Mathematische Zeitschrift (2023) 305:49
https://doi.org/10.1007/s00209-023-03374-8

Mathematische Zeitschrift

Three conjectures about character sums

Andrew Granville[1] · Alexander P. Mangerel[2]

Received: 25 December 2021 / Accepted: 15 September 2023

Abstract
We establish that three well-known and rather different looking conjectures about Dirichlet characters and their (weighted) sums, (concerning the Pólya–Vinogradov theorem for maximal character sums, the maximal admissible range in Burgess' estimate for short character sums, and upper bounds for $L(1, \chi)$ and $L(1 + it, \chi)$) are more-or-less "equivalent". We also obtain a new mean value theorem for logarithmically weighted sums of 1-bounded multiplicative functions.

Keywords Dirichlet character · Character sums · Multiplicative functions · Pretentious number theory · Halász's theorem

Mathematics Subject Classification 11L40 · 11M06

1 Introduction

1.1 Conjectures about weighted sums of Dirichlet characters

Let χ be a primitive character mod $q > 1$ and

$$S(\chi, N) := \sum_{n \leq N} \chi(n) \text{ for all } N \geq 1.$$

The three most widely-used, unconditionally proved estimates about characters sums are:

- The *Pólya–Vinogradov* theorem:

$$M(\chi) := \max_{1 \leq N \leq q} |S(\chi, N)| \leq c_1 \sqrt{q} \log q$$

✉ Alexander P. Mangerel
smangerel@gmail.com

Andrew Granville
andrew@dms.umontreal.ca

1 Départment de Mathématiques et Statistique, Université de Montréal, CP 6128 succ Centre-Ville, Montréal, QC H3C 3J7, Canada

2 Department of Mathematical Sciences, Durham University, Upper Mountjoy Campus, Stockton Road, Durham DH1 3LE, UK

for some explicit $c_1 > 0$;

- *Burgess's theorem* [3, 13]: For $N \geq q^{c_2}$ (and q *cube-free*)

$$|S(\chi, N)| = o(N)$$

for any $c_2 \geq \frac{1}{4}$; and

- The Dirichlet L-function $L(s, \chi)$ at $s = 1$ satisfies

$$|L(1, \chi)| = \left| \sum_{n \geq 1} \frac{\chi(n)}{n} \right| \leq c_3 \log q$$

for some explicit $c_3 > 0$. One can also show that for any fixed $T > 0$, there exists a constant $c_3(T) > 0$ such that if $t \in [-T, T]$ then $|L(1 + it, \chi)| \leq c_3(T) \log q$.

The Riemann hypothesis for $L(s, \chi)$ implies that one can take any $c_1, c_2, c_3 > 0$ but this has resisted unconditional proof. One unlikely but currently plausible 'obstruction' to establishing this unconditionally is the possibility that $\chi(p) = 1$ for all primes $p \leq q^c$, in which case $c_1, c_2, c_3 \gg c$, or indeed if χ is 1-pretentious for the primes up to q.[1]

Inspired by connections highlighted in [2, 5, 16] we show that improving any one of these bounds will, more-or-less, improve the others.

Theorem 1.1 *The following statements are equivalent:*

- *There exists $\kappa_1 > 0$ such that there are infinitely many primitive characters $\chi \pmod q$ for which $M(\chi) \geq \kappa_1 \sqrt{q} \log q$;*
- *There exists $\kappa_3 > 0$ such that there are infinitely many odd primitive characters $\psi \pmod r$ for which $|L(1, \psi)| \geq \kappa_3 \log r$.*

This follows from a more precise connection:

Corollary 1.1 *Suppose that χ is a primitive character mod q. We have $M(\chi) \gg \sqrt{q} \log q$ if and only if there exists a primitive character $\xi \pmod \ell$ with $\xi(-1) = -\chi(-1)$ and $\ell \ll 1$ for which $|L(1, \psi)| \gg \log q$, where ψ is the primitive (odd) character that induces $\chi\bar{\xi}$.*

In other words we prove that if $M(\chi) \gg \sqrt{q} \log q$ then χ is ξ-pretentious for some ξ of bounded conductor, and we will also establish a converse theorem.

Next we relate large $S(\chi, N)$-values with large $L(1 + it, \chi)$-values:

Theorem 1.2 *The following statements are equivalent:*

- *There exists $\kappa_2 > 0$ such that there are infinitely many primitive characters $\chi \pmod q$ for which there is an integer $N \in [q^{\kappa_2}, q]$ such that $|S(\chi, N)| \geq \kappa_2 N$;*
- *There exists $\kappa_3 > 0$ and $T > 0$ such that there are infinitely many primitive characters $\chi \pmod q$ for which there exists $t \in [-T, T]$ such that $|L(1 + it, \chi)| \geq \kappa_3 \log q$.*

If we restrict attention here to characters of bounded order then one can take $t = 0$. The precise connection is given in the following result.

Proposition 1.1 *Fix $c > 0$. Let χ be a primitive character mod q. There exists $t \in \mathbb{R}$ with $|t| \ll 1$ for which $|L(1 + it, \chi)| \geq c \log q$ if and only if there exist $\kappa = \kappa(c) > 0$ and $x \in [q^\kappa, q]$ for which $|S(\chi, x)| \gg_c x$. If $\chi^k = \chi_0$ for some $k \ll 1$ then we may take $t = 0$.*

[1] "Pretentiousness" will be defined in Sect. 3.

In other words we prove that if $|S(\chi, N)| \gg N$ for some $N > q^{\kappa}$ and $\kappa > 0$ then χ is n^{it}-pretentious for some bounded real number t, and we will also establish a converse theorem.

We can combine these results: If $M(\chi) \gg \sqrt{q}\log q$ and $|S(\chi, N)| \gg N$ for some $N > q^{\kappa}$, then χ is both ξ-pretentious and n^{it}-pretentious, which implies that ξ is n^{it}-pretentious, where ξ is a primitive character of bounded conductor. We will show that this implies $\xi = 1$ and $t = 0$, so that χ is an odd character that is 1-pretentious for the primes up to q.

Corollary 1.2 *Let χ be a primitive character modulo q. Assume $M(\chi) \geq c_1\sqrt{q}\log q$ and $|S(\chi, N)| \gg N$ for some $N \in [q^{c_2}, q]$, with $c_1, c_2 \gg 1$. Then $|L(1, \chi)| \gg \log q$, $\mathbb{D}(\chi, 1; q) \ll 1$ and χ is odd.*

Therefore such a putative character is the only obstruction to improving at least one of our three famous results unconditionally (that is, being able to take any $c_1 > 0$ in the Pólya–Vinogradov theorem, or being able to take any $c_2 > 0$ in Burgess's theorem, or being able to take any $c_3 > 0$ in bounds for $L(1, \chi)$).

Other new results on this topic will be discussed in Sect. 2.

1.2 Logarithmic averages of multiplicative functions

We prove our results on sums of characters by viewing characters as examples of multiplicative functions that take their values on the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| \leq 1\}$. Halász's theorem, which we will discuss in detail below, bounds the mean value of $f(n)$ for n up to x in terms of how "pretentious" f is. In particular, if f is real-valued then Hall and Tenenbaum [12] showed that

$$\sum_{n\leq x} f(n) \ll x\, e^{-\tau\mathbb{D}(f,1;x)^2} \text{ where } \mathbb{D}(f, 1; x)^2 = \sum_{p\leq x} \frac{1 - \mathrm{Re}(f(p))}{p}, \tag{1.1}$$

and

$$\tau = 0.3286\cdots = -\cos\theta \text{ where } \theta \in (0, \pi) \text{ satisfies } \sin\theta - \theta\cos\theta = \frac{\pi}{2}.$$

They gave an example where one attains equality in (1.1) (up to the inexplicit constant).

We give an analogous result for logarithmic averages of the form $\sum_{n\leq x} f(n)/n$ though, as discovered in [8], we do not need to restrict attention to real-valued f. Here we let $\lambda \in \mathbb{R}$ be such that

$$\int_0^1 |e(\theta) - \lambda| d\theta = 2 - \lambda,$$

so that $\lambda = 0.8221\ldots$.

Proposition 1.2 *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function.*
(a) *We have*

$$\sum_{n\leq x} \frac{f(n)}{n} \ll (\log x)(1 + \mathbb{D}(f, 1; x)^2)e^{-\lambda\,\mathbb{D}(f,1;x)^2} + \log\log x. \tag{1.2}$$

(b) *The exponent* λ *is "best possible" in a result of this kind, since there exists a multiplicative function* $f : \mathbb{N} \to \mathbb{U}$ *such that*

$$\left| \sum_{n \le x} \frac{f(n)}{n} \right| \asymp (\log x) e^{-\lambda \mathbb{D}(f,1;x)^2}. \tag{1.3}$$

The λ in the bound (1.2) improves on the $\frac{1}{2}$ in the bound given in Lemma 4.3 of [8].

We deduce Proposition 1.2(b) from Theorem 7.1 which establishes asymptotics for $\sum_{n \le N} f(n)/n$ for a class of multiplicative functions $f : \mathbb{N} \to \mathbb{U}$ for which $f(p) = g(\tau \log p)$ for each prime p for some fixed small real τ, where $g(t)$ is a 1-periodic function with "well-behaved" Fourier coefficients.

More details, as well as other new results on this topic, will be discussed in Sect. 3.

2 Connections between different sums of characters

2.1 Large character sums

Pólya gave the following Fourier expansion (see e.g., Lemma 1 of [17]) for character sums: for $\alpha \in [0, 1)$

$$\sum_{n \le \alpha q} \chi(n) = \frac{g(\chi)}{2\pi i} \sum_{1 \le |n| \le q} \frac{\bar{\chi}(n)}{n} \Big(1 - e(-n\alpha) \Big) + O(\log q), \tag{2.1}$$

where $e(t) := e^{2\pi i t}$ for $t \in \mathbb{R}$ and $g(\chi) := \sum_{a \pmod q} \chi(a) e(\frac{a}{q})$ is the Gauss sum. When χ is primitive we know that $|g(\chi)| = \sqrt{q}$ and

$$\sum_{1 \le |n| \le q} \frac{\bar{\chi}(n)}{n} = (1 - \chi(-1)) L(1, \bar{\chi}) + o(1), \tag{2.2}$$

and so to estimate the left-hand side of (2.1) for any α we are left to estimate the sums

$$\sum_{n \le q} \frac{\bar{\chi}(n) e(\pm n\alpha)}{n}.$$

Note that (2.2) is large if and only if χ is an odd character and $L(1, \chi)$ is large.

Fix $\frac{2}{\pi} < \Delta < 1$ and let $R_q := \exp(\frac{(\log q)^\Delta}{\log \log q})$. For any $\alpha \in [0, 1)$ we may obtain an approximation $|\alpha - \frac{b}{m}| < \frac{1}{m R_q}$, with $(b, m) = 1$ and $m \le R_q$, by Dirichlet's theorem.

If $r_q := (\log q)^{2-2\Delta} (\log \log q)^4 < m \le R_q$ then we say that α is on a *minor arc*. By straightforward modifications to the proof of Lemma 6.1 of [8], for such α we get

$$\sum_{n \le q} \frac{\chi(n) e(n\alpha)}{n} \ll \log \log q + \frac{(\log r_q)^{3/2}}{\sqrt{r_q}} \log q + \log R_q = o((\log q)^\Delta). \tag{2.3}$$

If $m \le r_q$ then we say that α is on a *major arc*. Let $N := \min\{q, \frac{1}{|m\alpha - b|}\}$, so that $R_q \le N \le q$. By Lemma 6.2 of [8],

$$\sum_{1 \le |n| \le q} \frac{\chi(n) e(n\alpha)}{n} = \sum_{1 \le |n| \le N} \frac{\chi(n) e(n \frac{b}{m})}{n} + O(\log \log q). \tag{2.4}$$

Therefore, if $M(\chi) \gg \sqrt{q}(\log q)^{\Delta}$ then either χ is odd and $L(1,\chi) \gg (\log q)^{\Delta}$, or $M(\chi) = |S(\chi,\alpha q)|$ where α lies on a major arc. The following proposition provides more detailed information in these cases.

Proposition 2.1 *Fix $\frac{2}{\pi} < \Delta < 1$ and let χ be a character mod q. We have*

$$M(\chi) \gg \sqrt{q}(\log q)^{\Delta}$$

if and only if there is a primitive character $\xi \pmod{\ell}$ with $\xi(-1) = -\chi(-1)$ and $\ell \le (\log q)^{2-2\Delta}(\log\log q)^4$ such that

$$\max_{1\le N\le q}\left|\sum_{n\le N}\frac{(\chi\bar{\xi})(n)}{n}\right| \gg \frac{\phi(\ell)}{\sqrt{\ell}}(\log q)^{\Delta}.$$

More precisely, in this case we have

$$M(\chi) \sim \tau_{\chi,\xi}\cdot\frac{\sqrt{q\ell}}{\pi\phi(\ell)}\cdot\max_{1\le N\le q}\left|\sum_{n\le N}\frac{(\chi\bar{\xi})(n)}{n}\right| \tag{2.5}$$

where $\tau_{\chi,1}\in[\frac{1}{2},3]$ and $\tau_{\chi,\xi} = \max\{1, |1-(\chi\bar{\xi})(2)|\}$ if $\xi\ne 1$.

Throughout, N_q denotes an integer value of N that maximizes the right-hand side of (2.5).

Using results from the next subsection, we will deduce Corollary 1.1 by showing that for $\psi = \chi\bar{\xi}$, when $\ell \ll 1$ we have

$$|L(1,\psi)| \gg \log q \text{ if and only if } \left|\sum_{n\le N_q}\frac{\psi(n)}{n}\right| \gg \log q. \tag{2.6}$$

However, there is not necessarily a correspondence between these two sums when they are slightly smaller. For example, if $\psi(p) = 1$ for all $p \le N := \exp((\log q)^{\tau})$ and $\psi(p) = -1$ for all $\exp((\log q)^{\tau}) < p \le q^{\delta}$ where $\frac{1}{2} < \tau < 1$ and $\delta > 0$ is some small fixed constant, then assuming ψ is non-exceptional (see (4.3)),

$$\left|\sum_{n\le N}\frac{\psi(n)}{n}\right| \asymp (\log q)^{\tau} \text{ while } |L(1,\psi)| \asymp \prod_{p\le q}\left|1-\frac{\psi(p)}{p}\right|^{-1} \asymp (\log q)^{2\tau-1},$$

which is much smaller. Moreover this (purported) example shows why *we cannot assume* that $N_q = q$ and that the largest sum is $\sim |L(1,\psi)|$.

Following an idea from [2], Proposition 2.1 has the following consequence for quadratic non-residues[2]

Corollary 2.1 $\Delta \in (\frac{2}{\pi}, 1)$. *Let n_q be the least quadratic non-residue modulo a prime $q \equiv 3 \pmod 4$ and suppose that $n_q \ge \exp((\log q)^{\Delta})$. For any odd, squarefree integer $\ell \le \left(\frac{\log n_q}{(\log q)^{\Delta}}\right)^2$ we have*

$$M\left(\left(\tfrac{\cdot}{\ell q}\right)\right) \ge (c\tau - o(1))\sqrt{q}\log n_q \text{ where } c = \frac{2(\sqrt{e}-1)}{\pi} = 0.41298\ldots,$$

$\tau = \frac{1}{2}$ if $\ell = 1$, otherwise $\tau = 1$ or 2 depending on whether $q \equiv 7$ or $3 \pmod 8$, respectively.

---

[2] It is worth recalling that the real primitive characters are given by $1(\cdot)$ as well as $(\frac{\cdot}{n})$ if $n > 1$, $(\frac{2n}{\cdot}\cdot)$ and $(\frac{4n}{\cdot})$ if $n \equiv 3 \pmod 4$, for each odd squarefree integer n.

