Temporal Reachability Dominating Sets: contagion in temporal graphs

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Abstract. SARS-CoV-2 was independently introduced to the UK at least 1300 times by June 2020. Given a population with dynamic pairwise connections, we ask if the entire population could be (indirectly) infected by a small group of k initially infected individuals. We formalise this problem as the Temporal Reachability Dominating Set (TaRDiS) problem on temporal graphs. We provide positive and negative parameterized complexity results in four different parameters: the number k of initially infected, the lifetime τ of the graph, the number of locally earliest edges in the graph, and the treewidth of the footprint graph $\mathcal{G}_\downarrow$. We additionally introduce and study the MaxMinTaRDiS problem, which can be naturally expressed as scheduling connections between individuals so that a population needs to be infected by at least k individuals to become fully infected. Interestingly, we find a restriction of this problem to correspond exactly to the well-studied Distance-3 Independent Set problem on static graphs.

Keywords: Temporal Graphs · Temporal Reachability · Treewidth · Polynomial Hierarchy

1 Introduction

Information and disease spread in real-world systems is often modeled using graphs. The time-sensitive nature of interactions between individuals is highlighted in *temporal graphs*, in which the set of vertices remains constant but the edge-set changes over time. These have been formalised in various ways; in Kempe, Kleinberg and Kumar's seminal work [29], a static graph G is extended with a *time-labeling function* $\lambda : E(G) \to \mathbb{N}^+$ assigning to each edge e a positive integer $\lambda(e)$ corresponding to the time at which it is active. A *temporal path* is then a static path where the edges are available in the order in which they are traversed. A vertex u is said to *reach* another vertex v if there is a temporal path from u to v.

Reachability and connectivity problems on temporal graphs have drawn significant interest in recent years. These have been studied in the context of network design [3,7,14] and transport logistics [24] (where maximizing connectivity and reachability at minimum cost is desired), and the study of epidemics [10,19,20,37] and malware spread [35](where it is not).

One metric closely related to a temporal graph's vulnerability to infection is its *maximum reachability*. That is, the largest number k of vertices which can be temporally reached by any vertex in the graph. In Enright et al.'s works [19,20], the problems of deleting and reordering edges in order to minimize k is shown to be NP-complete.

This framing of reachability asks what the worst-case spread is from a single source in the temporal graph. In reality, studied populations are often infected by several individuals. For example, SARS-CoV-2 had been independently introduced to the UK at least 1300 times by June 2020 [34]. We investigate how many sources are needed for the entire population to become infected - the Temporal Reachability Dominating Set (TaRDiS) problem. Later, we ask: if we can choose when connections between individuals occur and know k individuals will be initially infected (but not which ones) can we guarantee that the entire population will not be infected?

The answers to both of these questions will depend heavily on our model definition. In particular, the instantaneous transmission of infection through a large swath of the population, while realistic in some computer networks, is inconsistent with the spread of biological phenomena. Further, should multiple interactions between the same pair of individuals be allowed? Lastly, should it be possible for a single individual to simultaneously interact with several others?

These three definitional choices are dubbed *strictness*, *simpleness*, and *properness* respectively, by Casteigts et al. [12]. In that work, the authors identify the class of so-called *happy* temporal graphs (in which our last two questions are answered in the negative). As they note, hardness results on *happy* temporal graphs generalise to the other restrictions, in particular to the strict and non-strict settings.

1.1 Problem Setting

We begin with defining temporal graphs and related concepts. A *temporal graph* $\mathcal{G} = (V, E, \lambda)$ consists of a set of vertices V, a set of edges E and a function $\lambda : E \to [\tau]$[3]. We refer to λ as the *temporal assignment* of $\mathcal{G}$. The *lifetime* τ of a temporal graph is the value of the latest timestep. We abuse notation and write $\lambda(u, v)$ to mean $\lambda((u, v))$. For a static graph $G = (V, E)$, we denote the temporal graph (V, E, λ) by (G, λ). We also use $V(\mathcal{G}), E(\mathcal{G})$ to refer to the vertex and edge set of $\mathcal{G}$, respectively, and use $E_t(\mathcal{G})$ to refer to the set of edges active at time t. We say $\mathcal{G}$ and λ are *happy* if every vertex u is incident to at most 1 edge at a time[4]. The static graph $\mathcal{G}_\downarrow = (V, E)$ is called the *footprint* of $\mathcal{G}$.

[3] For a given $n \in \mathbb{N}^{>0}$ we denote by $[n]$ the set $\{1, 2, \dots, n\}$.

[4] In Casteigts et al.'s work [12], a temporal graph is happy if it is both simple (only one time per edge) and proper (every vertex incident to at most 1 edge at a time); under our definition, all temporal graphs are simple.

