_3^{[C']_r}+i\hat{\phi}_4^{[C']_r}\right)\right)} \\
& \times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\left(\hat{\phi}_2^{[C']_r}-i\hat{\phi}_1^{[C']_r}\right)-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\
& \times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\left(\hat{\phi}_4^{[C']_r}-i\hat{\phi}_3^{[C']_r}\right)-L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right]^2} .
\end{aligned}
\tag{A.52}
$$

The terms $e^{\frac{\pi r(1-k)^2L_2}{4kL_1}}$, $e^{-i\frac{\pi(1-k)}{k}\left(-i\frac{kL_2}{2\pi}\left(\hat{\phi}_1^{[C']_r}+i\hat{\phi}_2^{[C']_r}\right)\right)}$, $e^{\frac{\pi(1-\ell)^2L_3}{4\ell L_4}}$, and $e^{i\frac{\pi(1-\ell)}{\ell}\left(i\frac{\ell L_4}{2\pi}\left(\hat{\phi}_3^{[C']_r}+i\hat{\phi}_4^{[C']_r}\right)\right)}$ do not explicitly depend on C, C', m, n', and thus, it is convenient to drop them[24] and define the function $\Phi^{(p)}_{C'C}(x,\hat{\phi})$ as:

$$
\begin{aligned}
\Phi^{(p)}_{C'C}(x,\hat{\phi}) \equiv & \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \; \sum_{n'\in\mathbb{Z}} e^{\frac{i2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{i2\pi x_4}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \\
& \times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} e^{i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)} \\
& \times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\left(\hat{\phi}_2^{[C']_r}-i\hat{\phi}_1^{[C']_r}\right)-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\
& \times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\left(\hat{\phi}_4^{[C']_r}-i\hat{\phi}_3^{[C']_r}\right)-L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right]^2} .
\end{aligned}
\tag{A.53}
$$

We may also write $\Phi^{(p)}_{C'C}(x,\hat{\phi})$ in the form

$$
\begin{aligned}
\Phi^{(p)}_{C'C}(x,\hat{\phi}) = & \; e^{\frac{kL_1L_2}{2\pi r}\hat{\phi}_1^{[C']_r}\left(i\hat{\phi}_2^{[C']_r}+\hat{\phi}_1^{[C']_r}/2\right)} e^{\frac{\ell L_3L_4}{2\pi}\hat{\phi}_3^{[C']_r}\left(i\hat{\phi}_4^{[C']_r}+\hat{\phi}_3^{[C']_r}/2\right)} e^{-i\hat{\phi}_1^{[C']_r}x_1} e^{-i\hat{\phi}_3^{[C']_r}x_3} \\
\times & \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \; \sum_{n'\in\mathbb{Z}} e^{i\left(\frac{2\pi x_2}{L_2}+\frac{L_1k}{r}\hat{\phi}_1^{[C']_r}\right)\left(m+\frac{2C'-1-k}{2k}\right)} e^{i\left(\frac{2\pi x_4}{L_4}+\ell L_3\hat{\phi}_3^{[C']_r}\right)\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \\
& \times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} e^{i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)} \\
& \times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\hat{\phi}_2^{[C']_r}-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\
& \times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\hat{\phi}_4^{[C']_r}-L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right]^2} .
\end{aligned}
\tag{A.54}
$$

Finally, the fermion zero modes are given by (compare with (A.36))

$$
\lambda_{C'C}(x) = \sum_{p=0}^{\frac{r}{\gcd(k,r)}-1} \begin{bmatrix} \eta^{[C'+pk]_r} \\ 0 \end{bmatrix} \Phi^{(p)}_{C'C}(x,\hat{\phi}) \,. \tag{A.55}
$$

[24] One can show that they can be absorbed into the coefficients $\eta^{[C'+pk]_r}$ of the general solution of the Dirac equation, see (A.36) or (A.55) below.

JHEP09(2023)095

B A useful identity

Here, we evaluate the expression I_j^{ab} defined in (5.1), $j = 0, \ldots, \gcd(k,r)-1$, repeated here

$$I_j^{ab} = \sum_{C'=0}^{k-1} \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1} \delta_{C',[j+nr]_k} \sum_{p,p'=0}^{\frac{r}{\gcd(k,r)}-1} \frac{\mathcal{C}_a^{[C'+pk]_r}\, \mathcal{C}_b^{*\,[C'+p'k]_r}}{\sqrt{V}} \int_{\mathbb{T}^4} \sum_{B=0}^{\ell-1} \Phi^{(p)}_{C'B} \Phi^{(p')\,*}_{C'B}. \tag{B.1}$$

For convenience, we also repeat the expression for $\Phi^{(p)}$ (3.21):

$$\begin{aligned} \Phi^{(p)}_{C'B}(x,\hat{\phi}) = & \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \; \sum_{n'\in\mathbb{Z}} e^{\frac{i2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{i2\pi x_4}{L_4}\left(n'-\frac{2B-1-\ell}{2\ell}\right)} \\ & \times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} e^{i\frac{\pi(1-\ell)}{\ell}\left(B-\frac{1+\ell(2n'+1)}{2}\right)} \\ & \times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\left(\hat{\phi}_2^{[C']_r}-i\hat{\phi}_1^{[C']_r}\right)-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\ & \times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\left(\hat{\phi}_4^{[C']_r}-i\hat{\phi}_3^{[C']_r}\right)-L_3\left(\ell n'-\frac{2B-1-\ell}{2}\right)\right]^2}. \end{aligned} \tag{B.2}$$

To calculate I_j^{ab}, we now make a few observations, which help evaluate (B.1):

1. The integral over x_4 can be taken, yielding a factor of L_4 and the condition $\delta_{n',\tilde{n}'}$, where n' is the index of summation from $\Phi^{(p)}$ and $\tilde{n}'$ coming from $\Phi^{(p')\,*}$.

2. The sum over $B = 0, \ldots, \ell-1$ allows to extend the range of the x_3 integral from $-\infty, +\infty$, implying that the $\hat{\phi}_4$-dependence disappears.[25]

3. The integral over x_2 can also be taken, yielding an overall factor of L_2 and the constraint $\delta_{m,\tilde{m}}$, where m is from $\Phi^{(p)}$ and $\tilde{m}$ is from $\Phi^{(p)\,*}$. Note, in view of the definition of m ($\tilde{m}$) in (B.2), $m = \tilde{m}$ implies, recalling the range of p, p', that $p = p'$ and $m' = \tilde{m}'$. Thus, in the end of this step, we are left with an expression that contains only sums over C', n, p, and m', and only an integral over the x_1 direction of $\mathbb{T}^4$.

4. We also note that, for each j, only values of C' equal to $[j+nr]_k$ enter the sum (B.1) defining I_j^{ab}, with n taking values in the range given. Now is time to recall the relation (3.20) defining the independent holonomies. It shows that all these have the same $\hat{\phi}_\mu^{C'}$ and thus I_j^{ab} only depends on the $\gcd(k,r)$ independent φ_μ^j — as we explicitly indicate in (B.3) below.

[25] However, some factor of $\hat{\phi}_3$ remains which we will have to keep track of when evaluating the Gaussian integral over x_3.

JHEP09(2023)095

Explicitly performing the steps outlined in the above list, we obtain an intermediate result for (B.1),

$$
\begin{aligned}
I_j^{ab} &= \sqrt{V}\sqrt{\frac{\ell L_4}{2L_3}}\; e^{\frac{L_1L_2k}{2\pi r}(\varphi_1^j)^2}\; e^{\frac{L_3L_4\ell}{2\pi}(\varphi_3^j)^2} \\
&\times \sum_{n=0}^{\frac{k}{\gcd(k,r)}-1}\; \sum_{p=0}^{\frac{r}{\gcd(k,r)}-1} (\mathcal{C}_a\mathcal{C}_b^*)^{[[j+nr]_k+pk]_r} \\
&\times \sum_{m'\in\mathbb{Z}} \int_0^1 dx\; e^{-\frac{2\pi r L_1}{kL_2}\left(x-\frac{kp+[j+nr]_k}{r}+\frac{1+k}{2r}-\frac{k}{\gcd(k,r)}m'-\frac{kL_2}{2\pi r L_1}\varphi_2^j\right)^2}\,,
\end{aligned} \tag{B.3}
$$

which only contains a single integral over x_1, rescaled by L_1 and denoted by x. For brevity, we also denote $(\mathcal{C}_a\mathcal{C}_b^*)^A \equiv \mathcal{C}_a^A\, \mathcal{C}_b^{*\,A}$.

