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Consistent truncations from the geometry of sphere bundles

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Abstract: In this paper, we present a unified perspective on sphere consistent truncations based on the classical geometric properties of sphere bundles. The backbone of our approach is the global angular form for the sphere. A universal formula for the Kaluza-Klein ansatz of the flux threading the n-sphere captures the full nonabelian isometry group $\mathrm{SO}(n+1)$ and scalar deformations associated to the coset $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$. In all cases, the scalars enter the ansatz in a shift by an exact form. We find that the latter can be completely fixed by imposing mild conditions, motivated by supersymmetry, on the scalar potential arising from dimensional reduction of the higher dimensional theory. We comment on the role of the global angular form in the derivation of the topological couplings of the lower-dimensional theory, and on how this perspective could provide inroads into the study of consistent truncations with less supersymmetry.

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Contents

1 Introduction

Consistency of truncations, i.e. ensuring that any solution of a lower-dimensional theory obtained via Kaluza-Klein (KK) reduction automatically satisfies the higher-dimensional equations of motion, is of obvious practical interest. The spectrum of the lower-dimensional theory is determined by linearized KK analysis. However, nonlinear modifications away from the infinitesimal neighbourhood of the ground state are required in order to capture the interactions in the reduced theory and ensure that all higher-dimensional equations of motion are satisfied. The problem has received much recent (and not so-recent) attention, and many instances of consistent truncation have been elaborated in the literature. The conditions for a given truncation to be consistent are relatively straightforward when the reduction is on a group manifold, or a quotient theoreof: only the modes invariant under the group action should be kept [1].[1] However, the conditions for consistent truncation for more general internal manifolds are significantly less well understood.

This paper is an attempt to present a unified and synthetic approach to the construction of consistent truncations, based on the geometric properties of the fibre bundle employed in the reduction. We shall discuss here only the cases of sphere reductions (collected in table 1), where our formulae can be compared with known uplift results [4–35]. Our analysis retains the gauge fields for the full nonabelian isometry group of the sphere as well as scalar fields encoding its deformation. The justification and content of our approach can be summarized in three main points:

1. The popular approach, mostly based on variants of exceptional field theory (see e.g. [16, 31, 36–40]), solves the problem of combining the lower-dimensional fields into their higher-dimensional progenitors using clever applications of representations of duality symmetries. In contrast, the starting point of our analysis is a classical geometric object — the global angular form [41] e_n on an n-sphere bundle over the AdS space in question. The naive space of deformations on an n-sphere is given by $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$, and the only group-theoretic fact required is that the denominator and the numerator of this coset embed maximally into the duality group and its maximal compact subgroup, respectively.[2] In other words, our approach offers an alternative and in some ways simpler starting point for constructing a given consistent truncation ansatz.

   When restricted to each fibre, e_n is the generator of the top cohomology of the fibre, i.e. the volume form. The pull-back of its exterior derivative gives the Euler class of the sphere bundle (notably it vanishes for even n). More details can be found in section 2.1. Crucially, for any given sphere consistent truncation, the flux threading the internal sphere can be reconstructed from this object — or rather, as we shall see, from an appropriate incarnation of it, which we denote e'_n.

2. The embedding of e'_n into the consistent truncation ansatz (see section 2) allows us to see several important universal features.

   First, the flux through the undeformed n-sphere bundle is always of the form

$$\hat{\mathcal{F}}_n|_{\text{no defs}} = e'_n. \tag{1.1}$$

   This can be thought of as a common starting point for all the sphere consistent truncations. Away from the vacuum solution, the sphere is deformed and the solution

[1] In some cases the truncation includes a larger set of modes. Notably, the NS-NS sector of supergravity (and more generally the bosonic string) admits a consistent truncation on any group manifold G in which one retains the full set of gauge fields of the $G \times G$ isometry group of the bi-invariant metric on G [2, 3].

[2] In case of IIB reductions, one needs to incorporate SL(2) and U(1) factors, respectively. However, they can and will be ignored here.

needs to accommodate the associated scalar modes, which are encoded in a symmetric unimodular matrix, T. The inclusion of these scalars simply amounts to shifting e'_n by an exact form. On a case-by-case basis there can also be contributions from extra fields.

While an exact shift to e'_n would in principle still allow the addition of an infinite number of new terms, we demonstrate that the ambiguity can be fixed by considering the contributions of the kinetic terms for $\hat{\mathcal{F}}_n$ to the scalar potential. General supersymmetry arguments tell us that the scalar potential is quadratic in T (see section 3 for details). The part of the integral of $|\hat{\mathcal{F}}_n|^2$ on the n-sphere that yields no-derivative terms is not manageable, and in general is not polynomial in T. However this is not the only contribution to the scalar potential. The other comes from the reduction of the higher dimensional Hilbert-Einstein term. This contribution is even less pleasant, but it admits a choice of parameters from the deformed higher-dimensional metric that collects all non-polynomial (non-quadratic) contributions into the integral of a perfect square, while still leaving a free parameter. This contribution can be cancelled against the contribution from the flux kinetic term, provided the final free parameter is set to zero. Remarkably, by choosing the values for these free parameters as described above, we land precisely on the exact shift to e'_n and on the values for n and D (the dimension D of the full theory) which are required for consistent truncation.

As an aside, these statements can be compared with those in [36] based on the generalised parallelizability of spheres. There trivial $\mathrm{SO}(n+1)$ connections were considered. In that case, $e'_n = \mathrm{vol}_n$, and the shift to (1.1) is indeed exact, while the relevant parameters are fixed based on the requirement of off-shell supersymmetry, to the same values found here. Note that arguments based on the scalar potential of the lower-dimensional theory have also appeared in [42].

3. Equation (1.1) is already sufficient for obtaining lower-dimensional topological Chern-Simons couplings.[3] Hence one way of describing our approach would be by saying that it provides a consistent completion of the reduction in the topological sector. The latter in turn functions as a backbone for the full consistent Kaluza-Klein ansatz. This perspective could offer a fruitful counterpart to approaches rooted in exceptional duality groups, whose details are in general dependent on the dimensions and the amount of supersymmetry.

   For many AdS_5 theories with less supersymmetry, where the lower-dimensional symmetry based approaches are less powerful, the Chern-Simons couplings can and have been obtained by suitable modifications of e'_n [45–47]. In other words, in all these cases a new starting point similar to (1.1) is already available. While there are a number of technical challenges in completing to the full solution from this starting point, better understanding the general construction of e'_n with scalar deformations and its embedding into the consistent truncation ansatz are worth exploring.

[3] This is true e.g. for AdS_7 and AdS_5 reductions, as in [43, 44]. In AdS_4 the Chern-Simons couplings involve scalar fields, and require extra care.

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Higher dimensional theory	Sphere	Flux $\hat{\mathcal{F}}$	Dilaton	Gauge group	Extra fields
bosonic sector of $D = 11$ sugra	S^4	$\hat{G}_4$	no	SO(5)	5 three-forms
bosonic sector of $D = 11$ sugra	S^7	$\hat{*}\hat{G}_4$	no	SO(8)	35 pseudoscalars
SL(2, $\mathbb{R}$) singlet sector of $D = 10$ type IIB sugra	S^5	$\hat{F}_5$	no	SO(6)	none
bosonic sector of massive $D = 10$ type IIA sugra	S^6	$\hat{*}\hat{F}_4$	yes	ISO(7)	see section 2.3.4
(2.1) with $p = 2$ in D dim.	S^2	$\hat{F}_2$	yes	SO(3)	none
(2.1) with $p = 3$ in D dim.	S^3	$\hat{F}_3$	yes	SO(4)	1 two-form
(2.1) with $p = 3$ in D dim.	S^{D-3}	$\hat{*}\hat{F}_3$	yes	SO($D-2$)	none

Table 1. Consistent truncations on spheres considered in this section. For each case, the higher-dimensional theory is indicated, as well as the dimensionality of the sphere used in the truncation. The quantity $\hat{\mathcal{F}}$ denotes the D-dimensional flux that threads the sphere; in each case we indicate which field strength (or Hodge dual thereof) in the D-dimensional theory is identified with $\hat{\mathcal{F}}$. The column 'Dilaton' indicates whether the kinetic term for $\hat{\mathcal{F}}$ in the D-dimensional theory comes with a dilaton prefactor. We also indicate the gauge group of the lower-dimensional gauged model. Finally, the column 'Extra fields' collects the bosonic fields that have to be added for consistency, in addition to the gauge fields of SO($n+1$) and the scalars parametrizing the coset SL($n+1,\mathbb{R}$)/SO($n+1$).

