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Noninvertible anomalies in $\mathrm{SU}(N) \times \mathrm{U}(1)$ gauge theories

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Abstract: We study 4-dimensional $\mathrm{SU}(N) \times \mathrm{U}(1)$ gauge theories with a single massless Dirac fermion in the 2-index symmetric/antisymmetric representations and show that they are endowed with a noninvertible 0-form $\tilde{\mathbb{Z}}^{\chi}_{2(N\pm2)}$ chiral symmetry along with a 1-form $\mathbb{Z}_N^{(1)}$ center symmetry. By using the Hamiltonian formalism and putting the theory on a spatial three-torus $\mathbb{T}^3$, we construct the non-unitary gauge invariant operator corresponding to $\tilde{\mathbb{Z}}^{\chi}_{2(N\pm2)}$ and find that it acts nontrivially in sectors of the Hilbert space characterized by selected magnetic fluxes. When we subject $\mathbb{T}^3$ to $\mathbb{Z}_N^{(1)}$ twists, for N even, in selected magnetic flux sectors, the algebra of $\tilde{\mathbb{Z}}^{\chi}_{2(N\pm2)}$ and $\mathbb{Z}_N^{(1)}$ fails to commute by a $\mathbb{Z}_2$ phase. We interpret this noncommutativity as a mixed anomaly between the noninvertible and the 1-form symmetries. The anomaly implies that all states in the torus Hilbert space with the selected magnetic fluxes exhibit a two-fold degeneracy for arbitrary $\mathbb{T}^3$ size. The degenerate states are labeled by discrete electric fluxes and are characterized by nonzero expectation values of condensates. In an appendix, we also discuss how to construct the corresponding noninvertible defect via the "half-space gauging" of a discrete one-form magnetic symmetry.

Keywords: Anomalies in Field and String Theories, Discrete Symmetries, Global Symmetries

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Contents

1 Introduction

Symmetries are the backbone of quantum field theory (QFT); the spectrum of local and extended operators are organized in symmetry representations. The modern way to define symmetry is via its action on topological surfaces. A p-form symmetry acts on p-dimensional objects and is generated by operators supported on $p+1$-codimensional surfaces [1]. Traditionally, symmetries are connected to groups, and the operator that generates the symmetry is unitary. However, recent developments have highlighted the need to broaden the definition of symmetries to include actions generated by nonunitary operators. These symmetries are noninvertible since the corresponding operators do not have an inverse. Consequently, these operations do not form groups but can be comprehended as categories, offering a new perspective on the nature of symmetries.

Noninvertible symmetries were first identified and applied in 2-dimensional QFT, see, e.g., [2, 3]. The appreciation of the role of this new development in 4-dimensional QFT resulted in an avalanche of works on this topic (a non-comprehensive list is [4–29]).

One significant development in the field is the recognition that 4-dimensional quantum electrodynamics (QED) possesses a noninvertible symmetry [11, 12]. The idea is that QED with a single Dirac fermion has a classical $\mathrm{U}(1)_\chi$ chiral symmetry broken to the $\mathbb{Z}_2$ fermion number by the ABJ anomaly. The Noether current corresponding to $\mathrm{U}(1)_\chi$ is not conserved as it receives a contribution from the anomaly. However, one may define a conserved chiral current by subtracting a Chern-Simons term that encodes the anomaly. Next, an operator corresponding to $\mathrm{U}(1)_\chi$ is constructed by exponentiating the modified current and integrating it on a 3-surface. This operator is not gauge invariant. However, dressing it with a TQFT makes it gauge invariant. The resulting operator can be shown to generate a noninvertible symmetry for every rational value of the $\mathrm{U}(1)_\chi$ parameter. The construction in [11, 12] was further developed in [17, 18] by showing that the definition of the noninvertible operator can be extended to any real parameter of $\mathrm{U}(1)_\chi$ by coupling the theory to a scalar field living on the 3-surface.

Comprehending this novel structure in gauge theories is important in pursuing a deeper understanding of QFT. The present work examines $\mathrm{SU}(N)\times\mathrm{U}(1)$ gauge theory with a single massless Dirac fermion in a representation R. The theory has a classical $\mathrm{U}(1)_\chi$ symmetry broken by the ABJ anomaly in $\mathrm{SU}(N)$- and $\mathrm{U}(1)$-instanton backgrounds. Does this theory exhibit noninvertible symmetries, and if so, can they be utilized to establish exact nonperturbative statements? We demonstrate that the answer to this query is affirmative.

Unlike QED, when our theory is put on a general manifold, the chiral symmetry is reduced to $\mathbb{Z}^\chi_{2\mathrm{gcd}(T_R,d_R)}$, where T_R and d_R are the Dynkin index and dimension of R. Interestingly, we also show that the theory possesses a noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_R}$ discrete chiral symmetry, wherein $\mathbb{Z}^\chi_{2\mathrm{gcd}(T_R,d_R)}$ is an invertible part.[1] We establish the noninvertibility through a sequential process starting from the nonconservation of Noether's current of the chiral symmetry. Employing the Hamiltonian formalism and putting the theory on a 3-dimensional spatial torus $\mathbb{T}^3$, we construct a noninvertible symmetry operator. This setup provides a simple and explicit route to select the states on which the operator $\tilde{\mathbb{Z}}^\chi_{2T_R}$ acts nontrivially.

$\mathrm{SU}(N)\times\mathrm{U}(1)$ QCD-like theories are naturally endowed with an electric 1-form $\mathbb{Z}^{(1)}_N$ symmetry acting on the Wilson loops as well as a 1-form $\mathrm{U}(1)^{(1)}_m$ magnetic symmetry that characterizes sectors with definite magnetic fluxes. Then, it is natural to ask whether the theory exhibits a 't Hooft anomaly as we perform a $\tilde{\mathbb{Z}}^\chi_{2T_R}$ transformation in the background of $\mathbb{Z}^{(1)}_N$. To address this question, for definiteness[2] we consider a theory with fermions in the 2-index symmetric/anti-symmetric representation, which possesses a $\tilde{\mathbb{Z}}^\chi_{2(N\pm2)}$ noninvertible symmetry. We subject the theory to $\mathbb{Z}_N$ twists (2-form background fields for $\mathbb{Z}^{(1)}_N$) along the non-trivial cycles of $\mathbb{T}^3$. In the presence of the $\mathbb{Z}_N$ twists, the noninvertible symmetry projects onto sectors in Hilbert space with definite magnetic fluxes, easily identified in the Hamiltonian formalism. For N even, the algebra of $\tilde{\mathbb{Z}}^\chi_{2(N\pm2)}$ and $\mathbb{Z}^{(1)}_N$ fails to commute by a $\mathbb{Z}_2$ phase inside these sectors, revealing a mixed anomaly between $\mathbb{Z}^{(1)}_N$ and $\tilde{\mathbb{Z}}^\chi_{2(N\pm2)}$ symmetries (anomalies involving noninvertible symmetries have previously been considered in 1+1 dimensions, see [30–35] and references therein). This anomaly implies that the states in these special sectors must be 2-fold degenerate[3] on arbitrary size $\mathbb{T}^3$. Such degeneracies could be seen by examining the condensates in realistic lattice simulations.

