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SELLER COMPOUND SEARCH FOR BIDDERS*

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This article studies a seller's compound search for bidders by a deadline. We show that the optimal search outcomes can be implemented by a sequence of second-price auctions, characterized by declining reserve prices and increasing search intensities (sample sizes) over time. The monotonicity results are robust in both cases of short-lived and long-lived bidders. Furthermore, a seller with short-lived bidders sets lower reserve prices and searches more intensively than one with long-lived bidders. We also show that the inefficiency of an optimal search auction can stem from its inefficient search rule.

I. INTRODUCTION

It is puzzling that the dominant selling processes in many markets seem not competitive, where no obvious competition among buyers is observed. For instance, in mergers and acquisitions (M&As), it is well-documented that the dominant selling process is one-on-one negotiation.[1] Andrade *et al.* [2001] describe the prototypical M&As in the 1990s as friendly transactions, where normally there was just one bidder. Betton *et al.* [2008] also report that 95%

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[1] Depending on how many bidders to contact in the first stage, Boone & Mulherin [2009] classify the selling processes in M&As into three categories: *one-on-one negotiation*, where a seller contacts a single most likely buyer first; *private controlled sale* (auction), where a seller screens and first invites a small number of qualified bidders; and *public full-scale auction*, where a seller announces a public simultaneous auction, and all interested bidders can submit bids.

of their sample deals in the US market, during the period from 1980 to 2005, are noncompetitive negotiations. The observations go against the conventional wisdom that competition among buyers not only raises bid premiums but enhances allocative efficiency.

There have been various empirical explanations for this puzzle. Aktas *et al.* [2010] argue that negotiation in M&A is under the threat of following-up auctions. For example, if a seller fails to achieve a good deal at the negotiation stage, she may contact other bidders and run auctions in the following stages. Therefore, the pressure of following-up auctions will drive up bid premiums of the negotiation stage. This argument is supported by the empirical evidence that, in general, there is no significant difference in bid premiums across negotiation and auction in M&As (Boone & Mulherin [2007, 2008]).[2]

Besides the sequential nature, a typical M&A process also involves a deadline for completion and a search cost for contacting a potential bidder.[3] As the search is costly, a full-scale simultaneous search is usually not optimal. Likewise, constrained by a deadline, a one-by-one sequential search may not be optimal either, as too few bidders might be contacted. Therefore, a seller may conduct a compound search and search both sequentially and simultaneously, that is, she may contact a batch of bidders in each period.

In fact, we may take M&A as an example of a more general problem, where a searcher conducts a compound search for potential agents by a deadline. For example:

- Academic recruitment in the UK is normally driven by the REF deadline, where academic departments interview job candidates batch by batch.
- People in dating markets are usually under age-related pressures and would attend speed dating events, where they can meet many daters in a single event.
- In job search and school applications, a job searcher or a student normally sends out his applications batch by deadlines.
- In IPO roadshows, the underwriting firm and the management team travel across different cities and make promotions to potential investors before going public.

We investigate this kind of problem from the perspective of seller compound search. A (female) seller wants to allocate an indivisible product among a large number of potential (male) bidders within T periods. Bidders are grouped into different bidder samples, over which the seller may or may

[2] Alternatively, Boone & Mulherin [2007] state that many M&A deals classified as negotiation are auction in fact. After reconstructing a new sample using their measure, they still have half of the deals classified as noncompetitive negotiations.

[3] The search cost can be an information cost of the target firm (seller) due to the loss of its proprietary information to potential acquirers (bidders) in the process of due diligence.

not have complete control. The seller searches across the bidder samples till the deadline. The search rule involves a search schedule, for example, which sample to contact in the next period, and a stopping rule, for example, when to stop searching.

We consider both cases of *short-lived* and *long-lived* bidders. A short-lived bidder only participates in the stage transaction when invited, yet a long-lived bidder, once invited, will stay in the transaction thereafter. The two cases of short and long-lived bidders are analogous to sequential search with *no* and *full* recall, respectively.

The optimal search outcomes can be implemented by a compound search auction defined by $\{r_t, M_t\}_{t=1}^{T}$, that is, a sequence of reserve prices and bidder samples. The auction proceeds as follows: in period t, the seller invites a bidder sample M_t to a second-price auction with a reserve price r_t; if any bidder submits an effective bid, then the product is allocated according to the auction rule; if not, the seller then moves on to period $t+1$ and invites a new bidder sample M_{t+1}, and runs an auction among all the participating bidders with a new reserve price r_{t+1}; the seller continues with this process till the end of period T.

Our analysis generates several interesting results. First, an optimal search auction is characterized by decreasing reserve prices and increasing search intensities (sample sizes) over time. Moreover, the monotonicity results are robust in both cases of *short-lived* and *long-lived* bidders. Intuitively, an optimal reserve price in period t reflects the continuation value of following an optimal search rule from that point on, which gets smaller when the deadline T approaches. The other result of increasing search intensities suggests that a seller will contact a sample of the smallest size in the first period, and invite increasingly more bidders in later periods. This result may help explain why the dominant selling process in many important markets, for example, M&As, can be noncompetitive negotiation.

Second, we compare the optimal search auctions between short and long-lived bidders. Given a sequence of bidder samples, we show that the optimal reserve prices for short-lived bidders are lower than those for long-lived bidders in each period. On the other hand, given a sequence of reserve prices, a seller with short-lived bidders will search more intensively. The intuition is that a seller with short-lived bidders will lose her current bidders if she continues to search, and therefore, she is willing to set lower reserve prices and would search more intensively in the next period *ceteris paribus*.

Third, we show that an *efficient* search auction is also featured by decreasing reserve prices and increasing search intensities. Compared with the optimal auction, the efficient search auction has both lower reserve prices and search intensities *ceteris paribus*. The result indicates that the inefficiency of an optimal search auction may stem from its inefficient search rule.

Finally, we explore the seller's optimal choice of the collection of bidder samples. For short-lived bidders, we can derive the optimal collection of bidder samples by the standard method of backward induction. For long-lived bidders, we specify some conditions that are necessarily satisfied in optimum. With the aid of the formula of optimal search profit, we can derive the optimal collection of bidder samples.

The remainder of this paper is organized as follows. Section II provides a review of the related literature. Section III is model setup. Section IV investigates the optimal compound search problem with short-lived bidders. Section V studies the same problem with long-lived bidders. Section VI further studies an efficient compound search problem with long-lived bidders. Section VII is a short conclusion. All proofs appear in Appendices.

II. RELATED LITERATURE

Our paper is mainly related to the following strands of literature: (1) sequential search and search mechanisms, (2) negotiation versus auction, (3) sequential auctions and revenue management, and (4) auctions with buy-price options.

First, Weitzman [1979] studies so-called Pandora's problem of sequential search with *full* recall. In his model, Pandora faces a number of closed boxes, each containing a random prize; she needs to open the boxes sequentially, each at a search cost; her objective is to maximize the expected value of the prize discovered, net of the total search costs. The optimal search rule is as follows: (i) for each box, one can derive a unique cutoff value at which Pandora is indifferent between keeping this value and inspecting that box; (ii) the selection rule suggests that she should open the boxes in the order of descending cutoff values; (iii) the stopping rule indicates that she should stop searching whenever the value discovered is greater than the highest cutoff value of the remaining unopened boxes.[4]

When the search is bounded by a deadline, a searcher may adopt a compound search rule and sample multiple boxes in a single period. Gal *et al.* [1981] study compound search in labor markets by introducing a finite deadline into the sequential search model of Lippman & McCall [1976]. When the search is with *no* recall, they show that a searcher's optimal search rule is featured by decreasing reservation wages and increasing search intensities over time.[5] Morgan [1983] further studies the case of search with *full* recall and shows that the sequence of optimal search intensities is, in

[4] Armstrong [2017] provides some additional results of Weitzman's model and reviews its relevant applications and progress in the context of consumer search.

[5] Benhabib & Bull [1983] also study search intensity in job markets, where job offers are homogeneous and the search is with no recall. They derive similar monotonicity results on the optimal search rule as Gal *et al.* [1981], but they further consider the on-the-job search problem.

general, a stochastic process. This is because a searcher chooses search intensity adaptively in each period, depending on the realizations of the previous search outcomes. For the problem of sequential search with a deadline, Lee & Li [2021] develop a simple yet unified framework that fully characterizes the optimal search rules and values in both cases of full and no recall.

In the above literature, the targets for search are nonstrategic, for example, boxes. The literature on search mechanisms studies the search for strategic agents. McAfee & McMillan [1988] study a procurement auction where a buyer searches homogenous long-lived suppliers sequentially.[6] Crémer *et al.* [2007] examine a search mechanism where a seller searches heterogenous long-lived bidders. Using a mechanism design approach, they prove the insightful result that the seller's search for bidders can be reformulated as Pandora's problem. They show that the optimal search outcomes can be implemented by a sequential auction, with declining reserve prices and bidders invited one-by-one sequentially.

Our paper studies a seller's compound search problem with both *short* and *long-lived* bidders. For long-lived bidders, our analysis resembles that of Crémer *et al.* [2007]. For example, if we consider a bidder sample as a single *aggregate* bidder, our compound search problem is analogous to a sequential search problem. However, our paper differs from Crémer *et al.* [2007] in several aspects. (1) By studying a general compound search rule, we are able to examine how optimal search intensities (sample sizes) change over time. Our result shows that a seller becomes less selective (by setting a lower reserve price) and more aggressive (by searching more intensively) when the deadline approaches. (2) We study both cases of short and long-lived bidders. The comparative results suggest that a seller with short-lived bidders searches more intensively and sets lower reserve prices than one with long-lived bidders. (3) By comparing optimal and efficient search auctions, we show that the inefficiency of an optimal search auction can stem from its inefficient compound search rule.

Second, our paper is related to the persistent debates on the choice between negotiation and auction as an optimal selling mechanism. Bulow & Klemperer [1996] show that the value of competition by inviting one more bidder dominates the value of bargaining power. In a later paper, Bulow & Klemperer [2009] argue that a simultaneous auction can yield higher expected revenue than a sequential negotiation. In their model, bidders need to pay positive entry costs, and a seller is unable to commit to a take-it-or-leave-it offer. In a sequential negotiation, an already entered bidder can make a jump-bid

[6] In their model of procurement, a buyer seeks to buy an indivisible product from one of a set of producers, who are *ex-ante* homogeneous. The buyer searches the producers sequentially, each at a constant cost. They show that the optimal search mechanism is a combination of *constant* reservation-price search and auction.

to deter further entries of outside bidders, which may harm the seller. Therefore, a seller usually prefers a simultaneous auction over a sequential negotiation.[7]

But the empirical evidence does not support their results in general. For example, in M&As, the dominant selling process is one-on-one negotiation, not a competitive auction. Our paper proposes a possible explanation for this puzzle by modeling the selling process as a seller's compound search for bidders by a finite deadline.

Third, our paper is related to the growing literature on sequential auctions and revenue management. Said [2011] studies sequential auctions of multi-unit products with changing populations, yet in a different environment to our model. Liu *et al.* [2019] study sequential auctions in the case of limited commitment. Other recent literature on revenue management includes Board & Skrzypacz [2016] with forward-looking buyers in the case of full commitment, and Dilme & Li [2019], who study revenue management with the arrivals of strategic buyers in the case of no commitment. This paper studies a compound search auction with a changing population and full commitment.

Finally, our paper is also related to the literature on buy-price auctions. Reynolds & Wooders [2009] study a static buy-price auction with risk-averse bidders, and derive bidders' cutoff strategies similar to our paper. Zhang [2021] studies the optimal sequence of posted-price and auction in a sequential mechanism, where a population of short-lived bidders enters the market periodically. In each period, the seller chooses between a posted-price and an auction mechanism. He shows that, when there is a deadline and the auction cost is moderate, the optimal mechanism sequence takes the form of posted-prices and then auctions.