2 Global angular forms and sphere truncations

In this section we consider the sphere consistent truncations listed in table 1. Notice that for 11d supergravity on S^4 and S^7, for type IIB supergravity on S^5, and for massive type IIA supergravity on S^6 the consistent truncation applies to the full content of the model, including fermions. For simplicity, throughout this work we restrict to the bosonic sectors (and, for type IIB, to the SL(2, $\mathbb{R}$) singlet sector consisting of the Einstein frame metric and the self-dual five-form flux). We use D to denote the spacetime dimension of the higher-dimensional model, d for the dimension of the lower-dimensional, and n for the dimension of the sphere (so that clearly $D = d + n$). We use a hat for fields in D dimensions. The D-dimensional theory contains, among other fields, an n-form field strength that threads the S^n, which we denote $\hat{\mathcal{F}}_n$.

For the fourth, fifth, and sixth entry in table 1 the starting D-dimensional theory is a bosonic model with action

$$S_{(D)} = \int \left[\hat{R}\hat{*}1 - \frac{1}{2} d\hat{\phi} \wedge \hat{*} d\hat{\phi} - \frac{1}{2} e^{-a\hat{\phi}} \hat{F}_p \wedge \hat{*}\hat{F}_p \right] , \qquad (2.1)$$

where $\hat{R}\hat{*}1$ is the standard Einstein-Hilbert term, $\hat{\phi}$ is a real scalar (dilaton), and $\hat{F}_p$ is the

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closed field strength of a $(p-1)$-form gauge potential. The positive constant a is given by

$$a^2 = 4 - \frac{2(p-1)(D-p-1)}{D-2} = 4 - \frac{2(n-1)(D-n-1)}{D-2}\,, \tag{2.2}$$

which is the value required for the consistent truncation to be possible [35]. In the second step, we have observed that the dimensionality n of the sphere equals p if the flux threading the sphere is $\hat{F}_p$, and $D-p$ if it is its Hodge dual $\hat{*}\hat{F}_p$. In both cases we get the same value of a^2 as a function of D, n.

In each case, the modes retained in the truncation include the following bosonic fields in the d-dimensional supergravity theory:

- metric;
- gauge fields associated to the $\mathrm{SO}(n+1)$ isometry of the round sphere S^n;
- real scalars parametrizing the coset space $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$;
- a real dilaton, if there is a 'yes' in the pertinent column of table 1.

Crucially, consistency of the truncation might require that we keep additional modes in d dimensions, as reported in the last column of table 1.

Our main objects of interest are the uplift formulae for the D-dimensional metric, dilaton, and n-form flux $\hat{\mathcal{F}}_n$ that threads the sphere. In particular, we focus on the parts of the uplift formulae that only contain the $\mathrm{SO}(n+1)$ gauge fields and the $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ scalars. We observe a simple regular pattern, described in detail below, which extends the analysis of [36, 42] by considering simultaneously the $\mathrm{SO}(n+1)$ gauge fields and the $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ scalars.

The detailed forms of the uplift formulae depend on the specific case considered; we describe each case in turn in the following subsections. The general structure of the uplift formulae, however, can be presented uniformly for all cases. The D-dimensional metric, dilaton, and n-form flux $\hat{\mathcal{F}}_n$ take the form

$$d\hat{s}^2_D = Y^{c_1}(yTy)^{\frac{n-1}{D-2}}\left[ds^2_d + g^{-2}Y^{-\frac{2}{n+1}}\frac{1}{yTy}T^{-1}_{IJ}Dy^I Dy^J\right] + (\text{extra})\,, \tag{2.3}$$

$$e^{\frac{2s}{a}\hat{\phi}} = Y^{c_2}(yTy)^{-1} \times (\text{extra})\,, \tag{2.4}$$

$$\hat{\mathcal{F}}_n = e'_n + \frac{1}{\mathcal{V}_n n!}d\left[\frac{n}{n-1}\frac{1}{yTy}\epsilon y(Ty)(Dy)^{n-1}\right] + (\text{extra})\,, \tag{2.5}$$

$$e'_n = \frac{1}{\mathcal{V}_n n!}\epsilon y(Dy)^n + \frac{1}{\mathcal{V}_n n!}\sum_{j=1}^{j_{\max}}\frac{g^j}{j!}(n/2)_j \epsilon y F^j (Dy)^{n-2j}\,. \tag{2.6}$$

Our notation is as follows. We use ds^2_d for the metric in d dimensions, while Y is a positive scalar that encodes the d-dimensional dilaton, if present (if absent, the above formulae are understood with $Y \equiv 1$). The positive constant a was defined in (2.2) while the constants c_1, c_2 are given by

$$c_1 = \frac{2n}{(D-2)(n+1)}\,, \qquad c_2 = \frac{1}{a^2}\frac{4(D-n-2)(n-1)}{(D-2)(n+1)}\,. \tag{2.7}$$

The quantity s is a sign, determined by how the D-dimensional dilaton enters the kinetic term for the flux that threads the sphere, $S_{(D)} \sim \int e^{-sa\hat{\phi}}\hat{\mathcal{F}}_n \wedge \hat{*}\hat{\mathcal{F}}_n + \dots$. The symbol $\mathcal{V}_n$ stands for the volume of the round sphere S^n of radius 1,

$$\mathcal{V}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}, \tag{2.8}$$

while $(n/2)_j$ is the descending Pochhammer symbol, given by $(n/2)_0 := 1$ and

$$(n/2)_j = \frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\dots\left(\frac{n}{2}-j+1\right). \tag{2.9}$$

The indices I, J, $K = 1, \dots, n+1$ are vector indices of $\mathrm{SO}(n+1)$. Unless otherwise stated, they are raised and lowered with the $\mathrm{SO}(n+1)$ invariant tensor δ_{IJ} and its inverse δ^{IJ}. The quantities y^I are constrained coordinates on the sphere S^n. They are defined via the standard embedding of the round unit S^n in flat $\mathbb{R}^{n+1}$ in Cartesian coordinates, and therefore satisfy

$$\delta_{IJ} y^I y^J = 1. \tag{2.10}$$

The 1-forms Dy^I are defined as

$$Dy^I = dy^I + gA^{IJ}y_J, \tag{2.11}$$

where the 1-forms $A^{IJ} = A^{[IJ]}$ are the external d-dimensional gauge fields of $\mathrm{SO}(n+1)$, and g is a constant parameter. The associated field strength reads

$$F^{IJ} = dA^{IJ} + gA^{IK} \wedge A_K{}^J. \tag{2.12}$$

The quantities T_{IJ} are the entries of a symmetric, positive definite, unimodular matrix,

$$T_{IJ} = T_{JI}, \qquad \det T_{IJ} = 1. \tag{2.13}$$

The matrix T depends only on the external d-dimensional spacetime coordinates and parametrizes the $n(n+3)/2$ real scalars of the coset $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$.

In (2.3)-(2.6) we have introduced some shorthand notation used throughout this work. First of all, we write

$$yTy = y_I T^{IJ} y_J, \qquad (Ty)^I = T^{IJ} y_J. \tag{2.14}$$

Secondly, we have suppressed I indices and wedge products of differential forms in expressions involving the epsilon tensor of $\mathrm{SO}(n+1)$. For example,

$$\epsilon y(Ty)(Dy)^{n-1} = \epsilon_{IJK_1\dots K_{n-1}} y^I (Ty)^J Dy^{K_1} \wedge \dots \wedge Dy^{K_{n-1}}, \tag{2.15}$$

$$\epsilon y F^2 (Dy)^{n-4} = \epsilon_{LJ_1K_1J_2K_2I_1\dots I_{n-4}} y^L F^{J_1K_1} \wedge F^{J_2K_2} \wedge Dy^{I_1} \wedge \dots \wedge Dy^{I_{n-4}}. \tag{2.16}$$

As mentioned above, the quantity $\hat{\mathcal{F}}_n$ is the flux threading the S^n. We normalize $\hat{\mathcal{F}}_n$ so that its flux through S^n is one,

$$\int_{S^n} \hat{\mathcal{F}}_n = 1. \tag{2.17}$$

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The physical flux is in general an integer multiple of $\hat{\mathcal{F}}_n$. We comment on the structure of e'_n in greater detail below. Interestingly, e'_n does not contain the external scalars T_{IJ}. The latter enter $\hat{\mathcal{F}}_n$ only via the total derivative term in (2.5). This is an essential feature of the presentation (2.5) of the uplift formula for the n-form flux. It has been derived in [36] in the case in which the gauge fields of SO$(n+1)$ are set to zero.[4] Furthermore, we note that the d-dimensional dilaton Y does not enter the uplift formula for $\hat{\mathcal{F}}_n$.