This paper is organized as follows. In section 2, we demonstrate the origin of the noninvertible symmetry by carefully examining $\mathrm{SU}(N)\times\mathrm{U}(1)$ QCD-like theories put on $\mathbb{T}^3$ with general flux backgrounds and build the noninvertible operator of $\tilde{\mathbb{Z}}^\chi_{2(N\pm2)}$. Then, we show that this symmetry has a 't Hooft anomaly with the $\mathbb{Z}^{(1)}_N$ symmetry. In section 3, we discuss the implication of this anomaly in the magnetic sectors selected by $\tilde{\mathbb{Z}}^\chi_{2(N\pm2)}$ and exhibit the exact degeneracy. In the appendix, we show the equivalence of our operator construction to the "half-gauging" of a discrete subgroup of the magnetic one form symme-

[1] Throughout this paper, we use a tilde to distinguish noninvertible symmetries and operators. Note also that after gauging the vector U(1), the $\mathbb{Z}_2$ part (fermion number) of $\tilde{\mathbb{Z}}^\chi_{2T_R}$ is part of the gauge symmetry and not a global symmetry. With this in mind, we continue to denote the noninvertible symmetry by $\tilde{\mathbb{Z}}^\chi_{2T_R}$, as this subtlety does not affect our considerations of the anomaly and spectral degeneracy.

[2] The conclusions of the work described can be generalized to fermions in higher representations.

[3] Similar to the exact finite-volume degeneracies due to invertible 0-form/1-form anomalies [36].

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try $U(1)^{(1)}_m$, used in [11, 12] to construct a properly normalized defect yielding consistent Euclidean correlation functions. We conclude with a brief discussion in section 4.

2 Noninvertible symmetries and their anomalies

Consider $SU(N) \times U(1)$ gauge theory with a single-flavor massless Dirac fermion in a representation R. We use n, T_R, and d_R to denote the N-ality, Dynkin index, and the dimension of R, respectively, and focus mostly on $T_R = N \pm 2$ for the two-index symmetric (S)/antisymmetric (AS) representations of N-ality $n = 2$ and dimension $d_R = \frac{N(N\pm1)}{2}$ for S/AS. Yet, our construction can be easily generalized to theories with several flavors and fermions in higher representations. Classically, the theory is endowed with a $U(1)_\chi$ global chiral symmetry. The fermion charges (both are left-handed Weyl) under $(SU(N), U(1), U(1)_\chi)$ are

$$\psi_R \sim (R, 1, 1), \quad \psi_{\bar{R}} \sim (\bar{R}, -1, 1)\,. \tag{2.1}$$

The theory with the gauged U(1) has a $\mathbb{Z}_N^{(1)}$ 1-form electric symmetry, acting on both $SU(N)$ and U(1) Wilson loops, as well as a $U(1)^{(1)}_m$ 1-form magnetic symmetry which distinguishes the different U(1)-flux sectors.

We use A and a for the 1-form gauge fields of $SU(N)$ and U(1), respectively. The corresponding field strengths are F and f. The anomaly equation for the chiral $U(1)_\chi$ current is

$$\partial_\mu j^\mu_\chi - 2T_R \partial_\mu K^\mu_{SU(N)} - \frac{2d_R}{8\pi^2}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu a^\nu \partial^\lambda a^\sigma = 0, \tag{2.2}$$

where $K^\mu_{SU(N)}$ is the $SU(N)$ topological current. Its normalization is such that the integral of $K^0_{SU(N)} \equiv K^{CS}$ over a three-dimensional manifold changes by an integer under large gauge transformations. In other words, the operator

$$e^{-i2\pi \int_{\mathbb{T}^3} d^3x\, K^{CS}(A)} \tag{2.3}$$

is invariant under large gauge transformations. This operator (which shifts the θ-angle by 2π) will be important in what follows.

When the theory is defined on a spatial manifold with nontrivial 2-cycles, e.g., on $\mathbb{T}^3$, gauging U(1) breaks $\mathbb{Z}^\chi_{2T_R}$ of the $SU(N)$ theory further down to $\mathbb{Z}^\chi_{2\gcd(d_R,T_R)}$. The easiest way to see that is by compactifying the time direction, so we consider the Euclidean version of the theory on $\mathbb{T}^4$. Under a chiral transformation, $(\psi_R, \psi_{\bar{R}}) \to e^{i\alpha}(\psi_R, \psi_{\bar{R}})$, the measure changes by

$$\exp\left[i\alpha\left(2T_R c_2(F) + 2d_R c_2(f)\right)\right]\,, \tag{2.4}$$

where $c_2(F) \equiv \int_{\mathbb{T}^4} \frac{\mathrm{tr}\, F\wedge F}{8\pi^2} \in \mathbb{Z}$ and $c_2(f) \equiv \int_{\mathbb{T}^4} \frac{f\wedge f}{8\pi^2} \in \mathbb{Z}$ are the second Chern classes of $SU(N)$ and U(1). Then, demanding that the phase (2.4) is trivialized, and using Bézout's identity, which states that integers of the form $az_1 + bz_2$ are exact multiples of $\gcd(a, b)$, we arrive at our conclusion.[4] For example, when $\gcd(d_R, T_R) = 1$, there is no residual chiral

[4] As a side note, there is no solution of self-dual BPST instanton on $\mathbb{T}^4$ with unit topological charge and zero twists [37]. Adding a twist removes the obstruction to the existence of the solution. These twists are discussed below.

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symmetry and the only symmetry left over is the $\mathbb{Z}_2$ fermion number symmetry which stays intact, assuming the Lorentz symmetry is unbroken.

It is important to emphasize that $\mathbb{Z}^{\chi}_{2\mathrm{gcd}(d_R,T_R)}$ is a genuine invertible symmetry, which is represented by a unitary operator acting on the Hilbert space of the theory. What is the fate of $\mathbb{Z}^{\chi}_{2T_R}$ that is broken because of the U(1) instantons? Below, following the approach of [17, 18], we argue that this symmetry becomes noninvertible. We shall exhibit this and study the consequences using Hamiltonian quantization on $\mathbb{T}^3$ in a very explicit manner.

Consider the Hamiltonian quantization of the theory on a rectangular $\mathbb{T}^3$ of sides L_1, L_2, L_3, in the $A_0 = a_0 = 0$ gauge. The Hilbert space is constructed in terms of gauge fields $A = A_i dx^i$ and $a = a_i dx^i$, where $i = 1, 2, 3$ is the spatial index (below, we use $\hat{e}_i$ to denote a unit vector in the i-th direction). We often use x, y, z for x_1, x_2, x_3 when writing the components explicitly.

The physical Hilbert space is obtained from field-operator eigenstates after appropriate gauge averaging and imposing Gauss's law; a detailed description can be found in [36], see also the earlier works [38, 39]. The gauge fields obey boundary conditions on $\mathbb{T}^3$

$$\begin{aligned} A(x + \hat{e}_i L_i) &= \Gamma_i A(x) \Gamma_i^{-1}\,, \\ a(x + \hat{e}_i L_i) &= a(x) - d\omega_i(x)\,, \end{aligned} \tag{2.5}$$

given in terms of SU(N) and U(1) transition functions, Γ_i and ω_i, respectively [40]. We work in a gauge where the SU(N) transition functions Γ_i are constant unitary $N \times N$ matrices [41]. The fermions obey similar boundary conditions (for brevity, we write these for the N-ality two case, $n = 2$):

$$\begin{aligned} \psi_R(x + \hat{e}_i L_i) &= \Omega_i(x) \Gamma_i\, \psi_R(x)\, \Gamma_i^T\,, \\ \psi_{\bar{R}}(x + \hat{e}_i L_i) &= \Omega_i(x)^{-1} \Gamma_i^*\, \psi_{\bar{R}}(x)\, \Gamma_i^\dagger. \end{aligned} \tag{2.6}$$