The next step is to rearrange the sum (B.3) for I_j^{ab} by grouping together terms where the “moduli” product $(\mathcal{C}_a\mathcal{C}_b^*)^A$ has the same index. Recall that apriori A can take values in the range $A \in 0,\ldots,r-1$. However, it is important to realize not all allowed values of A appear in the sum defining I_j^{ab} for a given j. One numerically finds that for any given j, the index $A \equiv [[j+nr]_k+pk]_r$ takes only $\frac{r}{\gcd(k,r)}$ of its possible r values as n and p scan their possible values in the sum in (B.3).

To proceed further, we denote by S_j each of the $\gcd(k,r)$ sets of $\frac{r}{\gcd(k,r)}$ values that A can take for a given j:

$$
\begin{aligned}
S_j &= \left\{[[j+nr]_k+pk]_r, \text{for } n=0,\ldots\frac{k}{\gcd(k,r)}-1, \text{and } p=0,\ldots,\frac{r}{\gcd(k,r)}-1\right\}, \\
|S_j| &= \frac{r}{\gcd(k,r)}\,,
\end{aligned} \tag{B.4}
$$

where we stress that repeated values of $[[j+nr]_k+pk]_r$ are identified in S_j and that the set has $r/\gcd(k,r)$ elements. The sets S_j are straightforward to generate numerically in each case (we have used numerics extensively to obtain our final answer (B.6) below). A few examples might be useful:

$$
\begin{aligned}
k=5, r=4\ (\gcd(k,r)=1):&\quad S_0=\{0,1,2,3\}, \\
k=6, r=4\ (\gcd(k,r)=2):&\quad S_0=\{0,2\},\ S_1=\{1,3\}, \\
k=4, r=4\ (\gcd(k,r)=4):&\quad S_0=\{0\},\ S_1=\{1\},\ S_2=\{2\},\ S_3=\{3\}, \\
k=15, r=9\ (\gcd(k,r)=3):&\quad S_0=\{0,3,6\},\ S_1=\{1,4,7\},\ S_2=\{2,5,8\},
\end{aligned} \tag{B.5}
$$

while, e.g., for $k=9$, $r=9$ ($\gcd(k,r)=9$), all 9 sets S_j have a single element, similar to the $k=r=4$ case above. This illustrates a general feature of the $k=r$ case, which will be important in our studies of the moduli space.

The next step is the most important to obtain our final answer. For each different value of $A \in S_j$ that appears in I_j^{ab}, one also finds that $(\mathcal{C}_a\mathcal{C}_b^*)^A$ is multiplied by an integral $\int_0^1 dx$. The integral is, however, summed over $\frac{k}{\gcd(k,r)}$ times, each time with a different constant term appearing in the exponent in the integrand, due to the $(kp + [j + nr]_k)/r$ term. Remarkably, in each case one finds that, together with the sum over m', these constant terms take precisely the values needed to extend the range of the integration over x to the entire real line.[26] Performing the Gaussian integral over x, the final answer for I_j^{ab} is then remarkably simple

$$I_j^{ab} = \frac{\sqrt{V}}{2}\sqrt{\frac{\ell k L_2 L_4}{r L_1 L_3}}\, e^{\frac{L_1 L_2 k}{2\pi r}(\varphi_1^j)^2}\ e^{\frac{L_3 L_4 \ell}{2\pi}(\varphi_3^j)^2} \sum_{A_j \in S_j} (\mathcal{C}_a \mathcal{C}_b^*)^{A_j}\,. \tag{B.6}$$

The complexity is, of course, hidden away in the definition of the S_j sets from (B.4).

C Field strength and action of the multifractional instanton

Here, we compute the field strength $F_{\mu\nu}$, which we shall use to compute the action density and to verify that the action of the self-dual solution satisfies the relation $S = \frac{8\pi^2|Q|}{g^2}$. The non-zero components of $\mathcal{F}^{(0)}_{\mu\nu}$ are

$$\mathcal{F}^{(0)}_{13\,C',C} = -i\hat{D}_1 \mathcal{W}_{4C',C} + i\hat{D}_3 \mathcal{W}_{2\,C',C}\,, \quad \mathcal{F}^{(0)}_{14,C',C} = \hat{D}_1 \mathcal{W}_{4\,C',C} + i\hat{D}_4 \mathcal{W}_{2\,C',C}\,, \tag{C.1}$$

where $\mathcal{W}^{(0)}_{2\,C',C}$ and $\mathcal{W}^{(0)}_{4\,C',C}$ are from (4.21). The covariant derivatives $\hat{D}_\mu$ are given by

$$\hat{D}_\mu = \partial_\mu + i2\pi N \hat{A}_\mu + i\hat{\phi}_\mu^{[C']_r}\,, \tag{C.2}$$

or in terms of the components, with $\hat{\phi}_\mu^{[C']_r}$ from (3.20),

$$\begin{aligned} \hat{D}_1 &= \partial_1 + i\hat{\phi}_1^{[C']_r}\,, & \hat{D}_2 &= \partial_2 - i\frac{2\pi r x_1}{k L_1 L_2} + i\hat{\phi}_2^{[C']_r} \\ \hat{D}_3 &= \partial_3 + i\hat{\phi}_3^{[C']_r}\,, & \hat{D}_4 &= \partial_4 - i\frac{2\pi x_3}{\ell L_3 L_4} + i\hat{\phi}_4^{[C']_r}\,. \end{aligned} \tag{C.3}$$

One can check that the following identities hold

$$i\hat{D}_1 \Phi^{(p)}_{C',C} = \hat{D}_2 \Phi^{(p)}_{C',C}\,, \quad i\hat{D}_3 \Phi^{(p)}_{C',C} = \hat{D}_4 \Phi^{(p)}_{C',C}\,. \tag{C.4}$$

Then, one finds

$$\begin{aligned} -i\mathcal{F}^{(0)}_{14C',C} &= \mathcal{F}^{(0)}_{13C',C} = iV^{-1/4} \sum_{p=0}^{\frac{r}{\gcd(k,r)}-1} \left\{ -\mathcal{C}_4^{[C'+pk]_r} \mathcal{G}^{(p)}_{1,C',C}(x,\hat{\phi}) + \mathcal{C}_2^{[C'+pk]_r} \mathcal{G}^{(p)}_{3,C',C}(x,\hat{\phi}) \right\}, \\ \mathcal{F}^{(0)}_{12C',C} &= \mathcal{F}^{(0)}_{34C',C} = 0\,, \end{aligned} \tag{C.5}$$

[26] Admittedly, we have only numerical checks of this claim rather than an analytic proof. However, the checks are fairly easy to automate and the result is the same in each of the many cases we have studied.

where the functions $\mathcal{G}^{(p)}_{1,C',C}(x,\hat{\phi})$ and $\mathcal{G}^{(p)}_{3,C',C}(x,\hat{\phi})$ are defined as

$$
\begin{aligned}
\mathcal{G}^{(p)}_{1,C',C}(x,\hat{\phi}) &= \hat{D}_1 \Phi^{(p)}_{C',C}(x,\hat{\phi}) \\
&= -\frac{2\pi r}{kL_1L_2} \sum_{m=p+\frac{rm'}{\gcd(k,r)},m'\in\mathbb{Z}} \sum_{n'\in\mathbb{Z}} e^{\frac{i2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{i2\pi x_4}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \qquad \text{(C.6)}\\
&\quad \times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} e^{i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)} \\
&\quad \times \left(x_1 - \frac{kL_1L_2\hat{\phi}_2^{[C']_r}}{2\pi r} - \frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right) \\
&\quad \times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\left(\hat{\phi}_2^{[C']_r}-i\hat{\phi}_1^{[C']_r}\right)-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\
&\quad \times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\left(\hat{\phi}_4^{[C']_r}-i\hat{\phi}_3^{[C']_r}\right)-L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right]^2}, \qquad \text{(C.7)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{G}^{(p)}_{3,C',C}(x,\hat{\phi}) &= \hat{D}_3 \Phi^{(p)}_{C',C}(x,\hat{\phi}) \\
&= -\frac{2\pi}{\ell L_3L_4} \sum_{m=p+\frac{rm'}{\gcd(k,r)},m'\in\mathbb{Z}} \sum_{n'\in\mathbb{Z}} e^{\frac{i2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)} e^{\frac{i2\pi x_4}{L_4}\left(n'-\frac{2C-1-\ell}{2\ell}\right)} \qquad \text{(C.8)}\\
&\quad \times e^{-i\frac{\pi(1-k)}{k}\left(C'-\frac{1+k(1-2m)}{2}\right)} e^{i\frac{\pi(1-\ell)}{\ell}\left(C-\frac{1+\ell(2n'+1)}{2}\right)} \\
&\quad \times \left(x_3 - \frac{\ell L_3L_4\hat{\phi}_4^{[C']_r}}{2\pi} - L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right) \\
&\quad \times e^{-\frac{\pi r}{kL_1L_2}\left[x_1-\frac{kL_1L_2}{2\pi r}\left(\hat{\phi}_2^{[C']_r}-i\hat{\phi}_1^{[C']_r}\right)-\frac{L_1}{r}\left(km+\frac{2C'-1-k}{2}\right)\right]^2} \\
&\quad \times e^{-\frac{\pi}{\ell L_3L_4}\left[x_3-\frac{\ell L_3L_4}{2\pi}\left(\hat{\phi}_4^{[C']_r}-i\hat{\phi}_3^{[C']_r}\right)-L_3\left(\ell n'-\frac{2C-1-\ell}{2}\right)\right]^2}. \qquad \text{(C.9)}
\end{aligned}
$$