As pointed out above, retaining only the metric, the gauge fields A^{IJ}, the scalars T^{IJ}, and the dilaton Y (if there is a 'yes' entry in table 1) in the lower-dimensional model might not be consistent, and extra fields might be needed (see last column of table 1). When extra fields are present, they generically enter the uplift formulae for $d\hat{s}^2_D$, $\hat{\phi}$, and/or $\hat{\mathcal{F}}_n$. We have used the notation (extra) in (2.3)–(2.5) as a reminder of this caveat.

The quantity e'_n is closely related to the canonical global angular form for an S^n bundle. Therefore, before proceeding, we briefly review the salient aspects of global angular forms.

2.1 Brief review of global angular forms

We follow [44], see also the textbook [41]. Let E be a real oriented vector bundle of rank $n+1$ over a base manifold B. Suppose E is equipped with a connection and a metric. Let $S(E)$ denote the associated unit-sphere bundle: if $p \in B$ and we choose Cartesian coordinates y^I, $I = 1, \dots, n+1$ on the $\mathbb{R}^{n+1}$ fiber of E at p, the sphere fiber of $S(E)$ at p is described by (2.10). We use E_0 to denote the complement of the zero-section in E. One can prove that there exists a globally defined n-form on E_0, which we denote e_n,[5] with the following properties:

- The form e_n restricted to the fibers of $S(E)$ reduces to the standard volume form on S^n, normalized to integrate to 1,

$$\int_{S^n} e_n = 1\,. \tag{2.18}$$

- The exterior derivative of e_n is given by

$$de_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\pi^* \chi_{n+1}(E) & \text{if } n \text{ is odd.} \end{cases} \tag{2.19}$$

  Here π^* is the pullback by the projection $\pi : E_0 \to B$ onto the base space. The $(n+1)$-form $\chi_{n+1}(E)$ is a polynomial in the curvature of the bundle E, given explicitly below in (2.22), that represents the Euler characteristic of E as a cohomology class in $H^{n+1}(B;\mathbb{Z})$.

[4] The relative factor between $\epsilon y(Dy)^n$ and $\frac{1}{yTy}\epsilon y(Ty)(Dy)^{n-1}$ in (2.5), (2.6) is $\frac{n}{n-1}$, correcting a typo in equation (2.50) of [36] (the factor $2(d-2)!$ in the denominator of A' should be $(d-1)!$).

[5] In the mathematics literature the global angular form is usually normalized in such a way that it integrates to 2 on the sphere fibers, so that $e_n^{\text{here}} = \frac{1}{2} e_n^{\text{maths}}$. For n even, the closed form e_n^{maths} represents an *integral* cohomology class.

We can exhibit an explicit local expression for e_n in terms of the constrained coordinates y^I and the components F^{IJ} of the field strength of the SO$(n+1)$ connection on E [44],

$$e_n = \frac{1}{\mathcal{V}_n n!} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{g^j}{j!} (n/2)_j \epsilon y F^j (Dy)^{n-2j} . \tag{2.20}$$

The unit sphere volume $\mathcal{V}_n$ and the Pochhammer symbols $(n/2)_j$ have been introduced in (2.8), (2.9).

The term with $j=0$ in (2.20) of schematic form $\epsilon y (Dy)^n$ describes the volume form on the round S^n, normalized to integrate to 1. The derivative of the $j=0$ term generates a term with one F factor, schematically $\epsilon y F (Dy)^{n-1}$. The relative coefficient between the $j=0$ and $j=1$ term in (2.20) is engineered in such a way that this term linear in F cancels in de_n, leaving behind only one term with two F's. This can be verfied with the help of some Schouten identities and of the Bianchi identities

$$0 = DF^{IJ} := dF^{IJ} + gA^{IK} \wedge F_K{}^J + gA^{JK} \wedge F^I{}_K \,, \quad DDy^I = gF^{IJ} y_J \,. \tag{2.21}$$

The relative coefficient between the $j=1$ and $j=2$ terms in (2.20) is similarly engineered to guarantee the cancellation of the terms with two F's in de_n, leaving behind only a term with three F's. For even n this pattern of cancellations in de_n continues until we reach $de_n = 0$. For odd n, the pattern of cancellations proceeds until we get a de_n that is purely horizontal, i.e. without any Dy factors,

$$\begin{aligned} n = 2m-1: \quad & de_{2m-1} = -\pi^* \chi_{2m}(E) \,, \\ & \chi_{2m}(E) = \frac{g^m}{2^m m! (2\pi)^m} \epsilon_{I_1 J_1 \dots I_m J_m} F^{I_1 J_1} \wedge \dots \wedge F^{I_m J_m} = \mathrm{Pf}\left(\frac{gF}{2\pi}\right) . \end{aligned} \tag{2.22}$$

The quantity e_n' that enters the uplift formula (2.5) is constructed out of the canonical global angular form e_n. More precisely, depending on the specific case under examination from table 1, we go from e_n to e_n' by truncating the expansion of e_n in powers of the field strenghts F^{IJ}. In other words, e_n' is obtained by truncating the sum over j in (2.20). The details of the truncation are reported below for each case.

2.2 Interpretation of the uplift formula for $\hat{\mathcal{F}}_n$

Let us comment on the form (2.5) of the uplift formula for the flux $\hat{\mathcal{F}}_n$ threading the sphere.

If the gauge fields of SO$(n+1)$ are set to zero, and the scalar matrix T_{IJ} is set to the identity, the expression for $\hat{\mathcal{F}}_n$ clearly reduces to the volume form on the round S^n. This is a closed form that represents the generator of the cohomology group $H^n(S^n;\mathbb{Z})$. The physical flux is given by an integer times this generator, by virtue of flux quantization.

Let us imagine to turn on the scalar fields T_{IJ}, keeping the gauge fields zero for the moment. Formula (2.5) states that, under the continuous deformation parametrized by T_{IJ}, $\hat{\mathcal{F}}_n$ is shifted by an exact piece of the form $d[(yTy)^{-1} \epsilon y (Ty)(dy)^{n-1}]$. It follows that the de Rham cohomology class of $\hat{\mathcal{F}}_n$ is unmodified. This is to be expected, since a continuous deformation cannot change the integral value of the flux threading the sphere. The detailed form of the exact deformation of $\hat{\mathcal{F}}_n$ when the scalars T_{IJ} are turned on (but the gauge fields are zero) was derived in [36] using the notion of generalized parallelizability for spheres. In section 3 below we offer a different argument (similar to the considerations of [42]) that determines both the functional form and the numerical prefactor of the exact deformation.

We may alternatively start from the volume form on the round S^n and turn on the $\mathrm{SO}(n+1)$ gauge fields, keeping T_{IJ} fixed to be the identity matrix. In this case the total D-dimensional spacetime should be regarded as a sphere fibration. The 'naked' volume form $\epsilon y(dy)^n$ is no longer a well-defined n-form in spacetime. Rather, it must be promoted by means of the replacement $dy^I \to Dy^I$. The resulting n-form $\epsilon y(Dy)^n$ is indeed globally defined, yet it fails to be closed: its derivative takes the form $\epsilon F(Dy)^{n-1}$. The terms in the sum (2.6) with $j = 1, 2, \ldots, j_{\max}$ can be interpreted as corrective terms: as discussed in the previous section around (2.21), adding the term with $j = 1$ ensures that the non-closure of e'_n is of the form $\epsilon F^2(Dy)^{n-3}$; further adding the term with $j = 2$ yields a non-closure of the form $\epsilon F^3(Dy)^{n-5}$; and so on.

Finally, let us comment on the case in which both the gauge fields of $\mathrm{SO}(n+1)$ and the scalars T_{IJ} are turned on. The uplift formula (2.5), based on all the explicit examples we have studied, exhibits a particularly simple structure. Indeed, the dependence on the scalar fields T_{IJ} is entirely confined inside a total derivative, even after the gauge fields are activated. Furthermore, the $(n-1)$-form inside the total derivative is $(yTy)^{-1}\epsilon y(Ty)(Dy)^{n-1}$. This quantity is obtained from the exact deformation when the gauge fields as zero by means of the minimal replacement $dy^I \to Dy^I$ inside the total derivative. A priori, additional terms could have been added inside the total derivative after turning on the gauge fields, such as terms proportional to $\epsilon y(Ty)F(Dy)^{n-3}$. In all examples we have studied, however, we observe that such terms are not generated. In section 4, in the case of $D = 11$ supergravity on S^4, we offer a different perspective on this fact, based on a formula of Bott and Cattaneo [48].

2.3 Detailed uplift formulae

Let us now examine each case in table 1 in turn.