The U(1) and SU(N) transition functions obey cocycle conditions assuring that the fields satisfying (2.5), (2.6) are single-valued on the torus. The cocycle conditions obeyed by the transition functions are

$$\begin{aligned} \Gamma_i \Gamma_j &= e^{i \frac{2\pi}{N} n_{ij}}\, \Gamma_j \Gamma_i\,,\ n_{ij} = -n_{ji},\ n_{ij} \in \mathbb{Z}\ (\mathrm{mod} N), \\ \Omega_i(x + \hat{e}_j L_j) \Omega_j(x) &= e^{-i \frac{2\pi n}{N} n_{ij}}\, \Omega_j(x + \hat{e}_i L_i) \Omega_i(x)\,, \end{aligned} \tag{2.7}$$

or

$$\Omega_i(x) = e^{i\omega_i(x)},\ \text{with}\ \omega_i = \sum_{j=1}^{3} \pi \left(m_{ij} + \frac{n}{N} n_{ij} \right) \frac{x_j}{L_j},\ m_{ij} = -m_{ji} \in \mathbb{Z}\,. \tag{2.8}$$

Here, the (mod N) integers n_{ij} represent topological classes of 2-form $\mathbb{Z}_N^{(1)}$ background fields in the respective 2-planes and m_{ij} label integer U(1)-flux sectors, distinguished by their magnetic flux through the various 2-planes. The integer m_{ij} are charges under the global magnetic $\mathrm{U}(1)^{(1)}_m$ 1-form symmetry [1]. It is easily seen that all gauge and matter fields are single-valued on $\mathbb{T}^3$ when (2.7) are obeyed. The integers n_{ij} and m_{ij} label different flux sectors of the torus Hilbert space.

We next consider the global $\mathbb{Z}_N^{(1)}$ symmetry [1]. The generators of the 1-form $\mathbb{Z}_N^{(1)}$ symmetry act on the transition functions. We label by t_j and T_j the U(1) and SU(N) group elements representing the action of the generators of $\mathbb{Z}_N^{(1)}$ in the j-th direction on the U(1) and SU(N) transition functions, respectively. The action of $\mathbb{Z}_N^{(1)}$ is given by

$$\begin{aligned}\Omega_j &\to t_i(x+\hat{e}_j L_j)\,\Omega_j(x)\,t_i^{-1}(x) = e^{-i\frac{2\pi n}{N}\delta_{ij}}\Omega_j(x),\\ \Gamma_j &\to T_i(x+\hat{e}_j L_j)\,\Gamma_j\,T_i^{-1}(x) = e^{i\frac{2\pi}{N}\delta_{ij}}\Gamma_j.\end{aligned} \tag{2.9}$$

Put differently, $\mathbb{Z}_N^{(1)}$ transformations are represented by "improper" gauge transformations on the Hilbert space (in the first line, since both t_j and Ω_i are abelian, the Ω_j's can be dropped).

The explicit expressions for U(1) and SU(N) group elements $t_j(x)$ and $T_j(x)$ obeying (2.9) and generating the center symmetry can be worked out. The explicit form of $T_j(x)$ can be found in the literature.[5] The explicit form of $T_j(x)$ depends on both the choice of gauge for the transition functions and on the n_{ij} 2-form background (this is because $T_j(x)$ have nontrivial winding numbers for $n_{ij} \neq 0$ (mod$N$) [45]). We will not need the expression for T_j, but only the commutation relation of the SU(N) $\mathbb{Z}_N^{(1)}$ center symmetry generators and the gauge invariant operator (2.3),

$$T_i\, e^{-i2\pi\int_{T^3} d^3x K^{CS}(A)}\, T_i^{-1} = e^{-i2\pi\frac{\epsilon_{ijk}n_{jk}}{2N}}\; e^{-i2\pi i\int_{T^3} d^3 K^{CS}(A)}\,, \tag{2.10}$$

which is nontrivial in the presence of a 't Hooft twist; see [36] for a derivation. On the other hand, the center symmetry generators' action on the U(1) can be taken

$$t_i(x) = e^{-i\frac{2\pi n}{N}\frac{x_i}{L_i}} = e^{i\tilde{\lambda}^{(i)}(x)}\,, \text{ with } \tilde{\lambda}^{(i)}(x+\hat{e}_j L_j) = \tilde{\lambda}^{(i)}(x) - \delta^{ij}\frac{2\pi n}{N}\,. \tag{2.11}$$

Next, we construct the generator of the noninvertible chiral symmetry. To this end, we integrate the anomaly equation (2.2) and use it to define a conserved (but not gauge-invariant) $\mathrm{U}(1)_\chi$ symmetry operator on $\mathbb{T}^3$. Because of the boundary conditions in the space directions for the U(1) fields, whose transition functions (2.8) necessarily depend on x_i, there are a-dependent boundary terms.[6] We find, denoting $K^0(a) \equiv \frac{1}{8\pi^2}\epsilon^{ijk}a_i\partial_j a_k$:

$$0 = \int_{\mathbb{T}^3} d^3x\left[j^0_\chi - 2T_R K^{CS}(A) - 2d_R K^0(a)\right]\Big|_{x_0=0}^{x_0=L_0} - \frac{2d_R}{8\pi^2}\int_0^{L_0} dx_0 \int d^2S^i\epsilon^{ijk}a_j\partial_0 a_k\Big|_{x_i=0}^{x_i=L_i}. \tag{2.12}$$

Next, we note that $\partial_0 a_k$ is periodic on $\mathbb{T}^3$, while a_k itself obeys (2.5) with transition function ω_i of (2.8). It is easy to see that the second term above is also a total time derivative.

[5] See [42, 43] for SU(2) and [44] for SU(N).

[6] On the other hand, since the transition functions Γ_i are constant, no such boundary terms appear for SU(N).

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Integrating (2.12), we finally obtain that $Q_\chi(x_0 = L_0) = Q_\chi(x_0 = 0)$, where

$$\begin{aligned} Q_\chi &= \int_{\mathbb{T}^3} d^3x \left[j^0_\chi - 2T_R K^{CS}(A) - 2d_R K^0(a) \right] \\ &+ \frac{d_R}{4\pi}\left(m_{xy} + \frac{n}{N} n_{xy}\right) \left[\int_0^{L_y} \frac{dy}{L_y} \int_0^{L_z} dz a_z(x=0,y,z) + \int_0^{L_x} \frac{dx}{L_x} \int_0^{L_z} dz a_z(x,y=0,z) \right] \\ &+ \sum_{\text{cyclic}} (x \to y \to z \to x) \,. \end{aligned} \tag{2.13}$$

The last line above indicates that there are two more terms obtained from the term on the second line by cyclic rotation of x, y, z. Again, this is the operator Q_χ in the sector of Hilbert space with U(1) fluxes m_{ij} and $\mathbb{Z}_N^{(1)}$ fluxes n_{ij}.