III. THE MODEL

III(i). *Model Setup*

A (female) seller wants to allocate an indivisible product among a set $N = \{1, 2, \ldots, n\}$ of (male) bidders. She needs to complete the transaction within $T \leq n$ periods, and her value of the product is normalized to 0. Bidders' values of the product, V_i's, are independent draws from the distribution F on $[0, 1]$ with a density $f > 0$. F is common knowledge, yet the realization of V_i is bidder i's private information. We assume F is of

[7] Lu *et al.* [2019] study how to orchestrate costly information acquisition in an auction with a preshort-listing stage. Bidders are initially endowed with private signals that are positively correlated to their true values, and a bidder can learn his true value by paying an entry cost. They show that, under a sequential short-listing rule, the seller admits the most efficient remaining bidder in each round, provided that his conditional expected contribution to the virtual surplus is positive.

increasing failure rate (IFR) and hence the virtual value, $\psi(v) = v - \frac{1-F(v)}{f(v)}$, is strictly increasing. Both the seller and the bidders are risk-neutral, and we abstract from time discounting.[8]

A bidder can not submit a bid if not invited. To invite a bidder, the seller needs to incur a search cost $c \geq 0$, which is small enough such that a bidder is valuable, that is,

$$\int_{r^*}^{1} \psi(v) dF(v) > c, \tag{1}$$

where $\psi(r^*) = 0$. Note that the LHS of (1) is the maximum expected revenue the seller can obtain from a truthful bidder, and the RHS is just the search cost.

We consider the following compound search procedure for the seller. First, in period $t = 0$, the seller selects a collection $\mathcal{M} = \{M^1, M^2, \dots, M^T\}$ of bidder samples, such that $M^j \cap M^{j'} = \emptyset$ for $j \neq j'$ and $\bigcup_{j=1}^{T} M^j \subseteq N$. Second, in periods $t = 1, \dots, T$, the seller searches across the bidder samples in the form of a sequential auction. The auction is defined by $(\mathbf{r}, \mathbf{M})$, where $\mathbf{r} = (r_1, r_2, \dots, r_T)$ is a sequence of reserve prices and $\mathbf{M} = (M_1, M_2, \dots, M_T)$ is a sequence of bidder samples. Specifically, $\mathbf{M}$ specifies a *search schedule* which is a permutation of the collection $\mathcal{M}$. For example, if $M_t = M^j$, then the seller will search the bidder sample M^j in period t, if she intends to. Let $\mathcal{N}_t^c \equiv \mathcal{M} \setminus \{M_1, M_2, \dots, M_t\}$ denote the set of bidder samples in $\mathcal{M}$ that the seller has not searched till the end of period t. For convenience, denote $M_0 \equiv \emptyset$ and $\mathcal{N}_0^c \equiv \mathcal{M}$.

We further denote $m_t = |M_t|$ as the cardinality of M_t and $c_{M_t} \equiv m_t c$ the gross search cost of inviting the M_t bidders. Without causing confusion, we sometimes use the sequence of sample sizes, $\mathbf{M} = (m_1, m_2, \dots, m_T)$, to denote a search schedule. Intuitively, the level of *search intensity* in period t is measured by m_t. We say a seller searches more intensively in period t' than in period t, if and only if $m_{t'} \geq m_t$.

We consider both cases of *short-lived* and *long-lived* bidders. Denote B_t the set of bidders who can bid in the stage auction of period t. It then follows that

$$B_t = \begin{cases} M_t & \text{for short-lived bidders,} \\ N_t \equiv \bigcup_{\tau=0}^{t} M_\tau & \text{for long-lived bidders.} \end{cases}$$

We also denote $n_t = |N_t|$ as the cardinality of N_t. Given a collection $\mathcal{M}$, the rule of the search auction $(\mathbf{r}, \mathbf{M})$ is as follows

[8] The introduction of time discounting does not change the results qualitatively.

- At the beginning of period $t = 1$, the seller invites a sample $M_1 \in \mathcal{N}_0^c$ of bidders to the auction at the search cost c_{M_1}, and runs a second-price auction among the set B_1 of bidders, with a reserve price r_1;
- If an effective bid, for example, a bid higher than r_1, is submitted by any bidder $i \in B_1$ in period $t = 1$, then the transaction ends, and the payment and allocation are implemented according to the auction rule. Otherwise, the seller continues searching in period $t = 2$, by inviting a sample $M_2 \in \mathcal{N}_1^c$ of bidders at the search cost c_{M_2}. She then runs a second-price auction among the set B_2 of bidders, with a new reserve price r_2;
- The seller continues with this process, until the end of period T.

Here are some comments. First, we assume the seller announces the auction rule $(\mathbf{r}, \mathbf{M})$ in period 0, and then commits to it. Second, we consider the sealed-bid second price auction, where bidding true value is a weakly dominant strategy given the rule of the search auction.[9] Third, the individual rationality constraint is obviously satisfied.

We solve the optimal search problem in two steps. First, for any given collection $\mathscr{M}$ of bidder samples, we solve for the optimal search auction $(\mathbf{r}^*, \mathbf{M}^*)$ that maximizes the expected auction profit, that is, the expected auction revenue net of the total search costs. Second, we show how to derive the optimal collection $\mathscr{M}^*$ with some examples.

III(ii). *Compound Search Problem: A Reformulation*

The well-known result of Myerson [1981] shows that, in an *incentive feasible* mechanism, the maximum revenue a seller can obtain from a bidder sample M is equal to

$$\psi\left(V_M^{(1)}\right) = \psi\left(\max_{i \in M}\left\{V_i\right\}\right),$$

where $V_M^{(1)}$ denotes the first order statistics of the M bidders' values, which has a distribution $F_M^{(1)}(v) \equiv F^m(x)$. As in standard search models, we may think of a bidder sample M as a single "box" that contains a random value $\psi\left(V_M^{(1)}\right)$ and has a gross search cost c_M. Therefore, in an incentive feasible search auction, we can formulate the seller's compound search of T bidder samples in T periods as a sequential search of T "boxes" in T periods. Crémer *et al.* [2007, Theorem 1] has shown this result in the case of long-lived bidders, and reformulate their problem of sequential search without a deadline as Pandora's problem *à la* Weitzman [1979]. For short-lived bidders, the

[9] Note that an effective bid will terminate the transaction, and a participating bidder has just one chance to submit an effective bid. In the format of a second price auction, a bidder is indifferent between winning and losing conditional on just winning, and therefore, bidding true value is a weakly dominant strategy.

reformulation is even more straightforward as the incentive compatible conditions are simpler.

For a given collection $\mathscr{M}$ of bidder samples, a seller's search rule involves both a search schedule **M** and a stopping rule, for example, when to stop searching. We can formulate the search problem as a dynamic programming (DP) problem. Specifically, at the end of period t, suppose the seller has a fall-back revenue $\psi(v)$, and she then faces two options. If she stops, she can claim the fall-back revenue $\psi(v)$ and the product is allocated. If she continues searching, she needs to decide which bidder sample to invite in the next period.

For long-lived bidders, let $J_t(v)$ denote the value of having a fall-back revenue $\psi(v)$ at the end of period t. For $t \leq T$, the Bellman equation for this search problem is

$$(2) \qquad J_t(v) = \max_{M \in \mathcal{N}_t^c} \left\{ \psi(v), -c_M + \mathbb{E}J_{t+1} \left[\max \left\{ v, X_M^{(1)} \right\} \right] \right\},$$

where $\mathbb{E}$ is the expectation operator. The first term $\psi(v) \geq 0$ is the fall-back revenue the seller can claim by stopping searching,[10] and the other term is the maximum expected profit by continuing searching bidder sample M in the next period $t+1$. It is clear that $J_{T+1}(v) = 0$. Note that, at the end of period $t+1$, the new state variable is the maximum of v and $X_M^{(1)}$, as bidders are long-lived and the fall-back revenue $\psi(v)$ is still reclaimable in the next period. The case of long-lived bidders is analogous to sequential search with *full* recall.

For short-lived bidders, similarly, denote $\hat{J}_t(v)$ the value of having a fall-back revenue $\psi(v)$ at the end of period t. For $t \leq T$, the Bellman equation for this search problem is

$$(3) \qquad \hat{J}_t(v) = \max_{M \in \mathcal{N}_t^c} \left\{ \psi(v), -c_M + \mathbb{E}\hat{J}_{t+1} \left[\max \left\{ r^*, X_M^{(1)} \right\} \right] \right\}.$$

We also have $\hat{J}_{T+1}(v) = 0$. Note that, in the case of short-lived bidders, the seller cannot reclaim the current revenue $\psi(v)$ in later periods. Therefore, when the seller declines $\psi(v)$ and continues searching, her fall-back revenue turns to $0 = \psi(r^*)$. The new state variable at the end of period $t+1$ is then given by the maximum of r^* and $X_M^{(1)}$. The case of short-lived bidders is analogous to sequential search with *no* recall.

[10] The assumption of $\psi(v) \geq 0$ generally holds, as the seller can always stop searching and realize a zero revenue. In another word, without loss of generality, we can assume the fall-back value $v \geq r^*$.

IV. OPTIMAL SEARCH AUCTION: SHORT-LIVED BIDDERS

We start with the case of short-lived bidders, where Bellman equation (3) formulates the seller's search problem. Using backward induction, we can solve the optimal collection $\hat{\mathscr{M}}^*$ of bidder samples and the optimal search rule in one go. We denote $(\hat{m}_1^*, \dots, \hat{m}_T^*)$ the optimal search schedule, and $(\hat{\xi}_1^*, \dots, \hat{\xi}_T^*)$ the sequence of optimal cutoff values in this case. For example, at the end of period t, the seller is indifferent between stopping with a current revenue $\psi(\hat{\xi}_t^*)$ and continuing searching in the next period $t+1$, as shown in (3). Proposition 1 below fully solves the optimal compound search rule with short-lived bidders, which is featured by decreasing cutoff values and increasing search intensities.

Proposition 1 (optimal search with short-lived bidders). For compound search with short-lived bidders, the optimal cutoff value $\hat{\xi}_t^*$ and sample size $\hat{m}_t^*$ are given by

$$\text{(4)} \quad \psi\left(\hat{\xi}_t^*\right) = \max_{m_{t+1}} \left\{ \int_{\hat{\xi}_{t+1}^*}^1 \left[\psi(x) - \psi\left(\hat{\xi}_{t+1}^*\right)\right] dF^{m_{t+1}}(x) - cm_{t+1} \right\} + \psi\left(\hat{\xi}_{t+1}^*\right).$$

for any $t < T$ and $\psi\left(\hat{\xi}_T^*\right) = 0$. Moreover, the sequence of optimal cutoffs $\hat{\boldsymbol{\xi}}^* = \left(\hat{\xi}_1^*, \dots, \hat{\xi}_T^*\right)$ is decreasing and that of optimal sample sizes $\hat{\mathbf{M}}^* = \left(\hat{m}_1^*, \dots, \hat{m}_T^*\right)$ is increasing over time. That is, for all $t < T$, we have

$$\hat{\xi}_t^* > \hat{\xi}_{t+1}^*, \qquad \hat{m}_t^* \le \hat{m}_{t+1}^*.$$

Proof. Please refer to Appendix A for the omitted proofs. ∎

A simple transformation of (4) gives the following equivalent expression

$$\text{(5)} \quad \psi\left(\hat{\xi}_t^*\right) = \max_{m_{t+1}} \left\{ \int_0^1 \max\left\{ \psi\left(\hat{\xi}_{t+1}^*\right), \psi(x) \right\} dF^{m_{t+1}}(x) - cm_{t+1} \right\},$$

where the LHS is the seller's payoff by stopping and keeping the current revenue $\psi\left(\hat{\xi}_t^*\right)$, and the RHS is her maximum expected payoff by continuing searching in period $t+1$. The optimal cutoff values of $\hat{\xi}_t^*$ and $\hat{m}_t^*$ can be recursively derived from the last period T.