Proof Let $\xi = (\frac{\cdot}{\ell})$ and $\chi = (\frac{\cdot}{\ell q})$, so that $\chi\overline{\xi} = (\frac{\cdot}{q})1_{(\cdot,\ell)=1}$ and

$$\sum_{n\le x} \frac{(\chi\bar{\xi})(n)}{n} = \sum_{\substack{n\le x\\(n,\ell)=1}} \frac{(\frac{n}{q})}{n}.$$

Let $y = n_q - 1$ and $N = x = y^w$ where $w = e^{1/2}$. A y-*smooth integer* has all of its prime factors $\le y$, and any y-smooth integer here is a quadratic residue mod q. Therefore

$$\sum_{\substack{n\le x\\(n,\ell)=1}} \frac{(\frac{n}{q})}{n} \ge \sum_{\substack{n\le x\\P(n)\le y\\(n,\ell)=1}} \frac{1}{n} - \sum_{\substack{n\le x\\P(n)>y\\(n,\ell)=1}} \frac{1}{n} = 2\sum_{\substack{n\le x\\P(n)\le y\\(n,\ell)=1}} \frac{1}{n} - \sum_{\substack{n\le x\\(n,\ell)=1}} \frac{1}{n}$$

where $P(n)$ is the largest prime factor of n. Therefore, since $1_{(n,\ell)=1} = \sum_{d|(n,\ell)} \mu(d)$, and as $\ell \le y$ we deduce that

$$\sum_{\substack{n\le x\\(n,\ell)=1}} \frac{(\frac{n}{q})}{n} \ge \sum_{d|\ell}\mu(d)\left(2\sum_{\substack{n\le x\\d|n\\P(n)\le y}} \frac{1}{n} - \sum_{\substack{n\le x\\d|n}} \frac{1}{n}\right) = \sum_{d|\ell}\frac{\mu(d)}{d}\left(2\sum_{\substack{m\le x/d\\P(n)\le y}} \frac{1}{m} - \sum_{m\le x/d} \frac{1}{m}\right). \tag{2.7}$$

Let $\psi(x, y)$ be the number of y-smooth integers $\le x$. It is well-known that $\psi(y^u, y) = y^u\rho(u)(1 + O(\frac{1}{\log y}))$ as $y \to \infty$ for bounded u, with $\rho(u) = 1$ for $0 \le u \le 1$ and $\rho(u) = 1 - \log u$ for $1 \le u \le 2$. Therefore by partial summation we have

$$\sum_{\substack{n\le x\\P(n)\le y}} \frac{1}{n} = \int_{t=1}^{x} \frac{\psi(t, y)}{t^2}dt + O(1) = \int_{u=0}^{w} \rho(u)du \cdot \log y + O(1),$$

and

$$\int_{u=0}^{w} \rho(u)du = \int_{u=0}^{1} du + \int_{u=1}^{w} (1 - \log u)du = \frac{3}{2}w - 1,$$

so that

$$2\sum_{\substack{n\le x\\P(n)\le y}} \frac{1}{n} - \sum_{n\le x} \frac{1}{n} = (2w - 2)\log y + O(1).$$

Since $d \le \ell = y^{o(1)}$ we can use this in (2.7) with x replaced by x/d to obtain

$$\begin{aligned}\sum_{\substack{n\le x\\(n,\ell)=1}} \frac{(\frac{n}{q})}{n} &\ge \sum_{d|\ell}\frac{\mu(d)}{d}((2w - 2)\log y + O(1))\\ &= \frac{\phi(\ell)}{\ell}(2w - 2)\log y + O\left(\frac{\ell}{\phi(\ell)}\right) \sim \frac{\phi(\ell)}{\ell}(2\sqrt{e} - 2)\log n_q\\ &\gg \frac{\phi(\ell)}{\sqrt{\ell}}(\log q)^{\Delta}\end{aligned}$$

using the hypothesis on the size of ℓ. The hypothesis of the second part of Proposition 2.1 is therefore satisfied (with q replaced by ℓq), and so by (2.5) and the last displayed equation we have

$$M((\tfrac{\cdot}{\ell q})) \sim \tau_{\chi,\xi} \cdot \frac{\sqrt{q}\ell}{\pi\phi(\ell)} \cdot \max_{1\le N\le q} \left| \sum_{\substack{n\le N\\(n,\ell)=1}} \frac{(\frac{n}{q})}{n} \right| \gtrsim \frac{2(\sqrt{e}-1)\tau_{\chi,\xi}}{\pi} \cdot \sqrt{q}\log n_q$$

where $\tau_{\chi,1} \in [\frac{1}{2}, 3]$ and $\tau_{\chi,\xi} = \max\{1, |1-(\frac{2}{q})|\}$ if $\xi \neq 1$. □

3 Halász's Theorem and beyond

For multiplicative functions $f, g : \mathbb{N} \to \mathbb{U}$ and $x \ge 2$, we define the *pretentious distance*

$$\mathbb{D}(f, g; x) := \left(\sum_{p\le x} \frac{1-\mathrm{Re}(f(p)\bar{g}(p))}{p} \right)^{1/2}.$$

It is well-known that $\mathbb{D}$ satisfies the triangle inequality:

$$\mathbb{D}(f, h; x) \le \mathbb{D}(f, g; x) + \mathbb{D}(g, h; x) \text{ for all } f, g, h : \mathbb{N} \to \mathbb{U}. \tag{3.1}$$

With $2 \le y \le x$ we also write $\mathbb{D}(f, g; y, x)$ to work only with the primes in $(y, x]$. We say that f is g-*pretentious* (for the primes up to x) if $\mathbb{D}(f, g; x) \ll 1$; so if f is g-pretentious then $f(p) \approx g(p)$ frequently for $p \le x$.

3.1 Halász's theorem

For $T > 0$, $x \ge 2$ and a multiplicative function $f : \mathbb{N} \to \mathbb{U}$, we also define

$$M(f; x, T) := \min_{|t|\le T} \mathbb{D}(f, n^{it}; x)^2.$$

We let $t = t(f; x, T)$ be a real number in this range where we get equality. Halász's Theorem (see e.g., [7, Thm. 1]) states that if $1 \le T \le \log x$ then, for $M = M(f; x, T)$,

$$\sum_{n\le x} f(n) \ll (1+M)e^{-M}x + \frac{x}{T}. \tag{3.2}$$

If $f(n) = n^{it}$ with $|t| \le T$ then $M = 0$, which reflects the fact that $|\sum_{n\le x} n^{it}| \sim x/|1+it|$.

Halász's theorem shows that $\left|\sum_{n\le x} f(n)\right|$ is $o(x)$ if f is not n^{it}-pretentious for any $t \in \mathbb{R}$. Elementary estimates for $\zeta(s)$ to the right of the 1-line imply that

$$\mathbb{D}(1, n^{it}; x)^2 = \begin{cases} \log(1+|t|\log x) + O(1) & \text{if } |t| \le 100; \\ \log\log x + O(\log\log|t|) & \text{if } |t| \ge 100. \end{cases} \tag{3.3}$$

This shows that 1 is *not* n^{it}-pretentious unless $|t| \ll \frac{1}{\log x}$. Therefore if $|t_1|, |t_2| \le T := (\log x)^{O(1)}$ then n^{it_1} cannot be n^{it_2}-pretentious unless $|t_1 - t_2|\log x \ll 1$.

If $t(f; x, T)$ is not unique, say t_1 and t_2 both yield equality above, then (3.1) implies that $\mathbb{D}(n^{it_1}, n^{it_2}; x) \le \mathbb{D}(f, n^{it_1}; x) + \mathbb{D}(f, n^{it_2}; x) = 2\mathbb{D}(f, n^{it_1}; x)$. In particular if f is n^{it_1}-pretentious then f is not n^{it_2}-pretentious unless $t_2 = t_1 + O(\frac{1}{\log x})$.

3.2 Halász-type bounds for logarithmically weighted sums

If f is real-valued then we might expect that $t(f; x, T) = 0$ but there are examples where this is not so (which lead to the "best possible examples" in Hall and Tenenbaum's estimate (1.1)). The examples $f(n) = n^{it}$ show that there cannot be an upper bound in terms of $\mathbb{D}(f, 1; x)$ alone for arbitrary $f : \mathbb{N} \to \mathbb{U}$, though such bounds are given in [12] for f belonging to certain restricted families of multiplicative functions (most importantly those of bounded order).

In this article we will need bounds for the logarithmically weighted sums $\sum_{n\le x} f(n)/n$.

Proposition 3.1 *Let $x \ge 3$. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function with $M = M(f; x, 1)$, and let $t \in [-1, 1]$ minimize the expression*

$$\tau \mapsto \sum_{p\le x} \frac{2 - \mathrm{Re}((1 + f(p))p^{-i\tau})}{p}, \quad \tau \in [-1, 1].$$

If $|t| \le \frac{1}{\log x}$ then

$$\sum_{n\le x} \frac{f(n)}{n} \ll (1 + M)e^{-M} \log x + \log\log x. \tag{3.4}$$

If $|t| \ge \frac{1}{\log x}$ then

$$\sum_{n\le x} \frac{f(n)}{n} \ll \frac{1}{|t|}(1 + M + \log(|t| \log x))e^{-M} + \log\log x. \tag{3.5}$$

The bound (3.5) improves upon Theorem 2.4 in [6] and Theorem 1.4 in [15] whenever $|t| \gg \frac{\log\log x}{\log x}$ (though the latter can be used to replace $(1 + M)e^{-M}$ by just e^{-M}).

3.3 Deductions

Using Proposition 3.1 we now prove Corollary 1.1.

Proof of more than (2.6) Let $\psi = \chi\bar{\xi}$. By the definition of N_q,

$$\mathcal{L} := \left|\sum_{n\le N_q} \frac{\chi\bar{\xi}(n)}{n}\right| \ge \left|\sum_{n\le q} \frac{\chi\bar{\xi}(n)}{n}\right| = |L(1, \psi)| + O(1)$$

(see (4.1) below), and so (2.6) follows if $|L(1, \psi)| \gg \log q$.

Conversely, by (1.2) of Proposition 1.2 (which is a consequence of Proposition 3.1), we have

$$\mathcal{L} := \left|\sum_{n\le N_q} \frac{\chi\bar{\xi}(n)}{n}\right| \ll (\log N_q) \exp(-\{\lambda + o(1)\}\mathbb{D}(\psi, 1; N_q)^2)$$

so that $\exp(-\mathbb{D}(\psi, 1; N_q)^2) \ge (\frac{\log N_q}{\mathcal{L}})^{-1/\lambda+o(1)} \gg (\frac{\mathcal{L}}{\log N_q})^2$. Now as ψ is non-exceptional we have (see Sect. 4) that

$$|L(1, \psi)| \asymp \log q\, e^{-\mathbb{D}(\psi,1;q)^2} = \log q\, e^{-\mathbb{D}(\psi,1;N_q)^2-\mathbb{D}(\psi,1;N_q,q)^2}$$
$$\gg \log q \left(\frac{\mathcal{L}}{\log N_q}\right)^2 \left(\frac{\log N_q}{\log q}\right)^2 = \frac{\mathcal{L}^2}{\log q}$$

since $\mathbb{D}(\psi, 1; N_q, q)^2 \le 2\sum_{N_q<p\le q}\frac{1}{p} \le 2\log(\frac{\log q}{\log N_q}) + O(1)$. If $\mathcal{L} \gg \log q$ then this establishes (2.6); if $\mathcal{L} \gg (\log q)^\tau$ then this gives $|L(1,\psi)| \gg (\log q)^{2\tau-1}$ showing that the example given after (2.6) is, in some sense, "best possible". □

Proof of Corollary 1.1 Suppose that $M(\chi) \gg \sqrt{q}\log q$. Proposition 2.1 shows that there is ξ primitive of conductor $\ell \le \log q$ such that $\xi(-1) = -\chi(-1)$ and

$$\log q \ll \frac{M(\chi)}{\sqrt{q}} \ll \frac{\sqrt{\ell}}{\phi(\ell)}\left|\sum_{n\le N_q}\frac{(\chi\bar{\xi})(n)}{n}\right|.$$

The right-hand sum is $\ll \log q$, so $\ell \ll 1$. If $\chi\bar{\xi}$ is induced by a primitive character ψ (mod ℓ^*) with $\ell^* \mid \ell$ then by Lemma 4.4 of [8],

$$\sum_{n\le N_q}\frac{\psi(n)}{n} = \prod_{p\mid\frac{\ell}{\ell^*}}\left(1-\frac{\psi(p)}{p}\right)^{-1}\sum_{n\le N_q}\frac{(\chi\bar{\xi})(n)}{n} + O(1), \tag{3.6}$$

so $|\sum_{n\le N_q}\frac{\psi(n)}{n}| \gg \log q$. By (2.6), we obtain $|L(1,\psi)| \gg \log q$.

Conversely, suppose there is a primitive character ξ of conductor $\ell \ll 1$ with $\xi(-1) = -\chi(-1)$ such that $|L(1,\psi)| \gg \log q$, where ψ is the primitive character that induces $\chi\bar{\xi}$. By (2.6) and (3.6) we have $|\sum_{n\le N_q}\frac{(\chi\bar{\xi})(n)}{n}| \gg \frac{\phi(\ell)}{\sqrt{\ell}}\log q$, and so Proposition 2.1 implies that

$$M(\chi) \gg \frac{\sqrt{q\ell}}{\phi(\ell)}\left|\sum_{n\le N_q}\frac{(\chi\bar{\xi})(n)}{n}\right| \gg \sqrt{q}\log q,$$

as claimed. □

3.4 A generalization of Halász's theorem

Given a multiplicative function $f:\mathbb{N}\to\mathbb{C}$, let $F(s) := \sum_{n\ge1} f(n)/n^s$ denote its Dirichlet series, assumed to be analytic and non-zero for $\mathrm{Re}(s) > 1$. For such s we write $-\frac{F'}{F}(s) = \sum_{n\ge1}\Lambda_f(n)/n^s$.

For fixed $\kappa \ge 1$ we restrict attention to those multiplicative functions f for which $|\Lambda_f(n)| \le \kappa\Lambda(n)$, where Λ is the von Mangoldt function. A generalization of Halász's Theorem to such f (Theorem 1.1 of [11]) states that

$$\sum_{n\le x} f(n) \ll (1+M)e^{-M}x(\log x)^{\kappa-1} + \frac{x}{\log x}(\log\log x)^\kappa, \tag{3.7}$$

where M is defined by

$$e^{-M}(\log x)^\kappa = \max\left\{|s^{-1}F(s)|:\ s = 1+\tfrac{1}{\log x}+it \text{ with } |t|\le(\log x)^\kappa\right\}.$$

In Sect. 5 we will apply this result (with $\kappa = 2$) to the convolution $1 * f$, where $f:\mathbb{N}\to\mathbb{U}$ is multiplicative, in order to prove Proposition 3.1.

4 Large short character sums and large $L(1+it,\chi)$ values

In this section we will prove Proposition 1.1.