Exponentiating (2.13), we find the (non-gauge-invariant) operator representing $\mathbb{Z}^\chi_{2T_R}$:

$$X_{2T_R} = e^{i\frac{2\pi}{2T_R} Q_\chi} \,. \tag{2.14}$$

Let us now study the gauge transformation properties of X_{2T_R}. First, we note that because of the gauge invariance of (2.3), the invariance of (2.14) under SU(N) (large and small) gauge transformations is manifest. Next, consider U(1) gauge transformations with periodic[7] $e^{i\lambda}$:

$$a_i \to a_i - \partial_i \lambda, \text{ with } \lambda(x + \hat{e}_i L_i) = \lambda(x) + 2\pi n_i \,. \tag{2.15}$$

To study the transformation properties of (2.14) under U(1) transformations (2.15), we note that, with λ from (2.15), $-2\pi \int_{T^3} d^3x K^0(a)$ transforms as

$$-\frac{1}{4\pi} \int_{\mathbb{T}^3} d^3x \epsilon^{ijk} a_i \partial_j a_k \Big|_a^{a-d\lambda} = \frac{n_x}{2} \int dy dz (\partial_y a_z - \partial_z a_y) + \sum_{\text{cyclic}} (x \to y \to z \to x) \,. \tag{2.16}$$

Since, recalling (2.5), (2.7),

$$\int dy dz (\partial_y a_z - \partial_z a_y) = -2\pi \left(m_{yz} + \frac{n n_{yz}}{N} \right), \tag{2.17}$$

we find

$$\begin{aligned} -2\pi \int_{\mathbb{T}^3} d^3x K^0(a) \Big|_a^{a-d\lambda} &= -\frac{1}{4\pi} \int_{T^3} d^3x \epsilon^{ijk} a_i \partial_j a_k \Big|_a^{a-d\lambda} = -\pi n_x \left(m_{yz} + \frac{n n_{yz}}{N} \right) \\ &\quad + \sum_{\text{cyclic}} (x \to y \to z \to x) \,. \end{aligned} \tag{2.18}$$

Likewise, we also find the U(1) gauge transformation of the boundary terms in (2.13)

$$\int_0^{L_y} \frac{dy}{L_y} \int_0^{L_z} dz a_z(x=0,y,z) + \int_0^{L_x} \frac{dx}{L_x} \int_0^{L_z} dz a_z(x,y=0,z) \Big|_a^{a-d\lambda} = -4\pi n_z \,, \tag{2.19}$$

along with two identical relations obtained by cyclic permutations of x, y, z.

[7] For use below, we also recall from (2.11) that the generator t_j of the global $\mathbb{Z}_N^{(1)}$ in the j-th direction acts on the U(1) gauge field as a nonperiodic gauge transformation, i.e. is obtained from (2.15) upon replacing $n_j \to -\frac{n}{N}$, where, we remind the reader, n is the N-ality of the matter representation.

Thus, combining the U(1) gauge transformations (2.18), (2.19), with the expression for Q_χ from (2.13), we find the transformation of X_{2T_R}

$$X_{2T_R}[a - d\lambda] = X_{2T_R}[a]\; e^{-i2\pi \frac{d_R}{T_R}\left[n_x(m_{yz}+\frac{nn_{yz}}{N})+n_y(m_{zx}+\frac{nn_{zx}}{N})+n_z(m_{xy}+\frac{nn_{xy}}{N})\right]}\,, \tag{2.20}$$

showing explicitly that the operator (2.14) is not U(1) gauge invariant.

Thus, following [17, 18], to make X_{2T_R} gauge invariant, we sum over n_x, n_y, n_z, obtaining the noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_R}$ operator

$$\tilde{X}_{2T_R} = e^{i\frac{2\pi}{2T_R}Q_\chi} \sum_{n_x,n_y,n_z\in Z} e^{-i2\pi \frac{d_R}{T_R}\left[n_x(m_{yz}+\frac{nn_{yz}}{N})+n_y(m_{zx}+\frac{nn_{zx}}{N})+n_z(m_{xy}+\frac{nn_{xy}}{N})\right]}\,, \tag{2.21}$$

with Q_χ from (2.13). This equation shows that the operator is noninvertible and determines the sectors not annihilated by $\tilde{X}_{2T_R}$. To see this, we use the Poisson resummation formula

$$\sum_{n_x\in Z} e^{-i2\pi\frac{d_R}{T_R}(m_{yz}+\frac{nn_{yz}}{N})n_x} = \sum_{l_x\in Z} \delta\left(\frac{d_R}{T_R}\left(m_{yz}+\frac{nn_{yz}}{N}\right)-l_x\right). \tag{2.22}$$

For $\tilde{X}_{2T_R}$ to act nontrivially, i.e., not be set to zero, it must be that in each two-plane, the fluxes m_{ij}, n_{ij}, $i<j$, have to obey

$$\frac{d_R}{T_R}\left(m_{yz}+\frac{nn_{yz}}{N}\right) = l_x,\;\; l_x \in \mathbb{Z} \;\; \text{(plus cyclic)}. \tag{2.23}$$

It is easy to see that such integer-valued combinations of fluxes always exist (we shall see examples below). In the appendix, we construct the operator $\tilde{X}_{2T_R}$ using the "half-gauging" procedure of refs. [11, 12].

To summarize, here we have constructed a symmetry operator $\tilde{X}_{2T_R}$ of the noninvertible chiral symmetry. The operator of the noninvertible symmetry acts as a projection operator: it annihilates sectors of the torus Hilbert space whose fluxes do not obey (2.23) and acts as unitary operator in each flux sector obeying (2.23).[8]

We next study the commutator of the $\mathbb{Z}_N^{(1)}$ center symmetry transformation with the noninvertible $\tilde{X}_{2T_R}$. We denote the $\mathbb{Z}_N^{(1)}$ generator in the x-direction by $T_x t_x$ (for brevity omitting both hats over operators and the tensor product sign, with T_x, t_x obeying (2.9)). We use the commutation relation of (2.10) of T_i with the operator (2.3), as well the U(1) transformation law derived above, eq. (2.20), with only n_x nonzero and with the integer n_x replaced by $-\frac{n}{N}$ (as we reminded the reader in footnote 7). We find

$$\begin{aligned} T_x t_x\, \tilde{X}_{2T_R}\,(T_x t_x)^{-1} &= \tilde{X}_{2T_R} e^{-i2\pi\left[\frac{n_{yz}}{N}-\frac{d_R}{T_R}\frac{n}{N}(m_{yz}+n\frac{n_{yz}}{N})\right]} \\ &= \tilde{X}_{2T_R} e^{-i2\pi\left[\frac{n_{yz}}{N}-\frac{n}{N}l_x\right]}\,, \;\text{with } l_x \text{ from } (2.23). \end{aligned} \tag{2.24}$$

Equation (2.24) and its two cyclically permuted versions, constitute our main result. It shows that — provided both the phase on the r.h.s. is nontrivial and $\tilde{X}_{2T_R}$ is nonzero

[8] In backgrounds with m_{ij}, n_{ij} chosen to yield integer l_i (2.23), the operator can simply be defined by (2.14), since, as (2.20) shows, for such values of l_i, it is gauge invariant.

(i.e. (2.23) holds) — there is a mixed anomaly between the $\mathbb{Z}_N^{(1)}$ center symmetry and the noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_R}$ chiral symmetry of the SU(N) × U(1) theory. As both operators act nontrivially in the chosen integer-l_i $\mathbb{T}^3$ Hilbert space, the algebra (2.24) will be seen to imply exact degeneracies for any size $\mathbb{T}^3$.

In section 3, we shall show that there are cases where both the phase in (2.24) and $\tilde{X}_{2T_R}$ are nontrivial, i.e., (2.23) holds in the appropriate Hilbert space. As already familiar from [36], this will be seen to imply degeneracies in the Hilbert space between different "electric flux" states (eigenstates of $\mathbb{Z}_N^{(1)}$, i.e., of $T_x t_x$).