We can interpret the revenue $\psi\left(\hat{\xi}_t^*\right)$ as a reservation revenue for the seller, which is achievable by following an optimal search rule from period $t+1$ on. The reservation revenue is decreasing over time, as there is fewer trial opportunities for the seller to improve her payoff. On the other hand, when the deadline T approaches, as the seller has fewer opportunities to search, she would have stronger incentives to search more intensively.

The monotone properties of the optimal search rule match many real-world observations. For example, in M&As, it suggests that a seller will contact the smallest number of bidders and has the highest reservation revenue in the first stage. It may explain why negotiation can be the dominant selling-mechanism in M&As. Another example could be academic recruitment in the UK under the REF pressure. With the approaching of the REF deadline, UK universities would generally increase their recruitment intensities, in order to build up strong profiles of research outputs for REF submissions.

The optimal search outcomes in Proposition 1 can be implemented by the proposed search auction. The incentive problem for short-lived bidders is simple as a bidder will bid whenever his value is greater than the reserve price of the stage auction. Therefore, by setting a sequence of reserve prices $\hat{\mathbf{r}}^* = \hat{\boldsymbol{\xi}}^*$, the search auction $\left(\hat{\mathbf{r}}^*, \hat{\mathbf{M}}^*\right)$ implements the optimal outcomes of the compound search mechanism with short-lived bidders.

The proof of Proposition 1 provides an algorithm for deriving the optimal collection $\hat{\mathscr{M}}^*$ and the optimal sequence $\hat{\boldsymbol{\xi}}^*$ of cutoff values. To be clear, in the last period T, the optimal cutoff $\hat{\xi}^*_T = r^*$ and we can solve for the optimal sample size $\hat{m}^*_T$ that maximizes the expected auction profit in period T. With the solution of $\left(\hat{\xi}^*_T, \hat{m}^*_T\right)$, we can calculate the value of $\psi\left(\hat{\xi}^*_{T-1}\right)$ using (4), with which we then have the optimal cutoff value $\hat{\xi}^*_{T-1}$. From the RHS of (4), we then can solve for the optimal search intensity $\hat{m}^*_{T-1}$. Continuing with the process, we then fully solve the optimal compound search problem with short-lived bidders.

We next propose a simple example that may illustrate how to derive the optimal collection $\hat{\mathscr{M}}^*$ and cutoff values $\hat{\boldsymbol{\xi}}^*$ using the above algorithm.

Example 1 (2-period with 3 short-lived bidders). Let $F(x) = x$ and $c > c \approx 0.047$.[11] We denote the search schedule by $\hat{\mathbf{M}} = \left(\hat{m}_1, \hat{m}_2\right)$, where $\hat{m}_t$ is the bidder sample size in period t. We consider two candidates for optimal search schedule, $\hat{\mathbf{M}} = (1, 1)$ and $(1, 2)$.

We define ω_t, $t = 1, 2$, as the continuation value of following an optimal search procedure after the end of period t, and the optimal cutoff value $\hat{\xi}^*_t$ satisfies $\psi\left(\hat{\xi}^*_t\right) = \omega_t$, as implied by the Bellman equation of (3). It is clear that $\omega_2 = 0$ and $\hat{\xi}^*_2 = r^*$; and

$$\omega_1 = \max_{\hat{m}_2} \left\{ \int_{r^*}^{1} \psi(x) dF^{\hat{m}_2}(x) - \hat{m}_2 c \right\}, \tag{6}$$

[11] When $c < c$, it is optimal to invite all the three bidders in the first period, as $\hat{m}^*_2 = 3$ maximizes (6). Note that inviting all bidders in one period is never optimal with long-lived bidders.

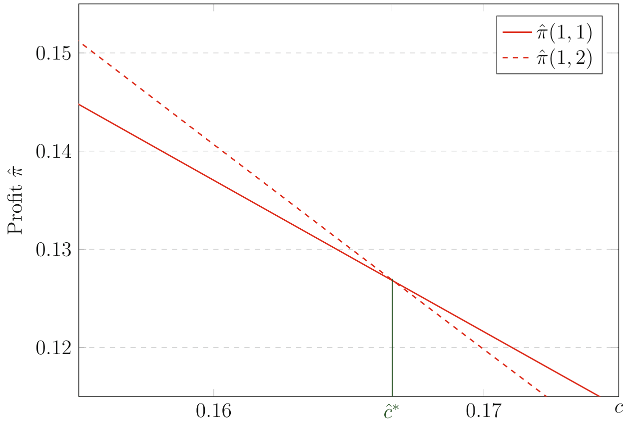

Figure 1
Optimal Search for Short-lived Bidders

Notes: See Example 1. When bidders are short-lived and $c \leq \hat{c}^* \approx 0.167$, the optimal search schedule is $\hat{\mathbf{M}}^* = (1, 2)$, as shown by the dashed red curve. If $c > \hat{c}^*$, however, the optimal search schedule turns to be $\hat{\mathbf{M}}^* = (1, 1)$, as shown by the solid red curve [Colour figure can be viewed at wileyonlinelibrary.com]

where the virtual value function $\psi(x) = 2x - 1$. Following an optimal search procedure of $\left\{ \left(\hat{\xi}_1^*, r^*\right), \left(\hat{m}_1^*, \hat{m}_2^*\right) \right\}$, the expected profit is

$$\hat{\pi}^* = \left[\int_{\hat{\xi}_1^*}^1 \psi(x) dF^{\hat{m}_1^*}(x) - \hat{m}_1^* c \right] + F^{\hat{m}_1^*}\left(\hat{\xi}_1^*\right) \left[\int_{r^*}^1 \psi(x) dF^{\hat{m}_2^*}(x) - \hat{m}_2^* c \right].$$

There are two cases, depending on the value of c.

- $c < c \leq \hat{c}^* = 1/6$, we have $\hat{m}_2^* = 2$ maximizes (6) with $\omega_1 = \frac{5}{12} - 2c$ and $\hat{\xi}_1^* = \frac{17}{24} - c$. The expected profit is $\hat{\pi}^* = c^2 - \frac{29c}{12} + \frac{289}{576}$.
- $\hat{c}^* < c < 1/4$, we have $\hat{m}_2^* = 1$ maximizes (6) with $\omega_1 = \frac{1}{4} - c$ and $\hat{\xi}_1^* = \frac{5}{8} - \frac{1}{2}c$. The expected profit is $\hat{\pi}^* = \frac{1}{64}(16c^2 - 104c + 25)$.

The results are illustrated by the two red curves in Figure 1, which shows how optimal $\left(\hat{m}_1^*, \hat{m}_2^*\right)$ changes with the search cost c. For example, when $c \geq \hat{c}^*$, the optimal sample collection $\hat{\mathcal{M}}^* = (1, 1)$. When $c < \hat{c}^*$, the optimal sample collection $\hat{\mathcal{M}}^* = (1, 2)$.

We further provide two results for comparison purpose in Example 2.

- $c = \frac{1}{16}$: it follows $\hat{m}_2^* = 2$, $\omega_1 = \frac{7}{24}$ and $\hat{\xi}_1^* = \frac{31}{48} \approx 0.646$;
- $c = \frac{5}{24}$: it follows $\hat{m}_2^* = 1$, $\omega_1 = \frac{1}{24}$, and $\hat{\xi}_1^* = \frac{25}{48} \approx 0.521$.

V. OPTIMAL SEARCH AUCTION: LONG-LIVED BIDDERS

For long-lived bidders, similar to the cutoff value of a box in Pandora's problem, we can define a unique cutoff value for any bidder sample $M \in \mathscr{M}$ with a sample size m, denoted by $\xi^*(m)$. The Bellman equation (2) then implies $\xi^*(m)$ satisfies

$$\psi(\xi^*(m)) = \int_0^1 \max\left\{\psi(\xi^*(m)), \psi(x)\right\} dF^m(x) - mc. \tag{7}$$

The LHS of (7) is the fall-back revenue $\psi(\xi^*(m))$, and the RHS is the net expected revenue if the seller searches the sample M in the next period and then stops.[12] For any given collection $\mathscr{M}$, the Pandora's rule then implies the following optimal search rule:

- Search order: At the end of period $t < T$, if a bidder sample is to be searched in period $t+1$, it must be the sample with the highest cutoff value in $\mathscr{N}_t^c$. The optimal search schedule, denoted by $\mathbf{M}^* = \left(m_1^*, \ldots, m_T^*\right)$, is such that

$$\xi^*\left(m_1^*\right) \geq \xi^*\left(m_2^*\right) \geq \cdots \geq \xi^*(m_T^*).$$

- Stopping rule: At the end of any period $t < T$, if the fall-back value v is greater than all the cutoff values of the remaining bidder samples, then stops searching; otherwise, continues searching in period $t+1$.

Next, we first characterize the optimal search rule in Sections V(i)–(iii) when a collection $\mathscr{M}$ is given. Second, we show how to derive the optimal collection $\mathscr{M}^*$ in Section V(iv), with an illustrative example. Finally, we provide some comparative results on the optimal search rules with short and long-lived bidders in Section V(v).

V(i). *Incentive Compatibility*

We first investigate the incentive compatible conditions for bidders, and will study equilibria in the form of cutoff strategies. A cutoff strategy is characterized by a vector of cutoff values, $\boldsymbol{\xi} = (\xi_t)_{1\leq t\leq T}$, such that a bidder will bid his true value v in period t if $v \geq \xi_t$, and wait otherwise. Given that bidders are *ex-ante* homogeneous, we will focus on symmetric equilibria where all the bidders adopt the same cutoff strategy in equilibrium.

Lemma 1 below characterizes a bidder's equilibrium cutoff strategy with decreasing cutoff values. Its proof also shows that there exists a one-to-one mapping between the sequence of reserve prices $\mathbf{r}$ and that of the cutoff

[12] Note that the cutoff value $\xi^*(m)$ in (7) is solely determined by the bidder sample M, which the seller will search in the right next period. It is known as the *one-step-ahead* property in sequential search problems with full recall.

values $\boldsymbol{\xi}$. Therefore, the search auction $(\mathbf{r}, \mathbf{M})$ can be equivalently represented by $(\boldsymbol{\xi}, \mathbf{M})$, which is used in our following discussions.

Lemma 1 (Bidders' equilibrium cutoff strategy). Given a compound search auction $(\boldsymbol{\xi}, \mathbf{M})$ with decreasing cutoff values, ξ_t is uniquely determined by

$$(8)\quad F^{(1)}_{N_t \backslash \{i\}}\left(\xi_t\right)\left(\xi_t - r_t\right) = F^{(1)}_{N_{t+1} \backslash \{i\}}\left(\xi_{t+1}\right)\left(\xi_{t+1} - r_{t+1}\right) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1} \backslash \{i\}}(x)dx,$$

for $t < T$, and $\xi_T = r_T$. Moreover, the reserve prices $(r_t)_{1 \le t \le T}$ is also decreasing in t.

Proof. The proof appears in Appendix B. ∎

It is intuitive that $\xi_T = r_T$, as it is the last period to submit a bid, and a bidder will bid whenever his value is greater than the reserve price. For $t < T$, condition (8) implies that $\xi_t \ge r_t$, and a bidder with a value higher than the reserve price r_t may wait. The intuition is that, with decreasing reserve prices, a bidder faces the trade-off between bidding now with a higher reserve price yet less competition, or bidding later with a lower reserve price yet more competition, as more bidders will enter the auction in the next period.

V(ii). *Optimal Cutoff Values*

Given a search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected auction profit is

$$(9)\qquad \pi\left(\boldsymbol{\xi}, \mathbf{M}\right) = \sum_{t=1}^{T} F^{(1)}_{N_{t-1}}\left(\xi_{t-1}\right)\left[R_t(N_t) - c_{M_t}\right],$$

where $F^{(1)}_{N_0}(\xi_0) \equiv 1$ and $R_t(N_t)$ is the expected revenue of the stage auction in period t, conditional on it happens. Note that, in the stage auction of period t, there are M_t strong bidders and N_{t-1} weak bidders. For example, the values of the M_t new bidders are independent draws from F, while those of the N_{t-1} weak bidders are truncated above from ξ_{t-1}. Substituting the bidders' equilibrium cutoff strategies (8) into (9), we then get the following expression of the expected auction profit.