Owing to the self-duality of the solution, we also have:

$$
\mathcal{F}^{(0)}_{23\,C',C} = \mathcal{F}^{(0)}_{14\,C',C}\,, \quad \mathcal{F}^{(0)}_{24\,C',C} = -\mathcal{F}^{(0)}_{13\,C',C}\,. \qquad \text{(C.10)}
$$

In the following, we calculate the action density $\mathrm{tr}\,[F_{\mu\nu}F_{\mu\nu}]$ of the twisted solution. Using (4.5), the square of the field strength is

$$
\begin{aligned}
F_{\mu\nu}F_{\mu\nu} &= \omega^2\left(\hat{F}^\omega_{\mu\nu}+F^s_{\mu\nu}\right)^2 + 4\pi\left(\hat{F}^\omega_{\mu\nu}+F^s_{\mu\nu}\right)\begin{bmatrix} \ell F^k_{\mu\nu} & \mathcal{F}^{k\times\ell}_{\mu\nu} \\ \mathcal{F}^{\dagger\ell\times k}_{\mu\nu} & -kF^\ell_{\mu\nu}\end{bmatrix} \\
&\quad + \begin{bmatrix} F^k_{\mu\nu}F^k_{\mu\nu}+\mathcal{F}^{k\times\ell}_{\mu\nu}\mathcal{F}^{\dagger\ell\times k}_{\mu\nu} & F^k_{\mu\nu}\mathcal{F}^{k\times\ell}_{\mu\nu}+\mathcal{F}^{k\times\ell}_{\mu\nu}F^\ell_{\mu\nu} \\ \mathcal{F}^{\dagger\ell\times k}_{\mu\nu}F^k_{\mu\nu}+F^\ell_{\mu\nu}\mathcal{F}^{\dagger\ell\times k}_{\mu\nu} & F^\ell_{\mu\nu}F^\ell_{\mu\nu}+\mathcal{F}^{\dagger\ell\times k}_{\mu\nu}\mathcal{F}^{k\times\ell}_{\mu\nu}\end{bmatrix}. \qquad \text{(C.11)}
\end{aligned}
$$

Then, the action density is given by the trace

$$
\begin{aligned}
\mathrm{tr}\,[F_{\mu\nu}F_{\mu\nu}] &= \mathrm{tr}\left[\omega^2\right]\left(\hat{F}^\omega_{\mu\nu}+F^s_{\mu\nu}\right)^2 \\
&\quad +4\pi\ell\left(\hat{F}^\omega_{\mu\nu}+F^s_{\mu\nu}\right)\mathrm{tr}_k\left[F^k_{\mu\nu}\right] - 4\pi k\left(\hat{F}^\omega_{\mu\nu}+F^s_{\mu\nu}\right)\mathrm{tr}_\ell\left[F^\ell_{\mu\nu}\right] \\
&\quad +\mathrm{tr}_k\left[F^k_{\mu\nu}F^k_{\mu\nu}+\mathcal{F}^{k\times\ell}_{\mu\nu}\mathcal{F}^{\dagger\ell\times k}_{\mu\nu}\right] + \mathrm{tr}_\ell\left[F^\ell_{\mu\nu}F^\ell_{\mu\nu}+\mathcal{F}^{\dagger\ell\times k}_{\mu\nu}\mathcal{F}^{k\times\ell}_{\mu\nu}\right]. \qquad \text{(C.12)}
\end{aligned}
$$

JHEP09(2023)095

To leading order in Δ we have:

$$\begin{aligned}
F^s_{\mu\nu} &= \Delta\left(\partial_\mu S^{\omega(0)}_\nu - \partial_\nu S^{\omega(0)}_\mu\right) ,\\
F^k_{\mu\nu} &= \Delta\left(\partial_\mu S^{k(0)}_\nu - \partial_\nu S^{k(0)}_\mu + i\mathcal{W}^{(0)k\times\ell}_\mu \mathcal{W}^{\dagger(0)\ell\times k}_\nu - i\mathcal{W}^{(0)k\times\ell}_\nu \mathcal{W}^{\dagger(0)\ell\times k}_\mu\right) ,\\
F^\ell_{\mu\nu} &= \Delta\left(\partial_\mu S^{\ell(0)}_\nu - \partial_\nu S^{\ell(0)}_\mu + i\mathcal{W}^{\dagger(0)\ell\times k}_\mu \mathcal{W}^{(0)k\times\ell}_\nu - i\mathcal{W}^{\dagger(0)\ell\times k}_\nu \mathcal{W}^{(0)k\times\ell}_\mu\right) ,\\
\mathcal{F}^{k\times\ell}_{\mu\nu} &= \sqrt{\Delta}\mathcal{F}^{(0)k\times\ell}_{\mu\nu} = \sqrt{\Delta}\left(\hat{D}_\mu \mathcal{W}^{(0)k\times\ell}_\nu - \hat{D}_\nu \mathcal{W}^{(0)k\times\ell}_\mu\right) .
\end{aligned} \tag{C.13}$$

Substituting (C.13) into (C.12), we find to $\mathcal{O}(\Delta)$:

$$\begin{aligned}
\mathrm{tr}\left[F_{\mu\nu}F_{\mu\nu}\right] = \mathrm{tr}\left[\omega^2\right]&\left(\hat{F}^\omega_{\mu\nu}\hat{F}^\omega_{\mu\nu} + 2\Delta(\partial_\mu \mathcal{S}^{\omega(0)}_\nu - \partial_\nu \mathcal{S}^{\omega(0)}_\mu)\hat{F}^\omega_{\mu\nu}\right)\\
&+4\pi\ell\Delta\hat{F}^\omega_{\mu\nu}\mathrm{tr}_k\left[\partial_\mu \mathcal{S}^{k(0)}_\nu - \partial_\nu \mathcal{S}^{k(0)}_\mu\right] - 4\pi k\Delta\hat{F}^\omega_{\mu\nu}\mathrm{tr}_\ell\left[\partial_\mu \mathcal{S}^{\ell(0)}_\nu - \partial_\nu \mathcal{S}^{\ell(0)}_\mu\right]\\
&+i4\pi\ell\Delta\hat{F}^\omega_{\mu\nu}\mathrm{tr}_k\left[\mathcal{W}^{(0)k\times\ell}_\mu \mathcal{W}^{\dagger(0)\ell\times k}_\nu - \mathcal{W}^{(0)k\times\ell}_\nu \mathcal{W}^{\dagger(0)\ell\times k}_\mu\right]\\
&-i4\pi k\Delta\hat{F}^\omega_{\mu\nu}\mathrm{tr}_\ell\left[\mathcal{W}^{\dagger(0)\ell\times k}_\mu \mathcal{W}^{(0)k\times\ell}_\nu - \mathcal{W}^{\dagger(0)\ell\times k}_\nu \mathcal{W}^{(0)k\times\ell}_\mu\right]\\
&+\Delta\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{\mu\nu}\mathcal{F}^{\dagger(0)\ell\times k}_{\mu\nu}\right] + \Delta\mathrm{tr}_\ell\left[\mathcal{F}^{\dagger(0)\ell\times k}_{\mu\nu}\mathcal{F}^{(0)k\times\ell}_{\mu\nu}\right] .
\end{aligned} \tag{C.14}$$