Before we continue with analyzing the implications of (2.24) for the finite-volume spectrum, let us make a connection of (2.24) with the Euclidean path integral. We denote the $\mathbb{T}^3$ Hilbert space by $\mathcal{H}_{l_i}$, with the understanding that U(1) (m_{ij}) and $\mathbb{Z}_N^{(1)}$ (n_{ij}) fluxes are chosen to yield an integer-l_i, so that $\tilde{X}_{2T_R}$ acts nontrivially. We now define the $T_x t_x$-twisted partition function via the Hamiltonian formalism as a trace[9] over states in $\mathcal{H}_{l_i}$ with a $\mathbb{Z}_N^{(1)}$ x-direction generator inserted in the partition function, i.e. $\mathcal{Z} \equiv \mathrm{tr}_{\mathcal{H}_{l_i}} e^{-\beta H} T_x t_x$. Then, we use the fact that $\tilde{X}_{2T_R}$ acts as an invertible unitary operator in $\mathcal{H}_{l_i}$, as well as the commutation relation (2.24), to obtain

$$\begin{aligned}\mathcal{Z} &= \mathrm{tr}_{\mathcal{H}_{l_i}} \left[e^{-\beta H} T_x t_x\right] = \mathrm{tr}_{\mathcal{H}_{l_i}} \left[e^{-\beta H} T_x t_x \tilde{X}_{2T_R} \tilde{X}^\dagger_{2T_R}\right] \\ &= e^{-i2\pi\left[\frac{n_{yz}}{N} - \frac{n}{N} l_x\right]} \ \mathrm{tr}_{\mathcal{H}_{l_i}} \left[e^{-\beta H} T_x t_x\right] . \end{aligned} \tag{2.25}$$

We conclude that the chiral symmetry $\tilde{X}_{2T_R}$ implies that $\mathcal{Z} = e^{-i2\pi\left[\frac{n_{yz}}{N} - \frac{n}{N} l_x\right]} \mathcal{Z}$, so that, if the phase is nontrivial, $\mathcal{Z}$ vanishes, unless fermion fields $(\psi_{\tilde{R}} \psi_R)^k$ are inserted to make $\mathcal{Z}$ nonzero.[10] To obtain a path integral interpretation of (2.25), we note that the twisted partition function (2.25) sums over SU(N) and U(1) gauge fields which obey the boundary conditions (2.5) in the $\mathbb{T}^3$ spatial directions, determined by n_{ij} and m_{ij}. The boundary conditions in the Euclidean time direction (of extent β) are twisted, by the insertion of T_x, leading to a nontrivial SU(N) twist $n_{x4} = 1$ in the x-time plane. Thus, the twisted partition function (2.25) sums over SU(N) field configurations with topological charges $-\frac{n_{14} n_{yz}}{N} + k = -\frac{n_{yz}}{N} + k$, with all possible integer k [46]. On the other hand, the insertion of t_x implies that the U(1) background obeys $a(x + \hat{e}_4 \beta) = a(x) + \frac{2\pi n}{N} \frac{dx}{L_1}$ and thus $f_{14} = -\frac{2\pi n}{N L_1 L_4}$. Recalling that the U(1) field strength in the yz plane is $\frac{2\pi}{L_2 L_3}(m_{yz} + \frac{n}{N} n_{yz})$), the U(1) topological charge is then equal to $\frac{n}{N}(m_{yz} + \frac{n}{N} n_{yz})$. Applying a chiral transformation with $\alpha = \frac{2\pi}{2T_R}$, and using the measure transformation (2.4) (with $c_2(F)$, $c_2(f)$ substituted by the fractional topological charges just mentioned) we obtain the phase $e^{-i2\pi\left[\frac{n_{yz}}{N} - \frac{d_R}{T_R} \frac{n}{N}(m_{yz} + \frac{n}{N} n_{yz})\right]}$, which, after using (2.23), is seen to be the same as in (2.25), as expected.

3 Hilbert space, magnetic sectors, and the 2-fold degeneracy

Here, we analyze the consequences of condition (2.23) and the algebra of (2.24). Condition (2.23) selects sectors in Hilbert space with definite U(1) magnetic fluxes in the 2-3, 3-1,

[9] The relations we derive below hold also if we insert $(-1)^F$ in the partition function.

[10] Below, we show that the phase in (2.25) can take values at most in $\mathbb{Z}_2$, and that $k = \frac{N \pm 2}{2}$, for S/AS fermions, for the values of N and choice of fluxes where the phase is nontrivial.

and 1-2 planes proportional to the integers l_x, l_y, l_z, respectively. In the following, we always set $l_y = l_z = 0$ (with $n_{xy} = m_{xy} = n_{zx} = m_{zx} = 0$) to reduce complexity and examine the theory for various values of $l_x = \frac{d_R}{T_R}(m_{yz} + \frac{n n_{yz}}{N})$. We also choose $n_{yz} \in \{0, 1, 2, \dots, N-1\}$. Since we are mainly concerned with symmetric (S)/antisymmetric (AS) fermions, we set the N-ality $n = 2$.

We start with sectors with a vanishing magnetic flux, i.e., we set $l_x = 0$, which translates into $m_{yz} + \frac{2}{N} n_{yz} = 0$. When N is odd, the only solution is the null solution $m_{yz} = n_{yz} = 0$. This yields a trivial phase in the algebra of (2.24). However, when $N = 2M$ is even, there are two solutions. First, the null one $m_{yz} = n_{yz} = 0$, giving a trivial phase in (2.24). The second is the new solution (we denote the phase in (2.24) by $e^{-i\alpha} = e^{-2\pi i(\frac{n_{yz}}{N} - \frac{2}{N} l_x)}$):

$$\text{S/AS}: \quad N = 2M, \; n_{yz} = M, \; m_{yz} = -1, \; l_x = 0, \alpha = \pi. \tag{3.1}$$

The $\mathbb{Z}_2$ phase in (2.24) implies that some electric flux states in a sector with a 0-magnetic flux are 2-fold degenerate.

Numerical tests reveal that this pattern continues in sectors with nonzero magnetic flux, $l_x \neq 0$. First, when N is odd, all allowed sectors have a trivial phase in the algebra of (2.24), indicating no kinematical constraints in these theories. For N even, $N = 2M$, one can use $l_x = \frac{d_R}{T_R}(m_{yz} + \frac{n n_{yz}}{N})$ to see that there exists integers $m_{yz} \in \mathbb{Z}$ and $n_{yz} \in \{0, 1, 2, \dots, N-1\}$ that satisfy the relation

$$M\,(2M \pm 1)(M m_{yz} + n_{yz}) = 2M(M \pm 1) l_x\,, \tag{3.2}$$

for S/AS fermions, respectively. Sectors with definite m_{yz} and n_{yz} that satisfy (3.2) will also yield at most a $\mathbb{Z}_2$ phase in the algebra (2.24), if they additionally satisfy

$$M^2\,(n_{yz} + (2M \pm 1) m_{yz}) \in M^2 (M \pm 1)(2\mathbb{Z} + 1)\,. \tag{3.3}$$

The $\mathbb{Z}_2$ phase implies that the states in these sectors have double-fold degeneracy.