Lemma 2 (expected auction profit). Given a compound search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected auction profit is

(10)
$$\pi(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} \int_{\xi_t}^{\xi_{t-1}} \psi(x) dF^{(1)}_{N_t}(x) + \sum_{t=1}^{T} F^{(1)}_{N_{t-1}}\left(\xi_{t-1}\right)\left[\int_{\xi_{t-1}}^{1} \psi(x) dF^{(1)}_{M_t}(x) - c_{M_t}\right],$$

where $\xi_0 \equiv 1$.

Different from the standard results of static auctions with symmetric bidders, (10) provides a nice formula of the expected revenue of a sequential auction net of search costs with asymmetric bidders. For example, the stage auction in period t happens with probability $F^{(1)}_{N_{t-1}}(\xi_{t-1})$, where there are M_t strong bidders and N_{t-1} weak bidders. When the highest value of the M_t strong bidders is below ξ_{t-1}, all the N_t bidders are competing with each other for the product. This corresponds to the first term of $\int_{\xi_t}^{\xi_{t-1}} \psi(x) dF^{(1)}_{N_t}(x)$ in (10). On the other hand, when the highest value of the M_t bidders is greater than ξ_{t-1}, then the N_{t-1} weak bidders are strictly dominated, and only the M_t bidders are competing for the product. This corresponds to the second term of $\int_{\xi_{t-1}}^{1} \psi(x) dF^{(1)}_{M_t}(x)$ in (10).

Proposition 2 (Optimal Cutoffs). Given a compound search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected auction profit $\pi(\boldsymbol{\xi}, \mathbf{M})$ is quasi-concave in ξ_t. Moreover, the sequence of optimal cutoff values, $(\xi_t^*)_{1\le t\le T}$, satisfies

$$c_{M_{t+1}} = \int_{\xi_t^*}^{1} \left[\psi(x) - \psi(\xi_t^*)\right] dF^{(1)}_{M_{t+1}}(x), \quad \text{for } 1 \le t < T, \tag{11}$$

and $\psi(\xi_T^*) = 0$ for $t = T$.

The optimal cutoff value in (11) also specifies the condition for optimal stopping, which is the same as (7). For instance, given the current fall-back revenue $\psi(\xi_t^*)$, the RHS of (11) is the increment in expected auction revenue if the seller continues searching the M_{t+1} bidders in period $t+1$ and then stops, while the LHS is the gross search cost $c_{M_{t+1}}$. If the RHS is smaller than $c_{M_{t+1}}$, then the seller will stop searching at the end of period t.[13]

V(iii). *Optimal Search Intensity*

We will next show that the optimal search intensities (sample sizes) are increasing over time. Therefore, these monotonicity properties of optimal cutoffs and optimal search intensities are robust across both cases of short-lived and long-lived bidders.

First, define $\xi^*(m)$ as the optimal cutoff value for searching a bidder sample M of sample size m. From condition (11), $\xi^*(m)$ is the implicit function defined by

$$mc = \int_{\xi^*(m)}^{1} \left[1 - F^m(x)\right] d\psi(x). \tag{12}$$

[13] The solution of ξ_t^* is unique, as the RHS of (11) is strictly decreasing in ξ_t^*. Moreover, ξ_t^* reveals a one-step-ahead property, in the sense that it just depends on the bidder sample M_{t+1} only. This property is based on the fact that the optimal cutoff values are decreasing, while the seller's fall-back values with long-lived bidders are necessarily increasing over time.

The following result shows $\xi^*(m)$ is decreasing in m.

Lemma 3. The optimal cutoff value $\xi^*(m)$ is strictly decreasing in m. That is, for any two sets of bidders, $M, M' \subseteq N$, if $m < m'$, then

$$\xi^*(m) > \xi^*(m').$$

The result is more striking than it first looks. Lemma 3 shows that, for homogeneous bidders, the benefit of increasing competition by inviting one more bidder is strictly dominated by the cost of doing that. In other words, removing one bidder from the sample will strictly increase the optimal cutoff value of that sample. Therefore, $\xi^*(m)$ achieves its maximum when the seller just samples 1 bidder. It also implies that, when a seller is not constrained by a finite deadline, it is optimal to search the bidders one-by-one sequentially.

We next show that the sequence of optimal search intensities is increasing over time. This result is a direct implication of Lemma 3 and the fact that, under an optimal search rule, the optimal cutoff value ξ_t^* is decreasing in t.

Proposition 3 (optimal search intensity). The sequence of optimal search intensity (sample size), denoted by $\mathbf{M}^* = \left(m_1^*, m_2^*, \ldots, m_T^*\right)$, is increasing over time, that is, for $1 \leq t < T$,

$$m_t^* \leq m_{t+1}^*.$$

Proposition 3 also confirms the result of increasing search intensity in the case of long-lived bidders. In other words, in a compound search, a seller will search increasingly more intensively from one period to the next. Again, the result may help explain why, in many important markets, the dominant selling process can be noncompetitive negotiation, where a seller just contacts one potential bidder in the first stage.

Finally, let us show how the proposed search auction $(\boldsymbol{\xi}^*, \mathbf{M}^*)$ can implement the optimal search outcomes. By comparing (11) with (12), it is clear that $\xi_t^* = \xi^*\left(m_{t+1}\right)$ for $1 \leq t < T$, and the result of (11) coincides with the optimal cutoff condition of (7). It then implies that the proposed search auction can implement the optimal search outcomes. To be specific, for any given collection $\mathscr{M}$ of bidder samples, the optimal search schedule $\mathbf{M}^* = \left(m_t^*\right)_{1 \leq t \leq T}$ is such that $m_t^* \leq m_{t+1}^*$ for $1 \leq t < T$. The sequence of optimal cutoff values, denoted by $\boldsymbol{\xi}^* = \left(\xi_t^*\right)_{1 \leq t \leq T}$, is determined as follows: for $1 \leq t < T$, $\xi_t^* = \xi^*\left(m_{t+1}^*\right)$, and $\xi_T^* = r^*$.

V(iv). *Optimal Sample Collection $\mathscr{M}^*$*

For any given collection $\mathscr{M}$ of bidder samples, let $\pi^*(\mathscr{M})$ denote the optimal expected auction profit by following an optimal search rule. By substituting

(11) into the expected auction profit of (10), we have the following formula of optimal search profit.

Lemma 4 (optimal auction profit). Given a collection $\mathscr{M}$ of bidder samples, a seller's optimal expected auction profit by following an optimal search rule is

$$\pi^*(\mathscr{M}) = \sum_{t=1}^{T} \int_{\xi^*\left(m_{t+1}^*\right)}^{\xi^*(m_t^*)} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x), \tag{13}$$

where $\xi^*(m_{T+1}) \equiv r^*$.

The formula (13) suggests a simple way to find the optimal collection $\mathscr{M}^*$ that maximizes $\pi^*(\mathscr{M})$. First, the requirement of increasing sample sizes implies that we can focus on monotone search schedules without loss of generality, that is, $m_t \leq m_{t+1}$. Second, as $\xi^*(m_T) \geq \xi^*\left(m_{T+1}\right) = r^*$ and $\xi^*(m)$ is strictly decreasing in m, there then exists an upper bound for m_T, denoted by $\overline{m}$. The maximum number of bidders in an $\mathscr{M}^*$ is then $\min\{n, \overline{m}T\}$. Using formula (13), we can find the optimal collection.

We next provide a simple example of 2-period with 3 long-lived bidders.

Example 2 (2-period with 3 long-lived bidders). Let $F(x) = x$ and $c \in (0, 1/4)$. The virtual value function is $\psi(x) = 2x - 1$, and the search schedule is $\mathbf{M} = (m_1, m_2)$. Proposition 5 suggests the optimal search schedule is either $\mathbf{M} = (1, 1)$ or $(1, 2)$, depending on the value of c. From formula (12), the optimal cutoff values for different sample sizes are given by:

- If $m = 1$, then $\xi^*(1)$ is the solution to $c = (1 - \xi)^2$, that is, $\xi^*(1) = 1 - \sqrt{c}$;
- If $m = 2$, then $\xi^*(2)$ is the solution to $2c = \frac{2}{3}(1 - \xi)^2(2 + \xi)$.

Case 1: If $c = \frac{1}{16}$, then $\xi^*(1) = \frac{3}{4}$ and $\xi^*(2) \approx 0.738$. For a search schedule of $\mathbf{M} = (1, 1)$, the expected profit, from (13), is

$$\pi(1,1) = \int_{\xi^*(1)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(1)} \left[1 - F^2(x)\right] d\psi(x) = \frac{29}{96} \approx 0.302.$$

The expected profit for a search schedule of $\mathbf{M} = (1, 2)$ is

$$\pi(1,2) = \int_{\xi^*(2)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(2)} \left[1 - F^3(x)\right] d\psi(x) \approx 0.365.$$

Therefore, the optimal collection of bidder samples is $\mathscr{M}^* = (1, 2)$.

Case 2: If $c = \frac{5}{24}$, then $\xi^*(1) \approx 0.544$ and $\xi^*(2) = 0.5$. For a search schedule of $(1, 1)$, the expected profit, again from (13), is

$$\pi(1,1) = \int_{\xi^*(1)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(1)} \left[1 - F^2(x)\right] d\psi(x) \approx 0.0634.$$

The expected profit for a search schedule of $(1, 2)$ is

$$\pi(1,2) = \int_{\xi^*(2)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(2)} \left[1 - F^3(x)\right] d\psi(x) \approx 0.0417.$$

Therefore, the optimal collection of bidder samples is $\mathscr{M}^* = (1, 1)$.

Figure 2 illustrates how the optimal sample collection changes with c. When $c \leq c^* \approx 0.164$, the optimal sample collection is $(1, 2)$, for example, the dashed blue curve, and when $c \geq c^*$, the optimal search schedule is $(1, 1)$, for example, the solid blue curve.

Figure 2 also illustrates the comparative results on optimal search with long-lived and short-lived bidders, as later to be shown in Proposition 4 and 5 in Section V(v). First, it shows that when $c^* < c < \hat{c}^*$, a seller with short-lived bidders searches more intensively than one with long-lived bidders in period $t = 2$. Second, it also shows that when the search schedule M is given, the cutoff value for short-lived bidders is smaller than that for long-lived bidders. For instance, when $c = \frac{1}{16}$, the optimal search schedules in both cases are the same, for example, $(1, 2)$. We have, from the results in Example 1,

$$\hat{\xi}_1^* = \frac{31}{48} \approx 0.646 < \xi_1^* \approx 0.738.$$

When $c = \frac{5}{24}$, the optimal schedule in both cases is $(1, 1)$, and we have

$$\hat{\xi}_1^* = \frac{25}{48} \approx 0.521 < \xi_1^* \approx 0.544.$$

V(v). *Short-lived versus Long-lived Bidders*

Comparing the Bellman equations of (2) and (3), we can show that, for a given search schedule **M**, the optimal cutoff value for short-lived bidders is smaller than that for long-lived bidders in each period. This is because a seller with short-lived bidders can not reclaim any offer declined before, and therefore, she is willing to accept a lower reserve prize.