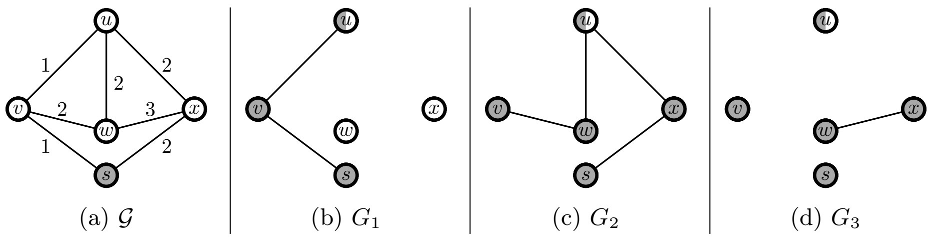


Fig. 1: Reachability and spread in a temporal graph from source s through snapshots. Vertices are shaded (half-shaded) when reached from s by a strict (nonstrict) temporal path. u is only reachable from s by a nonstrict path.

A *strict* (respectively *nonstrict*) *temporal path* from a vertex u to a vertex v is a static path from u to v consisting of edges $e_1, \dots, e_l$ such that $\lambda(e_i) < \lambda(e_{i+1})$ (resp. $\lambda(e_i) \leq \lambda(e_{i+1})$) for $i \in [1, l-1]$. A vertex u *temporally reaches* a vertex v (we sometimes say "reaches" for conciseness) if there is a temporal path from u to v. The *reachability set* $R_u(\mathcal{G})$ of a vertex u is the set of vertices reachable from u. We say a vertex u is reachable from a set S if for some $v \in S$, $u \in R_v$. A set of vertices T is *temporal reachability dominated* (or just *dominated*) by another set of vertices S if every vertex in T is reachable from S. Domination of and by single vertices is defined analogously. Strict and nonstrict spread are illustrated in Fig. 1. We differentiate between strict and nonstrict reachability by introducing a superscript $<$ or $\leq$ to the appropriate operators. For example, in Figure 2a, d is in $R_a^{\leq}$ and not $R_a^{<}$. We can now introduce our protagonist.

Definition 1 (TaRDiS). *In a temporal graph $\mathcal{G}$, a (strict)* temporal reachability dominating set *(TaRDiS) is a set of vertices S such that every vertex $v \in V(\mathcal{G})$ is temporally reachable from a vertex in S by a (strict) temporal path.*

Definition 2 (Sole Reachability Set). *We define the* sole reachability set *of a vertex v in a TaRDiS T as the set $SR(\mathcal{G}, T, v) = R_v(\mathcal{G}) \setminus (\cup_{u \in T \setminus \{v\}} R_u(\mathcal{G}))$. Equivalently, it is the set of vertices reachable from v and not from other vertices in T[5].*

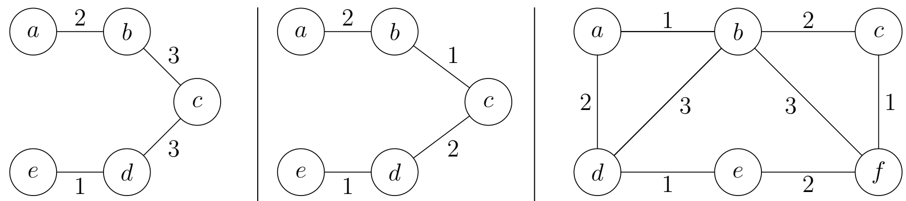
A minimum TaRDiS is a TaRDiS of fewest vertices in $\mathcal{G}$. Note that in a minimum TaRDiS, every vertex has a non-empty sole reachability set.

We now formally define our problems.

> **(Strict/Nonstrict) TaRDiS**
> *Input:* A temporal graph $\mathcal{G} = (V, E, \lambda)$ and an integer k.
> *Question:* Does $\mathcal{G}$ admit a (strict/nonstrict) TaRDiS of size at most k?

The restriction to *happy* inputs $\mathcal{G}$ is referred to as Happy TaRDiS and is a subproblem of both Strict TaRDiS and Nonstrict TaRDiS.

[5] When $\mathcal{G}$ is clear from context, we write R_u for $R_u(\mathcal{G})$ and $SR(T, v)$ for $SR(T, \mathcal{G}, v)$.



(a) $R_a^{\leq} \neq R_a^{<}$ and $R_e^{\leq} \neq R_e^{<}$.

(b) TaRDiS $\{a, e\}$ is not canonical.

(c) Every TaRDiS is canonical.

Fig. 2: Three temporal graphs, all admitting $\{a, e\}$ as a minimum TaRDiS.

> **(Strict/Nonstrict) MaxMinTaRDiS**
> *Input:* A static graph $H = (V, E)$ and integers k and τ.
> *Question:* Does there exists a temporal assignment $\lambda : E \rightarrow [\tau]$ such that every strict/nonstrict TaRDiS admitted by (H, λ) is of size at least k?