4.1 Truncations of $L(1, \chi)$

Let $t \in \mathbb{R}$. By partial summation we have

$$\left|\sum_{n>N} \frac{\chi(n)}{n^{1+it}}\right| \le \frac{|S(\chi, N)|}{N} + |1+it| \int_N^\infty \frac{|S(\chi, y)|}{y^2} dy \le (2+|t|)\frac{M(\chi)}{N},$$

so by the Pólya-Vinogradov theorem,

$$L(1+it, \chi) = \sum_{n\le N} \frac{\chi(n)}{n^{1+it}} + O\left((2+|t|)\frac{\sqrt{q}\log q}{N}\right). \tag{4.1}$$

We wish to also truncate the Euler product for $L(1+it, \chi)$ at q when $|t| \ll 1$, losing at most a constant multiple. The prime number theorem in arithmetic progressions tells us that there exist constants $A, c > 0$ such that if $L(s, \chi)$ has no exceptional zero then

$$\psi(x, \chi) = \sum_{n\le x} \chi(n)\Lambda(n) \ll x \exp\left(-c\frac{\log x}{\log q}\right) + \frac{x}{\log x} \tag{4.2}$$

for all $x \ge q^A$. By partial summation we deduce that if $B > A$ then

$$\sum_{p>q^B} \frac{\chi(p)}{p^{1+it}} = -(1+it)\int_{q^B}^\infty \frac{\psi(u, \chi)}{u^{2+it}\log u} du + O(1) \ll (1+|t|)\frac{e^{-cB}}{B} + 1.$$

Now let $B = \max\{2A, (1/c)\log(1+|t|)\}$ so that since $\sum_{q<p\le q^B} \chi(p)/p^{1+it} \le \sum_{q<p\le q^B} 1/p \ll \log B$, we have

$$\sum_{p>q} \frac{\chi(p)}{p^{1+it}} \ll \log B + (1+|t|)\frac{e^{-cB}}{B} + 1 \ll 1 + \log\log(1+|t|).$$

Hence, if χ is non-exceptional then for all $t \in \mathbb{R}$, $|t| \ll 1$,

$$|L(1+it, \chi)| \asymp \left|\prod_{p\le q}\left(1 - \frac{\chi(p)}{p^{1+it}}\right)^{-1}\right| \asymp (\log q)e^{-\mathbb{D}(\chi, n^{it}; q)^2}. \tag{4.3}$$

Taking $N = q$ in (4.1) and assuming $|t| \ll 1$, we have

$$L(1+it, \chi) = \sum_{n\le q} \frac{\chi(n)}{n^{1+it}} + o(1) = (1+it)\int_1^q \frac{S(\chi, u)}{u^{2+it}} du + o(1)$$
$$\ll \int_1^q \frac{|S(\chi, u)|}{u^2} du + o(1),$$

and so, for any $c > 0$, we have

$$|L(1+it, \chi)| \ll \left(c + \max_{q^c\le x\le q} \frac{1}{x}|S(\chi, x)|\right)\log q \tag{4.4}$$

using the bound $|S(\chi, u)| \le u$ for $u \le q^c$.

4.2 Exceptional characters

Landau proved that there exists an absolute constant $c > 0$ such that for any Q sufficiently large there is *at most* one $q \leq Q$, one primitive real character $\chi \pmod q$ and one real number $\beta \in (0, 1)$ such that

$$L(\beta, \chi) = 0 \text{ and } \beta \geq 1 - \frac{c}{\log Q}.$$

If such a triple (q, χ, β) exists then we call q an *exceptional modulus*, β an *exceptional zero* and χ an *exceptional character*.

If exceptional zeros exist there must be infinitely many of them (otherwise we can decrease c as needed). If $\{\beta_j\}_j$ is a sequence of exceptional zeros and $\{q_j\}_j$ is the corresponding set of exceptional moduli then

$$(1 - \beta_j) \log q_j \to 0 \text{ as } j \to \infty.$$

It is an important open problem to obtain effective lower bounds for $1 - \beta$. Siegel's theorem (see e.g., [14, Thm. 5.28]) states that if β is the largest real zero of $L(s, \chi)$ then $1-\beta \gg_\epsilon q^{-\epsilon}$ for any $\epsilon > 0$, but the implicit constant is ineffective unless $\epsilon \geq 1/2$.

If $L(s, \chi)$ has an exceptional zero then $\chi(p) = -1$ for many "small" primes. This suggests (but does not directly imply) the following result:

Lemma 4.1 *Suppose that χ is an exceptional character modulo q. Then:*

(a) $|L(1 + it, \chi)| = o(\log q)$ *when* $|t| \ll 1$, *and*
(b) *for fixed* $c > 0$ *we have* $|S(\chi, x)| = o_{q\to\infty}(x)$ *for all* $x \geq q^c$.

Proof (a) ($t = 0$): As χ is exceptional it must be real, and there is a $\beta \in (0, 1)$ such that $L(\beta, \chi) = 0$ with $\eta := (1 - \beta) \log q = o(1)$. By the truncation argument in (4.1),

$$L(\beta, \chi) = \sum_{n\leq q} \frac{\chi(n)}{n} n^{1-\beta} + O\left(\frac{M(\chi)}{q^\beta}\right) = \sum_{n\leq q} \frac{\chi(n)}{n} \left(1 + O(\eta)\right) + O(q^{\frac{1}{2}-\beta} \log q)$$
$$= \sum_{n\leq q} \frac{\chi(n)}{n} + O\left(\eta \log q\right)$$

since $\eta \gg q^{-o(1)}$. By (4.1) we deduce that for any $\epsilon > 0$,

$$|L(1, \chi)| = \left|\sum_{n\leq q} \frac{\chi(n)}{n}\right| + O(1/q^\epsilon) \ll \eta \log q \tag{4.5}$$

since $L(\beta, \chi) = 0$, which implies (a).

(b) We use the above to observe that

$$\frac{1}{q} \sum_{n\leq q} (1 * \chi)(n) = \frac{1}{q} \sum_{n\leq q} \chi(n) \left\lfloor \frac{q}{n} \right\rfloor = \sum_{n\leq q} \frac{\chi(n)}{n} + O(1) \ll \eta \log q + 1;$$

on the other hand we have

$$\frac{1}{q} \sum_{n\leq q} (1 * \chi)(n) \gg e^{-ue^{u/2}} \log q + O(1) \text{ where } u = \mathbb{D}(\chi, 1; q)^2$$

by [9, (3.5)], so that $\mathbb{D}(\chi, 1; q)^2 = u \geq \log\log(\frac{1}{\theta}) + O(1)$ where $\theta := \max\{\eta, \frac{1}{\log q}\}$.[3] If $x \in [q^c, q]$ then $\mathbb{D}(\chi, 1; x)^2 = \mathbb{D}(\chi, 1; q)^2 + O(1) \geq \log\log(\frac{1}{\theta}) + O(1)$. Therefore since χ is real, Hall and Tenenbaum's estimate (1.1) yields

$$|S(\chi, x)| \ll x e^{-\tau \mathbb{D}(\chi,1;x)^2} \ll \frac{x}{(\log(1/\theta))^\tau} = o(x).$$

(a) ($|t| \ll 1$): We insert the bound from (b) into (4.4), and let $c \to 0$ to deduce our result. □

Proof *(Proof of Proposition 1.1)* We may assume that χ is an unexceptional character, since the result follows vacuously when χ is exceptional by Lemma 4.1.

Now if $|L(1 + it, \chi)| \gg \log q$ for some $|t| \ll 1$ and if $c > 0$ is sufficiently small there exists $x \in [q^c, q]$ for which $|S(\chi, x)| \gg x$ by (4.4).

Now suppose that $|S(\chi, x)| \gg x$ for some $x \in [q^\kappa, q]$ for some fixed $\kappa \in (0, 1]$. For a sufficiently large constant T, Halász's Theorem (3.2) implies that $\mathbb{D}(\chi, n^{it}; x) \ll 1$ for some $|t| \leq T$, and so

$$\mathbb{D}(\chi, n^{it}; q)^2 = \mathbb{D}(\chi, n^{it}; x)^2 + \sum_{x<p\leq q} \frac{1 - \mathrm{Re}(\chi(p)p^{-it})}{p} \ll 1 + 2\log\left(\frac{\log q}{\log x}\right) \ll_\kappa 1. \tag{4.6}$$

Then (4.3) implies that $|L(1 + it, \chi)| \gg_\kappa \log q$ as χ is non-exceptional.

Suppose now that $\chi^k = \chi_0$ with $k \ll 1$. As $x > q^\kappa$ we note then that

$$x \ll |S(\chi, x)| \leq \#\{n \leq x : (n, q) = 1\} \ll \frac{\phi(q)}{q} x,$$

which implies that $\sum_{p|q} \frac{1}{p} \ll 1$. Next, repeatedly using the triangle inequality (3.1) for $\mathbb{D}$ together with (4.6),

$$\mathbb{D}(1, n^{ikt}; q) = \mathbb{D}(\chi^k, n^{ikt}; q) + O\left(1 + \sum_{p|q} \frac{1}{p}\right) \leq k\,\mathbb{D}(\chi, n^{it}; q) + O(1) \ll_\kappa 1.$$

By (3.3) we deduce that $\log(1 + k|t|\log q) \ll_\kappa 1$, so that $|t| \ll_\kappa \frac{1}{\log q}$. It follows then that $\mathbb{D}(\chi, 1; q)^2 = \mathbb{D}(\chi, n^{it}, q)^2 + O_\kappa(1) \ll 1$, and so $|L(1, \chi)| \asymp \log q\, e^{-\mathbb{D}(\chi,1;q)^2} \gg_\kappa \log q$ by (4.3). □

We would like to deduce that $|L(1, \chi)| \gg \log q$ from $|L(1+it, \chi)| \gg \log q$ with $|t| \ll 1$, for characters of higher order. This is not necessarily guaranteed, though we can prove the following.

Lemma 4.2 *Let χ be a complex character mod q and fix $T \geq 1$. If $|L(1 + it, \chi)| \gg \log q$ with $|t| \leq T$ then $|L(1 + it_0, \chi)| \gg \log q$ where $t_0 = t(\chi; q, T)$ and $|t - t_0| \ll \frac{1}{\log q}$.*

Proof Since χ is not real, it is non-exceptional. By (4.3) we see that

$$|L(1 + it, \chi)| \gg \log q \text{ if and only if } \mathbb{D}(\chi, n^{it}; q) \ll 1.$$

[3] One can obtain the better lower bound $\mathbb{D}(\chi, 1; q)^2 \geq \{2 + o(1)\}\log\log(\frac{1}{\eta})$ by summing [19, (3.23)] over all $m \ll \sqrt{\log 1/\eta}$ (note that their η is our $1/\eta$).

Let $t_0 = t(\chi; q, T)$ so that $\mathbb{D}(\chi, n^{it_0}; q) \le \mathbb{D}(\chi, n^{it}; q) \ll 1$, and therefore $|L(1+it_0, \chi)| \gg \log q$ by (4.3). Moreover

$$\mathbb{D}(1, n^{i(t-t_0)}; q) = \mathbb{D}(n^{it_0}, n^{it}; q) \le \mathbb{D}(\chi, n^{it}; q) + \mathbb{D}(\chi, n^{it_0}; q) \ll 1$$

by (3.1), and so we deduce that $|t - t_0| \ll \frac{1}{\log q}$ by (3.3). □

5 A variant of Halász's Theorem

In this section we will prove Propositions 3.1 and 1.2(a), our various upper bounds for logarithmic averages.

Proof of Proposition 3.1 As in the proof of Lemma 4.1,

$$\sum_{m\le x}(1*f)(m) = \sum_{n\le x} f(n)\left\lfloor \frac{x}{n}\right\rfloor = x\sum_{n\le x}\frac{f(n)}{n} + O(x).$$

Applying (3.7) to the mean value of $1*f$ with $\kappa = 2$, we then obtain

$$\sum_{n\le x}\frac{f(n)}{n} = \frac{1}{x}\sum_{m\le x}(1*f)(m) + O(1) \ll (1+M)e^{-M}\log x + 1$$

where $e^{-M}(\log x)^2 = |\frac{1}{s}\zeta(s)F(s)|$ with $s = 1+1/\log x + it$, for some real t, $|t| \le (\log x)^2$. We have $|\zeta(s)| \le \zeta(1+\frac{1}{\log x}) \ll \log x$,

$$|F(s)| \asymp \zeta(1+\tfrac{1}{\log x})\exp(-\mathbb{D}(f, n^{it}; x)^2) \asymp (\log x)\exp(-\mathbb{D}(f, n^{it}; x)^2) \le \log x; \tag{5.1}$$

and $M \ll \log\log x$. If $|t| \ge 1$ then $|\zeta(s)| \ll \log(2+|t|)$, and so

$$\left|\sum_{n\le x}\frac{f(n)}{n}\right| \ll (1+M)\frac{|F(s)|}{\log x}\frac{\log(2+|t|)}{1+|t|} \ll \log\log x.$$

We henceforth assume that $|t| \le 1$. We have

$$|\zeta(s)/s| \asymp \log y_t \text{ where } y_t = \min\{e^{1/|t|}, x\},$$

and

$$e^{-M} = \frac{|\zeta(s)F(s)/s|}{(\log x)^2} \asymp \exp\left(-\min_{|\tau|\le 1}(\mathbb{D}(1, n^{i\tau}; x)^2 + \mathbb{D}(f, n^{i\tau}; x)^2)\right).$$

By (3.3), $\mathbb{D}(1, n^{it}; x)^2 = \log(\frac{\log x}{\log y_t}) + O(1)$, so we deduce that

$$\sum_{n\le x}\frac{f(n)}{n} \ll \log y_t\ \exp(-\mathbb{D}(f, n^{it}; x)^2)\cdot\mathcal{L} \tag{5.2}$$

$$\asymp \mathcal{L}e^{-\mathcal{L}}\log x, \tag{5.3}$$

where we have set

$$\mathcal{L} := 1 + \mathbb{D}(f, n^{it}; x)^2 + \log(\tfrac{\log x}{\log y_t}) \ll \log\log x.$$

From (5.2) and $\mathbb{D}(f, n^{it}; x)^2 \ge M(f; x, 1)$ we obtain (3.4) when $|t| \le \frac{1}{\log x}$ and (3.5) when $\frac{1}{\log x} < |t| \le 1$. □

The next proof continues on using the results in the previous proof:

Proof of Proposition 1.2(a) If $|t| \le \frac{1}{\log x}$ then $\mathbb{D}(f, 1; x) = \mathbb{D}(f, n^{it}; x) + O(1)$, and in this case we also obtain (1.2) (for the previous proof) with the better constant 1 in place of λ.

For $\frac{1}{\log x} \le |t| \le 1$, we now prove the lower bound $\mathcal{L} \ge \lambda\,\mathbb{D}(f, 1; x)^2 + O(1)$. When we substitute this into (5.3), we obtain (1.2) since $y \mapsto ye^{-y}$ is a decreasing function for $y \ge 0$.

First, as $p^{-it} = 1 + O(|t| \log p)$ when $p \le y_t$, we obtain

$$\mathbb{D}(f, n^{it}; y_t)^2 - \mathbb{D}(f, 1; y_t)^2 = \sum_{p \le y_t} \frac{\mathrm{Re}(f(p)(1 - p^{-it}))}{p} \ll |t| \sum_{p \le y_t} \frac{\log p}{p} \ll 1.$$

The prime number theorem implies that

$$\begin{aligned}\sum_{y_t < p \le x} \frac{\mathrm{Re}(f(p)(p^{-it} - \lambda))}{p} \le \sum_{y_t < p \le x} \frac{|p^{-it} - \lambda|}{p} &= \left(\int_0^1 |e(u) - \lambda| du\right) \log\left(\frac{\log x}{\log y_t}\right) \\ &\quad + O(1) \\ &= (2 - \lambda) \sum_{y_t < p \le x} \frac{1}{p} + O(1), \qquad (5.4)\end{aligned}$$

using the definition of λ. Re-organised, this gives

$$\mathbb{D}(f, n^{it}; y_t, x)^2 + \log\left(\frac{\log x}{\log y_t}\right) + O(1) \ge \lambda\,\mathbb{D}(f, 1; y_t, x)^2 + O(1),$$

so that

$$\begin{aligned}\mathcal{L} &= (\mathbb{D}(f, n^{it}; y_t, x)^2 + \log(\tfrac{\log x}{\log y_t})) + \mathbb{D}(f, n^{it}; y_t)^2 + 1 \\ &\ge \lambda\,\mathbb{D}(f, 1; y_t, x)^2 + \mathbb{D}(f, 1; y_t)^2 + O(1) \ge \lambda\,\mathbb{D}(f, 1; x)^2 + O(1).\end{aligned}$$

□

6 Large character sums

6.1 Consequences of repulsion

Suppose that we are given a primitive character $\chi \pmod q$. Fix $A > 0$. For each primitive character $\psi \pmod \ell$ with $\ell \le (\log q)^A$ select $|t| \le 1$ for which $\mathbb{D}(\chi, \psi\, n^{it}; q)$ is minimized. Index the pairs (ψ, t) so that (ψ_j, t_j) is the pair that gives the j-th smallest distance $\mathbb{D}(\chi, \psi_j\, n^{it_j}; q)$ (breaking ties arbitrarily if needed). A simple modification of [1, Lem. 3.1] shows that for each $k \ge 2$ we have

$$\mathbb{D}(\chi, \psi_k\, n^{it_k}; q)^2 \ge (c_k + o(1)) \log\log q$$

where $c_k \ge 1 - \frac{1}{\sqrt{k}}$. As any $1 \le n \le N \le q$ has $P(n) \le q$, [15, Thm. 6.4] yields

$$\sum_{n \le N} \frac{(\chi\bar{\psi}_k)(n)}{n} = \sum_{\substack{n \le N \\ P(n) \le q}} \frac{(\chi\bar{\psi}_k)(n)}{n} \ll (\log q) e^{-\mathbb{D}(\chi, \psi_k n^{it_k}; q)^2} + 1 \ll (\log q)^{1 - c_k + o(1)}. \qquad (6.1)$$

Under the additional hypothesis that $\psi_1\psi_2$ is an even character (which follows if we restrict attention to ψ with $(\chi\psi)(-1) = -1$) we can take any $c_2 > 1 - \frac{2}{\pi}$ as we will see below in Lemma 6.1.