Proposition 4 (Short-Lived versus Long-Lived Bidders: Cutoff Value). For a given search schedule **M**, the optimal cutoff value for short-lived bidders is

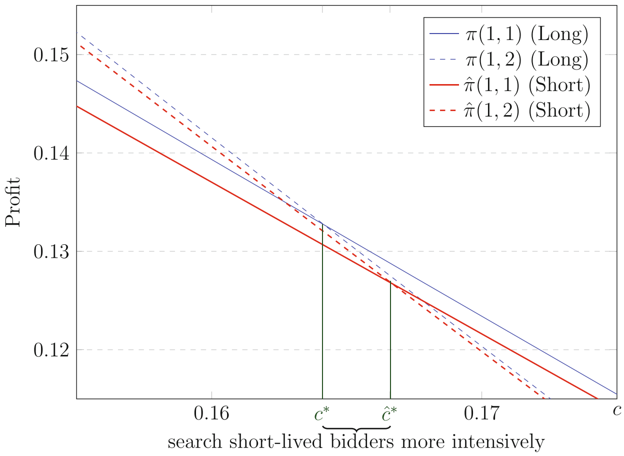

Figure 2

Long-Lived versus Short-Lived Bidders

Notes: See Example 2, where bidders are long-lived (as shown by the two blue curves). If c is in between $c^* \approx 0.164$ and $\hat{c}^* \approx 0.167$, the seller with short-lived bidders will invite more bidders in the second period than one with long-lived bidders [Colour figure can be viewed at wileyonlinelibrary.com]

smaller than that for long-lived bidders in each period. That is, for $0 \leq t < T-1$,

$$\hat{\xi}_t^* < \xi_t^*,$$

and $\hat{\xi}_T^* = \xi_T^* = r^*$ for $t = T$.

Another intuitive result is that, for given cutoff values, the optimal search intensity (sample size) for short-lived bidders is greater than that for long-lived bidders. For long-lived bidders, a higher fall-back revenue decreases the marginal value of search, and therefore dampen a seller's incentive to search bidders. In contrast, for short-lived bidders, a seller's fall-back revenue always turns to 0, as she cannot reclaim a previously declined revenue. Therefore, a seller with short-lived bidders will search bidders more intensively. Figure 2 in Example 2 also illustrates the comparative results on optimal cutoff and search intensity with long-lived and short-lived bidders.

Proposition 5 (short-lived versus long-lived bidders: search intensity). Given a sequence of cutoff values $\boldsymbol{\xi}$ that is declining in t, the optimal search intensity (sample size) for short-lived bidders is greater than that for long-lived bidders in each period $t = 1, \ldots, T$.

VI. EFFICIENT SEARCH AUCTION: LONG-LIVED BIDDERS

An optimal auction may lead to inefficient outcomes, for example, due to the possibility of no trade or a biased allocation where a bidder with the highest value does not win. Here, in our model of compound search auctions, the inefficiency may stem from the inefficiency of an optimal search rule. For example, a profit-maximizing seller may exclude some bidders who would otherwise be socially valuable, or she may also have excessive incentives to invite bidders sometime, or the optimal sequence of bidder samples can be different from an efficient one. In this section, we will examine efficient search auctions in the case of long-lived bidders.

VI(i). *Efficient Search Auction*

An efficient search auction maximizes the expected social welfare, which is equal to the value of the winning bidder net of the total search cost. Replacing the virtual value $\psi(v)$ by the true value v in (2), we can similarly set up the DP problem for welfare maximization, and conduct similar analysis. Specifically, given a compound search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, similar to (9), the *ex-ante* expected social welfare is

$$W(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} F^{(1)}_{N_{t-1}}\left(\xi_{t-1}\right)\left[W_t(N_t) - c_{M_t}\right], \tag{14}$$

where $W_t(N_t)$ is the expected value of the winning bidder in the stage auction of period t, conditional on it happens. It is evident that the incentive problem for the bidders remains the same as in Section V(i). By substituting (8) into the expected social welfare function (14), we can derive the following expression of the expected social welfare.

Lemma 5. Given a compound search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected social welfare is

$$W(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} \int_{\xi_t}^{\xi_{t-1}} x dF^{(1)}_{N_t}(x) + \sum_{t=1}^{T} F^{(1)}_{N_{t-1}}\left(\xi_{t-1}\right)\left[\int_{\xi_{t-1}}^{1} x dF^{(1)}_{M_t}(x) - c_{M_t}\right], \tag{15}$$

where $\xi_0 \equiv 1$.

The following result characterizes the cutoff values in an efficient search auction.

Proposition 6 (efficient cutoffs). Given a compound search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected social welfare $W(\boldsymbol{\xi}, \mathbf{M})$

is quasi-concave in ξ_t. The sequence of efficient cutoff values, $(\xi_t^{**})_{1\leq t\leq T}$, is the unique solution to

$$c_{M_{t+1}} = \int_{\xi_t^{**}}^{1} \left(x - \xi_t^{**}\right) dF_{M_{t+1}}^{(1)}(x), \text{ for } 1 \leq t < T, \tag{16}$$

and $\xi_T^{**} = 0$ for $t = T$.

The formula (16) for efficient cutoff value is identical to that for optimal cutoff value (11), except that the virtual value $\psi(v)$ is now replaced by the true value v. Similarly, we can define a function of the efficient cutoff for a bidder sample M, denoted by $\xi^{**}(m)$. Specifically, applying condition (16), $\xi^{**}(m)$ is the implicit function defined by

$$mc = \int_{\xi^{**}(m)}^{1} \left[1 - F^m(x)\right] dx. \tag{17}$$

It is easy to show that $\xi^{**}(m)$ is also strictly decreasing in m. For given bidder sample collection $\mathscr{M}$, denote $\mathbf{M}^{**} = (m_t^{**})_{1\leq t\leq T}$ as an efficient search schedule. Proposition 7 below shows that the sequence of efficient sample sizes is increasing over time.

Proposition 7 (efficient search intensity). The sequence of efficient search intensities (sampling sizes) is increasing over time, that is, for $t = 1, \ldots, T-1$,

$$m_t^{**} \leq m_{t+1}^{**}.$$

Comparing (16) and (17), we have $\xi_t^{**} = \xi^{**}(m_{t+1}^{**})$, and the compound search auction $(\boldsymbol{\xi}^{**}, \boldsymbol{M}^{**})$ then implements the efficient search outcomes. Given a collection $\mathscr{M}$ of bidder samples, we denote $W^{**}(\mathscr{M})$ as the maximum social welfare by following an efficient search rule. By substituting (16) into (15), it follows that

Lemma 6. Given a collection $\mathscr{M}$ of bidder samples, the maximum expected social welfare is

$$W^{**}(\mathscr{M}) = \sum_{t=1}^{T} \int_{\xi^{**}\left(m_{t+1}^{**}\right)}^{\xi^{**}(m_t^{**})} \left[1 - F_{N_t}^{(1)}(x)\right] dx \tag{18}$$

VI(ii). *Efficient versus Optimal Search Auction*

It is helpful to compare the cutoff values and the sample sizes between the optimal and the efficient search auction. The first result shows that, when the sequence of bidder samples is given, the optimal cutoff value is greater than the efficient one in each period.

Corollary 1. For a given search schedule $\mathbf{M}$, the optimal cutoff value is higher than the efficient one in each period, that is, for $1 \leq t \leq T$,

$$\xi^*(m_t) > \xi^{**}(m_t).$$

This result is reminiscent of the related results in static auctions. In a symmetric static auction, the optimal reserve price is $r^* > 0$, and the efficient reserve price is simply 0. Corollary 1 produces a similar result in the case of compound search auctions.

Second, as both $\xi^*(m)$ and $\xi^{**}(m)$ are strictly decreasing in m, we can define their inverse functions, denoted by $m^*(\xi_t)$ and $m^{**}(\xi_t)$ respectively, which roughly measure the optimal and efficient search intensity for a given cutoff value ξ_t.

Proposition 8 (efficient versus optimal search intensity). For a given sequence of declining cutoff values $\boldsymbol{\xi}$, the optimal search intensity (sample size) is greater than the efficient one in each period, that is, for $1 \leq t \leq T$,

$$m^*(\xi_t) > m^{**}(\xi_t).$$

Proposition 8 shows that, given a declining sequence of cutoff values, a profit-maximizing seller will search bidders more intensively in each period than a welfare-maximizing seller. Therefore, the expected total number of participating bidders is also larger than that in an efficient search auction, provided that the cutoff value is decreasing. This over-invitation result is reminiscent of the similar results in static search auctions (Szech [2011]; Li [2017]; Xu & Li [2019]).

In the end, we consider an simple example of efficient search auctions using the previous example of 2-period with 3 homogeneous bidders. Figure 3 illustrates how the efficient and the optimal search intensities vary with the search cost c. It clearly shows the new sources of inefficiency from an optimal search rule. For example, when $1/4 < c \leq 1/2$, a profit-maximizing seller will not invite any bidder, yet it is desirable for a welfare-maximizing seller to invite bidders. Second, when $c^{**} < c \leq c^*$, a profit-maximizing seller will choose a search schedule of $\mathbf{M}^* = (1, 2)$, while the efficient one is $\mathbf{M}^{**} = (1, 1)$.

Example 3 (2-period with 3 long-lived bidders). Let $F(x) = x$ and $c \in (0, 1/2]$. Denote the search schedule by $\mathbf{M} = (m_1, m_2)$. Proposition 7 suggests that the efficient search schedule is either $\mathbf{M} = (1, 1)$ or $(1, 2)$, depending on the value of c. From formula (17), the efficient cutoff value for different sample size is given by:

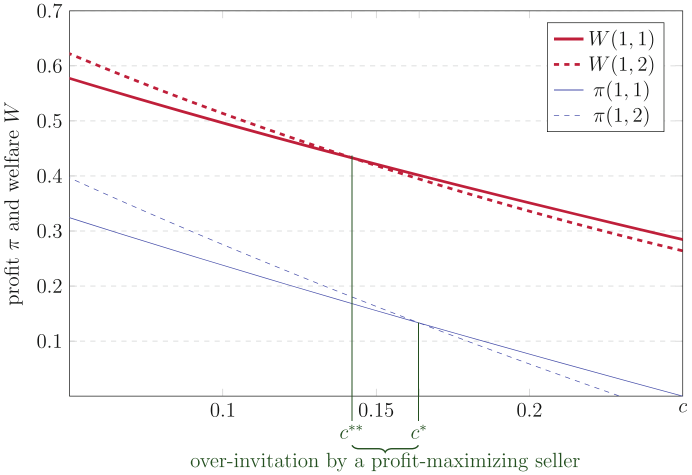

Figure 3
Optimal versus Efficient Auction

Notes: See Example 2 for expected profit and Example 3 for expected welfare. (1) **optimal search schedules**: when $c \le c^*$, the optimal search schedule is $(1, 2)$, as shown by the dashed blue curve; when $c^* < c \le 1/4$, it is $(1, 1)$, as shown by the solid blue curve; furthermore, when $c > 1/4$, a profit-maximizing seller never searches. (2) **efficient search schedules**: when $c \le c^{**}$, the efficient search schedule is $(1, 2)$, as shown by the dashed red curve; when $c^{**} < c \le 1/2$, it is $(1, 1)$, as shown by the solid red curve; moreover, when $c_2 > 1/2$, a welfare-maximizing seller never searches. Note that $c^{**} < c^*$. Therefore, when $c^{**} < c \le c^*$, a profit-maximizing seller searches more intensively in period 2 than a welfare-maximizing seller [Colour figure can be viewed at wileyonlinelibrary.com]

- If $m = 1$, then $\xi^{**}(1)$ is the solution to $c = \frac{1}{2}(1-\xi)^2$, for example, $\xi^{**}(1) = 1 - \sqrt{2c}$;
- If $m = 2$, then $\xi^{**}(2)$ is the solution to $2c = \frac{1}{3}(1-\xi)^2(2+\xi)$.

Case 1: If $c = \frac{1}{16}$, then $\xi^{**}(1) \approx 0.646$ and $\xi^{**}(2) \approx 0.622$. For a search schedule of $(1, 1)$, the expected social welfare, from (18), is

$$W(1,1) = \int_{\xi^{**}(1)}^{\xi^{**}(1)} [1 - F(x)]\, dx + \int_0^{\xi^{**}(1)} \left[1 - F^2(x)\right] dx \approx 0.556.$$

The expected welfare for a search schedule of $(1, 2)$ is

$$W(1,2) = \int_{\xi^{**}(2)}^{\xi^{**}(1)} [1 - F(x)]\, dx + \int_0^{\xi^{**}(2)} \left[1 - F^3(x)\right] dx \approx 0.593.$$

Therefore, the search schedule of $(1, 2)$ is efficient.