Then, using the trace properties $\mathrm{tr}_k[\mathcal{S}^{(0)k}_\mu] = \mathrm{tr}_\ell[\mathcal{S}^{(0)\ell}_\mu] = 0$, along with

$$\begin{aligned}
\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{\mu\nu}\mathcal{F}^{\dagger(0)\ell\times k}_{\mu\nu}\right] &= \mathrm{tr}_\ell\left[\mathcal{F}^{\dagger(0)\ell\times k}_{\mu\nu}\mathcal{F}^{(0)k\times\ell}_{\mu\nu}\right] , \qquad \text{(C.15)}\\
\mathrm{tr}_k\left[\mathcal{W}^{(0)k\times\ell}_\mu \mathcal{W}^{\dagger(0)\ell\times k}_\nu - \mathcal{W}^{(0)k\times\ell}_\nu \mathcal{W}^{\dagger(0)\ell\times k}_\mu\right] &= -\mathrm{tr}_\ell\left[\mathcal{W}^{\dagger(0)\ell\times k}_\mu \mathcal{W}^{(0)k\times\ell}_\nu - \mathcal{W}^{\dagger(0)\ell\times k}_\nu \mathcal{W}^{(0)k\times\ell}_\mu\right] ,
\end{aligned}$$

we find to $\mathcal{O}(\Delta)$

$$\begin{aligned}
\mathrm{tr}\left[F_{\mu\nu}F_{\mu\nu}\right] = \mathrm{tr}\left[\omega^2\right]&\left(\hat{F}^\omega_{\mu\nu}\hat{F}^\omega_{\mu\nu} + 2\Delta(\partial_\mu \mathcal{S}^{\omega(0)}_\nu - \partial_\nu \mathcal{S}^{\omega(0)}_\mu)\hat{F}^\omega_{\mu\nu}\right)\\
&+i4\pi N\Delta\hat{F}^\omega_{\mu\nu}\mathrm{tr}_k\left[\mathcal{W}^{(0)k\times\ell}_\mu \mathcal{W}^{\dagger(0)\ell\times k}_\nu - \mathcal{W}^{(0)k\times\ell}_\nu \mathcal{W}^{\dagger(0)\ell\times k}_\mu\right]\\
&+2\Delta\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{\mu\nu}\mathcal{F}^{\dagger(0)\ell\times k}_{\mu\nu}\right] .
\end{aligned} \tag{C.16}$$

In the following, we perform the calculation of the action setting $\mathcal{C}_4^{[C']_r} = 0$. Thus, recalling (5.4), we are particularly interested in the cases $r = 1$ and $r = k, k > 1$. However, the conclusion should hold in the general case. Keeping only the non-zero entries and using $-i\mathcal{F}^{(0)\beta}_{14} = \mathcal{F}^{(0)\beta}_{13}$ along with the self-duality property, we arrive at

$$\begin{aligned}
\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{\mu\nu}\mathcal{F}^{\dagger(0)\ell\times k}_{\mu\nu}\right] &= 2\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{13}\mathcal{F}^{\dagger(0)\ell\times k}_{13}\right] + 2\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{14}\mathcal{F}^{\dagger(0)\ell\times k}_{14}\right]\\
&\quad+2\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{23}\mathcal{F}^{\dagger(0)\ell\times k}_{23}\right] + 2\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{24}\mathcal{F}^{\dagger(0)\ell\times k}_{24}\right]\\
&= 8\mathrm{tr}_k\left[\mathcal{F}^{(0)k\times\ell}_{13}\mathcal{F}^{\dagger(0)\ell\times k}_{13}\right] .
\end{aligned} \tag{C.17}$$

Likewise:

$$\hat{F}^\omega_{\mu\nu}\mathrm{tr}_k\left[\mathcal{W}^{(0)k\times\ell}_\mu \mathcal{W}^{\dagger(0)\ell\times k}_\nu - \mathcal{W}^{(0)k\times\ell}_\nu \mathcal{W}^{\dagger(0)\ell\times k}_\mu\right] = -4i\hat{F}^\omega_{12}\mathrm{tr}_k\left[\mathcal{W}^{(0)k\times\ell}_2 \mathcal{W}^{\dagger(0)\ell\times k}_2\right] . \tag{C.18}$$

Thus, the action density is given by the expression

$$\begin{aligned}\mathrm{tr}\left[F_{\mu\nu}F_{\mu\nu}\right] = \mathrm{tr}\left[\omega^2\right]\left(\hat{F}^{\omega}_{\mu\nu}\hat{F}^{\omega}_{\mu\nu} + 2\Delta(\partial_\mu \mathcal{S}^{\omega}_{\nu} - \partial_\nu \mathcal{S}^{\omega}_{\mu})\hat{F}^{\omega}_{\mu\nu}\right)\\ +16\pi N\Delta \hat{F}^{\omega}_{12}\mathrm{tr}_k\left[\mathcal{W}_2^{(0)k\times\ell}\mathcal{W}_2^{\dagger(0)\ell\times k}\right]\\ +16\Delta\mathrm{tr}_k\left[\mathcal{F}_{13}^{(0)k\times\ell}\mathcal{F}_{13}^{\dagger(0)\ell\times k}\right]\,.\end{aligned} \tag{C.19}$$

The action is

$$S = \frac{2}{g^2}\int_{\mathbb{T}^4}\mathrm{tr}\left[F_{\mu\nu}F_{\mu\nu}\right]\,, \tag{C.20}$$

and upon integrating, the term $\partial_\mu \mathcal{S}^{(0)\omega}_{\nu}$ drops out because $\mathcal{S}^{(0)\omega}_{\nu}$ satisfies periodic boundary conditions. Thus, we finally have to $\mathcal{O}(\Delta)$:

$$S = S_0 + \frac{\Delta}{2g^2}\int_{\mathbb{T}^4} 16\pi N\hat{F}^{\omega}_{12}\mathrm{tr}_k\left[\mathcal{W}_2^{(0)k\times\ell}\mathcal{W}_2^{\dagger(0)\ell\times k}\right] + \frac{\Delta}{2g^2}\int_{\mathbb{T}^4} 16\mathrm{tr}_k\left[\mathcal{F}_{13}^{(0)k\times\ell}\mathcal{F}_{13}^{\dagger(0)\ell\times k}\right]\,,$$

where

$$\begin{aligned}S_0 &= \frac{1}{2g^2}\int_{\mathbb{T}^4}\mathrm{tr}\left[\omega^2\right]\left(\hat{F}^{\omega}_{\mu\nu}\hat{F}^{\omega}_{\mu\nu}\right) = \frac{1}{2g^2}\int_{\mathbb{T}^4}\mathrm{tr}\left[\omega^2\right]\left\{2\left(\hat{F}^{\omega}_{12}\hat{F}^{\omega}_{12} + \hat{F}^{\omega}_{34}\hat{F}^{\omega}_{34}\right)\right\}\\ &= (4\pi^2 Nk\ell)\frac{1}{g^2N^2}\left(\frac{r^2}{k^2}\frac{L_3L_4}{L_1L_2} + \frac{1}{\ell^2}\frac{L_1L_2}{L_3L_4}\right)\,.\end{aligned} \tag{C.21}$$

Using the definition of Δ (4.9) we readily find

$$S_0 = \frac{8\pi^2 r}{Ng^2} + \mathcal{O}(\Delta^2)\,. \tag{C.22}$$

Then, using $\hat{F}^{\omega}_{12} = -\frac{r}{kNL_1L_2}$, we have

$$S = S_0 + \frac{\Delta}{g^2}\left(-\frac{8\pi r}{kL_1L_2}\int_{\mathbb{T}^4}\mathrm{tr}_k\left[\mathcal{W}_2^{(0)k\times\ell}\mathcal{W}_2^{\dagger(0)\ell\times k}\right] + 8\int_{\mathbb{T}^4}\mathrm{tr}_k\left[\mathcal{F}_{13}^{(0)k\times\ell}\mathcal{F}_{13}^{\dagger(0)\ell\times k}\right]\right)\,. \tag{C.23}$$