For each integer $k \geq 1$ we define

$$\gamma_k := \frac{1}{k}\sum_{a=0}^{k-1} |\cos(\pi a/k)| = \frac{1}{k}\sum_{a=0}^{k-1} \big|1 + e\big(a/k\big)\big|.$$

Using the Fourier expansion

$$|1 + e(\alpha)| = \frac{4}{\pi}\left(1 - \sum_{d\neq 0} \frac{(-1)^d}{4d^2 - 1} e(d\alpha)\right), \quad \alpha \in \mathbb{R}/\mathbb{Z}, \tag{6.2}$$

we easily deduce that

$$\gamma_k = \frac{2}{\pi}\left(1 - 2\sum_{\substack{r\geq 1\\ k|r}} \frac{(-1)^r}{4r^2 - 1}\right) = \begin{cases} \frac{\operatorname{cosec}(\pi/2k)}{k} & \text{if } k \text{ is odd,} \\ \frac{\cot(\pi/2k)}{k} & \text{if } k \text{ is even;} \end{cases} \tag{6.3}$$

(see also Lemma 5.2 of [10]). Therefore $\gamma_k = \frac{2}{\pi} + O(\frac{1}{k^2})$, and the γ_k, with k even, increase towards $\frac{2}{\pi} = 0.6366\ldots$:

$$\gamma_2 = \tfrac{1}{2} < \gamma_4 = 0.6035\cdots < \gamma_6 = 0.6220\cdots < \gamma_8 = 0.6284\cdots$$

whereas the γ_k, with k odd, decrease towards $\frac{2}{\pi}$:

$$\gamma_1 = 1 > \gamma_3 = \tfrac{2}{3} > \gamma_5 = 0.6472\cdots > \gamma_7 = 0.6419\cdots > \gamma_9 = 0.6398\cdots$$

For $k > 1$, we have $\gamma_k \leq \frac{2}{3}$.

Lemma 6.1 *Fix $C > 0$ and let $m \leq (\log q)^C$. Let $t_1, t_2 \in \mathbb{R}$ be chosen such that $t := t_2 - t_1$ satisfies $|t| \ll 1$. Let χ_1 and $\chi_2 = \chi_1\xi$ be characters with modulus in $[q, q^2]$, where $\xi \pmod m$ is an odd character. Suppose that $\mathbb{D}(\chi_2, n^{it_2}; q) \geq \mathbb{D}(\chi_1, n^{it_1}; q)$. Then for any $\epsilon > 0$,*

$$\mathbb{D}(\chi_2, n^{it_2}; q)^2 \geq (1 - \tfrac{2}{\pi} - \epsilon\theta)\log\log q + O(\log\log\log q),$$

where $\theta := 1$ if ξ^j is exceptional for some $1 \leq j \leq \min\{\phi(m), \log\log q\}$, and $\theta := 0$ otherwise.

Proof Suppose that $\xi \pmod m$ has order k. We note that

$$\begin{aligned}\mathbb{D}(\chi_2, n^{it_2}; q)^2 &\geq \frac{1}{2}\Big(\mathbb{D}(\chi_2, n^{it_2}; q)^2 + \mathbb{D}(\chi_1, n^{it_1}; q)^2\Big) \\ &= \log\log q - \frac{1}{2}\mathrm{Re}\left(\sum_{p\leq q} \frac{\chi_1(p)p^{-it_1} + \chi_2(p)p^{-it_2}}{p}\right) + O(1) \\ &= \log\log q - \frac{1}{2}\mathrm{Re}\left(\sum_{p\leq q} \frac{\chi_1(p)p^{-it_1}(1 + \xi(p)p^{-it})}{p}\right) + O(1) \\ &\geq \log\log q - \frac{1}{2}\sum_{p\leq q} \frac{|1 + \xi(p)p^{-it}|}{p} + O(1) \end{aligned} \tag{6.4}$$

Using (6.2) we deduce that

$$\sum_{p\le q}\frac{|1+\xi(p)p^{-it}|}{p}=\frac{4}{\pi}\left(\log\log q-\sum_{1\le|d|\le D}\frac{(-1)^d}{4d^2-1}S_d\right)+O(1),$$

where $D:=\log\log q$ and

$$S_d:=\sum_{p\le q}\frac{\xi(p)^d}{p^{1+idt}}.$$

If k divides d then $\xi(p)^d=1_{p\nmid m}$, and so

$$S_d=\sum_{p\le q}\frac{1}{p^{1+idt}}+O\left(\sum_{p|m}\frac{1}{p}\right)=\sum_{p\le q}\frac{1}{p^{1+idt}}+O(\log\log\log\log q).$$

The sum over $p\le z:=\min\{q,e^{2\pi/|dt|}\}$ equals

$$\begin{aligned}\sum_{p\le z}\frac{1}{p}+O\left(|dt|\sum_{p\le z}\frac{\log p}{p}\right)&=\log(\min\{\log q,\tfrac{2\pi}{|dt|}\})+O(1)\\&=\log(\min\{\log q,\tfrac{1}{|t|}\})+O(\log\log\log q).\end{aligned}$$

The remaining set of primes $z<p\le q$ is non-empty only if $z=e^{2\pi/|dt|}$, which happens when $|dt|>\frac{2\pi}{\log q}$. Let $\sigma:=\frac{dt}{|dt|}\in\{-1,+1\}$. Applying the prime number theorem,

$$\sum_{z<p\le q}\frac{1}{p^{1+idt}}=\int_{e^{2\pi/|dt|}}^{q}u^{-idt}\frac{du}{u\log u}+O(1)=\int_1^X e(-\sigma v)\frac{dv}{v}+O(1)=O(1),$$

putting $v=\frac{|dt|\log u}{2\pi}$ and $X=\frac{\log q}{\log z}$ (≥ 1) and then integrating by parts. We deduce that if k divides d with $d\le D$ then $S_d=\log(\min\{\log q,\frac{1}{|t|}\})+O(\log\log\log q)$.

If k does not divide d let $\alpha:=\xi^d$, a non-principal character mod m, and let $T=dt$ so that $|T|\ll D$. For $\epsilon>0$ small, let

$$Y:=\begin{cases}\exp((\log m)^{10}+|T|) & \text{if } \alpha \text{ is non-exceptional},\\ \exp((\log q)^{\epsilon}) & \text{if } \alpha \text{ is exceptional}.\end{cases}$$

Trivially bounding the primes $p\le Y$ we deduce that

$$S_d=\sum_{Y<p\le q}\frac{\alpha(p)}{p^{1+iT}}+O(\log\log Y).$$

Since $\psi(y,\alpha)\ll\frac{y}{\log y}$ for $y\ge Y$ by the Siegel–Walfisz theorem when α is exceptional, and using (4.2) otherwise, this is

$$\begin{aligned}\int_Y^q\frac{d\psi(y,\alpha)}{y^{1+iT}\log y}+O\left(\frac{1}{Y}\right)&=\left[\frac{\psi(y,\alpha)}{y^{1+iT}\log y}\right]_Y^q+(1+iT)\int_Y^q\frac{\psi(y,\alpha)}{y^{2+iT}\log y}dy\\&\quad+O\left(\frac{1}{(\log Y)^2}\right)\\&\ll\frac{1}{(\log Y)^2}+T\int_Y^q\frac{1}{y(\log y)^2}dy\ll 1.\end{aligned}$$

We conclude that $S_d \ll \log\log Y$ whenever $k \nmid d$.

Putting these estimates into (6.3), we find that

$$\sum_{p\le q} \frac{|1+\xi(p)p^{-it}|}{p} = \frac{4}{\pi}\log\log q + 2\left(\gamma_k - \frac{2}{\pi}\right)\log(\min\{\log q, \tfrac{1}{|t|}\}) + O(\log\log\log q + \log\log Y),$$

and $\log\log Y \ll \epsilon\theta \log\log q + \log\log\log q$. Changing ϵ by a (possibly ineffective) constant factor if needed, we deduce from (6.4) that

$$\mathbb{D}(\chi_2, n^{it_2}; q)^2 \ge (1 - \tfrac{2}{\pi} - \epsilon\theta)\log\log q + (\tfrac{2}{\pi} - \gamma_k)\log(\min\{\log q, \tfrac{1}{|t|}\}) + O(\log\log\log q).$$

Now since ξ is odd, its order k must be even and therefore $\gamma_k \le \frac{2}{\pi}$; the result follows since the middle term is $\gg -O(1)$. □

For any $\epsilon > 0$ let $K > 1/\epsilon^2$ so that $1 - c_K < \epsilon$. As a consequence of (6.1) and Lemma 6.1, established just below, we have

$$\max_{1\le N\le q}\left|\sum_{n\le N}\frac{(\chi\bar{\psi})(n)}{n}\right| = \begin{cases} O((\log q)^{\epsilon}) & \text{if } \psi \ne \psi_k \text{ for all } k < K,\\ O_\epsilon((\log q)^{\frac{2}{\pi}+\epsilon}) & \text{if } \psi = \psi_k \text{ for some } k,\ 1 < k < K. \end{cases} \tag{6.5}$$

The implicit constant in the second case is effective as long as ψ^j is non-exceptional for all $j \ge 1$, and in this case $\frac{2}{\pi} + \epsilon$ can be replaced by $\frac{2}{\pi} + o(1)$.

6.2 Working with large character sums

In this subsection we set the stage for the proof of Proposition 2.1. We recall that $\Delta \in (\frac{2}{\pi}, 1)$ and $q \ge 3$ are given, and

$$R_q := \exp(\tfrac{(\log q)^{\Delta}}{\log\log q}), \quad r_q := (\log q)^{2-2\Delta}(\log\log q)^4.$$

Proposition 6.1 *For any given primitive character* $\chi \pmod q$ *there exists a primitive character* $\xi \pmod \ell$ *with* $\ell \le r_q$ *and* $(\chi\xi)(-1) = -1$ *such that if* $|\alpha - \frac{b}{m}| \le \frac{1}{mR_q}$ *with* $m \le r_q$ *and* $N := \min\{q, \frac{1}{|m\alpha - b|}\}$ *then*

$$\sum_{1\le|n|\le q}\frac{\chi(n)e(n\alpha)}{n} = 1_{\ell|m}\frac{2\eta g(\xi)}{\phi(m)}\prod_{p|\frac{m}{\ell}}(1 - (\bar{\chi}\xi)(p))\sum_{1\le n\le N}\frac{(\chi\bar{\xi})(n)}{n} + o((\log q)^{\Delta}),$$

where, whenever $\ell \mid m$ *we write* ℓ_q *to denote the largest divisor of* m/ℓ *that is coprime to* q, $m_q\ell_q = m/\ell$ *and*

$$\eta := \mu(m_q)\chi(\ell_q)\bar{\xi}(b)\xi(m_q) \in S^1 \cup \{0\}.$$

The proof of Proposition 6.1 is very similar to the proof of the main results in [4].

Proof By (2.4) we have

$$\sum_{1\le|n|\le q}\frac{\chi(n)e(n\alpha)}{n} = \sum_{1\le|n|\le N}\frac{\chi(n)e(n\frac{b}{m})}{n} + O(\log\log q). \tag{6.6}$$

With the intent of replacing exponentials by Dirichlet characters on the right-hand side, we split the nth summand according to the common factors of n with m. Therefore if $(n, m) = d$ with $m = cd$ and $n = rd$ we have

$$\begin{aligned}\sum_{1\le|n|\le N}\frac{\chi(n)}{n}e(n\tfrac{b}{m}) &= \sum_{cd=m}\frac{\chi(d)}{d}\sum_{\substack{1\le|r|\le N/d\\(r,c)=1}}\frac{\chi(r)}{r}e(r\tfrac{b}{c})\\ &= \sum_{cd=m}\frac{\chi(d)}{d\phi(c)}\sum_{\psi \pmod c}\bar{\psi}(b)g(\psi)\sum_{1\le|n|\le N/d}\frac{\chi\bar{\psi}(n)}{n}\\ &= 2\sum_{cd=m}\frac{\chi(d)}{d\phi(c)}\sum_{\substack{\psi \pmod c\\ \chi\psi(-1)=-1}}\bar{\psi}(b)g(\psi)\sum_{1\le n\le N/d}\frac{\chi\bar{\psi}(n)}{n},\end{aligned}$$

since if $(k, c) = 1$ then $e(\frac{k}{c}) = \frac{1}{\phi(c)}\sum_{\psi \pmod c}\bar{\psi}(k)g(\psi)$, and then noting the n and $-n$ terms cancel if $\chi\psi(-1) = 1$.

To control the size of $g(\psi)$ we split the sum over characters modulo c according to the primitive characters that induce them. Since each ψ factors as $\psi^*\psi_0^{(f)}$, where ψ^* is primitive modulo e and $\psi_0^{(f)}$ is principal modulo f with $ef = c$, the right-hand side of the above sum is

$$\begin{aligned}&2\sum_{efd=m}\frac{\chi(d)}{d\phi(ef)}\sideset{}{^*}\sum_{\substack{\psi^* \pmod e\\ \chi\psi^*(-1)=-1}}\bar{\psi}^*(b)g(\psi^*\psi_0^{(f)})\sum_{\substack{1\le n\le N/d\\(n,f)=1}}\frac{\chi\bar{\psi}^*(n)}{n}\\ &= 2\sum_{\substack{efd=m\\(e,f)=1}}\frac{\mu(f)\chi(d)}{d\phi(ef)}\sideset{}{^*}\sum_{\substack{\psi^* \pmod e\\ \chi\psi^*(-1)=-1}}\psi^*(f\bar{b})g(\psi^*)\sum_{\substack{1\le n\le N/d\\(n,f)=1}}\frac{\chi\bar{\psi}^*(n)}{n},\end{aligned} \tag{6.7}$$

since $g(\psi^*\psi_0^{(f)}) = \psi^*(f)\mu(f)g(\psi^*)$.