Case 2: If $c = \frac{5}{24}$, then $\xi^{**}(1) \approx 0.355$ and $\xi^{**}(2) = 0.256$. For a search schedule of $(1,1)$, the expected social welfare, from (18), is

$$W(1,1) = \int_{\xi^{**}(1)}^{\xi^{**}(1)} [1 - F(x)]\,dx + \int_{0}^{\xi^{**}(1)} \left[1 - F^2(x)\right] dx \approx 0.340.$$

The expected profit for a search schedule of $(1,2)$ is

$$W(1,2) = \int_{\xi^{**}(2)}^{\xi^{**}(1)} [1 - F(x)]\,dx + \int_{0}^{\xi^{**}(2)} \left[1 - F^3(x)\right] dx \approx 0.323.$$

Therefore, the search schedule of $(1,1)$ now is efficient.

Figure 3 plots how the efficient and optimal search schedule change with the unit search cost c. For example, when $c \leq c^{**} \approx 0.142$, the efficient search schedule is $(1,2)$; when $c^{**} < c \leq 1/2$, the efficient search schedule is $(1,1)$; and when $c > 1/2$, a welfare-maximizing seller will not conduct search at all. It also compares the efficient and the optimal search schedules connected to the two cutoff values of c^{**} and c^{*}. When $c^{**} < c \leq c^{*}$, a welfare-maximizing seller invites just 1 bidder in the second period, while a profit-maximizing seller invites 2 instead. It reveals the tendency of over-invitation by a profit-maximizing seller.

VII. CONCLUSION

This paper proposes a framework for studying seller compound search problems, where a seller searches for bidders batch by batch under a finite deadline. This model can be applied to a large variety of real-world problems, such as M&A selling processes, job recruiting campaigns, speed dating events, job/school applications, etc. Our main results show that an optimal compound search is characterized by decreasing reserve prices and increasing search intensities over time. The monotonicity results are robust in both cases of short and long-lived bidders, and across optimal and efficient search auctions. The result of increasing search intensities may help explain why negotiation could be a dominant selling process in many important markets, such as M&As, and why academic job markets in the UK become more active when the REF deadline approaches, and so on.

We show how the optimal search outcomes can be implemented by a compound search auction, and conduct several comparative analyses. First, across short and long-lived bidders, we show that a seller with short-lived bidders will set lower reserve prices and search more intensively than a seller with long-lived bidders *ceteris paribus*. This is because a seller with short-lived bidders can not reclaim any previously declined offer, and therefore, she is willing to set a lower reserve price and will search bidders more intensively.

Second, we show that an efficient search auction is featured by both lower reserve prices and greater search intensities than an optimal search auction, *ceteris paribus*. The result indicates a new source of inefficiency of the optimal search auction, for example, due to the inefficient search rule. For example, a profit-maximizing seller may exclude some socially valuable bidders, or has excessive incentives to invite bidders in certain stages of the transaction.

This paper contributes to the literature of search mechanism, where the targets for search are strategic agents. Compared with the existing literature, there are several distinct features of our model. First, we consider a model of compound search within a finite horizon, which enables us to investigate how optimal search intensities (sample sizes) vary over time. Second, we study both cases of short-lived and long-lived bidders, and provide some comparative results absent in the literature. Third, we identify a new source of inefficiency in optimal search auctions, for example, the inefficient search rule. We believe our framework can be conveniently applied to the study of a large variety of search problems, such as sequential matching in marriage markets, job recruitment by a deadline, or R&D tournament within a finite time horizon. We keep these extensions and explorations for future research.

APPENDIX A

OMITTED PROOFS

Proof of Proposition 1. We solve (3) by backward induction. Denote $Z_m^{(1)} = \max\left\{r^*, X_m^{(1)}\right\}$ and define

$$\omega_t = \max_m \left\{\mathbb{E}\hat{J}_{t+1}\left[Z_m^{(1)}\right] - mc\right\},$$

which is the continuation value of following an optimal search rule from the end of period t on. The Bellman equation (3) is then equivalent to

$$\hat{J}_t(v) = \max_m \left\{\psi(v), \mathbb{E}\hat{J}_{t+1}\left[Z_m^{(1)}\right] - mc\right\} = \max\left\{\psi(v), \omega_t\right\}, \tag{A1}$$

It is clear that $\omega_T = 0$, that is, the continuation value after the deadline T is 0. When $t = T-1$,

$$\omega_{T-1} = \max_m \left\{\mathbb{E}\hat{J}_T\left[Z_m^{(1)}\right] - mc\right\} = \max_m \left\{\mathbb{E}\max\left\{\psi(\overset{(1)}{\underset{m}{Z}}), \omega_T\right\} - mc\right\} > 0 = \omega_T.$$

When $t = T-2$, similarly,

$$\omega_{T-2} = \max_m \left\{\mathbb{E}\max\left\{\psi\left(Z_m^{(1)}\right), \omega_{T-1}\right\} - mc\right\} > \max_m \left\{\mathbb{E}\max\left\{\psi\left(Z_m^{(1)}\right), \omega_T\right\} - mc\right\} = \omega_{T-1},$$

where the inequality is from $\omega_{T-1} > \omega_T$. Continuing in this manner, we see that

$$\omega_t > \omega_{t+1} \text{ for } t = 0, 1, \dots, T-1.$$

The result of decreasing cutoff $\hat{\xi}_t^*$ is then implied by the fact that $\psi\left(\hat{\xi}_t^*\right) = \omega_t$.

Second, note that the optimal sample size m_t^* is the maximizer of

$$\varsigma(m, \hat{\xi}_t^*) = \mathbb{E}\max\left\{\psi\left(Z_m^{(1)}\right), \omega_t\right\} - mc = \psi(\hat{\xi}_t^*) + \int_{\hat{\xi}_t^*}^1 \left[\psi(x) - \psi\left(\hat{\xi}_t^*\right)\right] dF_m^{(1)}(x) - mc,$$

Combining with (A1), we then have the recurrence relation of (4). As $\psi(x)$ is increasing and $\hat{\xi}_t^*$ is independent of m, $\varsigma(m, \hat{\xi}_t^*)$ is then concave in m [Lemma 1][Szech 2011]. Furthermore,

$$\varsigma(m+1, \hat{\xi}_t^*) - \varsigma(m, \hat{\xi}_t^*) = \int_{\hat{\xi}_t^*}^1 F_m^{(1)}(x)\,[1 - F(x)]\,d\psi(x) - c$$

is decreasing in $\hat{\xi}_t^*$. The optimization condition for m_t is thus

$$\varsigma(m, \hat{\xi}_t^*) - \varsigma(m-1, \hat{\xi}_t^*) \geq 0 > \varsigma(m+1, \hat{\xi}_t^*) - \varsigma(m, \hat{\xi}_t^*).$$

Given that $\hat{\xi}_t^* > \hat{\xi}_{t+1}^*$, the concavity of $\varsigma(m, \hat{\xi}_t^*)$ in m then implies $\hat{m}_t^* \leq \hat{m}_{t+1}^*$.

Proof of Lemma 2. In the stage auction of period t, the set of participating bidders is $N_t = N_{t-1} \cup M_t$. Among them, M_t strong bidders' values are independent draws from F, and those of the other N_{t-1} weak bidders are independent draws from the truncated distribution $F(v\,|\xi_{t-1}) \equiv \Pr(V \leq v\,|\,V \leq \xi_{t-1})$.[14] Let $G_{N_t}^{(k)}$ denote the distribution of the k-th highest value of the N_t bidders. Based on the properties of order statistics, we then have

$$G_{N_t}^{(1)}(x) = F_{N_{t-1}}^{(1)}(v\,|\xi_{t-1})F_{M_t}^{(1)}(v),$$
$$G_{N_t}^{(2)}(x) = F_{N_{t-1}}^{(1)}(v\,|\xi_{t-1})F_{M_t}^{(2)}(v) + n_{t-1}\overline{F}(x\,|\xi_{t-1})F_{N_{t-1}-1}^{(1)}(v\,|\xi_{t-1})F_{M_t}^{(1)}(v),$$

where $n_{t-1} = |N_{t-1}|$, $\overline{F}(x\,|\xi_{t-1}) = 1 - F(x\,|\xi_{t-1})$ is the survival function, and $F_{N_{t-1}-1}^{(1)}(v\,|\xi_{t-1}) \equiv F^{n_{t-1}-1}(v\,|\xi_{t-1})$. The expected revenue of the stage auction in period t is thus

$$R_t(N_t) = r_t\left[G_{N_t}^{(2)}(\xi_t) - G_{N_t}^{(1)}(\xi_t)\right] + \int_{\xi_t}^1 x dG_{N_t}^{(2)}(x)$$

It is helpful to do the following transformation,

(A2)
$$R_t(N_t) = \left\{\xi_t\left[G_{N_t}^{(2)}(\xi_t) - G_{N_t}^{(1)}(\xi_t)\right] + \int_{\xi_t}^1 x dG_{N_t}^{(2)}(x)\right\} - (\xi_t - r_t)\left[G_{N_t}^{(2)}(\xi_t) - G_{N_t}^{(1)}(\xi_t)\right],$$

where the part in the curly braces is the expected revenue of a one-shot auction with a reserve price ξ_t. From Myerson [1981] and Kirkegaard [2012], it is equal to

(A3)
$$F_{N_{t-1}}^{(1)}(\xi_t\,|\xi_{t-1})\int_{\xi_t}^1 \psi(x)dF_{M_t}^{(1)}(x)$$
$$+ \int_{\xi_t}^{\xi_{t-1}}\left[\psi(v\,|\xi_{t-1})F_{M_t}^{(1)}(v) + \int_v^1 \psi(x)dF_{M_t}^{(1)}(x)\right]dF_{N_{t-1}}^{(1)}(v\,|\xi_{t-1}),$$

where $\psi(v\,|\xi_{t-1})$ is the virtual value of the N_{t-1} weak bidders. Substituting $\psi(v\,|\xi_{t-1}) = \psi(v) + \frac{\overline{F}(\xi_{t-1})}{f(v)}$ into (A3) and integrating by parts, we have the other expression of (A3) as

[14] Note that, if F is of IFR, so is the truncated distribution $F(v\,|\xi_{t-1})$.