Likewise, the variant of this problem in which the temporal assignment λ is required to be happy is referred to as Happy MaxMinTaRDiS. Note this is not a subproblem of Strict MaxMinTaRDiS or Nonstrict MaxMinTaRDiS.

Definition 3 (Locally earliest, Canonical TaRDiS). *In a temporal graph $\mathcal{G}$, an edge (u, v) is* locally earliest *if every other edge incident*[6] *to either u or v is at a time $t > \lambda(u, v)$. A* canonical TaRDiS *consists exclusively of vertices which are incident to a locally earliest edge.*

In Figure 2b, each of $\{b, d\}$ and $\{a, c, e\}$ is a TaRDiS, but only the former is canonical.

1.2 Our Contribution

At a high level, our work identifies the minimum lifetime τ for which each problem is computationally hard. This justifies the need for parameters other than τ in tractability results. We show existence of such algorithms when the parameters τ, k and $\mathrm{tw}(\mathcal{G}_\downarrow)$ are all bounded.

Our main results are highlighted in Table 1. For the case of happy temporal graphs, we exactly characterize the complexity of both TaRDiS and MaxMinTaRDiS with lifetime $\tau \leq 3$. Both problems are trivially solvable in linear time for $\tau \leq 2$. We show NP-completeness of Happy TaRDiS and Σ_2^P-completeness of Happy MaxMinTaRDiS when $\tau = 3$ - even when restricted to planar inputs[7].

[6] If $e = (u, v)$ is an edge, u is said to be incident to vertex v and to edge e. Also, u is incident to a set S of vertices or edges if and only if it is incident to some element of S.

[7] In this work, we say a temporal graph is planar if and only if its footprint is planar.

<table>
<tr><td></td><td colspan="3">TaRDiS</td><td colspan="3">MaxMinTaRDiS</td></tr>
<tr><td>Problem variant</td><td>Strict</td><td>Nonstrict</td><td>Happy</td><td>Strict</td><td>Nonstrict</td><td>Happy</td></tr>
<tr><td>$\tau = 1$</td><td>NP-c, W[2]-c (Lem. 6)</td><td>Linear (Lem. 5)</td><td rowspan="2">Linear (Lem. 5)</td><td rowspan="4">coNP-c (Lem. 7)</td><td>Linear (Lem. 5)</td><td rowspan="2">Linear (Lem. 5)</td></tr>
<tr><td>$\tau = 2$</td><td rowspan="3">NP-c (Lem. 6)</td><td rowspan="3">NP-c, W[1]-h (Thm. 3)</td><td>NP-c, W[1]-c (Thm. 4)</td></tr>
<tr><td>$\tau = 3$</td><td rowspan="2">NP-c (Thm. 1)</td><td rowspan="2">$\in \Sigma_2^P$ (Lem. 1)</td><td>$\boldsymbol{\Sigma_2^P}$-c (Thm. 2)</td></tr>
<tr><td>$\tau \geq 4$</td><td>NP-h and $\in \Sigma_2^P$ (Cor. 1 and Lem. 1)</td></tr>
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Table 1: A summary of results.

For MaxMinTaRDiS, membership of NP is nontrivial; even the existence of a polynomial-time verifiable certificate is uncertain. Interestingly, we show equivalence[8] of Nonstrict MaxMinTaRDiS restricted to inputs where $\tau = 2$ and Distance-3 Independent Set, which is NP-complete [22], in Section 4.

Having shown τ and planarity alone are insufficient for tractability, we show existence of an algorithm which is fixed-parameter tractable with respect to lifetime, k, and treewidth of the footprint of the input graph[9] combined. This is achieved in Section 5 by applying Courcelle's theorem [16]. In the full version of the paper, we also give an exact algorithm for TaRDiS on trees where the input graph can have unbounded lifetime.

1.3 Related Work

For a static graph $G = (V, E)$, a dominating set is a set of vertices S such that $\forall u \in G \setminus S \; \exists v \in S : (u, v) \in E$. The decision problem Dominating Set (given G and k, is there a dominating set in G of size k?) is $W[2]$-complete when parameterized by k [18]. That is, even if k is fixed it is unlikely there exists an algorithm solving the problem with running time $f(k) \cdot n^{O(1)}$. However, Dominating Set can be solved in polynomial time on graphs of bounded treewidth [8,17].

TaRDiS is exactly the problem of solving the directed variant of Dominating Set on a Reachability Graph [6]. This graph is also referred to as the transitive closure of journeys in a temporal graph and is shown to be efficiently computable by Whitbeck et al. [39]. Temporal versions of dominating set and other classical covering problems have been well studied [3,11,38], however these interpretations do not allow a chosen vertex to dominate beyond its neighbours.