2.3.1 $D = 11$ supergravity on S^4

11d supergravity can be consistently truncated on S^4 [4–6] to 7d maximal SO(5) gauged supergravity [49]. The flux that threads the sphere is clearly the closed G_4 flux of 11d supergravity,

$$\hat{\mathcal{F}}_4 \propto \hat{G}_4\,. \tag{2.23}$$

Recall that, by definition, $\hat{\mathcal{F}}_4$ is rescaled in such a way as to integrate to 1 on S^4. In this case there is no dilaton in the higher-dimensional theory, and thus no dilaton among the modes that are kept in seven dimensions. The bosonic content of 7d maximal SO(5) gauged supergravity consists of the metric, the SO(5) gauge fields, 20 real scalars parametrizing the coset $\mathrm{SL}(5,\mathbb{R})/\mathrm{SO}(5)$, and five 3-forms c_3^I, transforming in the vector representation of SO(5). The latter are the 'extra' fields reported in the last column of table 1.

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The complete uplift formulae for this consistent truncation are given in [6], see also [50]. By applying some Schouten identities, they can be recast in the following form,

$$d\hat{s}_{11}^2 = (yTy)^{1/3}\left[ds_7^2 + g^{-2}\frac{1}{yTy}T^{-1}_{IJ}Dy^I Dy^J\right], \tag{2.24}$$

$$\hat{\mathcal{F}}_4 = e_4 + \frac{1}{\mathcal{V}_4 4!}d\left[\frac{4}{3}\frac{1}{yTy}\epsilon y(Ty)(Dy)^3\right] + d(y_I c_3^I), \tag{2.25}$$

$$e_4 = \frac{1}{\mathcal{V}_4 4!}\left[\epsilon y(Dy)^4 + 2g\epsilon y F(Dy)^2 + g^2\epsilon y F^2\right]. \tag{2.26}$$

We observe that in this case the quantity e'_4 that enters the uplift formula for $\hat{\mathcal{F}}_4$ is exactly identified with the canonical global angular form e_4. In other words, the sum over j in (2.20) is not truncated. Moreover, let us point out that the extra three-forms c_3^I do not enter the uplift formula for the 11d metric, and enter the uplift formula for the 11d four-form flux in a very simple way, via a total derivative.

2.3.2 $D = 11$ supergravity on S^7

11d supergravity can be consistently truncated on S^7 [7–18] to 4d maximal SO(8) gauged supergravity [51, 52]. The flux that threads the sphere is now

$$\hat{\mathcal{F}}_7 \propto \hat{*}\hat{G}_4 . \tag{2.27}$$

The bosonic content of 4d maximal SO(8) gauged supergravity consists of the metric, the SO(8) gauge fields, and 70 real scalars parametrizing the coset $E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$. Out of these 70 scalars, 35 have positive intrinsic parity (proper scalars) and 35 have negative intrinsic parity (pseudoscalars). The 35 proper scalars parametrize the coset $\mathrm{SL}(8,\mathbb{R})/\mathrm{SO}(8)$. From this point of view, the remaining 35 pseudoscalars are regarded as 'extra' fields in the terminology of table 1.

The complete uplift formulae for the 11d metric and four-form flux for this consistent truncation are given in [18], building on [7–17]. In order to verify the general formulae (2.3), (2.5), the task at hand is: (i) turn off the 35 pseudoscalars, so that only the $\mathrm{SL}(8,\mathbb{R})/\mathrm{SO}(8)$ scalars remain; (ii) compute $\hat{G}_7$ by taking the Hodge dual of the $\hat{G}_4$ flux given in [18]. Some steps of these computations are reported in appendix A.3. The result reads

$$d\hat{s}_{11}^2 = (yTy)^{2/3}\left[ds_4^2 + g^{-2}\frac{1}{yTy}T^{-1}_{IJ}Dy^I Dy^J\right] + (\text{extra}), \tag{2.28}$$

$$\hat{\mathcal{F}}_7 = e'_7 + \frac{1}{\mathcal{V}_7 7!}d\left[\frac{7}{6}\frac{1}{yTy}\epsilon y(Ty)(Dy)^6\right] + (\text{extra}), \tag{2.29}$$

$$e'_7 = \frac{1}{\mathcal{V}_7 7!}\left[\epsilon y(Dy)^7 + \frac{7}{2}g\epsilon y F(Dy)^5\right]. \tag{2.30}$$

We notice that the 7-form e'_7 that enters the uplift formula is now distinct from the canonical global angular form e_7 given by (2.20). In the canonical e_7, we encounter terms with up to three F's, schematically $e_7 \sim \epsilon y(Dy)^7 + \epsilon y F(Dy)^5 + \epsilon y F^2(Dy)^3 + \epsilon y F^3 Dy$. In this case the base of the S^7 fibration is 4d spacetime, and thus the F^3 term vanishes for dimensional reasons. The F^2 term in e_7, however, survives on a 4d base space. It constitutes the difference between the canonical e_7 and its truncated counterpart e_7'.

It is worth emphasizing that in this consistent truncation the extra fields (the 35 pseudo scalars) enter both the uplift formula for the metric and seven-flux in non-trivial ways. While the uplift formulae retaining all 70 scalars are known explicitly [18], we were not able to recast them in a simple form in the same spirit as in (2.5). We leave this problem for future research.

2.3.3 $D = 10$ type IIB supergravity on S^5

10d type IIB supergravity can be consistently truncated on S^5 [19–31] to 5d maximal SO(6) gauged supergravity [53–55]. The flux that threads the sphere is the self-dual Ramond-Ramond five-form flux $\hat{F}_5$. To discuss the uplift formula for $\hat{F}_5$, we find it convenient to introduce a five-form $\hat{\mathcal{F}}_5$ that satisfies

$$\hat{\mathcal{F}}_5 + \hat{*}\hat{\mathcal{F}}_5 \propto \hat{F}_5 \,. \tag{2.31}$$

The bosonic content of 5d maximal SO(6) gauged supergravity consists of the metric, the SO(6) gauge fields, 42 real scalars parametrizing the coset $E_{6(6)}/\mathrm{USp}(8)$, and a collection of twelve real two-form potentials $B_2^{I\alpha}$, where $I = 1, \dots, 6$ is a fundamental index of $\mathrm{SL}(6,\mathbb{R})$ and $\alpha = 1, 2$ is a fundamental index of $\mathrm{SL}(2,\mathbb{R})$. Recall that $\mathrm{SL}(6,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ is a maximal subgroup of $E_{6(6)}$. The $\mathrm{SL}(2,\mathbb{R})$ factor is identified with the global $\mathrm{SL}(2,\mathbb{R})$ symmetry of classical type IIB supergravity in ten dimensions.

The complete uplift formulae for this consistent truncation are given in [31]. In this work, however, we restrict our attention to a subset of the bosonic fields of the 5d gauged supergravity, namely those that are inert under $\mathrm{SL}(2,\mathbb{R})$. In ten dimensions, the bosonic fields that are singlets of $\mathrm{SL}(2,\mathbb{R})$ are the Einstein frame metric and the five-form flux. They form a 10d consistent bosonic subsector of the full type IIB supergravity. This 10d bosonic theory can be consistently truncated on S^5 to the bosonic theory obtained from 5d maximal SO(6) supergravity by discarding the fermions, the two-forms $B_2^{I\alpha}$, and carving out a scalar submanifold $\mathrm{SL}(6,\mathbb{R})/\mathrm{SO}(6)$ out of the full $E_{6(6)}/\mathrm{USp}(8)$ scalar manifold. The uplift formulae for this truncation are given in [25] and can be rearranged to take the form[6]

$$d\hat{s}^2_{10} = (yTy)^{1/2}\left[ds_5^2 + g^{-2}\frac{1}{yTy}T^{-1}_{IJ}Dy^I Dy^J\right], \tag{2.32}$$

$$\hat{\mathcal{F}}_5 = e_5' + \frac{1}{\mathcal{V}_5 5!}d\left[\frac{5}{4}\epsilon y(Ty)(Dy)^4\right], \tag{2.33}$$

$$e_5' = \frac{1}{\mathcal{V}_5 5!}\left[\epsilon y(Dy)^5 + \frac{5}{2}g\epsilon y F(Dy)^3\right]. \tag{2.34}$$

We notice that the 5-form e_5' that enters the uplift formula is again distinct from the canonical global angular form e_5 given by (2.20). In the canonical e_5, we encounter terms with up to two F's, schematically $e_5 \sim \epsilon y(Dy)^5 + \epsilon y F(Dy)^3 + \epsilon y F^2 Dy$. The terms with two F's is not identically zero on external 5d spacetime, yielding indeed a non-zero difference between e_5 and e_5'.