(A4) $$\int_{\xi_t}^{\xi_{t-1}} \psi(x) d\frac{F_{N_t}^{(1)}(x)}{F_{N_{t-1}}^{(1)}\left(\xi_{t-1}\right)} + \int_{\xi_{t-1}}^{1} \psi(x) dF_{M_t}^{(1)}(x) + \frac{n_{t-1}\overline{F}(\xi_{t-1})}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} \int_{\xi_t}^{\xi_{t-1}} F_{N_t-1}^{(1)}(x)dx.$$

Second, from the cutoff condition (8) for bidders' equilibrium strategies, we have

(A5) $$\int_{\xi_t}^{\xi_{t-1}} F_{N_t-1}^{(1)}(x)dx = F_{N_{t-1}-1}^{(1)}(\xi_{t-1})(\xi_{t-1} - r_{t-1}) - F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t).$$

Moreover, the property of order statistics implies that

(A6) $$G_{N_t}^{(2)}(\xi_t) - G_{N_t}^{(1)}(\xi_t) = m_t \overline{F}(\xi_t) \frac{F_{N_t-1}^{(1)}(\xi_t)}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} + n_{t-1}\overline{F}(x\,|\xi_{t-1}) \frac{F_{N_t-1}^{(1)}(\xi_t)}{F_{N_{t-1}-1}^{(1)}(\xi_{t-1})}.$$

Substituting the results of (A3)-(A6) into (A2), we then have the *ex-ante* expected stage revenue in period t as follows:

$$\begin{aligned}
&F_{N_{t-1}}^{(1)}(\xi_{t-1}) R_t(N_t) \\
&\quad = \int_{\xi_t}^{\xi_{t-1}} \psi(x) dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}) \int_{\xi_{t-1}}^{1} \psi(x) \mathrm{d}F_{M_t}^{(1)}(x) \\
&\qquad + n_{t-1}\overline{F}(\xi_{t-1}) \left[F_{N_{t-1}-1}^{(1)}(\xi_{t-1})(\xi_{t-1} - r_{t-1}) - F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t)\right] \\
&\qquad - (\xi_t - r_t)\left[m_t \overline{F}(\xi_t) F_{N_t-1}^{(1)}(\xi_t) + n_{t-1}(F(\xi_{t-1}) - F(\xi_t)) F_{N_t-1}^{(1)}(\xi_t)\right] \\
&= \int_{\xi_t}^{\xi_{t-1}} \psi(x) dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}) \int_{\xi_{t-1}}^{1} \psi(x) \mathrm{d}F_{M_t}^{(1)}(x) \\
&\qquad + n_{t-1}\overline{F}(\xi_{t-1}) F_{N_{t-1}-1}^{(1)}(\xi_{t-1})(\xi_{t-1} - r_{t-1}) - n_t \overline{F}(\xi_t) F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t).
\end{aligned}$$

Summing all of them together, we then get the expected auction revenue of (10). ■

Proof of Proposition 2. For $t < T$, from (10), the partial derivative of $\pi(\boldsymbol{\xi}, \mathbf{M})$ with respect to ξ_t is

$$\begin{aligned}
\frac{\partial \pi}{\partial \xi_t} &= \psi(\xi_t) \left[f_{N_{t+1}}^{(1)}(\xi_t) - f_{N_t}^{(1)}(\xi_t)\right] \\
&\qquad + f_{N_t}^{(1)}(\xi_t) \left[\int_{\xi_t}^{1} \psi(x) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}}\right] - \psi(\xi_t) F_{N_t}^{(1)}(\xi_t) f_{M_{t+1}}^{(1)}(\xi_t) \\
&\qquad = f_{N_t}^{(1)}(\xi_t) \left[\int_{\xi_t}^{1} (\psi(x) - \psi(\xi_t)) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}}\right] = f_{N_t}^{(1)}(\xi_t) \cdot \eta(\xi_t),
\end{aligned}$$

where $f_{N_t}^{(1)}$ is the density function of $F_{N_t}^{(1)}$. Note that

$$\frac{\partial \eta}{\partial \xi_t} = -\psi'(\xi_t) \int_{\xi_t}^{1} dF_{M_{t+1}}^{(1)}(x) < 0,$$

and then $\frac{\partial \pi}{\partial \xi_t}$ changes its sign from positive to negative at most once. $\pi(\boldsymbol{\xi}, \mathbf{M})$ is then quasi-concave in ξ_t, and the first order necessary condition is also sufficient. The optimal condition then gives (11) for $t < T$. When $t = T$, $\frac{\partial \pi}{\partial \xi_T} = -\psi(\xi_T) f^{(1)}_{N_T}(\xi_T)$. It is obvious that π is concave in ξ_T given the IFR assumption, and thus $\psi(\xi^*_T) = 0$.

Proof of Lemma 3. From (12), we have

$$c = \int_{\xi^*(m)}^{1} \frac{1}{m} (1 - F^m(x))\, d\psi(x).$$

As $F(x) < 1$ for $x \in [0, 1)$, $(1 - F^m(x))/m$ is strictly decreasing in m. Therefore, when m increases, $\xi^*(m)$ must decrease so as to keep the above equality to hold. ■

Proof of Proposition 3. It is straightforward from Lemma 3.

Proof of Lemma 4. Given a collection $\mathscr{M}$ of bidder samples, Proposition 3 suggests the optimal search schedule is such that $m^*_t \le m^*_{t+1}$, for $1 \le t < T$. From (12) and (11), we have $\xi^*(m^*_{t+1}) = \xi^*_t$ for $1 \le t < T$, and define $\xi^*(m_{T+1}) \equiv r^*$ Substituting (11) into (10), we then get

$$\begin{aligned}
\pi^*(\mathscr{M}) &= \left[\int_{\xi^*_1}^{\xi_0} \psi(x) dF^{(1)}_{N_1}(x) - cm^*_1 \right] \\
&\quad + \sum_{t=2}^{T} \left[\int_{\xi^*_t}^{\xi^*_{t-1}} \psi(x) dF^{(1)}_{N_t}(x) + F^{(1)}_{N_{t-1}}\left(\xi^*_{t-1}\right) \int_{\xi^*_{t-1}}^{1} \psi\left(\xi^*_{t-1}\right) dF^{(1)}_{M_t}(x) \right] \\
&= \left[\int_{\xi^*_1}^{1} \psi(x) dF^{(1)}_{N_1}(x) - cm^*_1 \right] \\
&\quad + \sum_{t=2}^{T} \left[\psi\left(\xi^*_{t-1}\right) F^{(1)}_{N_{t-1}}\left(\xi^*_{t-1}\right) - \psi(\xi^*_t) F^{(1)}_{N_t}(\xi^*_t) - \int_{\xi^*_t}^{\xi^*_{t-1}} F^{(1)}_{N_t}(x) d\psi(x) \right] \\
&= \left[\int_{\xi^*_1}^{1} \psi(x) dF^{(1)}_{N_1}(x) - cm^*_1 \right] \\
&\quad + \psi(\xi^*_1) F^{(1)}_{N_1}(\xi^*_1) - \psi(\xi^*_1) + \sum_{t=2}^{T} \int_{\xi^*_t}^{\xi^*_{t-1}} \left[1 - F^{(1)}_{N_t}(x) \right] d\psi(x) \\
&= \int_{\xi^*(m_2)}^{\xi^*(m_1)} \left[1 - F^{(1)}_{N_1}(x) \right] d\psi(x) + \sum_{t=2}^{T} \int_{\xi^*(m_{t+1})}^{\xi^*(m_t)} \left[1 - F^{(1)}_{N_t}(x) \right] d\psi(x).
\end{aligned}$$

For the last equality, we apply the definition that $cm^*_1 = \int_{\xi^*(m^*_1)}^{1} \left[1 - F^{(1)}_{M_1}(x) \right] d\psi(x)$. ■

Proof of Proposition 4. Consider a given search schedule $\mathbf{M} = (M_1, \dots, M_T)$. For $t = T$, $\hat{J}_T(v) = \max\{\psi(v), 0\} = J_T(v)$, and therefore, $\hat{\xi}^*_T = \xi^*_T = r^*$. For $t = T - 1$, for long-lived bidders, from (2),

$$J_{T-1}(v) = \max \left\{ \psi(v), -c_{M_T} + \mathbb{E} J_T \left[\max \left\{ v, X^{(1)}_{M_T} \right\} \right] \right\},$$

and $v \geq r^*$ as $\psi(v) \geq 0$. For short-lived bidders, from (3),

$$\hat{J}_{T-1}(v) = \max\left\{\psi(v), -c_{M_T} + \mathbb{E}\hat{J}_T\left[\max\left\{r^*, X^{(1)}_{M_T}\right\}\right]\right\}.$$

As $\hat{J}_T(v) = J_T(v)$ and both are increasing function, it is then clear that $J_{T-1}(v) \geq \hat{J}_{T-1}(v)$ with equality only when $v = 0$. Repeating this process, we then conclude that $J_t(v) \geq \hat{J}_t(v)$, for $0 \leq t < T-1$. The indifference condition for cutoff value then implies $\hat{\xi}^*_t < \xi^*_t$, for $0 \leq t < T-1$.

Proof of Proposition 5. Recall the condition (7) for long-lived bidders and the recurrence equation (5) for short-lived bidders. Given a sequence of cutoff values $\boldsymbol{\xi}$ such that $\xi_t \geq \xi_{t+1}$, the above equations define the inverse real-value functions of $m^*_{t+1}(\xi_t)$ for long-lived bidders and $\hat{m}^*_{t+1}(\xi_t, \xi_{t+1})$ for short-lived bidders. That is, for given $\boldsymbol{\xi}$, $m^*(\xi_t)$ and $\hat{m}^*(\xi_t, \xi_{t+1})$ are respectively the optimal sample sizes for long-lived and short-lived bidders in period $t+1$. Our objective is to show $m^*_{t+1}(\xi_t) < \hat{m}^*_{t+1}(\xi_t, \xi_{t+1})$. We can define a new function

$$\rho(m, \xi) = \int_0^1 \max\{\psi(\xi), \psi(x)\}\, \mathrm{d}F^m(x) - mc$$

which is strictly concave in m (Szech [2011]), and obeys single-crossing difference in (m, ξ) given that, for $m' > m$, $\rho(m'', \xi) - \rho(m', \xi)$ is decreasing in ξ. The well-known result of Milgrom & Shannon [1994](Theorem 4) gives that

$$\tilde{m}(\xi) \equiv \arg\max_{\xi} \rho(m, \xi)$$

is strictly decreasing in ξ, and hence $\tilde{m}(\xi_t) < \tilde{m}(\xi_{t+1})$. In addition, from (7), it follows

$$\begin{aligned}\psi(\xi_t) &= \int_0^1 \max\left\{\psi(\xi_t), \psi(x)\right\} \mathrm{d}F^{m^*(\xi_t)}(x) - m^*(\xi_t)c \\ &\leq \int_0^1 \max\left\{\psi(\xi_t), \psi(x)\right\} \mathrm{d}F^{\tilde{m}(\xi_t)}(x) - \tilde{m}(\xi_t)c.\end{aligned}$$

Therefore, $m^*(\xi_t) \leq \tilde{m}(\xi_t) < \tilde{m}\left(\xi_{t+1}\right) = \hat{m}^*(\xi_t, \xi_{t+1})$, where the last equality is by (5).

Proof of Lemma 5. The conditional expected social welfare of the stage auction in period t is

$$W_t(N_t) = \int_{\xi_t}^{\xi_{t-1}} x dG^{(1)}_{N_t}(x) + \int_{\xi_{t-1}}^1 x dF^{(1)}_{M_t}(x).$$

where $G^{(1)}_{N_t}(x) = F^{(1)}_{N_{t-1}}\left(x \,\middle|\, \xi_{t-1}\right) F^{(1)}_{M_t}(x)$. Summing up all the terms of $F^{(1)}_{N_{t-1}}\left(\xi_{t-1}\right) W_t(N_t)$, we then get the result of (15). ■

Proof of Proposition 6. For $1 \leq t < T$, from (15), the derivative of $W(\boldsymbol{\xi}, \mathbf{M})$ with respect to ξ_t is

$$\frac{\partial W}{\partial \xi_t} = f^{(1)}_{N_t}(\xi_t)\left[\int_{\xi_t}^1 (x - \xi_t) dF^{(1)}_{M_{t+1}}(x) - c_{M_{t+1}}\right] = f^{(1)}_{N_t}(\xi_t)\tilde{\eta}(\xi_t) = 0.$$

Note that $\tilde{\eta}(\xi_t)$ is decreasing in ξ_t. Then W is quasi-concave in ξ_t, and the first order condition is also sufficient. Second, when $t = T$, the partial derivative of W with respect to ξ_t is $\frac{\partial W}{\partial \xi_T} = -\xi_T f^{(1)}_{N_t}(\xi_T) \le 0$, and therefore $\xi^{**}_T = 0$.