Fix $ef \mid m$ and $\psi^* \pmod e$. We extend the inner sum in (6.7) to all $n \le N$ as

$$\sum_{\substack{1\le n\le N/d\\(n,f)=1}}\frac{\chi\bar{\psi}^*(n)}{n} = \sum_{\substack{1\le n\le N\\(n,f)=1}}\frac{\chi\bar{\psi}^*(n)}{n} + O(\log(2d)). \tag{6.8}$$

By Lemma 4.4 of [8],

$$\sum_{\substack{1\le n\le N\\(n,f)=1}}\frac{\chi\bar{\psi}^*(n)}{n} = \prod_{p\mid f}\left(1-\frac{\chi\bar{\psi}^*(p)}{p}\right)\sum_{1\le n\le N}\frac{\chi\bar{\psi}^*(n)}{n} + O\Big((\log\log(2+f))^2\Big). \tag{6.9}$$

We next observe the identity

$$\sum_{\substack{fd=m/e\\(e,f)=1}}\frac{\mu(f)\chi(d)}{d\phi(ef)}\psi^*(f)\prod_{p\mid f}\left(1-\frac{\chi\bar{\psi}^*(p)}{p}\right) = \frac{\chi(e_q)\psi^*(m_q)\mu(m_q)}{\phi(m)}\prod_{p\mid\frac{m}{e}}(1-(\bar{\chi}\psi^*)(p)),$$

where now e_q is the largest divisor of m/e which is coprime to q, and $m_q e_q = m/e$. The main terms from (6.9) thus contribute

$$\frac{2}{\phi(m)} \sum_{e|m} \mu(m_q)\chi(e_q) \sum^*_{\substack{\psi^* \pmod e \\ \chi\psi^*(-1)=-1}} g(\psi^*)\psi^*(m_q\bar{b}) \prod_{p|\frac{m}{e}} (1-(\bar{\chi}\psi^*)(p)) \sum_{1\le n\le N} \frac{\chi\bar{\psi}^*(n)}{n} \tag{6.10}$$

in (6.7). By noting that $|g(\psi^*)| = \sqrt{e} = \sqrt{m/df}$ the contribution of the error terms from (6.8) and (6.9) in (6.7) is bounded by

$$\ll \sqrt{m} \sum_{fd|m} \frac{(\log 2f)(\log 2d)}{d^{3/2}\phi(f)f^{1/2}} \ll \sqrt{m} \le \sqrt{r_q} = o((\log q)^{\Delta}). \tag{6.11}$$

We now apply (6.5) with $N = \min\{q, \frac{1}{|m\alpha-b|}\}$. Set $\xi := \psi_1$, whose contribution,

$$\frac{2\mu(m_q)\chi(\ell_q)\xi(\bar{b}m_q)g(\xi)}{\phi(m)} \prod_{p|\frac{m}{\ell}} (1-(\bar{\chi}\xi)(p)) \sum_{1\le n\le N} \frac{\chi\bar{\xi}(n)}{n}$$

only appears in (6.10) if the conductor ℓ of ξ divides m. By (6.5) the contribution to (6.10) from all the characters $\psi \ne \psi_k$ for all $k < K$ is, for ϵ sufficiently small,

$$\ll (\log q)^{\epsilon} \sum_{e|m} \frac{\sqrt{e}\,\tau(m/e)\phi(e)}{\phi(m)} \ll \sqrt{m}(\log q)^{2\epsilon} \le \sqrt{r_q}(\log q)^{2\epsilon} \ll (\log q)^{1-\Delta+3\epsilon}$$
$$= o((\log q)^{\Delta}).$$

Since the coefficient in front of each individual sum over n in (6.10) is bounded, again by (6.5) the contribution of the main terms from all of the characters ψ_k with $1 < k < K$ is $\ll_\epsilon K \cdot (\log q)^{\frac{2}{\pi}+\epsilon} = o((\log q)^{\Delta})$, if ϵ is sufficiently small. We insert these estimates into (6.6) to obtain the result. □

Proof of Proposition 2.1 Let $\alpha \in [0, 1)$ be chosen so that $M(\chi) = |S(\chi, \alpha q)|$. Applying (2.1), we have

$$\frac{M(\chi)}{\sqrt{q}} = \frac{1}{2\pi}\left|\sum_{1\le|n|\le q} \frac{\chi(n)}{n} - \sum_{1\le|n|\le q} \frac{\chi(n)e(n\alpha)}{n}\right| + O(1).$$

Let $\xi := \psi_1$ once again, and let ℓ be its conductor. The proof is split up according to whether $\ell > 1$ or $\ell = 1$.

Case 1: Assume $\ell > 1$, so ξ is non-trivial. Suppose first that $|M(\chi)| \gg \sqrt{q}(\log q)^{\Delta}$. In light of (2.3), α is on a major arc, so there is $\frac{b}{m}$ such that $m \le r_q$ and $|\alpha - \frac{b}{m}| \le \frac{1}{mR_q}$, with $\ell \mid m$ by Proposition 6.1. Note that if we vary α in the interval $\left[\frac{b}{m} - \frac{1}{mR_q}, \frac{b}{m} + \frac{1}{mR_q}\right]$ then $N = N(\alpha) = \min\{q, \frac{1}{|m\alpha-b|}\}$ varies in the range $R_q \le N \le q$. As $\ell > 1$, Proposition 6.1 also shows that $\sum_{1\le|n|\le q} \frac{\chi(n)}{n} = o((\log q)^{\Delta})$, and moreover, writing $m_q\ell_q = m/\ell$ as before,

$$\frac{M(\chi)}{\sqrt{q}} = \frac{\sqrt{\ell}}{\pi\phi(m)} \prod_{p|\frac{m}{\ell}} |1-(\bar{\chi}\xi)(p)| \cdot \left|\sum_{1\le n\le N_q} \frac{(\chi\bar{\xi})(n)}{n}\right| + o((\log q)^{\Delta})$$

provided $\mu(m_q)\chi(\ell_q)\xi(m_q) \neq 0$.

Next, we find $m = d\ell$, given ξ and N_q, that maximizes

$$s_d := \frac{1}{\phi(d\ell)} \prod_{p|d} |1 - (\bar{\chi}\xi)(p)|.$$

Suppose that $p^e \| d$ and $D = d/p^e$. If $p \mid \ell$ then $s_d = s_D/p^e < s_D$ so we may assume that $p \nmid \ell$. In that case $s_d \leq 2s_D/\phi(p^e) \leq s_D$ unless $p^e = 2$. Hence $d = 1$ or 2 and $\phi(d\ell) = \phi(\ell)$, and so

$$\frac{M(\chi)}{\sqrt{q}} = \frac{\sqrt{\ell}}{\pi\phi(\ell)} \max\{1, |1 - (\bar{\chi}\xi)(2)|\} \left| \sum_{1 \leq n \leq N_q} \frac{(\chi\bar{\xi})(n)}{n} \right| + o((\log q)^{\Delta}),$$

which proves (2.5) when $\ell > 1$, and also that $|\sum_{n \leq N_q} \frac{(\chi\bar{\xi})(n)}{n}| \gg \frac{\phi(\ell)}{\sqrt{\ell}} (\log q)^{\Delta}$.

Conversely, assume that $\left|\sum_{n \leq N_q} \frac{(\chi\bar{\psi})(n)}{n}\right| \gg \frac{\phi(r)}{\sqrt{r}} (\log q)^{\Delta}$ for some primitive character ψ of conductor $r \leq r_q$ with $\psi(-1) = -\chi(-1)$. In view of (6.5), it follows that $\psi = \xi$ and $r = \ell$. The assumption also implies that $\log N_q + O(1) \geq (\log q)^{\Delta}$, so $N_q \geq R_q$.

Selecting $\beta \in [\frac{1}{\ell} - \frac{1}{\ell R_q}, \frac{1}{\ell} + \frac{1}{\ell R_q}]$ so that $N(\beta) = N_q \in [R_q, q]$ and applying Proposition 6.1,

$$\left| \sum_{1 \leq |n| \leq q} \frac{\chi(n)e(n\beta)}{n} \right| = \frac{2\sqrt{\ell}}{\phi(\ell)} \left| \sum_{1 \leq n \leq N_q} \frac{\chi\bar{\xi}(n)}{n} \right| + o((\log q)^{\Delta}) \gg (\log q)^{\Delta}. \tag{6.12}$$

Combining (2.1) with (6.12) and a second application of Proposition 6.1, we get

$$M(\chi) \geq |S(\chi, \beta q)| \geq \frac{\sqrt{q}}{2\pi} \left| \sum_{1 \leq |n| \leq q} \frac{\chi(n)e(n\beta)}{n} \right| - \frac{\sqrt{q}}{2\pi} \left| \sum_{1 \leq |n| \leq q} \frac{\chi(n)}{n} \right| + O(\log q) \gg \sqrt{q}(\log q)^{\Delta},$$

as required.

Case 2: Assume now that $\ell = 1$ and ξ is trivial so that χ is odd. If α is on a minor arc then from (2.1) and (2.3) we get

$$\frac{M(\chi)}{\sqrt{q}} = \frac{1 - \chi(-1)}{2\pi} \left| \sum_{n \leq q} \frac{\chi(n)}{n} \right| + o((\log q)^{\Delta}). \tag{6.13}$$

On the other hand, if α is on a major arc then by Proposition 6.1,

$$\frac{M(\chi)}{\sqrt{q}} = \frac{1 - \chi(-1)}{2\pi} \max_{\substack{R_q \leq N \leq q \\ 1 \leq m \leq r_q}} \left| \sum_{n \leq q} \frac{\chi(n)}{n} - \frac{\chi(\ell_q)\mu(m_q)}{\phi(m)} \prod_{p|m} (1 - \bar{\chi}(p)) \sum_{1 \leq n \leq N} \frac{\chi(n)}{n} \right| + o((\log q)^{\Delta}).$$

The coefficient of the sum up to N is ≤ 2 (which is attained if $m = 2$ and $\chi(2) = -1$), so that by the triangle inequality

$$\frac{M(\chi)}{\sqrt{q}} \leq \frac{3}{\pi} \left| \sum_{n \leq N_q} \frac{\chi(n)}{n} \right| + o((\log q)^{\Delta}). \tag{6.14}$$

We obtain this as an equality when $\chi(2) = -1$ with $N_q = q$ and $m = 2$.

By (6.13) and then by taking $m = 1$ and $N = N_q$ above, we obtain

$$\frac{M(\chi)}{\sqrt{q}} \geq \frac{1}{\pi} \max \left\{ \left| \sum_{n \leq q} \frac{\chi(n)}{n} \right|, \left| \sum_{n \leq q} \frac{\chi(n)}{n} - \sum_{n \leq N_q} \frac{\chi(n)}{n} \right| \right\} + o((\log q)^{\Delta})$$
$$\geq \frac{1}{2\pi} \left| \sum_{n \leq N_q} \frac{\chi(n)}{n} \right| + o((\log q)^{\Delta}). \tag{6.15}$$

(6.14) and (6.15) imply (2.5). Together these bounds also yield that $M(\chi) \gg \sqrt{q}(\log q)^{\Delta}$ if and only if $|\sum_{n \leq N_q} \frac{\chi(n)}{n}| \gg (\log q)^{\Delta}$. □

Proof of Corollary 1.2 Assume that $M(\chi) \geq c_1\sqrt{q} \log q$ and $|S(\chi, N)| \gg N$ for some $N \in [q^{c_2}, q]$. By Corollary 1.1 and Proposition 1.1, there is $|t| \ll 1$ and $\ell \ll 1$ such that, simultaneously,

$$|L(1 + it, \chi)| \gg \log q \text{ and } |L(1, \chi\bar{\xi})| \gg |L(1, \psi)| \gg \log q,$$

where ξ is primitive modulo ℓ and $\xi(-1) = -\chi(-1)$, and ψ is the primitive character that induces $\chi\bar{\xi}$. By (4.1),

$$\left| \sum_{n \leq q} \frac{\chi(n)n^{-it}}{n} \right|, \left| \sum_{n \leq q} \frac{\chi(n)\bar{\xi}(n)}{n} \right| \gg \log q.$$

Applying (1.2) of Proposition 3.1 we obtain $\mathbb{D}(\chi, n^{it}; q), \mathbb{D}(\chi, \xi; q) \ll 1$, so by (3.1),

$$\mathbb{D}(\xi, n^{it}; q) \leq \mathbb{D}(\chi, \xi; q) + \mathbb{D}(\chi, n^{it}; q) \ll 1.$$

Therefore $\ell = 1$, else we let $Y := \exp((\log(2\ell))^{10} + |t|) \ll 1$, and apply (4.2) and partial summation as in the proof of Lemma 6.1 to get

$$\sum_{p \leq q} \frac{\xi(p)p^{-it}}{p} = \sum_{Y < p \leq q} \frac{\xi(p)p^{-it}}{p} + O(\log\log Y) \ll 1,$$

which implies that $\mathbb{D}(\xi, n^{it}; q)^2 = \log\log q + O(1)$, a contradiction.

We deduce that ξ is trivial so that $\chi(-1) = \chi\xi(-1) = -1$ and that $|L(1, \chi)| \gg \log q$. By Lemma 4.1, χ must be non-exceptional, so (4.3) gives $|L(1, \chi)| \asymp \log q \, e^{-\mathbb{D}(\chi, 1; q)^2}$ and we deduce that $\mathbb{D}(\chi, 1; q) \ll 1$. □

7 A class of examples

7.1 The set-up

Let $g : \mathbb{R} \to \mathbb{U}$ be a 1-periodic function with $g(0) = 1$ and Fourier expansion

$$g(t) = \sum_{n \in \mathbb{Z}} g_n e(nt)$$

so that

$$g_n := \hat{g}(n) := \int_0^1 g(u)e(-nu)du \text{ for all integers } n,$$

and therefore $|g_n| \le \int_0^1 |g(u)|du \le 1$ for all n. We will assume that $|g_n| \ll |n|^{-3}$ for all integers $n \ne 0$ (so that $\{g_n\}_n$ is absolutely summable).[4]

Write $\gamma_0 = g_0 + 1$ and $\gamma_n = g_n$ for all integers $n \ne 0$. Then $\sum_{n\in\mathbb{Z}} \mathrm{Re}(\gamma_n) = \sum_{n\in\mathbb{Z}} \mathrm{Re}(g_n) + 1 = \mathrm{Re}(g(0)) + 1 = 2$ so that $\mu := \max_n \mathrm{Re}(\gamma_n) > 0$. Let $\mathcal{L} := \{\ell \in \mathbb{Z} : \mathrm{Re}(\gamma_\ell) = \mu\}$, which is a non-empty set, and finite as $|g_n| \ll |n|^{-3}$. Moreover there exists $\delta > 0$ such that $\mathrm{Re}(g_n) \le \mu - \delta$ for all $n \notin \mathcal{L}$.

Fix $t \in (0, 1]$. We define a multiplicative function $f = f_t : \mathbb{N} \to \mathbb{U}$ at primes p by

$$f_t(p) := g\left(\frac{t \log p}{2\pi}\right) \in \mathbb{U}, \tag{7.1}$$

and inductively on prime powers p^m, $m \ge 2$, via the convolution formula

$$f_t(p^m) := \frac{1}{m} \sum_{1\le j\le m} f_t(p^{m-j}) g\left(\frac{t \log p^j}{2\pi}\right). \tag{7.2}$$

Under these assumptions we will prove the following estimate:

Theorem 7.1 *Let $t \in [-1, 1]$ be such that $|t|$ is small but $|t| \gg (\log X)^{-\epsilon}$ for all $\epsilon > 0$. Then*

$$\sum_{n\le X} \frac{f_t(n)}{n} = (1 + O(|t|)) \sum_{\ell\in\mathcal{L}} \frac{X^{i\ell t}}{i\ell' t} \frac{C_\ell (it \log X)^{\gamma_\ell - 1}}{\Gamma(\gamma_\ell)}$$

where $C_\ell := \prod_{k\ne 0} k^{-g_{\ell-k}}$, and $\ell' = 1$ if $\ell = 0$ and $\ell' = \ell$ otherwise.

One can make the weaker assumption that $|g_n| \ll 1/|n|^{1+\epsilon}$ for all integers $n \ne 0$, and obtain the weaker, but satisfactory, error term $O(|t|^{\epsilon/2})$ in place of $O(|t|)$.

Henceforth fix t and use $f = f_t$. By (7.2) and induction on $m \ge 1$ we have

$$|f(p^m)| \le \frac{1}{m} \sum_{1\le j\le m} |f(p^{m-j})| \le 1,$$

so that f indeed takes values in $\mathbb{U}$. If $F(s)$ is the Dirichlet series of f for $\mathrm{Re}(s) > 1$ then $F(s)$ is analytic and non-vanishing in that half-plane, and so $-\frac{F'}{F}(s)$ is also analytic there. The convolution identity (7.2) implies that

$$-\frac{F'}{F}(s) = \sum_{n\ge 1} g\left(\frac{t \log n}{2\pi}\right) \frac{\Lambda(n)}{n^s}.$$

Integrating $-\frac{F'}{F}(s)$ termwise, we see that when $\mathrm{Re}(s) > 1$,

$$\log F(s) = \sum_{n\ge 1} \frac{\Lambda_f(n)}{n^s \log n} = \sum_{p^k} \frac{g(\frac{t \log p^k}{2\pi})}{kp^{ks}} = \sum_{m\in\mathbb{Z}} g_m \sum_{p^k} \frac{p^{ikmt}}{kp^{ks}} = \sum_{m\in\mathbb{Z}} g_m \log \zeta(s - imt),$$

swapping orders of summation using the absolute summability of $\{g_m\}_m$. For $\mathrm{Re}(s) > 1$, we may thus write

$$F(s) = \prod_{m\in\mathbb{Z}} \zeta(s - imt)^{g_m}.$$

[4] The proof works provided $|g_n| \ll 1/|n|^{2+\epsilon}$ for all integers $n \ne 0$.