[6] The quantity we denote $\hat{\mathcal{F}}_5$ is denoted $\hat{*}\hat{G}_{(5)}$ in [25], where it is given in equation (4).

2.3.4 Massive $D = 10$ type IIA supergravity on S^6

Massive type IIA supergravity can be consistently truncated on S^6 to maximal dyonically gauged ISO(7) supergravity [32–34]. The bosonic fields of this 4d theory consists of the metric, electric gauge fields for the gauge group $\mathrm{ISO}(7) = \mathrm{SO}(7) \ltimes \mathbb{R}^7$ of dimension $21 + 7$, 70 real scalars parametrizing the coset $E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$, as well as 7 magnetic gauge fields and 7 2-form potentials. The presence of magnetic gauge fields and 2-form potentials is due to a magnetic gauging of the $\mathbb{R}^7$ subgroup of ISO(7) [33, 56].

The full uplift formulae for this consistent truncation are given in [34]. We want to study the terms in these formulae that contain the 4d metric, the electric gauge fields associated to the SO(7) subgroup of ISO(7), as well as a subset of the scalar fields, consisting of the scalars T_{IJ} in the coset $\mathrm{SL}(7,\mathbb{R})/\mathrm{SO}(7)$, together with one extra real scalar Y. All other bosonic fields of the truncation as regarded as 'extra' in the terminology of table 1. In analogy to the consistent truncations based on the bosonic action (2.1), discussed below in section 2.3.5, we may think of Y as a 4d dilaton. The precise embedding of T_{IJ} and Y into $E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$ is described in appendix A.4.

Let us focus on the uplift formulae for the 10d Einstein frame metric, the 10d dilaton, and the 10d flux that threads the S^6. The latter is identified with

$$\hat{\mathcal{F}}_6 \propto e^{\frac{1}{2}\hat{\phi}} \hat{*} \hat{F}_4 \,, \tag{2.35}$$

where $\hat{F}_4$ is the Ramond-Ramond 4-form field strength. Notice the appearance of a dilaton prefactor: it is due to the fact that the 10d equations of motion take the form $d(e^{\frac{1}{2}\hat{\phi}} \hat{*} \hat{F}_4) = \dots$. The proportionality constant in (2.35) is determined by our choice of normalization for $\hat{\mathcal{F}}_6$ (which integrates to 1 on S^6).

The task at hand is to start from the uplift formulae of [34], in the simplified setting in which we set to zero the 4d fields we are not keeping track of. After obtaining the expression for $\hat{F}_4$, we can compute $e^{\frac{1}{2}\hat{\phi}} \hat{*} \hat{F}_4$. Some details of this derivation are reported in appendix A.4. We find the following uplift formulae,

$$d\hat{s}^2_{10} = Y^{\frac{3}{14}} (yTy)^{\frac{5}{8}} \left[ds^2_4 + g^{-2} \frac{Y^{-\frac{2}{7}}}{yTy} T^{-1}_{IJ} Dy^I Dy^J \right] + (\text{extra}) \,, \tag{2.36}$$

$$e^{4\hat{\phi}} = Y^{\frac{20}{7}} (yTy)^{-1} \times (\text{extra}) \,, \tag{2.37}$$

$$\hat{\mathcal{F}}_6 = e'_6 + \frac{1}{\mathcal{V}_6 6!} d \left[\frac{6}{5} \epsilon y (Ty)(Dy)^5 \right] + (\text{extra}) \,, \tag{2.38}$$

$$e'_6 = \frac{1}{\mathcal{V}_6 6!} \left[\epsilon y (Dy)^6 + 3 g \epsilon y F (Dy)^4 \right] . \tag{2.39}$$

Interestingly, all Y and dY terms eventually drop out from the expression for $\hat{\mathcal{F}}_6$, which only contains the SO(7) gauge fields and the $\mathrm{SL}(7,\mathbb{R})/\mathrm{SO}(7)$ scalars. The quantity e'_6 is a truncation of the canonical global angular form e_6 as given in (2.20). Indeed, on a 4d base space the canonical e_6 contains three terms, schematically $e_6 \sim \epsilon y (Dy)^6 + \epsilon y F (Dy)^4 + \epsilon y F^2 Dy^2$, while e'_6 lacks the F^2 term.

The powers of yTy and Y in (2.36), (2.37) are in agreement with the general expressions (2.3), (2.4). The sign s is $+1$. This is due to the fact that we are adopting the conventions of [34] for the action of massive $D = 10$ type IIA supergravity — see (A.1) therein. In particular, the kinetic term for the Ramond-Ramond 4-form flux is of the form $e^{+\frac{1}{2}\hat{\phi}}\hat{F}_4 \wedge \hat{*}\hat{F}_4$. In terms of the dual flux $\hat{\mathcal{F}}_6 \propto e^{\frac{1}{2}\hat{\phi}}\hat{*}\hat{F}_4$, this term reads schematically $e^{-\frac{1}{2}\hat{\phi}}\hat{\mathcal{F}}_6 \wedge \hat{*}\hat{\mathcal{F}}_6$, from which we infer $s = +1$.

2.3.5 Consistent truncations based on (2.1)

These three cases are discussed in [35], where the full uplift formulae are given. The uplift formulae for the D-dimensional metric and dilaton take the form[7]

$$d\hat{s}^2_D = Y^{c_1}(yTy)^{\frac{n-1}{D-2}}\left[ds^2_d + g^{-2}Y^{-\frac{2}{n+1}}\frac{1}{yTy}T^{-1}_{IJ}Dy^I Dy^J\right], \quad e^{\frac{2s}{a}\hat{\phi}} = Y^{c_2}(yTy)^{-1}. \quad (2.40)$$

The constants c_1, c_2 are as in (2.7). The sign s is $+1$ for the reductions on S^2, S^3 and -1 for the reduction on S^{D-3}. This can be seen from (2.1). For the reductions on S^2, S^3 the flux that threads the sphere is the 'electric' flux $\hat{F}_p$, whose kinetic term is of the form $e^{-a\hat{\phi}}\hat{F}_p \wedge \hat{*}\hat{F}_p$. For the reduction on S^{D-3} the flux threading the sphere is the 'magnetic' flux $e^{-a\hat{\phi}}\hat{*}\hat{F}_3$. Written in terms of $e^{-a\hat{\phi}}\hat{*}\hat{F}_3$, the kinetic term for $\hat{F}_p$ in (2.1) takes the form $e^{+a\hat{\phi}}(e^{-a\hat{\phi}}\hat{*}\hat{F}_3) \wedge \hat{*}(e^{-a\hat{\phi}}\hat{*}\hat{F}_3)$, from which we observe the anticipated flip in sign in the exponent of the dilaton prefactor. We adopt the same normalization for Y as in [35]. The kinetic term for Y in the lower-dimensional action takes the form

$$S_{(d)} \supset \int -\frac{1}{2}c_5 Y^{-2}dY \wedge *dY\,, \qquad c_5 = \frac{4(D-n-2)}{a^2(D-2)(n+1)}\,. \quad (2.41)$$

The canonically normalized lower-dimensional dilaton ϕ is related to Y by $Y = e^{\frac{1}{\sqrt{c_5}}\phi}$.

Let us now discuss the uplift formulae for the flux threading the sphere, in each of the three cases. For the reduction on S^2, $\hat{\mathcal{F}}_2 \propto \hat{F}_2$ with

$$\hat{\mathcal{F}}_2 = e_2 + \frac{1}{\mathcal{V}_2 2!}d\left[2\frac{1}{yTy}\epsilon_{IJK}y^I(Ty)^J Dy^K\right], \quad (2.42)$$

$$e_2 = \frac{1}{\mathcal{V}_2 2!}\epsilon_{IJK}y^I(Dy^J \wedge Dy^K + gF^{JK})\,. \quad (2.43)$$

The notation $\mathcal{V}_n$ was introduced in (2.8). This result is given in equation (55) of [35] in terms of the quantity denoted T^{ij} there, which corresponds to $Y^{1/3}T_{IJ}$ in our notation. Remarkably, all occurrences of Y and its derivatives drop away from the expression for $\hat{\mathcal{F}}_2$. We also notice that e_2 is the canonical global angular form (2.20), satisfying $de_2 = 0$.