Finally, the remaining integrals are given by (we set all holonomies to 0, as the final answer will not depend on them):

$$\begin{aligned}\int_{\mathbb{T}^4}\mathrm{tr}_k\left[\mathcal{W}_2^{(0)k\times\ell}\mathcal{W}_2^{\dagger(0)\ell\times k}\right] = \sqrt{L_1L_2L_3L_4}\sum_{C=1}^{\ell}\sum_{C'=1}^{k}|\mathcal{C}_2^{[C']_r}|^2\\ \times\sum_{m=p+\frac{rm'}{\gcd(k,r)},\,m'\in\mathbb{Z}}\int_0^1 d\tilde{x}_1 e^{-\frac{2\pi rL_1}{kL_2}\left(\tilde{x}_1-\frac{2mk+2(j+nr)-1-k}{2r}\right)^2}\\ \times\sum_{n'\in\mathbb{Z}}\int_0^1 d\tilde{x}_3 e^{-\frac{2\pi L_3}{\ell L_4}\left(\tilde{x}_3-\frac{(2n'+1)\ell-(2C-1)}{2}\right)^2}\,,\end{aligned} \tag{C.24}$$

JHEP09(2023)095

and

$$\begin{aligned}
&\int_{\mathbb{T}^4} \mathrm{tr}_k \left[\mathcal{F}_{13}^{(0)k\times\ell}\mathcal{F}_{13}^{\dagger(0)\ell\times k}\right] \\
&= \frac{4\pi^2}{\ell^2}\sqrt{\frac{L_1L_2L_3}{L_4^3}}\sum_{C=1}^{\ell}\sum_{C'=1}^{k}|\mathcal{C}_2^{[C']_r}|^2 \\
&\quad\times \sum_{m=p+\frac{rm'}{\gcd(k,r)},\, m'\in\mathbb{Z}} \int_0^1 d\tilde{x}_1 e^{-\frac{2\pi r L_1}{kL_2}\left(\tilde{x}_1-\frac{2mk+2(j+nr)-1-k}{2r}\right)^2} \\
&\quad\times \sum_{n'\in\mathbb{Z}}\int_0^1 d\tilde{x}_3\left(\tilde{x}_3-\frac{((2n'+1)\ell-(2C-1))}{2}\right)^2 e^{-\frac{2\pi L_3}{\ell L_4}\left(\tilde{x}_3-\frac{(2n'+1)\ell-(2C-1)}{2}\right)^2}.
\end{aligned} \tag{C.25}$$

Now, collecting terms of $\mathcal{O}(\Delta)$ and using $r\ell L_3L_4 = kL_1L_2$, thus ignoring corrections $\mathcal{O}(\Delta^2)$, we find:

$$\begin{aligned}
S \;=\; &S_0+8\pi\sqrt{\frac{r}{\ell k}}\frac{\Delta}{g^2}\sum_{C=1}^{\ell}\sum_{C'=1}^{k}|\mathcal{C}_2^{[C']_r}|^2 \\
&\times \sum_{m=p+\frac{rm'}{\gcd(k,r)},m'\in\mathbb{Z}} \int_0^1 d\tilde{x}_1 e^{-\frac{2\pi r L_1}{kL_2}\left(\tilde{x}_1-\frac{2mk+2(j+nr)-1-k}{2r}\right)^2} \\
&\times\sum_{n'}\int_0^1 d\tilde{x}_3\left\{-1+\frac{4\pi}{\ell}\frac{L_3}{L_4}\left(\tilde{x}_3-\frac{((2n'+1)\ell-(2C-1))}{2}\right)^2\right\}e^{-\frac{2\pi L_3}{\ell L_4}\left(\tilde{x}_3-\frac{(2n'+1)\ell-(2C-1)}{2}\right)^2}.
\end{aligned} \tag{C.26}$$

One can check (using Mathematica) that:[27]

$$\sum_{C=1}^{\ell}\sum_{n}\int_0^1 d\tilde{x}_3\left\{-1+\frac{4\pi}{\ell}\frac{L_3}{L_4}\left(\tilde{x}_3-\frac{((2n+1)\ell-(2C-1))}{2}\right)^2\right\}e^{-\frac{2\pi L_3}{\ell L_4}\left(\tilde{x}_3-\frac{(2n+1)\ell-(2C-1)}{2}\right)^2}=0, \tag{C.27}$$

and thus, we conclude that, as expected

$$S = S_0 + \mathcal{O}(\Delta^2) = \frac{r}{N}\frac{8\pi^2}{g^2} + \mathcal{O}(\Delta^2)\,, \tag{C.28}$$

i.e. the action of the multifractional instanton is, to the order in Δ we are working on, equal to $\frac{r}{N}$ times the BPST instanton action.

D Blow up of the gauge invariant local densities along the noncompact moduli of the $k \neq r$ solution

To determine the gauge invariant density (6.5), we need to solve for $\mathcal{S}_\nu^{(0)\omega}$. To this end, we use (4.17) (or the equivalent forms (4.22), (4.23)). Acting on these equations with $\partial = \sigma^\nu\partial_\nu$ and using the identity $\sigma^\nu\bar{\sigma}^\mu + \sigma^\mu\bar{\sigma}^\nu = 2\delta_{\mu\nu}$, we find the expression

$$\Box\mathcal{S}^{(0)\omega} = -\frac{i}{\pi\ell k}\sigma^\nu\partial_\nu\mathcal{Y}\,, \tag{D.1}$$

[27] One can show that (C.27) is true by converting the combined infinite sum and the integral over the unit interval to an infinite integral.

where (once more, for brevity, we omit the $k \times \ell$ and $\ell \times k$ superscripts)

$$\mathcal{Y} = \begin{bmatrix} \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] & -2\mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_4^{\dagger(0)} \right] \\ -2\mathrm{tr}_k \left[\mathcal{W}_4^{(0)} \mathcal{W}_2^{\dagger(0)} \right] & -\mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] \end{bmatrix} . \tag{D.2}$$

Equating the components of (D.1), we arrive at the following set of equations:

$$\begin{aligned} & i\pi\ell k \Box \left(\mathcal{S}_4^{(0)\omega} + i\mathcal{S}_3^{(0)\omega} \right) \\ & \quad = (\partial_4 + i\partial_3)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] - 2\,(i\partial_1 + \partial_2)\, \mathrm{tr}_k \left[\mathcal{W}_4^{(0)} \mathcal{W}_2^{\dagger(0)} \right] , \\ & i\pi\ell k \Box \left(\mathcal{S}_4^{(0)\omega} - i\mathcal{S}_3^{(0)\omega} \right) \\ & \quad = -\,(\partial_4 - i\partial_3)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] - 2\,(i\partial_1 - \partial_2)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_4^{\dagger(0)} \right] , \\ & i\pi\ell k \Box \left(i\mathcal{S}_1^{(0)\omega} + \mathcal{S}_2^{(0)\omega} \right) \\ & \quad = -2\,(\partial_4 + i\partial_3)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_4^{\dagger(0)} \right] - (i\partial_1 + \partial_2)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] , \\ & i\pi\ell k \Box \left(i\mathcal{S}_1^{(0)\omega} - \mathcal{S}_2^{(0)\omega} \right) \\ & \quad = -2\,(\partial_4 - i\partial_3)\, \mathrm{tr}_k \left[\mathcal{W}_4^{(0)} \mathcal{W}_2^{\dagger(0)} \right] + (i\partial_1 - \partial_2)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] . \end{aligned} \tag{D.3}$$

Thus, we find

$$\begin{aligned} & \pi\ell k \Box \mathcal{S}_4^{(0)\omega} \\ & = \partial_3 \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] - (\partial_1 - i\partial_2) \mathrm{tr}_k \left[\mathcal{W}_4^{(0)} \mathcal{W}_2^{\dagger(0)} \right] - (\partial_1 + i\partial_2) \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_4^{\dagger(0)} \right] , \\ & -\pi\ell k \Box \mathcal{S}_3^{(0)\omega} \\ & = \partial_4 \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] - (i\partial_1 + \partial_2) \mathrm{tr}_k \left[\mathcal{W}_4^{(0)} \mathcal{W}_2^{\dagger(0)} \right] + (i\partial_1 - \partial_2) \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_4^{\dagger(0)} \right] . \end{aligned} \tag{D.4}$$

and

$$\begin{aligned} \left(\partial_3 \mathcal{S}_4^{(0)\omega} - \partial_4 \mathcal{S}_3^{(0)\omega} \right) = (\pi\ell k \Box)^{-1} \Big\{ & \left(\partial_3^2 + \partial_4^2 \right) \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_2^{\dagger(0)} - \mathcal{W}_4^{(0)} \mathcal{W}_4^{\dagger(0)} \right] \\ & + (-\partial_1\partial_3 - \partial_2\partial_4 + i\partial_2\partial_3 - i\partial_1\partial_4)\, \mathrm{tr}_k \left[\mathcal{W}_4^{(0)} \mathcal{W}_2^{\dagger(0)} \right] \\ & + (-\partial_1\partial_3 - \partial_2\partial_4 - i\partial_2\partial_3 + i\partial_1\partial_4)\, \mathrm{tr}_k \left[\mathcal{W}_2^{(0)} \mathcal{W}_4^{\dagger(0)} \right] \Big\} . \end{aligned} \tag{D.5}$$