We will work with the finite truncations of this product,

$$F_N(s) := \prod_{|m|\le 2N} \zeta(s-imt)^{g_m}.$$

The proof of Theorem 7.1 relies on a technical contour integration argument complicated by the possibility that the zeros and poles of $\zeta(s-imt)$ might contribute essential singularities whenever $g_m \neq 0$. The following key technical lemma will be proved in Sect. 1.

For given $\tau \in \mathbb{R}$ we define

$$\sigma(\tau) := \frac{c}{\log(2+|\tau|)},$$

where $c > 0$ is chosen sufficiently small so that $\zeta(\sigma + i\tau) \neq 0$ whenever $\sigma \ge 1 - \sigma(\tau)$.

Lemma 7.1 *Let $t \in [-1, 1]$ be such that $|t|$ is small but $|t| \gg (\log X)^{-\epsilon}$ for all $\epsilon > 0$. Fix $A \ge 2$, let $N := \lceil \frac{(\log X)^A}{|t|} \rceil$ and $T := (N + \frac{1}{2})|t|$. Also let $r_0 := \frac{1}{4}\min\{\sigma(3T), |t|\}$.*

(a) *If $s = \sigma + i\tau$ with $\sigma \ge \frac{1}{\log X}$ and $|\tau| \le T$ then*

$$F(s+1) = F_N(s+1) + O((\log X)^{-2}).$$

(b) *We have*

$$\max_{|\tau|\le T} |F_N(1 - r_0 + i\tau)| \ll_\epsilon (\log X)^\epsilon.$$

(c) *Let $\eta \in \{-1, +1\}$. Then*

$$\max_{-r_0\le\sigma\le r_0} |F_N(1 + \sigma + i\eta T)| \ll_\epsilon (\log X)^\epsilon.$$

(d) *If $|t|$ is sufficiently small then for any $\ell \in \mathbb{Z}$,*

$$\prod_{\substack{|k|\le 2N\\ k\neq \ell}} \zeta(1 - i(k-\ell)t)^{g_k} = (1 + O(|t|))C_\ell (it)^{g_\ell - 1}.$$

More generally when $|n| \le N$ and $|s| \le 2r_0$,

$$\prod_{\substack{|k|\le 2N\\ k\neq n}} |\zeta(1 + s - i(k-n)t)^{g_k}| \ll_\epsilon (\log X)^\epsilon.$$

Proof of Theorem 7.1 Let $c_0 := \frac{1}{\log X}$, $A = 2$ and N and T be as in Lemma 7.1 so that $T \ge (\log X)^2$. By a quantitative form of Perron's formula [20, Cor. 2.4], we have

$$\sum_{n\le X} \frac{f(n)}{n} = \frac{1}{2\pi i}\int_{(c_0)} F(s+1)\frac{X^s}{s}ds = \frac{1}{2\pi i}\int_{c_0 - iT}^{c_0+iT} F(s+1)\frac{X^s}{s}ds + O\left(\frac{1}{\log X}\right).$$

By Lemma 7.1(a),

$$\sum_{n\le X} \frac{f(n)}{n} = \frac{1}{2\pi i}\int_{c_0 - iT}^{c_0+iT} F_N(s+1)\frac{X^s}{s}ds + O\left(\frac{1}{\log X}\right). \tag{7.3}$$

We now deform the path $[c_0 - iT, c_0 + iT]$ into a contour intersecting with the critical strip within the common zero- and pole-free regions of $\{\zeta(s-int)\}_{|n|\le 2N}$. Since $|\text{Im}(s) - nt| \le 3T$ we see that $\zeta(s+1-int) \neq 0$ for all $|n| \le 2N$ and $|\text{Im}(s)| \le T$ whenever $\text{Re}(s) \ge -\sigma(3T)$.

Let $\mathcal{H}$ denote the Hankel contour[5] [20, p. 179] of radius $\frac{1}{\log X}$, and let $r_0 := \frac{1}{4}\min\{|t|, \sigma(3T)\}$. For each $|n| \le N$ we write

$$\mathcal{H}_n := (\mathcal{H} + int) \cap \{\sigma + i\tau \in \mathbb{C} : \sigma \ge -r_0\}.$$

We glue the paths $\{\mathcal{H}_n\}_{|n|\le N}$ together and to the horizontal lines $[-r_0 + iT, c_0 + iT]$ and $[-r_0 - iT, c_0 - iT]$ using the line segments

$$L_n := (-r_0 + in|t|, -r_0 + i((n+1)|t|)) \text{ for } -N \le n \le N-1,$$
$$B_1 := [-r_0 - iT, -r_0 - iN|t|), \qquad B_2 := (-r_0 + iN|t|, -r_0 + iT]$$

Denote this concatenated path by Γ_N and define the contour

$$\Gamma := [c_0 - iT, c_0 + iT] \cup [c_0 + iT, -r_0 + iT] \cup \Gamma_N \cup [-r_0 - iT, c_0 - iT],$$

traversed counterclockwise. Since $F_N(s+1)/s$ is analytic in the interior of the component cut out by Γ, the residue theorem implies that

$$\frac{1}{2\pi i}\int_{c_0 - iT}^{c_0 + iT} F_N(s+1)\frac{X^s}{s}ds = \mathcal{M} + \mathcal{R}, \tag{7.4}$$

where $\mathcal{M} := \frac{1}{2\pi i}\sum_{|n|\le N}\int_{\mathcal{H}_n}\frac{F_N(s+1)}{s}X^s ds$ is the contribution from the Hankel contours, and

$$\mathcal{R} := \frac{1}{2\pi i}\left(\int_{B_1}\frac{F_N(s+1)}{s}X^s ds + \int_{B_2}\frac{F_N(s+1)}{s}X^s ds + \sum_{-N\le n\le N-1}\int_{L_n}\frac{F_N(s+1)}{s}X^s ds\right)$$
$$-\frac{1}{2\pi i}\left(\int_{-r_0 - iT}^{c_0 - iT}\frac{F_N(s+1)}{s}X^s ds - \int_{-r_0 + iT}^{c_0 + iT}\frac{F_N(s+1)}{s}X^s ds\right).$$

Along the segments L_n and B_j, where $\mathrm{Re}(s+1) = 1 - r_0$, we apply Lemma 7.1(b) to obtain

$$\sum_{-N\le n\le N-1}\left|\int_{L_n}\frac{F_N(s+1)}{s}X^s ds\right| + \sum_{j=1,2}\left|\int_{B_j}\frac{F_N(s+1)}{s}X^s ds\right|$$
$$\ll_\epsilon X^{-r_0}(\log X)^\epsilon\left(\sum_{|n|\le N}\frac{1}{r_0 + |nt|} + \frac{1}{T}\right) \ll \frac{1}{\log X}.$$

Along the horizontal segments we use Lemma 7.1(c) to give

$$\left|\int_{-r_0 \pm iT}^{c_0 \pm iT}\frac{F_N(s+1)}{s}X^s ds\right| \ll_\epsilon \frac{(\log X)^\epsilon}{T} \ll \frac{1}{\log X}.$$

Thus, $\mathcal{R} \ll \frac{1}{\log X}$, and it remains to treat $\mathcal{M}$. For each $|n| \le N$ note that $\mathcal{H}_n = \mathcal{H}_0 + int$, and so by a change of variables,

$$\mathcal{M}_n := \frac{1}{2\pi i}\int_{\mathcal{H}_n}\frac{F_N(s+1)}{s}X^s ds = \frac{X^{int}}{2\pi i}\int_{\mathcal{H}_0}\frac{G_n(s)}{s + int}\frac{X^s}{s^{g_n}}ds,$$

[5] That is, for $r := \frac{1}{\log X}$ the contour consisting of the circular segment $\{s \in \mathbb{C} : |s| = r, \arg(s) \in (-\pi, \pi)\}$ (omitting the point $s = -r$) together with the lines $\mathrm{Re}(s) \le -r$ covered twice, once at argument $+\pi$ and once at argument $-\pi$, traversed counterclockwise.

where we set

$$G_n(s) := [s\zeta(s+1)]^{g_n} \prod_{\substack{|m|\le 2N \\ m\ne n}} \zeta(s+1-i(m-n)t)^{g_m}.$$

G_n is analytic near 0, and when $|s| \le \frac{1}{2}\min\{|t|, \sigma(3T)\} = 2r_0$ we can write

$$G_0(s) = \sum_{j\ge 0} \mu_{0,j}(t)s^j, \quad \frac{G_n(s)}{s+int} = \sum_{j\ge 0} \mu_{n,j}(t)s^j \quad \text{if } n \ne 0.$$

The functions $\mu_{n,j}(t)$ are determined by Cauchy's integral formula as

$$\mu_{n,j}(t) = \frac{1}{2\pi i}\int_{|s|=r} \frac{G_n(s)}{(s+int)^{1_{n\ne 0}}}\frac{ds}{s^{j+1}}, \quad 0 < r \le 2r_0. \tag{7.5}$$

Note in particular that

$$\mu_{0,0}(t) = \prod_{\substack{|m|\le 2N \\ m\ne 0}} \zeta(1-imt)^{g_m}, \quad \mu_{n,0}(t) = \frac{1}{int}\prod_{\substack{|m|\le 2N \\ m\ne n}} \zeta(1-i(m-n)t)^{g_m} \text{ if } n \ne 0,$$

while for $j \ge 1$ we take $r = 2r_0$ and apply Lemma 7.1(d) in (7.5) to get[6] for all $|n| \le N$,

$$|\mu_{n,j}(t)| \le r^{-j}\max_{|s|=r} \frac{\prod_{\substack{|k|\le 2N \\ k\ne n}} |\zeta(s+1-i(k-n)t)^{g_k}|}{|s+int|^{1_{n\ne 0}}} \ll_\epsilon \frac{2^j(\log X)^\epsilon}{\min\{|t|,\sigma(3T)\}^j(1_{n=0}+|nt|)}. \tag{7.6}$$

Integrating over $\mathcal{H}_0$ (noting that $|s| \le r/2$ for all $s \in \mathcal{H}_0$) and applying [20, Cor. 0.18], when $n \ne 0$ we obtain

$$\begin{aligned}
\mathcal{M}_n &= \frac{X^{int}\mu_{n,0}(t)}{2\pi i}\int_{\mathcal{H}_0} X^s s^{-g_n}\,ds + O_\epsilon\left(\frac{(\log X)^\epsilon}{|nt|\min\{|t|,\sigma(3T)\}}\int_{\mathcal{H}_0} X^{\mathrm{Re}(s)}|s|^{1-\mathrm{Re}(g_n)}|ds|\right) \\
&= \frac{X^{int}\mu_{n,0}(t)}{\Gamma(g_n)}\left((\log X)^{g_n-1} + O(X^{-r_0})\right) \\
&\quad + O_\epsilon\left(\frac{(\log X)^{2\epsilon}}{|n|t}\left(\int_{-r_0}^{-\frac{1}{\log X}} |\sigma|^{1-\mathrm{Re}(g_n)}X^\sigma\,d\sigma + (\log X)^{\mathrm{Re}(g_n)-2}\right)\right) \\
&= \frac{X^{int}}{int}\frac{(\log X)^{\gamma_n-1}}{\Gamma(\gamma_n)}\left(\prod_{\substack{|k|\le 2N \\ k\ne n}} \zeta(1-i(k-n)t)^{g_k} + O_\epsilon\left(\frac{1}{(\log X)^{1-\epsilon}}\right)\right)
\end{aligned}$$

and similarly

$$\mathcal{M}_0 = \frac{(\log X)^{\gamma_0-1}}{\Gamma(\gamma_0)}\left(\prod_{\substack{|k|\le 2N \\ k\ne 0}} \zeta(1-ikt)^{g_k} + O_\epsilon\left(\frac{1}{(\log X)^{1-\epsilon}}\right)\right).$$

6 Here and below, we repeatedly use the fact that if $z \in \mathbb{C}\backslash\{0\}$ then, choosing any appropriate branch of complex argument, we have $|z^{g_n}| = |z|^{\mathrm{Re}(g_n)}\exp(-\mathrm{Im}(g_n)\cdot\arg(z)) \asymp |z|^{\mathrm{Re}(g_n)}$ for all $n \in \mathbb{Z}$, as $\sup_{n\in\mathbb{Z}}|g_n| \le 1$ and $|\arg(z)|$ is uniformly bounded.

We next focus on the products of ζ-values. When $n = \ell \in \mathcal{L}$, Lemma 7.1(d) gives

$$\prod_{\substack{|k|\le 2N\\ k\ne \ell}} \zeta(1-i(k-\ell)t)^{g_k} = (1+O(|t|))C_\ell(it)^{g_\ell-1}.$$

We saw above that $\mathrm{Re}(\gamma_n) \le \mu-\delta$ for all $n \notin \mathcal{L}$. Combining this with Lemma 7.1(d) and the estimates $|t|^{\mathrm{Re}(g_\ell)-1} \ge 1$ (since $|g_\ell| \le 1$ for all ℓ) and $1/\Gamma(g_n) \ll 1$ uniformly (since $1/\Gamma$ is entire), when $\ell \notin \mathcal{L}$ we obtain

$$\sum_{\substack{|n|\le N\\ n\ne \ell}} \prod_{\substack{|k|\le 2N\\ k\ne n}} |\zeta(1-i(k-n)t)^{g_k}| \frac{(\log X)^{\mathrm{Re}(g_n)}}{|\Gamma(g_n+1_{n=0})|(1_{n=0}+|nt|\log X)}$$
$$\ll_\epsilon \frac{(\log X)^{\mu-1-\delta+\epsilon}}{|t|} \ll \frac{(|t|\log X)^{\mu-1-\delta+\epsilon}}{|t|}.$$

Accounting for the error term for $\mathcal{M}_\ell$ and using $\delta < 1$, it follows that

$$\mathcal{M} = \Big(1+O(|t|)\Big)\sum_{\ell\in\mathcal{L}} \frac{X^{i\ell t}}{i\ell' t}\frac{C_\ell(it\log X)^{\gamma_\ell-1}}{\Gamma(\gamma_\ell)} + O_\epsilon\Big(\frac{(|t|\log X)^{\mu-1+\epsilon}}{|t|}\cdot\frac{1}{(\log X)^\delta}\Big).$$

Setting $\delta' := \frac{1}{2}\delta$ and taking $\epsilon < \delta'$ we obtain

$$\mathcal{M} = \Big(1+O\Big(|t|+\frac{1}{(\log X)^{\delta'}}\Big)\Big)\sum_{\ell\in\mathcal{L}} \frac{X^{i\ell t}}{i\ell' t}\frac{C_\ell(it\log X)^{\gamma_\ell-1}}{\Gamma(\gamma_\ell)}.$$

The proof is completed upon combining this estimate with our bound for $\mathcal{R}$ in (7.4), and then using (7.3). □

The values of $f_t(p)$ at primes p are crucial in obtaining the shape of the asymptotic formula in Theorem 7.1, not the values $f_t(p^m)$ with $m \ge 2$ at prime powers, as the following Corollary shows:

Corollary 7.1 *Assume the hypotheses of Theorem 7.1. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f(p) = f_t(p)$ for all primes p, and define a multiplicative function h so that $f := f_t * h$. Then*

$$\sum_{n\le X}\frac{f(n)}{n} = \sum_{\ell\in\mathcal{L}} \frac{X^{i\ell t}}{i\ell' t}\frac{C_\ell(it\log X)^{\gamma_\ell-1}}{\Gamma(\gamma_\ell)}(H(1+i\ell t)+O(|t|)+o(1)),$$

where $H(s) := \sum_{n\ge1}\frac{h(n)}{n^s}$; moreover, for each $\ell \in \mathcal{L}$ we have $H(1+i\ell t) \ne 0$ unless $f(2^k) = -2^{ik\ell t}$ for all $k \ge 1$.