[8] We say that two problems X and Y are *equivalent* if they have the same language - that is, an instance I is a yes-instance of X if and only if the same instance I is a yes-instance of Y. Where X has a language consisting of triples (G, k, τ) and Y has a language of tuples (G, k), we may say that Y is equivalent to X with τ fixed to some value.

[9] Intuitively, the treewidth $\mathrm{tw}(G)$ of a graph G represents how "treelike" G is. We refer the interested reader to Chapter 7 in [17].

Furthermore, many other problems looking to optimally assign times to the edges of a static graph have been studied [2,20,30]. TaRDiS also generalises Temporal Source, which asks whether a single vertex can infect every other vertex in the graph. This is equivalent to the graph being a member of the class $\mathcal{J}^{1\forall}$ in the temporal graph classification given by Casteigts [11].

In research on networks, broadcasting refers to transmission to every device. In a typical model, there is a single source (which is input rather than chosen) in a graph which does not vary with time, in a setting where communication rather than computation is at a premium [33]. Broadcasting-based questions deviating from this standard have been studied as well. Namely, the NP-hardness of computing optimal broadcasting schedules for one or several sources [21] [28], broadcasting in ad-hoc networks or time-varying graphs [13], and the choice of multiple sources (originators) for broadcasting in minimum time in a static graph [15] [25]. To our knowledge, ours is the first work to focus on the hardness of choosing multiple sources in a temporal graph to minimize the number of sources, in an offline setting.

2 Preliminary results

Proofs of the following results can be found in the full version of the paper on arXiv.

Lemma 1. *Each variant[10] of TaRDiS is contained in NP, and each variant of MaxMinTaRDiS is contained in Σ_2^P.*

We now explore some properties of Happy TaRDiS and Happy MaxMinTaRDiS.

Lemma 2. *In a happy temporal graph, there always exists a minimum TaRDiS which is also canonical. It follows that the number of locally earliest edges upper-bounds the size of the minimum TaRDiS[11].*

Note that a temporal graph which has no locally earliest edges is necessarily not happy, and cannot admit any canonical TaRDiS. The converse does not hold; in Fig. 2c, $\mathcal{G}$ is not canonical (b is incident to two edges at time 3), but every TaRDiS is canonical, since every node is incident to a locally earliest edge.

Lemma 3. *Happy TaRDiS is FPT (Fixed-Parameter Tractable) in the number of locally earliest edges.*

We now note the relationship between the proper edge colourings of a static graph and the happy temporal assignments it admits.

Lemma 4. *A static graph H admits a happy time-labeling function $\lambda : E(G) \to [\tau]$ if and only if H is τ-edge colourable.*

[10] In this work, by "each variant" we refer to the Strict, Nonstrict and Happy variants of the problems introduced in Section 1.1

[11] This mirrors Lemma 54 from [5].

Corollary 1. *Happy MaxMinTaRDiS restricted to instances with $k = 0$ asks only if there exists a happy labeling λ with lifetime τ for the input graph G. This is equivalent to the Edge colouring problem with τ colours. Edge colouring is NP-complete for 3 or more colours [27], so Happy MaxMinTaRDiS is NP-hard for any $\tau \geq 3$.*

The following two lemmas consider the case where the lifetime of the graph is 1. We stress the contrast between the strict and nonstrict settings under this restriction.

Lemma 5. *When $\tau = 1$, the size of a minimum nonstrict TaRDiS depends only on the number of connected components in $\mathcal{G}_{\downarrow}$. Hence, Nonstrict TaRDiS and Nonstrict MaxMinTaRDiS can be computed in linear time in this case.*

Further, when $\tau \leq 2$ the problems Happy TaRDiS and Happy MaxMinTaRDiS are solvable in linear time.

Lemma 6. *Dominating Set is a special case of Strict TaRDiS, namely when $\tau = 1$.*

Given that the classical problem Dominating Set is a special case of TaRDiS, our problem Strict MaxMinTaRDiS inherits the following hardness result.

Lemma 7. *For any static graph H and $k \in \mathbb{N}^+$, (H, k) is a yes-instance of Strict MaxMinTaRDiS if and only if $(H, k-1)$ is a no-instance of Dominating Set. Hence Strict MaxMinTaRDiS is coNP-complete.*

The intuition follows from the idea that, for Strict MaxMinTaRDiS, it is always optimal to assign every edge the same time.

3 Hardness

Strict TaRDiS and Strict MaxMinTaRDiS are computationally hard even for lifetime $\tau = 1$ (Lemmas 6 and 7). Under the same restriction, Happy TaRDiS, Happy MaxMinTaRDiS and Nonstrict TaRDiS are all solvable in linear time (Lemma 5). In this section, we identify the minimum lifetime τ such that the problem becomes intractable for each of these problems.