We are interested in the case $r > 1$ and $\gcd(k, r) = 1$. Let us consider the example $r = 2, k = 3$. Then, using the parameterization of (5.9), taking the upper sign for definiteness,

$$\mathcal{C}_2^0 = u\,, \quad \mathcal{C}_2^1 = u\,, \quad \mathcal{C}_4^0 = -iu\,, \quad \mathcal{C}_4^1 = iu\,. \tag{D.6}$$

we find[28]

$$\begin{aligned}
\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_2^{\dagger(0)}\right] &= u^2\sum_{C=1}^{\ell}\sum_{C'=1}^{k}\left[\Phi^0_{C',C}+\Phi^1_{C',C}\right]\left[\Phi^{*0}_{C',C}+\Phi^{*1}_{C',C}\right] \\
&= u^2\sum_{C=1}^{\ell}\sum_{C'=1}^{k}|\Phi^0_{C',C}|^2+|\Phi^1_{C',C}|+\Phi^0_{C',C}\Phi^{*1}_{C',C}+\Phi^{*0}_{C',C}\Phi^1_{C',C}\,, \quad \text{(D.7)} \\
\mathrm{tr}_k\left[\mathcal{W}_4^{(0)}\mathcal{W}_4^{\dagger(0)}\right] &= u^2\sum_{C=1}^{\ell}\sum_{C'=1}^{k}\left[\Phi^0_{C',C}-\Phi^1_{C',C}\right]\left[\Phi^{*0}_{C',C}-\Phi^{*1}_{C',C}\right] \\
&= u^2\sum_{C=1}^{\ell}\sum_{C'=1}^{k}|\Phi^0_{C',C}|^2+|\Phi^1_{C',C}|-\Phi^0_{C',C}\Phi^{*1}_{C',C}-\Phi^{*0}_{C',C}\Phi^1_{C',C}\,, \quad \text{(D.8)}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}_k\left[\mathcal{W}_4^{(0)}\mathcal{W}_2^{\dagger(0)}\right] = iu^2\sum_{C=1}^{\ell}\Big[&\left(-\Phi^0_{1,C}+\Phi^1_{1,C}\right)\left(\Phi^{*0}_{1,C}+\Phi^{*1}_{1,C}\right) \\
&+\left(\Phi^0_{2,C}-\Phi^1_{2,C}\right)\left(\Phi^{*0}_{2,C}+\Phi^{*1}_{2,C}\right) \\
&+\left(-\Phi^0_{3,C}+\Phi^1_{3,C}\right)\left(\Phi^{*0}_{3,C}+\Phi^{*1}_{C',C}\right)\Big]\,, \quad \text{(D.9)} \\
\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_4^{\dagger(0)}\right] = -iu^2\sum_{C=1}^{\ell}\Big[&\left(\Phi^0_{1,C}+\Phi^1_{1,C}\right)\left(-\Phi^{*0}_{1,C}+\Phi^{*1}_{1,C}\right) \\
&+\left(\Phi^0_{2,C}+\Phi^1_{2,C}\right)\left(\Phi^{*0}_{2,C}-\Phi^{*1}_{2,C}\right) \\
&+\left(\Phi^0_{3,C}+\Phi^1_{3,C}\right)\left(-\Phi^{*0}_{3,C}+\Phi^{*1}_{3,C}\right)\Big]\,. \quad \text{(D.10)}
\end{aligned}$$

It is not hard, using (A.34), to check that the combinations

$$\begin{aligned}
&|\Phi^0_{C',C}(x)|^2, |\Phi^1_{C',C}(x)|^2, \left(-\Phi^0_{1,C}+\Phi^1_{1,C}\right)(x)\left(\Phi^{*0}_{1,C}+\Phi^{*1}_{1,C}\right)(x), \quad \text{(D.11)} \\
&\left(\Phi^0_{2,C}-\Phi^1_{2,C}\right)(x)\left(\Phi^{*0}_{2,C}+\Phi^{*1}_{2,C}\right)(x), \left(-\Phi^0_{3,C}+\Phi^1_{3,C}\right)(x)\left(\Phi^{*0}_{3,C}+\Phi^{*1}_{C',C}\right)(x)
\end{aligned}$$

satisfy periodic boundary conditions.[29] Then, we use the Fourier transform of these com-

[28] The sums over C' and C should be really thought of as being over $0,\ldots,k-1$ and $0,\ldots,\ell-1$, respectively, to be consistent with the main body of the paper. We apologize to the reader for this slight mismatch.

[29] However, the component that carries the subscript C', C is sent to one with subscript $C'-r, C+1$. Nevertheless, the combinations that give the gauge invariant density are periodic. Also, from the linearity of the Fourier analysis of the Fourier-transformed components below, the superposition of the various terms makes sense. The difficulty in the analysis below is that numerical convergence is hard to achieve.

binations, namely,

$$
\begin{aligned}
|\Phi^0_{C',C}(x)|^2 &= \sum_{p_\mu\in\mathbb{Z}} e^{i\frac{2\pi p_\mu x)_\mu}{L_\mu}} \mathcal{X}_{0;C',C}(p)\,,\\
|\Phi^1_{C',C}(x)|^2 &= \sum_{p_\mu} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}} \mathcal{X}_{1;C',C}(p)\,,\\
\Phi^0_{C',C}(x)\Phi^{*1}_{C',C}(x) &= \sum_{p_\mu} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}} \mathcal{X}_{2;C',C}(p)\,,\\
\left(-\Phi^0_{1,C}+\Phi^1_{1,C}\right)(x)\left(\Phi^{*0}_{1,C}+\Phi^{*1}_{1,C}\right)(x) &= \sum_{p_\mu} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}} \mathcal{X}_{3;C}(p)\,,\\
\left(\Phi^0_{2,C}-\Phi^1_{2,C}\right)(x)\left(\Phi^{*0}_{2,C}+\Phi^{*1}_{2,C}\right)(x) &= \sum_{p_\mu} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}} \mathcal{X}_{4;C}(p)\,,\\
\left(-\Phi^0_{3,C}+\Phi^1_{3,C}\right)(x)\left(\Phi^{*0}_{3,C}+\Phi^{*1}_{3,C}\right)(x) &= \sum_{p_\mu} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}} \mathcal{X}_{5;C}(p)\,.
\end{aligned}
\tag{D.12}
$$

to find

$$
\begin{aligned}
\mathrm{tr}_k\left[\mathcal{W}_4^{(0)}\mathcal{W}_4^{\dagger(0)}\right] &= u^2\sum_{p_\mu} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}}\mathcal{H}(p) \qquad\qquad (\text{D.13})\\
&\equiv u^2\sum_{p_\mu,C,C'} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}}\left(\mathcal{X}_{0;C',C}(p)+\mathcal{X}_{1;C',C}(p)-\mathcal{X}_{2;C',C}(p)-\mathcal{X}^*_{2;C',C}(p)\right)\,.
\end{aligned}
$$