One expects that $\mathcal{L}$ typically contains just one element, $\{\ell\}$, and so an asymptotic is given by this formula for all large X if $H(1+i\ell t) \ne 0$ (that is, if $f(2^k) \ne -2^{ik\ell t}$ for some $k \ge 1$). If $\mathcal{L}$ contains more than one element first note that $H(1+i\ell t) = 0$ for at most one value of ℓ, so we have a sum of main terms of similar magnitude. For $X \in [Z, Z^{1+o(1)}]$ we get a formula of the form $(\sum_{\ell\in\mathcal{L}} c_\ell X^{i\ell t} + o(1))(\log Z)^{\mu-1}$ (where the c_ℓ depend on Z but not X) and such a finite length trigonometric polynomial will have size $o(1)$ for a logarithmic measure 0 set of X-values (that is for $X \in [Z, YZ]$ where $|t|\log Y \to \infty$ with $\log Y = o(\log Z)$).

Proof Each $h(p)=0$ so that $h(n)=0$ unless n is powerful. Since each $|f_t(p^k)|, |f(p^k)| \le 1$, we deduce by induction that $|h(p^k)| \le 2^{k-1}$ for each $k \ge 2$. We begin by assuming that each $h(2^k)=h(3^k)=0$, and so if $(n,6)=1$ then $|h(n)| \le n^\kappa$ where $\kappa = \frac{\log 2}{\log 5} (< \frac{1}{2})$.

As $h(n)=0$ unless n is powerful and $(n,6)=1$, we have

$$\sum_{n>N} \frac{|h(n)|}{n} \ll N^{\kappa-\frac{1}{2}} \text{ and } \sum_{b\ge 1} \frac{|h(b)|\log b}{b} < \infty.$$

Select $A>0$ so that $A(\frac{1}{2}-\kappa) > 2-\mu$. Then with $M := (\log X)^A$ we have $\sum_{b>M} \frac{|h(b)|}{b} = o((\log X)^{\mu-2})$, and by Theorem 7.1,

$$\begin{aligned}
\sum_{n\le X} \frac{f(n)}{n} &= \sum_{\substack{ab\le X\\ b\le M}} \frac{f_t(a)h(b)}{ab} + \sum_{\substack{ab\le X\\ b>M}} \frac{f_t(a)h(b)}{ab}\\
&= \sum_{b\le M} \frac{h(b)}{b} \sum_{a\le X/b} \frac{f_t(a)}{a} + O\left(\sum_{a\le X} \frac{1}{a} \sum_{M<b\le X/a} \frac{|h(b)|}{b}\right)\\
&= (1+O(|t|)) \sum_{\ell\in\mathcal{L}} \frac{X^{i\ell t}}{i\ell' t} \frac{C_\ell (it\log X)^{\gamma_\ell -1}}{\Gamma(\gamma_\ell)} \sum_{b\le M} \frac{h(b)}{b^{1+i\ell t}} \left(1-\frac{\log b}{\log X}\right)^{\gamma_\ell-1}\\
&\quad + o((\log X)^{\mu-1}).
\end{aligned}$$

The claimed formula follows since

$$\begin{aligned}
\sum_{b\le M} \frac{h(b)}{b^{1+i\ell t}} \left(1-\frac{\log b}{\log X}\right)^{\gamma_\ell-1} &= H(1+i\ell t) + O\left(\sum_{b>M} \frac{|h(b)|}{b} + \frac{1}{\log X} \sum_{b\le M} \frac{|h(b)|\log b}{b}\right)\\
&= H(1+i\ell t) + o(1).
\end{aligned}$$

Now suppose that $h(3^k)$ is not necessarily 0. The key issue is

$$\begin{aligned}
\sum_{n>N} \frac{|h(n)|}{n} &= \sum_{k\ge 0} \frac{|h(3^k)|}{3^k} \sum_{\substack{n>N/3^k\\ (n,6)=1}} \frac{|h(n)|}{n} \ll \sum_{k\ge 0} \frac{|h(3^k)|}{3^k} (N/3^k)^{\kappa-\frac{1}{2}}\\
&\le N^{\kappa-\frac{1}{2}} \sum_{k\ge 0} \frac{2^{k-1}}{(3^{\kappa+\frac{1}{2}})^k} \ll N^{\kappa-\frac{1}{2}}
\end{aligned}$$

since $3^{\kappa+\frac{1}{2}} > 2$.

Our assumptions guarantee that the sum for $H(s)$ converges on the 1-line. Is $H(1+i\tau) \ne 0$ for $\tau \in \mathbb{R}$? We see that the Euler factors converge on the 1-line and indeed

$$\left|\sum_{k\ge 0} \frac{h(p^k)}{p^{k(1+i\tau)}}\right| \ge 1 - \sum_{k\ge 1} \frac{|h(p^k)|}{p^k} \ge 1 - \sum_{k\ge 2} \frac{2^{k-1}}{p^k} = 1 - \frac{2}{p(p-2)} > 0$$

for each prime $p \ge 3$.

Now suppose that $h(2^k)$ is not necessarily 0. The analogous argument works for any $h(.)$ for which there exists $\epsilon > 0$ such that $|h(2^k)| \ll (2^k)^{1-\epsilon}$. To establish this we first assume that each $|f_t(2^k)| = 1$ so write $f_t(2^k) = e(\theta_k)$ and $g(\frac{t\log 2^j}{2\pi}) = r_j e(\gamma_j)$ with $0 \le r_j \le 1$. Then (7.2) becomes $m\, e(\theta_m) = \sum_{1\le j\le m} r_j e(\theta_{m-j} + \gamma_j)$. This implies that $r_j = 1$ and $\theta_m = \theta_{m-j} + \gamma_j (\text{mod } 1)$ for $1 \le j \le m$. Now $\theta_0 = \gamma_0 = 0$ and so $\theta_m = \gamma_m = m\gamma_1 (\text{mod } 1)$.

But then $\sum_{k\geq 0} f_t(2^k)/2^{ks} = \sum_{k\geq 0}(e(\gamma_1)/2^s)^k = (1-e(\gamma_1)/2^s)^{-1}$ and so if $k \geq 1$ then $h(2^k) = f(2^k) - e(\gamma_1)f(2^{k-1})$ and so each $|h(2^k)| \leq 2$.

Otherwise there exists a minimal $k \geq 1$ such that $|f_t(2^k)| < 1$; let $\delta = 1 - |f_t(2^k)| \in (0, 1]$. Now select $\alpha > 0$ for which $\delta(\alpha - 1) = \alpha^k(2-\alpha)$ so that $1 < \alpha < 2$. We claim that $|h(2^m)| \leq \kappa\alpha^m$ for all $m \geq 0$, where $\kappa := \max_{0\leq m\leq k} |h(2^m)|/\alpha^m$. This is trivially true for $m \leq k$; otherwise for $m > k$ we have (as $h(1) = 1, h(2) = 0$)

$$|h(2^m)| = \left| f(2^m) - \sum_{j=0}^{m-1} f_t(2^{m-j})h(2^j) \right| \leq 2 + \sum_{\substack{j=2\\ j\neq m-k}}^{m-1} |h(2^j)| + (1-\delta)|h(2^{m-k})|$$

$$\leq \sum_{\substack{j=0\\ j\neq m-k}}^{m-1} \kappa\alpha^j + (1-\delta)\kappa\alpha^{m-k} < \kappa\alpha^m \left(\sum_{i\geq 1} \alpha^{-i} - \delta\alpha^{-k} \right) = \kappa\alpha^m$$

as $2 \leq \kappa + \kappa\alpha$, by induction, using the definition of α.

Finally we wish to determine whether the Euler factor of $H(1 + i\ell t)$ at 2 equals 0. This equals the Euler factor for f at 1 divided by the Euler factor for f_t at 1. Since each $|f_t(2^k)| \leq 1$ the denominator is bounded; since $|f(2^k)| \leq 1$ we have $\sum_{k\geq 0} \frac{f(2^k)}{2^{k(1+i\ell t)}} = 0$ if and only if $f(2^k) = -2^{ik\ell t}$ for all $k \geq 1$. □

7.2 Our specific example

We now use Theorem 7.1 to construct a multiplicative function f satisfying the conclusion of Proposition 1.2(b). We will use the auxiliary 1-periodic function

$$g(u) = \frac{e(u) - \lambda}{|e(u) - \lambda|} = \frac{|e(u) - \lambda|}{e(-u) - \lambda} = \sum_{n\in\mathbb{Z}} g_n e(nu).$$

We see that g takes values on S^1 with $g(0) = 1$. We will verify the following properties of $\{g_n\}_n$ in the appendix (Sect. 1), which shows that g satisfies the assumptions required to apply Theorem 7.1.

Lemma 7.2 *For the* $\{g_n\}_n$ *defined just above we have:*

(a) $g_n \in \mathbb{R}$ *for all* n,
(b) $|g_n| \ll (\frac{2\lambda}{1+\lambda^2})^{|n|} \leq 0.99^{|n|}$ *for all* $n \in \mathbb{Z}$,
(c) $g_{-n} < g_n$ *for all* $n \geq 1$,
(d) $g_1 = 0.7994\ldots$, *and there is* $\delta > 0$ *such that* $g_n \leq g_1 - \delta$ *or all* $n \neq 0, 1$, *and*
(e) $g_0 < g_1 - 1$.

Deduction of Proposition 1.2(b) Let x be large. Let $t \in [\frac{1}{\log\log x}, 1]$ be small, and set $y_t := e^{\frac{1}{t}}$ and $f = f_t$. For small enough t, Theorem 7.1 yields

$$\left| \sum_{n\leq x} \frac{f(n)}{n} \right| \asymp t^{-1}(t\log x)^{g_1 - 1} \asymp \log x \exp\left((g_1 - 2) \sum_{y_t < p \leq x} \frac{1}{p} \right). \tag{7.7}$$

Using the definition of λ,

$$2 - \lambda = \int_0^1 \frac{|e(u) - \lambda|}{e(-u) - \lambda}(e(-u) - \lambda)du = \int_0^1 g(u)e(-u)du - \lambda \int_0^1 g(u)du = g_1 - \lambda g_0,$$

so that $g_1 - 2 = \lambda(g_0 - 1)$. By partial summation and the prime number theorem we have

$$\begin{aligned}\mathbb{D}(f, 1; y_t, x)^2 &= \sum_{y_t < p \le x} \frac{1 - \mathrm{Re}(g(\frac{t}{2\pi}\log p))}{p} \\ &= \mathrm{Re}\bigg(\int_0^1 \Big(1 - g(u)\Big)du\bigg) \log\left(\frac{\log x}{\log y_t}\right) + O(1) \\ &= (1 - g_0) \sum_{y_t < p \le x} \frac{1}{p} + O(1).\end{aligned}$$

Combining these last few observations we deduce that

$$\left|\sum_{n \le x} \frac{f(n)}{n}\right| \asymp \log x \exp\Bigg(\lambda(g_0 - 1) \sum_{y_t < p \le x} \frac{1}{p}\Bigg) \asymp \log x \exp\Big(-\lambda\, \mathbb{D}(f, 1; y_t, x)^2\Big).$$

Finally, since g is Lipschitz and $g(0) = 1$,

$$|g(\tfrac{t}{2\pi}\log p) - 1| \ll t \log p \text{ for all } p \le y_t,$$

and so by Mertens' theorem,

$$\mathbb{D}(f, 1; y_t)^2 = \sum_{p \le y_t} \frac{1 - \mathrm{Re}(g(\frac{t}{2\pi}\log p))}{p} \ll t \sum_{p \le y_t} \frac{\log p}{p} \ll 1.$$

Combining these last two estimates, (1.3) follows. □

Acknowledgements We would like to thank Dimitris Koukoulopoulos and K. Soundararajan for helpful discussions. We are also grateful to the anonymous referee for their useful comments on a previous version of the paper. A.G. is partially supported by grants from NSERC (Canada). Most of this paper was completed while A.M. was a CRM-ISM postdoctoral fellow at the Centre de Recherches Mathématiques.

Funding See the acknowledgment.

Availability of data and materials Not applicable.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethics approval Not applicable.

Consent to participate Not applicable.

Consent for publication Not applicable.

Code availability Not applicable.

Appendix A: Auxiliary results towards Proposition 1.2(b)

We establish the technical Lemmas 7.1 and 7.2, used in the proofs of Proposition 1.2(b) and Theorem 7.1.

A.1 On the Fourier Coefficients of *g*

Proof of Lemma 7.2 (a) Since $g(u) = \bar{g}(-u)$, a term-by-term comparison of the Fourier series of each shows that $g_n = \bar{g}_n$ and thus $g_n \in \mathbb{R}$ for each $n \in \mathbb{Z}$.
(b) We will prove a numerically sharper bound, which will be used in part (d). Note that

$$g(u) = \frac{e(u) - \lambda}{\sqrt{1+\lambda^2}} \cdot \left(1 - \frac{\lambda}{1+\lambda^2}(e(u) + e(-u))\right)^{-1/2}.$$

As $2\lambda < 0.99(1+\lambda^2)$, Taylor expanding the bracketed expression gives

$$\begin{aligned} g(u) &= \frac{e(u)-\lambda}{\sqrt{1+\lambda^2}} \sum_{j\geq 0} \binom{-1/2}{j} (-1)^j \left(\frac{\lambda}{1+\lambda^2}\right)^j \sum_{0\leq i\leq j} \binom{j}{i} e((2i-j)u) \\ &= \frac{e(u)-\lambda}{\sqrt{1+\lambda^2}} \sum_{n\in\mathbb{Z}} e(nu) \sum_{\substack{j\geq |n| \\ 2|(n+j)}} \binom{j}{(j+n)/2} 2^{-2j} \binom{2j}{j} \left(\frac{\lambda}{1+\lambda^2}\right)^j \\ &= \sum_{m\in\mathbb{Z}} e(mu) \frac{h_{m-1} - \lambda h_m}{\sqrt{1+\lambda^2}}, \end{aligned}$$

where we have set

$$h_n := \sum_{\substack{j\geq |n| \\ 2|(j+n)}} 2^{-2j} \binom{2j}{j} \binom{j}{(j+|n|)/2} \left(\frac{\lambda}{1+\lambda^2}\right)^j \tag{A.1}$$

$$= \sum_{l\geq 0} \binom{|n|+2l}{|n|+l} \binom{2(|n|+2l)}{|n|+2l} \left(\frac{\lambda}{4(1+\lambda^2)}\right)^{|n|+2l}. \tag{A.2}$$

It follows that

$$g_n = \frac{h_{n-1} - \lambda h_n}{\sqrt{1+\lambda^2}} \text{ for all } n \in \mathbb{Z}.$$

We use Stirling's approximation in the form $\sqrt{2\pi n}(n/e)^n \leq n! \leq e^{\frac{1}{12}} \sqrt{2\pi n}(n/e)^n$ for $n \in \mathbb{N}$ (see [18]), and bound $\binom{j}{(j+|n|)/2}$ by a central binomial coefficient as $\binom{j}{(j+|n|)/2} \leq \frac{1}{2^\nu}\binom{j+\nu}{(j+\nu)/2}$, where $\nu = 1_{2\nmid j}$. When $n \neq 0$ this gives

$$\begin{aligned} 2^{-2j}\binom{2j}{j}\binom{j}{(j+|n|)/2} &\leq 2^{-2j}\left(\frac{e^{\frac{1}{12}}}{\sqrt{\pi j}} 2^{2j}\right)\left(\frac{e^{\frac{1}{12}}}{2^\nu \sqrt{\pi(j+\nu)/2}} 2^{j+\nu}\right) \\ &\leq \frac{e^{\frac{1}{6}}\sqrt{2}}{\pi j} 2^j, \text{ if } 2 \mid (j+n). \end{aligned}$$