3.1 NP-completeness of Happy TaRDiS with lifetime 3

Happy TaRDiS can trivially be solved in linear time when the input has lifetime $\tau \leq 2$ by Lemma 5. Here we show that the problem immediately becomes NP-complete for inputs where $\tau = 3$, even when $\mathcal{G}$ is planar.

Theorem 1. *Happy TaRDiS is NP-complete, even restricted to instances where $\mathcal{G}$ is planar and lifetime is 3.*

Proof (sketch). Our reduction is from the NP-complete problem Planar exactly 3–bounded 3–SAT, which asks whether the input Boolean formula ϕ is satisfiable [36]. We are guaranteed that ϕ is a planar formula in 3-CNF with each variable appearing exactly thrice and each literal at most twice. Let ϕ be an instance of Planar exactly 3–bounded 3–SAT consisting of clauses $\{c_1, \ldots, c_m\}$ over variables $X = \{x_1, \ldots, x_n\}$. We first create an auxiliary formula ϕ' by modifying every 2-clause of ϕ to include the special literal $\perp$. For example, the clause $(x_i \vee \neg x_j)$ becomes $(x_i \vee \neg x_j \vee \perp)$ in ϕ'. Note that ϕ' admits a satisfying assignment in which $\perp$ evaluates to `False` if and only if ϕ admits a satisfying assignment.

We will produce a happy temporal graph $\mathcal{G}$ which admits a TaRDiS of size exactly $k = 2n + 2m$ if and only if ϕ is satisfiable.

We create two *literal vertices* p_i^a and $\overline{p_i^a}$ (resp. n_i^a and $\overline{n_i^a}$) corresponding to the ath positive (resp. negative) appearance of the variable x_i, and connect these two vertices by an edge at time 3. Literal vertices $\perp^a$ and $\overline{\perp^a}$ correspond to $\perp$. Then for each variable x_i in ϕ we produce the *variable gadget* on vertices V_i as shown in Figure 3 and connect it by edges at time 2 to the literal vertices $\overline{p_i^a}$ and $\overline{n_i^a}$ corresponding to variable i. This gadget has the property that any canonical TaRDiS S of size k must contain, for each $i \in [n]$, exactly two vertices from V_i, including exactly one of the four vertices $T_i^1, T_i^2, F_i^1, F_i^2$, and thereby encodes a truth assignment to the variable x_i. Note that T_i^1 (resp. F_i^1) reaches all positive (resp. negative) literal vertices for x_i, and no other literal vertices.

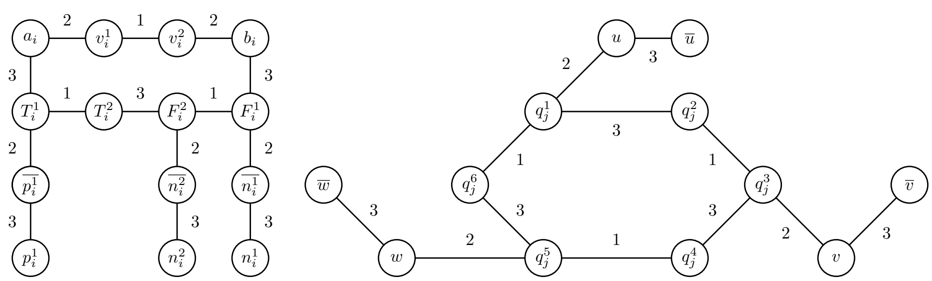

Fig. 3: Gadgets and adjacent *literal vertices* in the reduction from Planar exactly 3–bounded 3–SAT to Happy TaRDiS. Left: the variable gadget for x_i, which appears twice negatively. Right: the clause gadget for the clause c_j. If, for example, clause $c_j = (x_5 \vee \neg x_{17} \vee x_{20})$ is the first positive appearance of x_5 and x_{20} and the second negative appearance of x_{17} in ϕ, then $u = p_5^1$, $v = n_{17}^2$ and $w = p_{20}^1$.

Then, for any three literal vertices u, v, w corresponding to the same clause c_j, each of these is connected by an edge at time 2 to one of the *clause vertices* $Q_j = \{q_j^1, \ldots, q_j^6\}$ as shown in Figure 3. Any canonical TaRDiS S must include at least two vertices from Q_j. Further, if a canonical TaRDiS S has size k, then S includes exactly two vertices from Q_j and one of u, v, w is not reached from

$S \cap Q_j$, meaning it must be reached from $S \cap V_i$ for some $i \in [n]$. In other words, $Q_j \cap S$ has size two if and only if the truth assignment corresponding to S satisfies c_j. Note that literal vertices corresponding to the special literal $\perp$ are reachable *only* from clause vertices, and so behave identically to literals set to `False`, as intended.