The function $\mathcal{H}(p)$, the Fourier transform of $\mathrm{tr}_k\left[\mathcal{W}_4^{(0)}\mathcal{W}_4^{\dagger(0)}\right]$ modulo u^2, will play an important role below. In addition, we find

$$
\begin{aligned}
&(\pi\ell k\Box)^{-1}\left\{\left(\partial_3^2+\partial_4^2\right)\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_2^{\dagger(0)}-\mathcal{W}_4^{(0)}\mathcal{W}_4^{\dagger(0)}\right]\right\}\\
&\quad=\frac{2u^2}{\pi\ell k}\sum_{p_\mu,C,C'}\frac{\left[\frac{p_3^2}{L_3^2}+\frac{p_4^2}{L_4^2}\right]e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}}}{\frac{p_1^2}{L_1^2}+\frac{p_2^2}{L_2^2}+\frac{p_3^2}{L_3^2}+\frac{p_4^2}{L_4^2}}\left(\mathcal{X}_{2;C',C}(p)+\mathcal{X}^*_{2;C',C}(p)\right)\,,
\end{aligned}
\tag{D.14}
$$

and

$$
\begin{aligned}
&(\pi\ell k\Box)^{-1}\Big\{(-\partial_1\partial_3-\partial_2\partial_4+i\partial_2\partial_3-i\partial_1\partial_4)\,\mathrm{tr}_k\left[\mathcal{W}_4^{(0)}\mathcal{W}_2^{\dagger(0)}\right]\\
&\qquad+(-\partial_1\partial_3-\partial_2\partial_4-i\partial_2\partial_3+i\partial_1\partial_4)\,\mathrm{tr}_k\left[\mathcal{W}_2^{(0)}\mathcal{W}_4^{\dagger(0)}\right]\Big\}\\
&=\frac{u^2}{\pi\ell k}\sum_{p_\mu,C}\frac{e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}}}{\frac{p_1^2}{L_1^2}+\frac{p_2^2}{L_2^2}+\frac{p_3^2}{L_3^2}+\frac{p_4^2}{L_4^2}}\Big\{\left(-i\frac{p_1p_3}{L_1L_3}-i\frac{p_2p_4}{L_2L_4}-\frac{p_2p_3}{L_2L_3}+\frac{p_1p_4}{L_1L_4}\right)\\
&\qquad\times(\mathcal{X}_{3;C}(p)+\mathcal{X}_{4;C}(p)+\mathcal{X}_{5;C}(p))\\
&\qquad+\left(i\frac{p_1p_3}{L_1L_3}+i\frac{p_2p_4}{L_2L_4}-\frac{p_2p_3}{L_2L_3}+\frac{p_1p_4}{L_1L_4}\right)\\
&\qquad\times\left(\mathcal{X}^*_{3;C}(p)+\mathcal{X}^*_{4;C}(p)+\mathcal{X}^*_{5;C}(p)\right)\Big\}\,.
\end{aligned}
\tag{D.15}
$$

Finally, one can also define the Fourier components of $\mathrm{tr}[F_{34}F_{34}](x)$:

$$\mathrm{tr}[F_{34}F_{34}](x) = \sum_{p_\mu \in \mathbb{Z}} e^{i\frac{2\pi p_\mu x_\mu}{L_\mu}} \mathcal{Q}(p)\,. \tag{D.16}$$

Using (6.5), we find, apart from an additive constant:

$$\begin{aligned}
\frac{\mathcal{Q}(p)}{u^2 \hat{F}^\omega_{34}\Delta} = 8\pi N \mathcal{H}(p) &+ \frac{4}{\pi \ell k}\sum_{C,C'} \frac{\left[\frac{p_3^2}{L_3^2}+\frac{p_4^2}{L_4^2}\right]}{\frac{p_1^2}{L_1^2}+\frac{p_2^2}{L_2^2}+\frac{p_3^2}{L_3^2}+\frac{p_4^2}{L_4^2}}\left(\mathcal{X}_{2;C',C}(p)+\mathcal{X}^*_{2;C',C}(p)\right)\\
&+\frac{2}{\pi\ell k}\sum_C \frac{1}{\frac{p_1^2}{L_1^2}+\frac{p_2^2}{L_2^2}+\frac{p_3^2}{L_3^2}+\frac{p_4^2}{L_4^2}}\left\{\left(-i\frac{p_1p_3}{L_1L_3}-i\frac{p_2p_4}{L_2L_4}-\frac{p_2p_3}{L_2L_3}+\frac{p_1p_4}{L_1L_4}\right)\right.\\
&\qquad\times(\mathcal{X}_{3;C}(p)+\mathcal{X}_{4;C}(p)+\mathcal{X}_{5;C}(p))\\
&\qquad+\left(i\frac{p_1p_3}{L_1L_3}+i\frac{p_2p_4}{L_2L_4}-\frac{p_2p_3}{L_2L_3}+\frac{p_1p_4}{L_1L_4}\right)\\
&\qquad\left.\times\left(\mathcal{X}^*_{3;C}(p)+\mathcal{X}^*_{4;C}(p)+\mathcal{X}^*_{5;C}(p)\right)\right\}\,.
\end{aligned} \tag{D.17}$$

We need to check whether the expression on the R.H.S. vanishes for all values of p_μ. The easiest check to perform is to choose $p_\mu = (0, p_2, 0, 0)$. With this choice, all terms vanish except $\mathcal{H}(p)$, the Fourier transform of $\mathrm{tr}_k\left[\mathcal{W}_4^{(0)}\mathcal{W}_4^{\dagger(0)}\right]$ modulo u^2. One can check numerically that $\mathcal{H}(p)$ is non-vanishing, indicating that the gauge-invariant density $\mathrm{tr}[F_{34}F_{34}](x)$ increases indefinitely as $u\to\infty$.

E Fermion zero modes on the deformed-$\mathbb{T}^4$, for $k = r$

In this appendix, we solve for the fermion zero modes in the background (4.1), which we rewrite in the familiar k/ℓ block matrix form, using the notation of (3.5):

$$A_\mu = \begin{pmatrix} \|\left(2\pi\ell\,(A^\omega_\mu - \frac{z_\mu}{L_\mu}) + \phi^{C'}_\mu\right)\delta_{C'B'} + \epsilon^2\,\mathcal{S}_{\mu\,C'B'}\| & \|\epsilon\,\mathcal{W}_{\mu\,C'B}\| \\ \|\epsilon\,(\mathcal{W}^\dagger_\mu)_{CB'}\| & \|-2\pi k\,(A^\omega_\mu - \frac{z_\mu}{L_\mu})\delta_{CB} + \epsilon^2\,\mathcal{S}_{\mu\,CB}\| \end{pmatrix}. \tag{E.1}$$

Here we consider exclusively the $k = r$ case, where:

1. A^ω_μ is the constant flux background $A^\omega_1 = A^\omega_3 = 0$, $A^\omega_2 = -\frac{x_1}{NL_1L_2}$, $A^\omega_4 = -\frac{x^3}{N\ell L_3 L_4}$.

2. $\phi^{C'}_\mu$ are the $r-1$ allowed holonomies in $SU(k=r)$ (thus obeying $\sum_{C'}\phi^{C'}_\mu = 0$) from (3.5) and z_μ are the holonomies in the U(1)-direction ω, eq. (2.6).

   We also recall that these are, after computing the commutator in the Weyl equation, combined into the r independent $\hat{\phi}^{C'}_\mu$ of eqs. (3.19), (3.20) with no constraint on the trace.

3. $\mathcal{W}_\mu$ is leading order $k = r$ solution. Thus, $\mathcal{W}_3 = \mathcal{W}_4 = 0$ and $\mathcal{W}_1 = -i\mathcal{W}_2$, and with $\mathcal{W}_2$ given by (4.21), with the r coefficients $\mathcal{C}^A_2$ fixed by solving eq. (5.4).

JHEP09(2023)095

4. The components of $\mathcal{S}_\mu$ are obtained by solving (4.22), (4.23) (recall that they obey the tracelessness condition $\sum_{C'} S_{\mu\, C'C'} + \sum_C S_{\mu\, CC} = 0$).

5. Finally, to remind us of the powers of $\sqrt{\Delta}$ appearing in the leading order solution for $\mathcal{W}_\mu$ and $\mathcal{S}_\mu$, we introduced a parameter ϵ ($\equiv 1$).