Let us flesh this out in detail. For definiteness, we consider S/AS fermions, which yield a nontrivial phase in (2.24) as well as involve only a minimal SU(N) 't Hooft twist, $n_{yz} = 1$ (as opposed to the $l_x = 0$ solutions (3.1), which must have $n_{yz} = N/2$ to produce an anomaly). For both S/AS fermions, these minimal-twist solutions, yielding an integer l_x and a $\mathbb{Z}_2$ phase in (2.24), must have $N = 4p + 2$. Using again $e^{-i\alpha} = e^{-2\pi i(\frac{n_{yz}}{N} - \frac{2}{N} l_x)}$, examples of m_{yz} that give the $\mathbb{Z}_2$ phase are:

$$\begin{aligned} \text{S}: \quad & N{=}4p{+}2, \quad n_{yz}{=}1, \quad m_{yz}{=}{-}2p{-}1, \quad l_x{=}{-}p(3{+}4p), \qquad \alpha{=}2\pi\left(\frac{1}{2}{+}2p\right), \\ \text{AS}: \quad & N{=}4p{+}2, \quad n_{yz}{=}1, \quad m_{yz}{=}{-}2p{-}1, \quad l_x{=}{-}(1{+}p)(1{+}4p), \quad \alpha{=}2\pi\left(\frac{3}{2}{+}2p\right). \end{aligned} \tag{3.4}$$

Thus, we now focus on the flux backgrounds (3.4), enumerate the degenerate "electric flux" sectors[11] in the corresponding Hilbert spaces $\mathcal{H}_{l_x, l_y = l_z = 0}$, and discuss some of their

[11] These states can be explicitly constructed in the semiclassical limit of a small $\mathbb{T}^3$ [39, 47] (or a small $\mathbb{T}^2 \in \mathbb{T}^3$, with the $\mathbb{T}^2$ spanning the y, z directions [48, 49]) by focusing on the lowest energy states. We shall not do this here in complete detail, but see Footnote 13.

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properties. The transition functions for SU(N) and U(1) obeying the cocycle conditions with n_{yz}, m_{yz} given in (3.4) are

$$\begin{aligned} \Gamma_x &= 1,\ \Gamma_y = P,\ \Gamma_z = Q,\ \text{where } PQ = e^{i\frac{2\pi}{N}} QP\,, \\ \omega_x &= 1,\ \omega_y = -\frac{\pi z}{L_3}\frac{4p(p+1)}{2p+1},\ \omega_z = \frac{\pi y}{L_2}\frac{4p(p+1)}{2p+1}\,. \end{aligned} \tag{3.5}$$

The center symmetry generators for SU(N) and U(1), obeying (2.9), can be taken to be[12]

$$\begin{aligned} T_x &= T_x(y,z)\ (\text{recall}\,, T_x^N|\text{phys}\rangle = |\text{phys}\rangle), T_y = Q^{-1}, T_z = P, \\ t_k &= e^{-i\frac{2\pi}{2p+1}\frac{x_k}{L_k}}\,. \end{aligned} \tag{3.6}$$

Since $T_x t_x$ is a symmetry, eigenstates of the Hamiltonian can be labeled by its eigenvalues, $T_x t_x|E, e_x\rangle = |E, e_x\rangle e^{i\frac{2\pi}{N}e_x}$, with $e_x \in \mathbb{Z}$ (mod$N$). On the other hand, the algebra (2.24), $T_x t_x \tilde{X}_{2T_R}(T_x t_x)^{-1} = -\tilde{X}_{2T_R}$, implies that, in $\mathcal{H}_{l_x, l_y=l_z=0}$, with l_x from (3.4),

$$\tilde{X}_{2T_R}|E, e_x\rangle_{\mathcal{H}_{l_x,l_y=l_z=0}} \sim |E, e_x + \frac{N}{2}(\text{mod}\, N)\rangle_{\mathcal{H}_{l_x,l_y=l_z=0}}. \tag{3.7}$$

Thus, $\tilde{X}_{2T_R}$ maps an eigenstate of the Hamiltonian of energy E and flux e_x to another eigenstate of the same energy, but with flux $e_x + \frac{N}{2}$ (mod$N$) (hence, a $T_x t_x$ eigenvalue differing by $e^{i\pi}$, as per (2.24); we also note that the phase in the action of $\tilde{X}$ also depends on whether the state is bosonic of fermionic).[13]

To further characterize the degenerate flux states, we will show that the degenerate states (3.7) have nonvanishing expectation values of a condensate, which we write schematically as $(\psi_{\bar{R}}\psi_R)^{\frac{N\pm 2}{2}}$. These expectation values take opposite values in the two degenerate flux states. To this end, we now go back to our twisted partition function (2.25). Since in $\mathcal{H}_{l_x,l_y=l_z=0}$ the phase is in $\mathbb{Z}_2$, in order to obtain a nonzero phase we must insert $(\psi_{\bar{R}}\psi_R)^{\frac{N\pm 2}{2}}$,

[12] For SU(N), these are the ones from [47]. Briefly, we remind the reader that T_x cannot be taken to be constant, since, as already discussed, a twist of the partition function in the time direction by T_x, recall (2.25), leads to fractional topological charge on the $\mathbb{T}^4$, equal to $\frac{1}{N}$ + integer. This implies that T_x has fractional winding number [45], as a map from $\mathbb{T}^3$ to the gauge group, with its N-th power being a large gauge transformation. Thus, on physical states T_x obeys $T_x^N = 1$. Explicit expressions for T_1 can be found in the literature (see [42, 43] for SU(2) and [44] for SU(N)) but are not needed here.

[13] As promised, on a small $\mathbb{T}^3$, the lowest flux states can be worked out classically. For the bosonic backgrounds, the lowest-energy gauge field backgrounds are

$$\begin{aligned} A^{(l)} &= -iT_1^l dT_1^{-l},\ l = 0, \ldots, N-1, \\ a^{(l)} &= \frac{4\pi p(p+1)}{L_3 L_2(2p+1)}(ydz - zdy) + \frac{2\pi l}{(2p+1)L_1}dx\,. \end{aligned} \tag{3.8}$$

The fundamental SU(N) winding Wilson loops then take values $W_y = W_z = 0, W_x = e^{i\frac{2\pi}{N}l}$. Using the backgrounds (3.8), solving for the fermions, imposing Gauss's law, and averaging over gauge transformations, one can construct the N classically-degenerate states in $\mathcal{H}_{l_x,l_y=l_z=0}$. It is already clear from (3.8) that these N states are obtained by the action of $T_x t_x$ from each other. The electric flux states from eq. (3.7) are a discrete Fourier transform thereof. The anomaly implies that the N-fold degeneracy will be lifted and that only the pairwise degeneracy will remain quantum mechanically. Extending this small-torus explicit analysis further, along the lines of [48, 49], as well as similar studies for other backgrounds, e.g., (3.1), are left for the future.

a gauge invariant object which transforms with a $\mathbb{Z}_2$ phase under $\psi_{R,\bar{R}} \to e^{i\frac{2\pi}{2(N\pm2)}}\psi_{R,\bar{R}}$. We have

$$\begin{aligned}\langle(\psi_{\bar{R}}\psi_R)^{\frac{N\pm2}{2}}\rangle &= \operatorname{tr}_{\mathcal{H}_{l_i}}\left[e^{-\beta H}(\psi_{\bar{R}}\psi_R)^{\frac{N\pm2}{2}}T_x t_x\right]\\ &= \sum_{E,e_1=0,\ldots,N-1} e^{-\beta E(e_1)}e^{i\frac{2\pi}{N}e_1}\langle E,e_1|(\psi_{\bar{R}}\psi_R)^{\frac{N\pm2}{2}}|E,e_1\rangle\,. \end{aligned} \tag{3.9}$$

Next, recall that $\tilde{X}^\dagger_{2T_R}\tilde{X}_{2T_R} = 1$ in $\mathcal{H}_{l_i}$, remembering that we are in a definite magnetic flux sector, where the chiral symmetry operator $\tilde{X}_{2T_R}$ acts in an invertible manner, and using (3.7), we find that gauge-invariant condensates obey

$$\langle E,e_1|(\psi_{\bar{R}}\psi_R)^k|E,e_1\rangle = e^{i\frac{2\pi}{N\pm2}k}\,\langle E,e_1+\frac{N}{2}|(\psi_{\bar{R}}\psi_R)^k|E,e_1+\frac{N}{2}\rangle\,. \tag{3.10}$$

In particular, (3.10) shows that the condensate appearing in (3.9) takes opposite values in the degenerate states. In the twisted partition function (3.9), this minus sign is cancelled by change of the phase $e^{i\frac{2\pi}{N}e_1}$ (from the action of $T_x t_x$). Thus, we can restrict the evaluation of (3.9) by summing over half the e_1 sectors:

$$\langle(\psi_{\bar{R}}\psi_R)^{\frac{N\pm2}{2}}\rangle = 2\sum_{E,e_1=0,\ldots\frac{N}{2}-1} e^{-\beta E(e_1)}e^{i\frac{2\pi}{N}e_1}\langle E,e_1|(\psi_{\bar{R}}\psi_R)^{\frac{N\pm2}{2}}|E,e_1\rangle\,. \tag{3.11}$$

These nonzero expectation values can be computed semiclassically at a small torus and shown not to vanish, similar to [50].