For the reduction on S^3 we find similarly $\hat{\mathcal{F}}_3 \propto \hat{F}_3$ with

$$\hat{\mathcal{F}}_3 = e_3 + \frac{1}{\mathcal{V}_3 3!}d\left[\frac{3}{2}\frac{1}{yTy}\epsilon_{IJKL}y^I(Ty)^J Dy^K \wedge Dy^L\right] + f_3\,, \quad (2.44)$$

$$e_3 = \frac{1}{\mathcal{V}_3 3!}\epsilon_{IJKL}y^I\left(Dy^J \wedge Dy^J \wedge Dy^K + \frac{3}{2}gF^{JK} \wedge Dy^L\right). \quad (2.45)$$

The quantity f_3 is the field strength of a 2-form potential in the $d = D-3$ dimensional theory. This 2-form potential is an extra field needed for the consistency of the truncation,

[7] The matrix T in our expressions is always unimodular. It corresponds to the matrix denoted $\tilde{T}$ in [35].

see table 1. Up to normalization, f_3 is the same as the field $F_{(3)}$ in the notation of [35]. We observe once more that all factors of Y and dY drop from the uplift formula for the flux threading the sphere. The quantity e_3 coincides in this case with the canonical (2.20). Indeed, the derivative of e_3 is purely horizontal,

$$de_3 = -\frac{g^2}{8(2\pi)^2}\epsilon_{IJKL}F^{IJ}\wedge F^{KL}\,. \tag{2.46}$$

Recall that the D-dimensional flux $\hat{\mathcal{F}}_3$ is closed. Its closure holds because the extra 2-form potential with field strength f_3 has a non-trivial Bianchi identity involving the SO(4) gauge fields [35], with df_3 exhibiting the same structure as the r.h.s. of (2.46). In fact, in computing $d\hat{\mathcal{F}}_3$ we verify an exact cancellation between de_3 and df_3.

We finally turn to the reduction on S^{D-3}. The flux threading the sphere is in this case given by

$$\hat{\mathcal{F}}_{D-3} \propto e^{-a\hat\phi}\hat{*}\hat{F}_3\,. \tag{2.47}$$

We have defined the dual flux $\hat{\mathcal{F}}_{D-3}$ by including a suitable power of the D-dimensional dilaton. This is motivated by the fact that the EOM for $\hat{F}_3$ derived from the action (2.1) reads

$$d(e^{-a\hat\phi}\hat{*}\hat{F}_3) = 0\,. \tag{2.48}$$

Thus, in D dimensions $\hat{F}_3$ is closed off-shell and $\hat{\mathcal{F}}_{D-3}$ is closed on-shell.

The uplift formula for $\hat{\mathcal{F}}_{D-3}$ is given in (47) of [35], which can be written as

$$\hat{\mathcal{F}}_{D-3} = e'_{D-3} + \frac{1}{\mathcal{V}_{D-3}(D-3)!}d\left[\frac{D-3}{D-4}\frac{1}{yTy}\epsilon y(Ty)(Dy)^{D-4}\right], \tag{2.49}$$

$$e'_{D-3} = \frac{1}{\mathcal{V}_{D-3}(D-3)!}\left[\epsilon y(Dy)^{D-3} + \frac{D-3}{2}\epsilon yF(Dy)^{D-5}\right]. \tag{2.50}$$

We are adopting a compact notation analogous to that in (2.15). In analogy with the other cases discussed in this section, we verify that all occurrences of Y and dY drop out from the uplift formula for $\hat{\mathcal{F}}_{D-3}$. In this case, we find generically a truncated version e'_{D-3} of the canonical global angular form e_{D-3}, because the sum over j in (2.20) is truncated after the first term. (For $D \leq 6$ the sum over j in the canonical global angular form e_{D-3} stops anyway at 1; thus for $D \leq 6$ we actually have $e'_{D-3} = e_{D-3}$.) The fact that we truncate the sum over j at $j = 1$ implies that, for general D, de'_{D-3} consists of one term quadratic in F, of the schematic form $\epsilon F^2(Dy)^{D-6}$. The lower-dimensional theory, however, lives in $d = D-(D-3) = 3$ dimensions, implying that $de'_{D-3} = 0$ for dimensional reasons. We this conclude that the form of the uplift formula (2.49) is such that $d\hat{\mathcal{F}}_{D-3} = 0$ holds identically, without use of the 3d equations of motion. The closure of $\hat{F}_3$, on the other hand, holds after using them.

3 Reduction of the action and exact flux deformation

In this section we give an argument that explains the form of the exact piece in the uplift formula (2.5) for the flux that threads the n-sphere. The argument is based on the derivation of the scalar potential of the lower-dimensional model from the D-dimensional action integrated on S^n (see also [42]).

3.1 Models without dilaton

Let us first consider the cases in which, in the D-dimensional theory, the kinetic term for the flux threading the sphere has no dilaton prefactor. The relevant terms in the D-dimensional action are the Einstein-Hilbert term and the kinetic term for $\hat{\mathcal{F}}_n$,

$$S_{(D)} = \int \left[\hat{R} \hat{*} 1 - \frac{1}{2 g_{\mathcal{F}}^2} \hat{\mathcal{F}}_n \wedge \hat{*} \hat{\mathcal{F}}_n \right] . \tag{3.1}$$

Recall that, in our conventions, $\hat{\mathcal{F}}_n$ is normalized to integrate to 1 on S^n. As a result, its kinetic term is not necessarily canonically normalized in D-dimensions, but rather comes with a coupling constant $g_{\mathcal{F}}^{-2}$. The following argument does not depend on the value of $g_{\mathcal{F}}^{-2}$. We also notice that we have suppressed the overall D-dimensional Newton's constant from the action (3.1).

Our goal is to integrate (3.1) on S^n to derive couplings in the d-dimensional effective action. More precisely, we seek to determine the potential for the scalars T_{IJ}. To this end, we may ignore the terms with the gauge fields of $\mathrm{SO}(n+1)$, and further consider the simpler case in which the profile for the T_{IJ} scalars in d dimensions is constant,

$$A_{IJ} = 0 \, , \qquad dT_{IJ} = 0 \, . \tag{3.2}$$

To perform the dimensional reduction of the Einstein-Hilbert term in (3.1) we need an ansatz for the D-dimensional metric, capturing the contributions of the T_{IJ} scalars. We use the ansatz

$$d\hat{s}_D^2 = (yTy)^{b_1} \left[ds_d^2 + g^{-2} (yTy)^{b_2} T_{IJ}^{-1} dy^I dy^J \right] , \tag{3.3}$$

where b_1, b_2 are constant parameters. In due course, they will be fixed to their expected values $b_1 = (n-1)/(D-2)$, $b_2 = -1$, cfr. (2.3). The D-dimensional Ricci scalar and volume element of the metric (3.3) can now be evaluated by means of a straightforward computation. We keep track of terms that contribute to the d-dimensional scalar potential and to the d-dimensional Einstein-Hilbert term. Further details of the derivation are reported in appendix B.1. The result reads

$$\begin{aligned} \int_{S^n} \hat{R} \hat{*} 1 \supset {} & g^{-n} \sqrt{-g_d} R[g_d] \int_{S^n} d^n \xi \sqrt{\mathring{g}} (yTy)^{\frac{1}{2} b_1 (D-2) + \frac{1}{2} b_2 n + \frac{1}{2}} \\ & + g^{2-n} \sqrt{-g_d} \int_{S^n} d^n \xi \sqrt{\mathring{g}} (yTy)^{\frac{1}{2} b_1 (D-2) + \frac{1}{2} b_2 (n-2) + \frac{1}{2}} \mathcal{G} \, . \end{aligned} \tag{3.4}$$

Here ξ^m are local coordinates on S^n, g_d, $R[g_d]$ are the determinant and Ricci scalar of the d-dimensional metric, and $\mathring{g}$ is the determinant of the metric on the round unit sphere. We have introduced the shorthand notation

$$\mathcal{G} = \frac{(\mathrm{Tr}\, T)^2}{yTy} - \frac{\mathrm{Tr}\, T^2}{yTy} - \frac{2\, (\mathrm{Tr}\, T)\, (yT^2y)}{(yTy)^2} + \frac{2\, yT^3y}{(yTy)^2} + \mathcal{K} \left[\frac{(yT^2y)^2}{(yTy)^3} - \frac{(yT^3y)}{(yTy)^2} \right] , \tag{3.5}$$

where the constant $\mathcal{K}$ is given in terms of D, n, b_1, b_2 by

$$\mathcal{K} = -2 b_2 b_1 (D-2)(n-1) - b_1^2 (D-2)(D-1) - b_2^2 (n-2)(n-1) \, . \tag{3.6}$$

In the quantities yT^2y and yT^3y we are suppressing $\mathrm{SO}(n+1)$ indices as in (2.14).