Our goal is to solve the Weyl equation $\partial_\mu \bar\sigma^\mu \lambda + i\bar\sigma^\mu [A_\mu, \lambda] = 0$ in the $k = r$ background (E.1), using a series expansion in ϵ, to leading order. We take λ also in the block-diagonal form (3.8), obeying (3.9):

$$\lambda = \begin{bmatrix} ||\lambda_{C'B'}|| & ||\lambda_{C'B}|| \\ ||\lambda_{CB'}|| & ||\lambda_{CB}|| \end{bmatrix}, \; C', B' \in \{0, \ldots k-1\}, \; C, B \in \{0, \ldots \ell - 1\}\,. \tag{E.2}$$

Newt, write the Weyl equation, using the quaternionic notation of section 4: $\bar\partial = \partial_\mu \bar\sigma_\mu$ and $\bar A = \bar\sigma_\mu A_\mu$ (and similar for all other vectors in (E.1), with $\bar\sigma_\mu$ defined in Footnote 5) and obtain the following equations for the components of λ of (E.2), with a sum over repeated indices B, B' implied:

$$\begin{aligned}
\bar\partial \lambda_{C'D'} &= -i\epsilon\bar\sigma_\mu(\mathcal{W}_{\mu\, C'B}\lambda_{BD'} - \lambda_{C'B}(\mathcal{W}^\dagger)_{\mu\, BD'}) - i\epsilon^2\bar\sigma_\mu(\mathcal{S}_{\mu\, C'B'}\lambda_{B'D'} - \lambda_{C'B'}\mathcal{S}_{\mu\, B'D'}),\\
\bar\partial \lambda_{CD} &= -i\epsilon\bar\sigma_\mu((\mathcal{W}^\dagger)_{\mu\, CB'}\lambda_{B'D} - \lambda_{CB'}\mathcal{W}_{\mu\, B'D}) - i\epsilon^2\bar\sigma_\mu(\mathcal{S}_{\mu\, CB}\lambda_{BD} - \lambda_{CB}\mathcal{S}_{\mu\, BD}),\\
\bar\partial \lambda_{C'D} &= -i(2\pi N \bar A^\omega + \bar{\hat\phi}^{C'})\lambda_{C'D} - i\epsilon\bar\sigma_\mu(\mathcal{W}_{\mu\, C'B}\lambda_{BD} - \lambda_{C'B'}\mathcal{W}_{\mu\, B'D})\\
&\quad - i\epsilon^2\bar\sigma_\mu(\mathcal{S}_{\mu\, C'B'}\lambda_{B'D} - \lambda_{C'B}\mathcal{S}_{\mu\, BD}),\\
\bar\partial \lambda_{CD'} &= i(2\pi N \bar A^\omega + \bar{\hat\phi}^{C'})\lambda_{CD'} - i\epsilon\bar\sigma_\mu((\mathcal{W}^\dagger)_{\mu\, CB'}\lambda_{B'D'} - \lambda_{CB}(\mathcal{W}^\dagger)_{\mu\, BD'})\\
&\quad - i\epsilon^2\bar\sigma_\mu(\mathcal{S}_{\mu\, CB}\lambda_{BD'} - \lambda_{CB'}\mathcal{S}_{\mu\, B'D'})\,.
\end{aligned} \tag{E.3}$$

We now observe that we can consistently solve (E.3) in a series expansion in ϵ, assigning the following (leading-order only shown) ϵ-scaling of the various components of λ:

$$\begin{aligned}
\lambda_{C'D'} &= \epsilon^0 \lambda_{C'D'} + \mathcal{O}(\epsilon^2)\,,\\
\lambda_{CD} &= \epsilon^0 \lambda_{CD} + \mathcal{O}(\epsilon^2)\,,\\
\lambda_{C'D} &= \epsilon\, \lambda_{C'D} + \mathcal{O}(\epsilon^3)\,,\\
\lambda_{CD'} &= \epsilon\, \lambda_{CD'} + \mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{E.4}$$

Substituting into (E.3) and keeping only the leading terms in ϵ in each equation, we find the following equations for the leading order (in $\sqrt{\Delta}$) fermion zero modes in the background (E.1):

$$\begin{aligned}
\bar\partial \lambda_{C'D'} &= 0,\\
\bar\partial \lambda_{CD} &= 0,\\
(\bar\partial + i(2\pi N \bar A^\omega + \bar{\hat\phi}^{C'}))\lambda_{C'D} &= -i(\mathcal{W}_{\mu\, C'B}\, \bar\sigma_\mu \lambda_{BD} - \bar\sigma_\mu \lambda_{C'B'}\mathcal{W}_{\mu\, B'D}),\\
(\bar\partial - i(2\pi N \bar A^\omega + \bar{\hat\phi}^{C'}))\lambda_{CD'} &= -i(\mathcal{W}^*_{\mu\, B'C}\, \bar\sigma_\mu \lambda_{B'D'} - \bar\sigma_\mu \lambda_{CB}\mathcal{W}^*_{\mu\, D'B}).
\end{aligned} \tag{E.5}$$

JHEP09(2023)095

Now, we recall that the first two equations were already solved in section 3.4.1. From eq. (3.17), taken with $k = r$, we have the diagonal zero mode solutions

$$\begin{aligned} \lambda_{\alpha\ B'C'} &= \delta_{B'C'}\ \theta_\alpha^{C'}, \\ \lambda_{\alpha\ BC} &= -\delta_{BC}\ \frac{1}{\ell}\sum_{C'} \theta_\alpha^{C'}, \end{aligned} \tag{E.6}$$

where we momentarily restored the spinor index α. We first define the spinor

$$\eta^{C'} \equiv \theta^{C'} + \frac{1}{\ell}\sum_{B'=0}^{k-1} \theta^{B'}, \tag{E.7}$$

and then plug (E.6) into the last two equations in (E.5) to obtain:

$$\begin{aligned} (\bar\partial + i(2\pi N \bar A^\omega + \bar{\hat\phi}^{C'}))\lambda_{C'D} &= i\mathcal{W}_{\mu\ C'D}\ \bar\sigma_\mu\ \eta^{C'}, \\ (\bar\partial - i(2\pi N \bar A^\omega + \bar{\hat\phi}^{C'}))\lambda_{CD'} &= -i\mathcal{W}^*_{\mu\ D'C}\ \bar\sigma_\mu\ \eta^{D'}. \end{aligned} \tag{E.8}$$

We now recall that for the $k = r$ solution, $\mathcal{W}_3 = \mathcal{W}_4 = 0$ and $\mathcal{W}_1 = -i\mathcal{W}_2$, hence

$$\begin{aligned} \mathcal{W}_{\mu\ C'D}\ \bar\sigma_\mu &= (-i\bar\sigma_1 + \bar\sigma_2)\mathcal{W}_{2\ C'D} = \mathcal{W}_{2\ C'D}\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{W}^*_{\mu\ D'C}\ \bar\sigma_\mu &= (i\bar\sigma_1 + \bar\sigma_2)\mathcal{W}^*_{2\ D'C} = \mathcal{W}^*_{2\ D'C}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \end{aligned} \tag{E.9}$$

and that $\mathcal{W}_{2\,C',C} = V^{-1/4}\mathcal{C}_2^{C'}\Phi^{(0)}_{C',C}(x,\hat\phi)$, where $\mathcal{C}_2^{C'}$ is as determined in section 5.

The equation for $\lambda_{C'D}$ then takes the form, using the derivatives from (C.3) and noting that the equations for each $C' = 1,\dots,r$ decouple:

$$\begin{aligned} (\hat D_4 - i\hat D_3)\lambda_{1\ C'D} - (i\hat D_1 + \hat D_2)\lambda_{2\ C'D} &= \bar\eta_2^{C'}\ \Phi^{(0)}_{2\ C'D}, \\ (-i\hat D_1 + \hat D_2)\lambda_{1\ C'D} + (\hat D_4 + i\hat D_3)\lambda_{2\ C'D} &= 0, \end{aligned} \tag{E.10}$$

where we absorb various inessential constants in the redefined $\bar\eta_2^{C'}$ coefficient. The solution of these equations is given by the function $\mathcal{G}^{(0)}_{3\ C'D}$ defined in (C.9), explicitly

$$\begin{aligned} \lambda_{1\ C'D} &= \bar\eta_2^{C'}\mathcal{G}^{(0)}_{3\ C'D}, \\ \lambda_{2\ C'D} &= 0. \end{aligned} \tag{E.11}$$

Similarly, one finds that the other zero mode is

$$\begin{aligned} \lambda_{1\ CD'} &= 0, \\ \lambda_{2\ CD'} &= \bar\eta_1^{D'}\mathcal{G}^{*\,(0)}_{3\ D'C}. \end{aligned} \tag{E.12}$$

Thus, there are in total $2r$ zero modes labeled by $\bar\eta^{C'}_{1,2}$, with $C' = 1,\dots,r$. The x-dependence of the zero mode labeled by a given C' is governed only by the holonomies $\hat\phi^{C'}_\mu$, similar to the bosonic case discussed earlier.

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JHEP09(2023)095