To summarize, above we constructed the doubly-degenerate states using the background (3.4), in the $N = 4p+2$ theory, as an example. However, based on the $\mathbb{Z}_2$-valued anomaly, similar descriptions involving degenerate states with opposite values of the relevant condensate hold in all even-N cases. In particular, the doubly-degenerate flux states corresponding to (3.1) can also be explicitly worked out.

Before we discuss these cases, let us contrast the findings in the $SU(N)\times U(1)$ theory on $\mathbb{T}^3$ with those in the $SU(N)$ theory, also on $\mathbb{T}^3$. Consider the $SU(4p+2)$ theory (i.e., with even N not divisible by 4). It has a $\mathbb{Z}_2^{(1)}$ center symmetry and an invertible discrete chiral symmetry $\mathbb{Z}^\chi_{2(N\pm2)}$. These have a $\mathbb{Z}_2$-valued mixed anomaly in appropriate $\mathbb{Z}_2^{(1)}$ backgrounds on $\mathbb{T}^3$. This anomaly implies, as in [36], an exact two-fold degeneracy in the twisted Hilbert space of the $SU(4p+2)$ theory, on any torus size. This is similar to the degeneracy of the $SU(4p+2)\times U(1)$ theory on $\mathbb{T}^3$ discussed in this paper. We stress, however, that the latter theory has $\gcd(d_R,T_R)=1$, and hence no genuine chiral symmetry. The degeneracy we found is, thus, due to the noninvertible chiral $\tilde{\mathbb{Z}}^\chi_{2T_R}$ symmetry.

Consider now the case when N is divisible by 4. In the $SU(N)$ theory this mixed anomaly is trivial on $\mathbb{T}^3$ and hence one cannot use the $\mathbb{Z}_2^{(1)}$ 1-form symmetry to argue for an exact degeneracy on a finite-size torus. In the $SU(N)\times U(1)$ theory, however, for any even N, we showed that there is an anomaly between the $\mathbb{Z}_N^{(1)}$ center and noninvertible chiral symmetries in the (3.1) background, leading to an exact two-fold degeneracy at any size $\mathbb{T}^3$.

The case of minimal dimension condensate occurs if we take $N=4$ and an antisymmetric tensor. Here, the anomaly predicts equal and opposite values of the bilinear fermion condensate $\psi_{\bar{R}}\psi_R$ in the two states that are degenerate at any finite volume, in the appropriately twisted background. Since the degeneracy is present at any finite volume, should the condensate remain nonzero in the infinite volume limit, this predicts the $\mathbb{Z}_4 \to \mathbb{Z}_2$ (noninvertible) chiral symmetry breaking in the thermodynamic limit.

We stress that the use of appropriate twists — the m_{ij}, n_{ij} with integer l_i, i.e. the ones that reveal the anomaly — at finite volume is simply a tool to probe the gauge dynamics. At least in the theory with a nonzero mass gap, the infinite volume limit is expected to be independent of the boundary conditions and the degeneracies revealed are expected to persist in the thermodynamic limit.

4 Discussion

Here, we found that the anomaly establishes the 2-fold degeneracy on arbitrary-size $\mathbb{T}^3$. Yet, one eventually wants to see what happens as we take the thermodynamic limit by sending the volume of $\mathbb{T}^3$ to infinity.

To speculate on what could happen in $SU(N) \times U(1)$ theory, let us again return to its cousin, the $SU(N)$ gauge theory with S/AS fermions. The latter has a global $U(1)$ baryon number and $U(1)_\chi$ chiral symmetry. As usual, quantum effects break $U(1)_\chi$ down to, now, the invertible $\mathbb{Z}^\chi_{2(N\pm2)}$. The 0-form faithful global symmetry of this theory is $\frac{U(1)}{\mathbb{Z}_{\frac{N}{p}}} \times \mathbb{Z}^\chi_{2(N\pm2)}$, with $p = \gcd(N, 2)$. The fact that the quotient group is nontrivial means that we can activate a 't Hooft flux in the center of $SU(N)$ accompanied by a flux in $U(1)$ such that the cocycle conditions are always obeyed on general four-dimensional manifolds. The authors constructed these fluxes in vector-like theories, dubbed as the baryon-color (BC) fluxes, in [51, 52] (also see [53–57] for applications, and [58] for the construction and applications of these fluxes in chiral gauge theories). The partition function acquires a $\mathbb{Z}_{N\pm2}$ phase as we apply a $\mathbb{Z}^\chi_{2(N\pm2)}$ rotation in the background of the BC flux. This phase was interpreted as an anomaly of $\mathbb{Z}^\chi_{2(N\pm2)}$ in the BC background. Assuming that the theory has a mass gap and forms hadrons in the IR, the anomaly is interpreted to imply the existence of $N \pm 2$ degenerate vacua [53]. One expects to see this degeneracy emerge on a finite-volume manifold (larger than the inverse strong-scale) and persist in the thermodynamic limit.

As we argued in this work, gauging the $U(1)$ baryon symmetry endows the theory with a $\mathbb{Z}_2$ anomaly phase when N is even, implying that at any finite volume there is exact 2-fold degeneracy. Thus, 2 degenerate vacua are guaranteed to survive the infinite volume limit. If the $SU(N) \times U(1)$ theory also has $N \pm 2$ degenerate vacua in the infinite volume limit (as the $SU(N)$ theory is believed to) the exact $N \pm 2$-fold degeneracy should be revealed in the thermodynamic limit.

One might, of course, wonder whether a stronger phase (and stronger constraints, as in [59]) can be exhibited if we subject the $SU(N) \times U(1)$ theory to a gravitational background. In other words, it would be interesting to investigate whether there is a mixed anomaly between the noninvertible $\tilde{\mathbb{Z}}^\chi_{2(N\pm2)}$ and gravity. If the gravitational anomaly does not produce a stronger phase beyond $\mathbb{Z}_2$, the implications in the thermodynamic limit of the $SU(N) \times U(1)$ theory are of interest. By weakly gauging $U(1)$ in the $SU(N)$ theory, one anticipates the presence of $N \pm 2$ nearly degenerate vacua. However, the dynamics of the

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U(1) gauge field may affect the precise degeneracy. This intriguing investigation remains open for future study.