To perform the integration of the $\hat{\mathcal{F}}_n$ kinetic term in (3.1) over the n-sphere, we need an ansatz capturing the dependence of $\hat{\mathcal{F}}_n$ on the scalars T_{IJ}. Since we are working under the simplifying assumptions (3.2), $\hat{\mathcal{F}}_n$ must be proportional to the volume form of the round metric on S^n, where the proportionality factor may depend on y^I and T_{IJ},

$$\hat{\mathcal{F}}_n = \frac{1}{n!} f(y,T)\epsilon_{I_0 I_1 \dots I_n} y^{I_0} dy^{I_1} \wedge \cdots \wedge dy^{I_n} \,. \tag{3.7}$$

The reduction is reported in appendix B.1. The result reads

$$\int_{S^n} \hat{\mathcal{F}}_n \wedge \hat{*}\hat{\mathcal{F}}_n = g^n \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}} (yTy)^{\frac{b_1 D}{2} - b_1 n - \frac{b_2 n}{2} - \frac{1}{2}} f(y,T)^2 \,. \tag{3.8}$$

In the d-dimensional action the Einstein-Hilbert term should not be multiplied by a non-trivial function of T_{IJ}. This leads us to set the exponent of yTy in the first line of (3.4) to zero,

$$\frac{1}{2} b_1 (D-2) + \frac{1}{2} b_2 n + \frac{1}{2} = 0 \,. \tag{3.9}$$

For the integrand in the second line of (3.4) we then find the quantity $(yTy)^{-b_2}\mathcal{G}$. The matrix T_{IJ} is unimodular, but it is convenient to regard the quantities yTy and $\mathcal{G}$ as functions of an arbitrary non-singular symmetric matrix. In this sense, we can formally consider their behavior under a constant rescaling $T_{IJ} \to \lambda T_{IJ}$. We see that yTy and $\mathcal{G}$ are homogeneous of degree 1 in T, implying that the combination $(yTy)^{-b_2}\mathcal{G}$ has degree $1-b_2$. We know, however, that the scalar potential of the d-dimensional theory should be a quadratic function of T. This is known to be the case for all examples in table 1 with a 'no' in the dilaton column. These considerations lead us to set $1-b_2 = 2$. In combination with (3.9) this fixes b_1 and b_2 to their expected values,

$$b_1 = \frac{n-1}{D-2} \,, \qquad b_2 = -1 \,. \tag{3.10}$$

The overall integrand in the second line of (3.4) then becomes $(yTy)\mathcal{G}$.

To proceed, we observe that we are free to add to $(yTy)\mathcal{G}$ any total divergence on S^n, without affecting the value of the integral. Exploiting this freedom, (3.4) can be brought to the form (see appendix B.1 for details)

$$\begin{aligned}
&\int_{S^n} \hat{R} \hat{*} 1 \supset g^{-n} \mathcal{V}_n \sqrt{-g_d} R[g_d] + g^{2-n} \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}}\, \mathcal{I} \,, \\
&\mathcal{I} = 2\left[\frac{yT^2y}{yTy} - \frac{4-k}{8} \mathrm{Tr}\, T\right]^2 - \left(1 + \frac{k}{2}\right) \mathrm{Tr}\,(T^2) + \left(\frac{1}{2} + \frac{k}{4} - \frac{k^2}{32}\right) (\mathrm{Tr}\, T)^2 \,,
\end{aligned} \tag{3.11}$$

where we have introduced the shorthand notation

$$k := \frac{(n-1)(D-n-1)}{D-2} - 2 \,. \tag{3.12}$$

Note that inside the integrand $\mathcal{I}$ all dependence on y^I is now collected into a perfect square. Moreover, using (3.10), (3.12) we can rewrite (3.8) in the form

$$\int_{S^n} \hat{\mathcal{F}}_n \wedge \hat{*}\hat{\mathcal{F}}_n = g^n \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}} (yTy)^{2+k} f(y,T)^2 \,. \tag{3.13}$$

Upon integrating $\mathcal{I}$ against the volume form $\sqrt{\mathring{g}}$, we obtain a function of T_{IJ} which represents the contribution of the D-dimensional Einstein-Hilbert term to the d-dimensional scalar potential. The resulting function of T_{IJ} is manifestly homogeneous of degree 2 under $T_{IJ} \to \lambda T_{IJ}$. It is not, however, a quadratic function. In fact, it is not a rational function of the entries of T_{IJ}. This is due to the contribution from the square $[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T]^2$ in the integrand. This is point is clarified in appendix B.3. Non-quadratic contributions to the scalar potential must drop out when the Einstein-Hilbert contribution is added to (3.13). Comparison of (3.11), (3.13) immediately suggests a simple and natural mechanism to achieve this goal: the term $[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T]^2$ in $\mathcal{I}$ is cancelled against $(yTy)^{2+k} f(y,T)^2$ in (3.13) at the level of the *integrand*. The remaining terms in $\mathcal{I}$ are a y-independent quadratic function of T.

For the rest of this section, the aforementioned cancellation between $[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T]^2$ and $(yTy)^{2+k} f(y,T)^2$ is our working assumption. Therefore, we set

$$f(y,T) = 2g_{\mathcal{F}} g^{1-n} (yTy)^{-\frac{k}{2}} \left[\frac{yT^2y}{(yTy)^2} - \frac{4-k}{8} \frac{\operatorname{Tr} T}{yTy} \right] . \tag{3.14}$$

A necessary condition for the identification (3.14) to hold is that the integral of the r.h.s. on S^n with measure $\sqrt{\mathring{g}}$ be independent of T. Indeed, the integral of the l.h.s. must be independent of T, because turning on T must deform $\hat{\mathcal{F}}_n$ by an exact piece (the quantized flux through S^n cannot be modified by a continuous deformation). In appendix B.2 we prove that a necessary condition for the integral of the r.h.s. of (3.14) to be independent of T is

$$k = 0 . \tag{3.15}$$

Crucially, this condition is also sufficient to ensure that the integral of $f(T,y)$ is independent of T. This follows from the observation that

$$\epsilon y (dy)^n + d\left[\frac{n}{n-1} \frac{1}{yTy} \epsilon y (Ty)(dy)^{n-1} \right] = -\frac{2}{n-1} \left[\frac{yT^2y}{(yTy)^2} - \frac{1}{2} \frac{\operatorname{Tr} T}{yTy} \right] \epsilon y (dy)^n . \tag{3.16}$$

The integral of the l.h.s. over S^n is manifestly independent of T by virtue of Stokes' theorem.

To summarize, our assumptions are: (1) turning on T deforms the flux by an exact piece; (2) the perfect square in the Einstein-Hilbert and the contribution from the kinetic term for $\hat{\mathcal{F}}_n$ cancel against each other at the level of the S^n integrand. Under these assumptions, we determine uniquely the form of $f(y,T)$ and therefore of $\hat{\mathcal{F}}_n$ via (3.7), up to the overall normalization. The latter is readily fixed by requiring $\int_{S^n} \hat{\mathcal{F}}_n = 1$. The result is

$$\hat{\mathcal{F}}_n = \frac{1}{\mathcal{V}_n n!} \epsilon y (dy)^n + \frac{1}{\mathcal{V}_n n!} d\left[\frac{n}{n-1} \frac{1}{yTy} \epsilon y (Ty)(dy)^{n-1} \right] , \tag{3.17}$$

in perfect agreement with (2.5) and [36]. Both the functional form of the exact deformation of the flux, as well as its numerical coefficient relative to the undeformed flux, are determined.[8]

Remarkably, in the process of our derivation we have found the condition $k = 0$, which is a constraint on the possible values of n, D. In fact, this condition selects precisely the

[8] More precisely, the $(n-1)$-form inside the total derivative is completely fixed up to shifts by closed $(n-1)$-forms, which clearly would not modify the expression for $\hat{\mathcal{F}}_n$.

values for n, D for which a consistent truncation is possible with the field content we are considering. Setting (3.12) equal to zero, one finds that the only integer solutions with $n < D$ are given by

$$(n, D) = (1, 2), (4, 11), (5, 10), (7, 11)\,. \tag{3.18}$$

While the first of these would describe reduction on a circle from 2d to 1d, the remaining three correspond to familiar consistent truncations. This is a striking result. Based on the requirement that the d-dimensional scalar potential be quadratic in the scalars T_{IJ} and that the flux be deformed by an exact piece, we are able to back out the possible dimensions of the sphere and the lower-dimensional model. Similar conclusions were drawn in [42] based on slightly different arguments.[9]

3.2 Models with dilaton

The analysis of the previous section can be repeated for the cases in which the kinetic term for $\hat{\mathcal{F}}_n$ comes with a non-trivial dilaton prefactor. The relevant terms in the D-dimensional action are now

$$S_{(D)} = \int \left[\hat{R}\hat{*}1 - \frac{1}{2g_{\mathcal{F}}^2} e^{-sa\hat{\phi}} \hat{\mathcal{F}}_n \wedge \hat{*}\hat{\mathcal{F}}_n - \frac{1}{2} d\hat{\phi} \wedge \hat{*} d\hat{\phi} \right], \tag{3.19}$$

where a is a positive constant and s is a sign. Below we will see how the value (2.2) for a is selected. Notice that the D-dimensional dilaton $\hat{\phi}$ has a canonically normalized kinetic term. Our goal is to determine the scalar potential of the d-dimensional theory. As a result, we may work under the simplifying assumptions

$$A^{IJ} = 0\,, \qquad dT_{IJ} = 0\,, \qquad dY = 0\,, \tag{3.20}$$

where Y encodes the d-dimensional dilaton.