This reduction produces an instance $(\mathcal{G}, k)$ of Happy TaRDiS from an instance ϕ of Planar exactly 3–bounded 3–SAT such that $\mathcal{G}$ admits a TaRDiS of size at most k if and only if ϕ is satisfiable. We have membership of NP from Lemma 1. □

3.2 Other hardness results

Theorem 2. *Happy MaxMinTaRDiS is Σ_2^P-complete.*

Σ_2^P-completeness implies it is strictly harder than NP-complete problems such as Satisfiability unless the polynomial hierarchy collapses to the first level. By Lemma 1, the problem is contained in Σ_2^P. Σ_2^P-hardness is proven in the arXiv version of the paper by reduction from Restricted Planar Satisfiability, a restricted variant of the canonical problem for that class. Interestingly, the proof does not easily generalize to $\tau \geq 4$. Consequently, we have only NP-hardness from Lemma 4 (NP is the class Σ_1^P of the polynomial hierarchy), leaving open the exact complexity of this restriction.

Theorem 3. *Nonstrict TaRDiS is NP-complete and W[2]-hard with respect to k for every $\tau \geq 2$. This holds even on graphs with degree at most 4.*

The proof (available in the full version) is by reduction from Set Cover, which is known to be NP-complete [9] and W[2]-hard [17].

4 Nonstrict MaxMinTaRDiS

Here we consider the restriction of Nonstrict MaxMinTaRDiS to instances with lifetime 2. We show the problem to be equivalent to the Distance 3 Independent Set (D3IS) decision problem.

Definition 4. *A distance-3-independent set (D3IS) of a static graph H is a set $S \subseteq V(H)$ such that for all distinct $u, v \in S$, $d(u, v) \geq 3$.*

The decision problem D3IS asks, for some input graph H and integer k, whether H admits a D3IS of size $\geq k$. We aim to show that a static graph H and integer k are a yes-instance of Nonstrict MaxMinTaRDiS with lifetime 2 if and only if the same graph H and integer k are a yes-instance of D3IS.

We begin by showing that existence of a maximal D3IS of size k in a graph H implies that we can find a temporal assignment $\lambda : E(H) \to \{1, 2\}$ such that a minimum TaRDiS in (H, λ) is of cardinality k. Given such a D3IS S of H, we assign $\lambda(u, v) = 1$, when $u \in S$ or $v \in S$, and $\lambda(u, v) = 2$, otherwise.

Lemma 8. *Under λ as described above, S is a minimum TaRDiS of (H, λ).*

Proof. We first show that S is a TaRDiS. We can assume without loss of generality that H is a single connected component. Suppose for contradiction some vertex u is not reachable from any vertex in S. Note that every vertex in S trivially reaches its neighbours. So, by construction of λ, u is incident only to edges at time 2. Since we have assumed that H consists of a single connected component, there must be a static path in H from each vertex in S to u. Let z be the closest vertex in S to u. Then the shortest path from z to u must have length 2 and consist of an edge assigned time 1 followed by an edge assigned time 2 (else S is not maximal). Hence the shortest path from z to u is a nonstrict temporal path and $u \in R_z$, contradicting our assumption. Hence S is a TaRDiS of the constructed instance.

We now show minimality of S. By construction of λ, every vertex $v \in S$ is temporally reachable only from its closed neighbourhood $N[v]$. No temporal path originating outside $N[v]$ can include any edge incident to v since any such path must contain an edge assigned time 2 before the final edge which is assigned time 1. Since S forms a D3IS, $N[u] \cap N[v] = \emptyset$ for all u, $v \in S$. Therefore, for (H, λ) to be temporal reachability dominated, there must at least be a vertex from the neighbourhood of each vertex in S. These are disjoint sets, so any TaRDiS must have cardinality at least k. Hence S is a minimum TaRDiS of (H, λ). □

Definition 5. *We call a TaRDiS S on a temporal graph $\mathcal{G}$* independent *if, for all vertices u, v in S with $u \neq v$, u does not reach v. Equivalently, S is independent if and only if every vertex in S is in its own sole reachability set under S.*

Lemma 9. *If a temporal graph $\mathcal{G}$ admits an independent nonstrict TaRDiS S, then S is a D3IS in the footprint graph $\mathcal{G}_\downarrow$.*

Proof. Consider two vertices $u, v \in S$ at distance $d(u, v)$ from one another in $\mathcal{G}_\downarrow$. If u and v are adjacent, then u and v reach each other, which would contradict independence of S. If $d(u, v) = 2$ then there is at least one vertex $w \in N[u] \cap N[v]$, and either $\lambda(u, w) \leq \lambda(w, v)$ or $\lambda(u, w) > \lambda(w, v)$. So one of u and v must reach the other. Hence any two vertices in S must be distance at least 3 from one another, the definition of a D3IS. □

Lemma 10. *If some temporal graph $\mathcal{G}$ with lifetime 2 has a minimum nonstrict TaRDiS of cardinality k, then $\mathcal{G}_\downarrow$ admits a D3IS of size k.*

Proof. Our proof is constructive; given a minimum TaRDiS S, we show existence of an *independent* minimum TaRDiS S^* of equal cardinality and then apply Lemma 9.