Finally, we comment on the $SU(N) \times U(1)$ theory at finite temperature. The presence of the identified mixed anomaly implies that it is necessary for either the 1-form symmetry $\mathbb{Z}_N^{(1)}$, the 0-form symmetry $\tilde{\mathbb{Z}}^{\chi}_{2(N\pm2)}$, or both symmetries to be broken [60–62]. Actually, since $\mathbb{Z}_N^{(1)}$ acts on the U(1) Wilson lines, we expect it to be broken at zero and finite temperature. Consequently, the anomaly is consistently matched, leading us to only expect the restoration of the broken $\tilde{\mathbb{Z}}^{\chi}_{2(N\pm2)}$ symmetry and the breaking of the $\mathbb{Z}_2^{(1)}$ subgroup of the 1-form symmetry for even-N at some finite temperature.

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A The noninvertible defect via "half-gauging" of $\mathbb{Z}^{(1)}_{T_R} \subset U(1)^{(1)}_m$

Here, we consider the construction of the noninvertible defect for[14] $\tilde{\mathbb{Z}}^{\chi}_{2T_R}$ using the "half-gauging" procedure of refs. [11, 12] instead of the sum over gauge orbits of [17, 18]. The advantage of the former construction is that it gives rise to well-defined Euclidean correlation functions involving the noninvertible defect, considered more generally than as an operator inserted at a particular time. In particular, this allows inserting the defect at a particular location in space, giving rise to well-defined Hilbert spaces twisted by the noninvertible symmetry.

Our discussion below makes use of the techniques described explicitly in ref. [11] and is restricted to the case $\gcd(d_R, T_R) = 1$. For our $SU(N) \times U(1)$ theories with a two-index S/AS Dirac fermion, this is the case of even-N not divisible by 4, leaving the generalization to the more general case for future work (although a generalization to $\gcd(d_R, T_R) > 1$ should be possible [11]). In order to define the defect, we consider the gauging of the $\mathbb{Z}^{(1)}_{T_R}$ subgroup of the magnetic $U(1)^{(1)}_m$ 1-form symmetry. As shown in [11], this gauging produces the same theory but with a discrete shift of the U(1) theta angle $\theta \to \theta - \frac{2\pi d_R}{T_R}$. This shift, as per our eq. (2.4), is undone by a $\mathbb{Z}_{2T_R}$ chiral rotation of the fermions $(\psi_R, \psi_{\bar{R}})$. We conclude that the $SU(N) \times U(1)$ theory is invariant under the above gauging of $\mathbb{Z}^{(1)}_{T_R} \subset U(1)^{(1)}_m$.

As in [11, 12], the upshot of the half-gauging procedure is to define a defect, replacing our eq. (2.21) for the operator $\tilde{X}_{2T_R}$ by the following object, which we label, for brevity, by the same letter

$$\tilde{X}_{2T_R} = e^{i\frac{2\pi}{2T_R}\int\limits_{t=0} d^3x j^0_\chi - i2\pi \int\limits_{t=0} d^3x K^{CS}(A) + \int\limits_{t=0} \mathcal{A}^{T_R, d_R}[\frac{da}{T_R}]}\,, \tag{A.1}$$

[14] Once again, we acknowledge our abuse of notation: as per the remark of Footnote 1, the $\mathbb{Z}_2$ fermion number subgroup of $\mathbb{Z}^{\chi}_{2T_R}$ is part of the U(1) gauge group.

where, as in the main text, a is the dynamical U(1) gauge field and A is the SU(N) gauge field with $K^{CS}[A]$ entering as in (2.3).

The 3d defect TQFT $\mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]$ is defined via an integral over a 4d bulk with a boundary, which is here taken to be the $t=0$ plane:

$$e^{\int\limits_{t=0} \mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]} = \int \mathcal{D}(b,c)\, e^{\int\limits_{t\geq 0}\left(\frac{i d_R}{4\pi T_R}\, da\wedge da + \frac{i}{2\pi}\, b^{(2)}\wedge da + \frac{i T_R}{2\pi}\, b^{(2)}\wedge dc^{(1)} + \frac{i k T_R}{4\pi}\, b^{(2)}\wedge b^{(2)}\right)} . \tag{A.2}$$

The fields $(b^{(2)}, c^{(1)})$ define the 2-form $\mathbb{Z}_{T_R}$ gauge field (used in the half-gauging procedure) and obeying the Dirichlet boundary condition $b^{(2)} = 0$ at $t = 0$, and k is the modular inverse of d_R, i.e. $k d_R = 1$ (modT_R). Defining the defect via the "half-gauging" procedure replaces the sum over gauge copies of the gauge noninvariant U(1) Chern-Simons term by the TQFT $\mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]$ and produces a well defined Euclidean defect.

The $\mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]$ theory has a $\mathbb{Z}^{(1)}_{T_R}$ global symmetry with an anomaly d_R, and the partition function $e^{\int\limits_{t=0} \mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]}$ of (A.2) has the transformation properties of $e^{-i\frac{d_R T_R}{4\pi}\int\limits_{t\geq 0} \frac{da}{T_R}\wedge\frac{da}{T_R}}$. In particular, under regular gauge transformations $\delta a = d\omega$ with $\oint d\omega = 2\pi\mathbb{Z}$,

$$\delta\, e^{\int\limits_{t=0} \mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]} = e^{i\frac{d_R}{T_R}\int\limits_{t=0} d\omega\wedge\frac{da}{2\pi}}\, e^{\int\limits_{t=0} \mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]} , \tag{A.3}$$

which implies, together with the gauge invariance of $\mathcal{A}^{T_R,d_R}[\frac{da}{T_R}]$, that the defect vanishes unless $\frac{d_R}{T_R}\oint da = 2\pi\mathbb{Z}$. This is precisely the condition (2.23) of the main text, in the absence of an electric $\mathbb{Z}^{(1)}_N$ center symmetry background. Thus, we have succeeded in defining a noninvertible defect associated with the $\tilde{\mathbb{Z}}^{\chi}_{2T_R}$ chiral symmetry.

The question left open, then, is to show that the definition of $\tilde{X}_{2T_R}$ from (A.1) reproduces the mixed anomaly with center symmetry exhibited in eq. (2.24). This would necessitate generalizing the half-gauging procedure to backgrounds with fractional $\oint da \in \frac{2\pi\mathbb{Z}}{N}$.[15] In the language of defects, one has to consider defects associated with the (invertible) electric $\mathbb{Z}^{(1)}_N$ 1-form symmetry of the SU(N) $\times$ U(1) theory and determine the fusion rules of these codimension-two defects with the noninvertible defect defined by (A.1). The fusion rules of $\tilde{X}_{2T_R}$ with these codimension-two $\mathbb{Z}^{(1)}_N$-defects of two different orientations are relevant to the anomaly (2.24). The first are "perpendicular" to $\tilde{X}_{2T_R}$, with a worldvolume in the x-t plane and in the language of this paper correspond to the turning on of n_{yz} from (2.24). The other type of codimension-two defects implementing the $\mathbb{Z}^{(1)}_N$ symmetry are "parallel" to $\tilde{X}_{2T_R}$, with world volume in the y-z plane,[16] implementing the one-form symmetry transformation with parameter $n_x = -\frac{n}{N}$ due to $T_x t_x$ of (2.24). The careful study of the various defects mentioned should make finding these fusion rules possible, but we leave this interesting question to future studies. We believe, however, that the $\mathbb{T}^4$ path-integral discussion after eq. (2.25) gives strong support for our finding (2.24) — while, admittedly, leaving the study of the more general interesting situations mentioned above open.

[15] Thus, incorporating the electric $\mathbb{Z}^{(1)}_N$ center-symmetry background, denoted by n_{yz} in (2.23), (2.24).

[16] We note that examples of fusion rules for similar parallel codimension-two symmetry defects with duality/triality defects were studied in [9].

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