Setting $c := \frac{2\lambda}{1+\lambda^2}$, we find using the first expression for h_n in (A.1) that

$$0 \le h_m \le \sum_{\substack{j \ge |m| \\ 2|(j+m)}} \frac{e^{\frac{1}{6}}\sqrt{2}}{\pi j} 2^j \left(\frac{\lambda}{1+\lambda^2}\right)^j \le \frac{e^{\frac{1}{6}}\sqrt{2}}{(1-c^2)\pi} \frac{c^{|m|}}{|m|}, \quad m \ne 0.$$

Using the second expression for h_n in (A.1), when $|n| \ge 1$ we get

$$\begin{aligned} &h_{n-1} - \lambda h_n \\ &= \sum_{l \ge 0} \binom{|n|-1+2l}{|n|-1+l} \binom{2(|n|-1+2l)}{|n|-1+2l} \left(\frac{\lambda}{4(1+\lambda^2)}\right)^{|n|-1+2l} \\ &\left(1 - \frac{2|n|-1+4l}{2(|n|+l)} \frac{\lambda^2}{1+\lambda^2}\right) \\ &\le \left(1 - \frac{2|n|-1}{2|n|} \frac{\lambda^2}{1+\lambda^2}\right) h_{n-1}. \end{aligned}$$

Together with the previous bound, it follows that when $|n| \ge 2$,

$$|g_n| \le \frac{e^{\frac{1}{6}}\sqrt{2}}{(1-c^2)\pi\sqrt{1+\lambda^2}} \left(1 - \frac{2|n|-1}{2|n|} \frac{\lambda^2}{1+\lambda^2}\right) \frac{c^{|n|-1}}{|n|-1}, \tag{A.3}$$

and the bound $|g_n| \ll c^{|n|}$ immediately follows.
(c) We deduce from (A.1) that $h_n = h_{-n}$ for any $n \ge 1$. Thus,

$$\sqrt{1+\lambda^2} g_n = \begin{cases} h_{n-1} - \lambda h_n & \text{if } n \ge 1, \\ h_{|n|+1} - \lambda h_{|n|} & \text{if } n \le 0. \end{cases} \tag{A.4}$$

To prove $g_{-n} < g_n$ for all $n \ge 1$ we need only show that $h_{n+1} < h_{n-1}$ for all $n \ge 1$. To see this, note that

$$\begin{aligned} h_{n+1} &= \left(\frac{\lambda}{2(1+\lambda^2)}\right)^2 \sum_{l \ge 0} \binom{n-1+2l}{n-1+l} \binom{2(n-1+2l)}{n-1+2l} \left(\frac{\lambda}{4(1+\lambda^2)}\right)^{n-1+2l} \\ &\quad \frac{(2n+4l)^2-1}{(n+l+1)(n+l)} \\ &\le 16\left(\frac{\lambda}{2(1+\lambda^2)}\right)^2 \sum_{l \ge 0} \binom{n-1+2l}{n-1+l} \binom{2(n-1+2l)}{n-1+2l} \left(\frac{\lambda}{4(1+\lambda^2)}\right)^{n-1+2l}, \end{aligned}$$

and thus $h_{n+1} \le \left(\frac{2\lambda}{1+\lambda^2}\right)^2 h_{n-1} < h_{n-1}$, as required.
(d) Since $g_{-1} < g_1$, and $g_{-n} < g_n$ for $n \ge 2$ by (c), it is enough to show that there is $\delta > 0$ such that $g_n \le g_1 - \delta$ for all $n \ge 2$.

The upper bound (A.3) is decreasing with n, and one may verify that it gives $\le 0.7 < g_1 - \frac{1}{20}$ for $n = 10$. Thus, we clearly have $g_n \le g_1 - \frac{1}{20}$ for all $n \ge 10$.

On the other hand, a computer-assisted calculation shows that $g_n \le g_1 - \frac{1}{4}$, say, for all $2 \le n \le 9$:

$g_1 = 0.7994\ldots$, $g_2 = 0.2848\ldots$, $g_3 = 0.1659\ldots$ $g_4 = 0.1102\ldots$, $g_5 = 0.0778\ldots$,
$g_6 = 0.0568\ldots$, $g_7 = 0.0423\ldots$, $g_8 = 0.0321\ldots$, $g_9 = 0.0246\ldots$.

The claim thus follows with $\delta = \frac{1}{20}$.
(e) Though this may be verified by computer, we give a proof. From (A.4),

$$g_1 - 1 - g_0 = \frac{1}{\sqrt{1+\lambda^2}}(h_0 - \lambda h_1 - (h_1 - \lambda h_0)) - 1 = \frac{1+\lambda}{\sqrt{1+\lambda^2}}(h_0 - h_1) - 1.$$

It is enough to show that $h_0 - h_1 > \frac{\sqrt{1+\lambda^2}}{1+\lambda}$. To see this, we write

$$h_0 - h_1 = \sum_{l \geq 0} 2^{-2l} \binom{2l}{l} \binom{4l}{2l} \left(\frac{\lambda}{1+\lambda^2}\right)^l \left(1 - \frac{\lambda}{2(1+\lambda^2)} \frac{4l+1}{l+1}\right) \geq 1 - \frac{\lambda}{2(1+\lambda^2)},$$

using $\frac{\lambda}{2(1+\lambda^2)} \frac{4l+1}{l+1} \leq \frac{2\lambda}{1+\lambda^2} < 1$ and positivity to restrict to the term $l = 0$. But we see that

$$1 - \frac{\lambda}{2(1+\lambda^2)} > \left(1 - \frac{2\lambda}{(1+\lambda)^2}\right)^{1/2} = \frac{\sqrt{1+\lambda^2}}{1+\lambda},$$

since, setting $t = \frac{2\lambda}{(1+\lambda)^2}$, we have

$$1 - \frac{\lambda}{2(1+\lambda^2)} = 1 - \frac{t}{4} \cdot \left(1 + \frac{2\lambda}{1+\lambda^2}\right) \geq 1 - \frac{t}{2} > (1-t)^{1/2},$$

as required. □

A.2 On products of shifted zeta functions

Proof of Lemma 7.1 Throughout, set $G := \sum_{n \in \mathbb{Z}} |g_n| < \infty$ (since we assumed that $|g_n| \ll 1/(1+|n|)^3$) and $F_N(s) := \prod_{|n| \leq 2N} \zeta(s - int)^{g_n}$.
(a) Let $\sigma \geq \frac{1}{\log X}$. Since $|f(n)| \leq 1$,

$$|F(1+\sigma+i\tau)| \leq \zeta(1+\sigma) \ll \log X \text{ for all } |\tau| \leq T.$$

When $|n| > 2N$, $|\tau - nt| \geq (2N+1)|t| - T \geq T$, so that also

$$\zeta(1+\sigma+i(\tau-nt)) \ll \log(2+|\tau-nt|) \ll \log(2|n|) \text{ for all } |\tau| \leq T.$$

Now $|\zeta(2+2\sigma)/\zeta(1+\sigma)| < |\zeta(1+\sigma+i(\tau-nt))| < |\zeta(1+\sigma)|$ and so if $|g_n| < \frac{1}{\log\log X}$ then

$$\zeta(1+\sigma+i(\tau-nt))^{-g_n} = 1 + O(g_n \log\log X). \tag{A.5}$$

This holds for $|n| > 2N$ as we assumed that $|g_n| \ll |n|^{-3}$. Since $N \geq (\log X)^A$ for some $A \geq 2$ it follows that

$$\begin{aligned} &\max_{|\tau| \leq T} |F_N(1+\sigma+i\tau) - F(1+\sigma+i\tau)| \\ &\ll \max_{|\tau| \leq T} \left(|F(1+\sigma+i\tau)| \cdot \left| \exp\left(\sum_{|n|>2N} \log \left| \zeta(1+\sigma+i(\tau-nt))^{-g_n} \right| \right) - 1 \right| \right) \\ &\ll (\log X) \sum_{n > 2N} |n|^{-3} \log\log(2n) \ll_\epsilon (\log X) N^{-2+\epsilon} \ll (\log X)^{-2}, \end{aligned} \tag{A.6}$$

as required.
(b) Let $|\tau| \leq T$ and $|n| \leq 2N$, and put $\sigma := 1 - r_0$. If $|\sigma - 1 + i(\tau - nt)| \leq 1$ then

$$|\zeta(\sigma + i(\tau-nt))|^{\pm 1} \ll |\sigma - 1 + i(\tau - nt)|^{-1} \leq r_0^{-1}.$$

Otherwise, $|\zeta(\sigma+i(\tau-nt))|^{\pm 1} \ll \log(2+|\tau-nt|) \ll \log T$. Thus, for any $|\tau| \le T$,

$$|F_N(\sigma+i\tau)| \ll \prod_{\substack{|n|\le 2N \\ |-r_0+i(\tau-nt)|\le 1}} r_0^{-|g_n|} \cdot \prod_{\substack{|n|\le 2N \\ |-r_0+i(\tau-nt)|>1}} (\log T)^{|g_n|} \ll (r_0^{-1}\log T)^G.$$

Since $\min\{\sigma(3T), |t|\}^{-1}\log T \ll_\epsilon (\log X)^\epsilon$, the claim follows.
(c) Assume $\eta=+1$; the claim with $\eta=-1$ is completely analogous. Note that $|T-kt| \ge |t|/2$ for all $|k|\le 2N$. If $|\sigma+i(T-kt)|\le 1$ then $|kt| \ge T/2 \ge N|t|/2$, i.e., $|k|\ge N/2$, and then

$$|\zeta(1+\sigma+i(T-kt))^{g_k}| \ll |T-kt|^{-|g_k|} \ll |t|^{-|g_k|}.$$

Splitting the product as in (b) and using $|g_n| \ll 1/(1+|n|)^3$ for each $|\sigma|\le r_0$ we get

$$\begin{aligned}|F_N(1+\sigma+iT)| &\ll \prod_{|n|\ge \frac{N}{2}} |t|^{-|g_n|} \cdot \prod_{\substack{|n|\le 2N \\ |\sigma+i(T-nt)|>1}} (\log T)^{|g_n|} \\ &\ll \exp\Bigg(\log(\tfrac{1}{|t|}) \sum_{|n|\ge\frac{N}{2}} |n|^{-3}\Bigg)(\log T)^G \ll (\log\log x)^G,\end{aligned}$$

and the claim follows.
(d) Observe that if $|m|\le |t|^{-1}$ then $imt\zeta(1+imt) = 1+O(|mt|)$; otherwise, if $|mt|\ge 1$ then when $|t|$ is small enough m is large and we may Taylor expand

$$[imt\zeta(1+imt)]^{g_{\ell-m}} = 1 + O\Big(|g_{\ell-m}|\log(2+|mt|)\Big),$$

similarly to (A.5). Thus, since $|g_{\ell-m}| \ll (1+|m-\ell|)^{-3}$ we have

$$\begin{aligned}&\Bigg|\prod_{m\ne 0}[imt\zeta(1+imt)]^{g_{\ell-m}} - 1\Bigg| \\ &\ll |t| \sum_{|m|\le|t|^{-1}} |g_{\ell-m}||m| + \sum_{|m|>|t|^{-1}} |g_{\ell-m}|\log(2+|mt|) \\ &\ll |t| \sum_{|m|\le|t|^{-1}} (1+|m-\ell|)^{-2} + \sum_{|m|>|t|^{-1}} \frac{\log(2+|m|)}{(1+|m-\ell|)^3} \ll |t|.\end{aligned}$$

Since also $|t| \ge \frac{1}{\log X}$, it follows that (handling the range $|k|>2N$ as in (a))

$$\begin{aligned}\prod_{\substack{|k|\le 2N \\ k\ne\ell}} \zeta(1-i(k-\ell)t)^{g_k} &= \left(1+O\left(\frac{1}{(\log X)^2}\right)\right)\prod_{m\ne 0}(imt)^{-g_{\ell-m}}\prod_{n\ne 0}[int\zeta(1+int)]^{g_{\ell-n}} \\ &= \Big(1+O(|t|)\Big)C(it)^{g_\ell-1}\end{aligned}$$

since $\sum_{m\in\mathbb{Z}} g_m = g(0) = 1$, and the first claim is proved.

Suppose more generally that $|n| \leq N$, and that $|s| \leq \frac{1}{2}\min\{|t|, \sigma(3T)\} = 2r_0$. Whenever $k \neq n$ we have $|s - i(k-n)t| \geq |t|/2$. Thus, arguing as in (c),

$$\prod_{\substack{|k|\leq 2N\\ k\neq n}} |\zeta(s+1-i(k-n)t)^{g_k}| \ll \prod_{\substack{|k|\leq 2N\\ k\neq n\\ |s-i(k-n)t|\leq 1}} |t|^{-|g_k|} \prod_{\substack{|k|\leq 2N\\ k\neq n\\ |s-i(k-n)t|>1}} (\log T)^{|g_k|} \ll (|t|^{-1}\log T)^G,$$

and since $|t| \gg (\log X)^{-\epsilon}$ for any $\epsilon > 0$ the claim follows. □

References

1. Balog, A., Granville, A., Soundarajan, K.: Multiplicative functions in arithmetic progressions. Ann. Math. Qué. **37**, 3–30 (2013)
2. Bober, J.W., Goldmakher, L.: Pólya–Vinogradov and the least quadratic nonresidue. Math. Ann. **366**, 853–863 (2016)
3. Burgess, D.A.: On character sums and primitive roots. Proc. Lond. Math. Soc. **12**(3), 179–192 (1962)
4. de la Bretèche, R., Granville, A.: Exponential sums with multiplicative coefficients and applications. Trans. Am. Math. Soc. **(to appear)**
5. Fromm, E., Goldmakher, L.: Improving the Burgess bound via Pólya–Vinogradov. Proc. Am. Math. Soc. **147**, 461–466 (2019)
6. Goldmakher, L.: Multiplicative mimicry and improvements of the Pólya–Vinogradov inequality. Algebra Number Theory **6**(1), 123–163 (2012)
7. Granville, A., Soundararajan, K.: Decay of mean values of multiplicative functions. Can. J. Math. **55**(6), 1191–1230 (2003)
8. Granville, A., Soundararajan, K.: Large character sums: pretentious characters and the Pólya–Vinogradov inequality. J. Am. Math. Soc. **20**(2), 357–384 (2007)
9. Granville, A., Soundararajan, K.: Negative values of truncations to $L(1, \chi)$. In: Clay Math. Proc., vol. 7, pp. 141–148. Amer. Math. Soc., Providence (2007b)
10. Granville, A., Harper, A., Soundararajan, K.: Mean values of multiplicative functions over function fields. Res. Number Theory **1**, 1–25 (2015)
11. Granville, A., Harper, A., Soundararajan, K.: A new proof of Halasz's theorem, and its consequences. Compos. Math. **155**, 126–163 (2019)
12. Hall, R.R., Tenenbaum, G.: Effective mean value estimates for complex multiplicative functions. Math. Proc. Camb. Philos. Soc. **110**, 337–351 (1991)
13. Hildebrand, A.: A note on Burgess' character sum estimate. C.R. Math. Acad. Sci. Can. **8**(1), 35–37 (1986)
14. Iwaniec, H., Kowalski, E.: Analytic Number Theory. AMS Colloquium Publications vol. 53, Providence (2004)
15. Lamzouri, Y., Mangerel, A.P.: Large odd order character sums and improvements to the Pólya–Vinogradov inequality. Trans. Am. Math. Soc. arXiv:1701.01042 [math.NT] **(to appear)**
16. Mangerel, A.P.: Short character sums and the Pólya–Vinogradov inequality. Q. J. Math. **71**(4), 1281–1308 (2020)
17. Montgomery, H.L., Vaughan, R.C.: Exponential sums with multiplicative coefficients. Invent. Math. **43**, 69–82 (1977)
18. Robbins, H.: A remark on Stirling's formula. Am. Math. Mon. **62**(1), 26–29 (1955)
19. Tao, T., Teräväinen, J.: The Hardy–Littlewood–Chowla conjecture in the presence of a Siegel zero. arxiv:2109.06291 [math.NT] **(preprint)**
20. Tenenbaum, G.: Introduction to Analytic and Probabilistic Number Theory. Graduate Studies in Mathematics, vol. 163, AMS Publications, Providence (2015)