First, we justify some simplifying assumptions about $\mathcal{G}$. Since TaRDiS can be computed independently in disconnected components of $\mathcal{G}_\downarrow$, we will assume $\mathcal{G}_\downarrow$ is connected. Further, if $E_1(\mathcal{G}) = \emptyset$ or $E_2(\mathcal{G}) = \emptyset$ we may choose any single vertex from $\mathcal{G}$ to be a minimum TaRDiS which is also independent; hence we assume $E_1(\mathcal{G}) \neq \emptyset$ and $E_2(\mathcal{G}) \neq \emptyset$. Also, recall that $SR(S, x) \neq \emptyset$ for all $x \in S$ by minimality of S.

We construct S^* by replacing every vertex x in S such that $x \notin SR(S,x)$ with some vertex x^* with the property that $R_{x^*} = R_x$ and $x^* \in SR(S,x)$, as follows. Let $y \neq x$ be a vertex in S such that y reaches x. The path from y to x cannot arrive at time 1, else S is not minimal as $R_x = R_y$ and $S \setminus \{x\}$ is a TaRDiS. Choose x^* to be the closest vertex to x in $SR(S,x)$. We know such a vertex exists by minimality of S. We claim that the path from x to x^* arrives at time 1 and so $R_x = R_{x^*}$. To see this, suppose otherwise. The path must begin with at least one edge at time 1, otherwise x^* would be reachable from y. If the path arrives at time 2, then the last vertex on the path reached at time 1 is closer to x than x^*, and is in $SR(S,x)$. Else, some other vertex in S reaches x^*.

This concludes our construction of S^* as an independent minimum TaRDiS. By Lemma 9, S^* is also a D3IS. □

Combining Lemmas 8 and 10 gives us the following theorem.

Theorem 4. *Nonstrict MaxMinTaRDiS with lifetime $\tau = 2$ is equivalent to Distance 3 Independent Set.*

Interestingly, the same does not hold for $\tau \geq 3$. A counterexample is shown in Figure 2c, where the minimum TaRDiS is larger than the maximum D3IS of the footprint.

Eto et al. [22] show that D3IS is NP-complete even on planar, bipartite graphs with maximum degree 3. They also show D3IS to be $W[1]$-hard on chordal graphs with respect to the size of the distance 3 independent set. D3IS is also shown to be APX-hard on r-regular graphs for all integers $r \geq 3$ and admit a PTAS on planar graphs by Eto et al. [23]. Agnarsson et al. [1] show that D3IS is tractable on interval graphs, trapezoid graphs and circular arc graphs. Thus, these results also apply to Nonstrict MaxMinTaRDiS when $\tau = 2$.

5 Tractability

We show tractability by expressing our problems in extended monadic second-order (EMSO) logic and applying Courcelle's theorem [16].

5.1 TaRDiS

We show tractability of our problems by expressing them in EMSO logic and applying the variant of Courcelle's theorem given by Arnborg et al. [4]. This result states that an optimisation problem which is definable in EMSO can be solved in linear time when parameterised by treewidth and length of the formula. Full expressions can be found in the full paper on arXiv. We provide a sketch showing some key components. Langer et al. provide definitions of first-order, MSO and EMSO logic in their survey [32].

Lemma 11. *Temporal reachability is expressible in MSO logic.*

Proof (Sketch). To encode the temporal nature of a temporal graph $\mathcal{G}$, we use an auxiliary static graph $\mathcal{S}$. Our auxiliary graph is an adaptation of the one given by Haag et al. [26, Theorem 23]. They use this graph to construct an MSO formula for Temporal Feedback Edge Set. The vertices of $\mathcal{S}$ consist of the disjoint union of $V(\mathcal{G})$, $E(\mathcal{G})$, and the set of timesteps $[\tau]$. The edges of $\mathcal{S}$ consist of the following binary relations. A relation R is written as $R(e, v)$ to simplify the expression $(e, v) \in R$.

- the incidence relation $\mathrm{inc} \subseteq E \times V$ where $\mathrm{inc}(e, v) \iff v \in e$;
- the presence relation $\mathrm{pres} \subseteq E \times \tau$ where $\mathrm{pres}(e, t) \iff (e, t) \in \varepsilon$.

As shown by Haag et al., $|\mathcal{S}| \in O(\tau + |\mathcal{G}|)$ and the treewidth of $\mathcal{S}$ is bounded by a function of $\mathrm{tw}(\mathcal{G}_\downarrow) + \tau$.