Proof of Lemma 6. From (16) and (17), we have $\xi^{**}(m^{**}_{t+1}) = \xi^{**}_t$ for $1 \le t < T$. For $t = T$, we define $\xi^{**}\left(m_{T+1}\right) = \xi^{**}_T = 0$. We then have $W^{**}(\mathscr{M})$ equal to

$$\begin{aligned}
&\left[\int_{\xi_1^{**}}^{\xi_0} x dF^{(1)}_{N_1}(x) - cm_1^{**}\right] \\
&\quad + \sum_{t=2}^{T}\left[\int_{\xi_t^{**}}^{\xi_{t-1}^{**}} x dF^{(1)}_{N_t}(x) + F^{(1)}_{N_{t-1}}(\xi^{**}_{t-1})\int_{\xi^{**}_{t-1}}^{1} \xi^{**}_{t-1} dF^{(1)}_{M_t^{**}}(x)\right] \\
&= \left[\int_{\xi_1^{**}}^{1} x dF^{(1)}_{N_1}(x) - cm_1^{**}\right] \\
&\quad + \sum_{t=2}^{T}\left[\xi^{**}_{t-1}F^{(1)}_{N_{t-1}}(\xi^{**}_{t-1}) - \xi^{**}_t F^{(1)}_{N_t}(\xi^{**}_t) - \int_{\xi_t^{**}}^{\xi_{t-1}^{**}} F^{(1)}_{N_t}(x)dx\right] \\
&= \left[\xi_1^{**} - \xi_1^{**}F^{(1)}_{N_1}(\xi_1^{**}) + \int_{\xi_1^{**}}^{\xi^{**}(m_1^{**})}\left[1 - F^{(1)}_{N_1}(x)\right]dx\right] \\
&\quad + \sum_{t=2}^{T}\left[\xi^{**}_{t-1}F^{(1)}_{N_{t-1}}\left(\xi^{**}_{t-1}\right) - \xi^{**}_t F^{(1)}_{N_t}(\xi^{**}_t) + \int_{\xi_t^{**}}^{\xi_{t-1}^{**}}\left[1 - F^{(1)}_{N_t}(x)\right]dx - \left(\xi^{**}_{t-1} - \xi^{**}_t\right)\right] \\
&= \sum_{t=1}^{T}\int_{\xi^{**}(M^{**}_{t+1})}^{\xi^{**}(M^{**}_t)}\left[1 - F^{(1)}_{N_t}(x)\right]dx,
\end{aligned}$$

where in the third equality, we substitute $cm_1^{**} = \int_{\xi^{**}(m_1^{**})}^{1}\left[1 - F^{(1)}_{M_1^{**}}(x)\right]dx$. ■

Proof of Proposition 7. It is due to the fact that $\boldsymbol{\xi}^{**}$ is decreasing over time.

Proof of Corollary 1. For $1 \le t < T$, $\boldsymbol{\xi}^{*}$ and $\boldsymbol{\xi}^{**}$ are given by (11) and (16) respectively. If we define

$$\tilde{\eta}(v) = \int_v^1 (x - v)dF^{(1)}_{M_t}(x) - c_{M_t} \quad \text{and} \quad \eta(v) = \int_v^1 \left[\psi(x) - \psi(v)\right]dF^{(1)}_{M_t}(x) - c_{M_t},$$

then both $\tilde{\eta}(v)$ and $\eta(v)$ are decreasing in v. Note that

$$\eta(v) - \tilde{\eta}(v) = \int_v^1 \left[\frac{1 - F(v)}{f(v)} - \frac{1 - F(x)}{f(x)}\right]dF^{(1)}_{M_t}(x) > 0,$$

due to the IFR assumption. It then follows that $\xi^{*}(m_t) > \xi^{**}(m_t)$. Finally, for $t = T$, we already know $r^{*} = \xi^{*}_T > \xi^{**}_T = 0$.

Proof of Proposition 8. From (12) and (17), it follows that, for a given ξ,

$$c = \int_{\xi}^{1} \frac{1 - F^{m^*}(x)}{m^*} \cdot \psi'(x)dx = \int_{\xi}^{1} \frac{1 - F^{m^{**}}(x)}{m^{**}} dx.$$

As $\psi'(x) > 1$ from the IFR assumption and $[1 - F^m(x)]/m$ is decreasing in m, we then get the result.

APPENDIX B

BIDDERS' EQUILIBRIUM CUTOFF STRATEGY

For a bidder $i \in N_t$ with value v, let $\overline{U}_{i,t}(v)$ denote his maximum expected payoff at the beginning of period t. It is clear that $\overline{U}_{i,T+1}(v) = 0$. Let $U_{i,t}^b(v)$ denote bidder i's expected payoff of submitting an effective bid in the stage auction of period t, where there are a set N_t of participating bidders. It then follows that, in any period $t \leq T$,

$$U_{i,t}^b(v) = F_{N_t \setminus \{i\}}^{(1)}(\xi_t)(v - r_t) + \mathbb{I}_{\{v \geq \xi_t\}} \int_{\xi_t}^{v} (v - x) dF_{N_t \setminus \{i\}}^{(1)}(x), \tag{B1}$$

where $\mathbb{I}$ is an indicator function, and $F_{N_t \setminus \{i\}}^{(1)}$ is the distribution of the highest value of the set $N_t \setminus \{i\}$ of bidders. On the RHS of (B1), the first term is the expected payoff when no other bidders in N_t bid, and the second term is that when some other bidders in N_t submit bids in the stage auction.

The envelope theorem then gives

$$\overline{U}_{i,t}(v) = \max\{U_{i,t}^b(v), \overline{U}_{i,t+1}(v)\}, \tag{B2}$$

which is nondecreasing, convex, and right-hand differentiable for all $v \in (0, 1]$.[15] The properties of $\overline{U}_{i,t}(v)$ imply that a bidders' optimal strategy is necessarily in the form of cut-off strategies, and hence our assumption of cutoff strategies is without loss of generality. Applying envelope theorem, Lemma 7 below shows that there exists a one-to-one mapping between the sequence of reserve prices, $\mathbf{r}$, and that of the cutoff values, $\boldsymbol{\xi}$.

Lemma 7. Given a compound search auction $(\mathbf{r}, \mathbf{M})$, in each period $t \leq T$, there exists a unique ξ_t such that each bidder $i \in N_t$ bids if and only if his value $v \geq \xi_t$. Furthermore,

$$\overline{U}_{i,t}(v) = \begin{cases} \overline{U}_{i,t+1}(v) & \text{if } v < \xi_t, \\ \overline{U}_{i,t+1}(\xi_t) + \int_{\xi_t}^{v} F_{N_t \setminus \{i\}}^{(1)}(x)dx & \text{if } v \geq \xi_t. \end{cases} \tag{B3}$$

Proof of Lemma 7. It is obvious from (B2) that $\overline{U}_{i,t}(0) = \overline{U}_{i,t+1}(0) = 0$. As the product must be sold at a positive probability in any period $t \leq T$, there exists

[15] The derivation of the cut-off strategy is standard. It also appears in the literature of buy-price auction (Reynolds & Wooders [2009]; Chen *et al.* [2017]) and sequential auctions with information acquisition costs (Crémer *et al.* [2009]). Here we apply the envelope theorem.

a $v^\circ \in (0,1]$ such that $\overline{U}_{i,t}(v^\circ) > \overline{U}_{i,t+1}(v^\circ)$. To prove (B3), we will first show that $\overline{U}_{i,t}(v) > \overline{U}_{i,t+1}(v)$ for any $v \geq v^\circ$. Suppose, for a contradiction, that $\overline{U}_{i,t}(v) = \overline{U}_{i,t+1}(v)$ for some $v \geq v^\circ$. Let $\overline{v} = \min\{v \geq v^\circ \mid \overline{U}_{i,t}(v) = \overline{U}_{i,t+1}(v)\}$, which is well-defined as $\overline{U}_{i,t}$ and $\overline{U}_{t+1}$ are continuous. Then for any $\tilde{v} \in [v^\circ, \overline{v})$, it must be $\overline{U}_{i,t}(\tilde{v}) = U^b_t(v) > \overline{U}_{i,t+1}(\tilde{v})$ and hence $\overline{U}'_{i,t}(\tilde{v}) = F^{(1)}_{N_t\setminus\{i\}}(\tilde{v})$ from (B1), which is in turn strictly greater than $F^{(1)}_{N_{t+1}\setminus\{i\}}(\tilde{v}) \geq \overline{U}'_{i,t+1}(\tilde{v})$. It then contradicts the continuity of $\overline{U}_{i,t}(v)$ and $\overline{U}_{i,t+1}(v)$ and hence $\overline{U}_{i,t}(v) > \overline{U}_{i,t+1}(v)$ for all $v \geq v^\circ$. Then, $\xi_t \equiv \max\{v \mid \overline{U}_{i,t}(v) = \overline{U}_{i,t+1}(v)\}$ is uniquely defined and the standard payoff equivalence argument yields the bidder's payoff function as (B3). ∎

Note that, if $v < \xi_t$, a bidder will not bid in period t and hence his maximum expected payoff remains unchanged till the beginning of the next period, that is, for any $v < \xi_t$, $\overline{U}_{i,t}(v) = \overline{U}_{i,t+1}(v)$, as shown in (B3). Similarly, an "incumbent" bidder $i \in N_{t-1}$ has a higher expected payoff than a "newly solicited" bidder $j \in M_t$, only when i's value is greater than ξ_{t-1}. As long as the incumbent bidder i remains in period t, his expected payoff function should be equal to newly solicited bidders. That is, for any $i \in N_{t-1}$, $j \in M_t$, and any $v \leq \xi_{t-1}$,

$$\overline{U}_{i,t}(v) = \overline{U}_{j,t}(v),$$

and hence the cutoff ξ_t does not depend on when the bidder has been invited, as long as the cutoff is decreasing (i.e., $\xi_{t-1} \geq \xi_t$).

With the preparation, we provide the proof of Lemma 1 as below.

Proof of Lemma 1. For $t < T$, a bidder $i \in N_t$ with the cutoff value ξ_t is indifferent between bidding and waiting in period t, and therefore $\overline{U}_{i,t}(\xi_t) = U^b_{i,t}(\xi_t) = \overline{U}_{i,t+1}(\xi_t)$. As $\xi_t \geq \xi_{t+1}$, he then prefers bidding to waiting in period $t+1$, which implies $\overline{U}_{t+1}(\xi_t) = U^b_{t+1}(\xi_t)$. It follows from (B1) that, for $t < T$, $U^b_{i,t}(\xi_t) = F^{(1)}_{N_t\setminus\{i\}}(\xi_t)(\xi_t - r_t)$, and

$$\begin{aligned} U^b_{i,t+1}(\xi_t) &= F^{(1)}_{N_{t+1}\setminus\{i\}}\left(\xi_{t+1}\right)\left(\xi_t - r_{t+1}\right) + \mathbb{I}_{\{\xi_t \geq \xi_{t+1}\}} \int_{\xi_{t+1}}^{\xi_t} (\xi_t - x) dF^{(1)}_{N_{t+1}\setminus\{i\}}(x) \\ &= F^{(1)}_{N_{t+1}\setminus\{i\}}\left(\xi_{t+1}\right)\left(\xi_{t+1} - r_{t+1}\right) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1}\setminus\{i\}}(x) dx. \end{aligned}$$

We then prove the result (8) as $U^b_{i,t}(\xi_t) = U^b_{i,t+1}(\xi_t)$. Moreover, $\overline{U}_{i,T+1}(v) = 0$ implies that $\xi_T = r_T$. Finally we show the reserve prices $\{r_t\}_{1\leq t\leq T}$ are also decreasing in t. From (8),

$$\begin{aligned} F^{(1)}_{N_t\setminus\{i\}}(\xi_t)(\xi_t - r_t) &= F^{(1)}_{N_{t+1}\setminus\{i\}}\left(\xi_{t+1}\right)\left(\xi_{t+1} - r_{t+1}\right) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1}\setminus\{i\}}(x) dx \\ &\leq F^{(1)}_{N_{t+1}\setminus\{i\}}\left(\xi_t\right)\left(\xi_{t+1} - r_{t+1}\right) + F^{(1)}_{N_{t+1}\setminus\{i\}}(\xi_t)\left(\xi_t - \xi_{t+1}\right) \\ &= F^{(1)}_{N_{t+1}\setminus\{i\}}(\xi_t)(\xi_t - r_{t+1}). \end{aligned}$$

The result then implies $r_t \geq r_{t+1}$, as desired. ∎

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