The formula $\mathrm{path}(v, w)$ tests whether there is a temporal path from v to $w \in V$. In the strict case we use the subformula $\mathrm{ttadj}(u, v, t)$ given by Haag et al.[26, Theorem 23]. This uses the incidence and presence relations on $\mathcal{S}$ to test if vertices u and v are adjacent at time t. For nonstrict temporal paths, we test whether two vertices are in the same connected component of $\mathcal{G}_t$ instead of adjacency. This uses a variation of the formula $\mathrm{conn}(v, w, E')$ by Haag et al. which tests if there is a path using edges in the set E' from v to w. The formula for path is

$$\mathrm{path}(v, w) = \exists v_0, \ldots, v_\tau \in V : v = v_0 \wedge v_\tau = w \wedge \bigwedge_{t=0}^{\tau-1} (v_t = v_{t+1} \vee \mathrm{a}(v_t, v_{t+1}, t+1))$$

where $\mathrm{a}(v_t, v_{t+1}, t + 1)$ is a place-holder for the subformula testing adjacency or connectedness for strict and nonstrict temporal paths respectively. $\square$

Lemma 12. *Both strict and nonstrict TaRDiS are expressible in MSO.*

We use the reachability formulae given in Lemma 11 to express TaRDiS in MSO. This is done by a formula which tests whether there exist k vertices such that all vertices in $\mathcal{G}$ are reachable from them.

We note that strict reachability and thus Strict TaRDiS are expressible in FO logic, which is a strict subset of EMSO. Combining the tractability result given by Arnborg et al. [4] with Lemma 12 gives the following theorem.

Theorem 5. *TaRDiS is fixed-parameter tractable with respect to lifetime, k and treewidth of the footprint of the temporal graph.*

5.2 MaxMinTaRDiS

We now show that MaxMinTaRDiS is expressible in EMSO logic, thus proving the following theorem.

Theorem 6. *MaxMinTaRDiS is fixed-parameter tractable when parameterised by lifetime, k and treewidth of the graph.*

Proof (Sketch). Our EMSO formula follows from the MSO formula for TaRDiS. Since the temporal assignment is not part of the input, we must encode it as a partition of edges into sets which correspond to the time at which they are active. Therefore, to check if an edge e is active at time t, we simply need to check if $e \in S_t$. We use variations of the formulae in Lemmas 11 and 12 to test whether a given set is a TaRDiS. For MaxMinTaRDiS it is necessarily to enforce minimality of a given TaRDiS and existence of a temporal assignment that we can check these properties on. These are done using the following subformulae over the sets X_1, X_2 of vertices and $S_1, \ldots, S_\tau$ of edges respectively:

$$\mathrm{geq}(X_1, X_2) = \exists k : \mathrm{card}(k, X_1) \wedge \neg \mathrm{card}(k+1, X_2)$$

where $\mathrm{card}(k, X)$ tests if a set X contains at least k distinct elements, and

$$\mathrm{part}(S_1, \ldots, S_\tau) = \forall e \in E : \bigvee_{1 \le i \le \tau} e \in S_i$$

which can be adjusted if we require a happy or simple temporal assignment. Full formulae expressing MaxMinTaRDiS are in the full version of the paper on arXiv. Hence the theorem holds. □

6 Conclusions and open questions

In this paper, we introduce the TaRDiS and MaxMinTaRDiS problems and study their parameterized complexity. We show a bound on the lifetime τ and a restriction to planar inputs combined are insufficient to obtain tractability[12], and moreover tightly characterize the minimum lifetime τ for which each problem becomes intractable. Further, we show that τ, k and the treewidth of the input (temporal) graph combined are sufficient to yield tractability in all cases by leveraging Courcelle's theorem.

These results leave open the questions of the exact complexity of Nonstrict MaxMinTaRDiS with lifetime $\tau \geq 3$ and Happy MaxMinTaRDiS with lifetime $\tau \geq 4$. An interesting extension of our work would be to find approximability results for these problems. From the parameterized side, it remains to be shown whether parameterization by a structural parameter of the footprint (e.g. treewidth) alone is sufficient to obtain obtain tractability for any of the considered variants. Another interesting dimension is the comparison of Nonstrict MaxMinTaRDiS and Happy MaxMinTaRDiS when τ is lower-bounded by a function of the number of edges m. With the constraint $\tau = m$ the two problems become equivalent, and their computational complexity in this case is an interesting open question. Analogously to t-Dominating Set [31], t-TaRDiS, in which t individuals must be reached provides a natural generalisation of our problem and the potential for parameterization by t.

[12] The result in Theorem 1 generalizes to Strict TaRDiS and Nonstrict TaRDiS; D3IS and Dominating Set are NP-complete on planar graphs.

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