The metric ansatz must be modified, to take into account Y. We adopt the following modification of (3.3),

$$d\hat{s}_D^2 = (yTy)^{b_1} Y^{b_1'} \left[ds_d^2 + g^{-2} (yTy)^{b_2} Y^{b_2'} T_{IJ}^{-1} dy^I dy^J \right], \tag{3.21}$$

where b_1, b_2, b_1', b_2' are constant parameters, to be fixed momentarily. The reduction of the D-dimensional Einstein-Hilbert term can be performed analogously to the previous section. The result reads

$$\begin{aligned} \int_{S^n} \hat{R}\hat{*}1 \supset{}& g^{-n} Y^{\frac{b_1' D}{2} + \frac{b_2' n}{2} - b_1'} \sqrt{-g_d} R[g_d] \int_{S^n} d^n\xi \sqrt{\mathring{g}} (yTy)^{\frac{1}{2}b_1(D-2) + \frac{1}{2}b_2 n + \frac{1}{2}} \\ &+ g^{2-n} Y^{\frac{b_1' D}{2} + \frac{b_2' n}{2} - b_1' - b_2'} \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}} (yTy)^{\frac{1}{2}b_1(D-2) + \frac{1}{2}b_2(n-2) + \frac{1}{2}} \mathcal{G}\,, \end{aligned} \tag{3.22}$$

with the same $\mathcal{G}$ as in (3.5).

[9] Compared to [42], our argument is based on a cancellation at the level of the *integrand* on S^n. Thus, we do not have to evaluate explicitly S^n integrals of rational functions of y, T, which in general yield non-rational functions of the entries of T, see the example (B.41) in appendix B.1.

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The reduction of the kinetic term for $\hat{\mathcal{F}}_n$ is also analogous to the previous case. We adopt the same ansatz for the flux as (3.7), except that now we allow f to be a function of Y, too,

$$\hat{\mathcal{F}}_n = \frac{1}{n!} f(y,T,Y)\epsilon_{I_0 I_1 \dots I_n} y^{I_0} dy^{I_1} \wedge \dots \wedge dy^{I_n} \,. \tag{3.23}$$

We also use the following ansatz for the D-dimensional dilaton,

$$e^{s\hat{\phi}} = (yTy)^{b_3} Y^{b'_3} \,, \tag{3.24}$$

where b_3, b'_3 are real constant parameters. Reducing the kinetic term for $\hat{\mathcal{F}}_n$ yields

$$\begin{aligned} &\int_{S^n} e^{-sa\hat{\phi}} \hat{\mathcal{F}}_n \wedge \hat{*}\hat{\mathcal{F}}_n = \qquad\qquad (3.25)\\ &= g^n Y^{\frac{b'_1 D}{2} - b'_1 n - \frac{b'_2 n}{2} - ab'_3} \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}} (yTy)^{\frac{b_1 D}{2} - b_1 n - \frac{b_2 n}{2} - \frac{1}{2} - ab_3} f(y,T,Y)^2 \,. \end{aligned}$$

Finally, we have to take into account the contributions to the d-dimensional scalar potential originating from the kinetic term for $\hat{\phi}$. We have

$$\begin{aligned} -\frac{1}{2}\int_{S^n} d\hat{\phi} \wedge \hat{*} d\hat{\phi} &= 2g^{2-n} b_3^2 Y^{\frac{b'_1 D}{2} - b'_1 + \frac{b'_2 n}{2} - b'_2} \times \qquad\qquad (3.26)\\ &\times \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}}\, (yTy)^{\frac{b_1 D}{2} - b_1 + \frac{b_2 n}{2} - b_2 + \frac{1}{2}} \left[\frac{(yT^2y)^2}{(yTy)^3} - \frac{(yT^3y)}{(yTy)^2} \right] . \end{aligned}$$

In order to get a canonical Einstein-Hilbert term in d dimensions from (3.22), we set

$$\frac{1}{2} b_1 (D-2) + \frac{1}{2} b_2 n + \frac{1}{2} = 0 \,, \qquad \frac{b'_1 D}{2} + \frac{b'_2 n}{2} - b'_1 = 0 \,. \tag{3.27}$$

Exactly as in the previous section, we can examine how the second line of (3.22) scales with T, impose an overall scaling of degree 2, and thus obtain another linear relation between b_1 and b_2. The net result is to fix these parameters as before, see (3.10). Based for instance on the results of [35], we know that the scalar potential, written as a function of unimodular T and Y, is also homogeneous in rescalings of Y, with degree $2/(n+1)$. We may alternatively regard this condition as a way of (partially) fixing ambiguities related to redefinitions of Y. If we demand that the second line of (3.22) be homogeneous of degree $2/(n+1)$ in Y we get a second linear relation in b'_1, b'_2, which combined with (3.27) gives us

$$b'_1 = \frac{2n}{(D-2)(n+1)} \,, \qquad b'_2 = -\frac{2}{n+1} \,. \tag{3.28}$$

These values guarantee that (3.21) matches with our previous expression (2.3).

Having fixed b_1, b_2 according to (3.10), we can mimick the same steps as in the previous section. By adding a suitable total divergence on S^n, we can eliminate the yT^3y terms from the sum of (3.22) and (3.26), obtaining the simpler form

$$\begin{aligned} &\int_{S^n} \left[\hat{R}\hat{*}1 - \frac{1}{2} d\hat{\phi} \wedge \hat{*} d\hat{\phi} \right] = g^{-n} \mathcal{V}_n \sqrt{-g_d} R[g_d] + g^{2-n} Y^{\frac{2}{n+1}} \sqrt{-g_d} \int_{S^n} d^n\xi \sqrt{\mathring{g}}\, \tilde{\mathcal{I}} \,, \\ &\tilde{\mathcal{I}} = 2\left[\frac{yT^2y}{yTy} - \frac{4-\tilde{k}}{8} \mathrm{Tr}\, T \right]^2 - \left(1 + \frac{\tilde{k}}{2}\right) \mathrm{Tr}\,(T^2) + \left(\frac{1}{2} + \frac{\tilde{k}}{4} - \frac{\tilde{k}^2}{32}\right) (\mathrm{Tr}\, T)^2 \,, \qquad (3.29) \end{aligned}$$

where we have introduced

$$\tilde{k} = \frac{(n-1)(D-n-1)}{D-2} - 2 + 2b_3^2 \,. \tag{3.30}$$

Remarkably, the functional form of the integrand in (3.29) is exactly the same as in (3.11), up to the replacement $k \to \tilde{k}$.

Making use of (3.10), (3.28), and (3.30), the contribution (3.25) from the flux kinetic term takes the form

$$\begin{aligned} &-\frac{1}{2g_{\mathcal{F}}^2}\int_{S^n} e^{-sa\hat{\phi}}\hat{\mathcal{F}}_n \wedge \hat{*}\hat{\mathcal{F}}_n = \\ &\quad = -\frac{1}{2g_{\mathcal{F}}^2} g^n Y^{\frac{2n(D-n-1)}{(D-2)(n+1)} - ab_3'} \sqrt{-g_d}\int_{S^n} d^n\xi \sqrt{\mathring{g}} (yTy)^{2+\tilde{k}-ab_3-2b_3^2} f(y,T,Y)^2 \,. \end{aligned} \tag{3.31}$$

Once again, a natural simple mechanism suggests itself to eliminate non-rational functions of T originating from the square in $\tilde{\mathcal{I}}$ in (3.29): a cancellation against f^2 at the level of the integrand. We thus proceed assuming

$$f(y,T,Y) = 2g_{\mathcal{F}} g^{1-n} Y^{\frac{ab_3'}{2} + \frac{D-Dn+n^2+n-2}{(D-2)(n+1)}} (yTy)^{-\frac{\tilde{k}}{2}+\frac{1}{2}ab_3+b_3^2}\left[\frac{yT^2y}{(yTy)^2} - \frac{4-\tilde{k}}{8}\frac{\operatorname{Tr}T}{yTy}\right]. \tag{3.32}$$

As in the previous section, we assume that $f(y,T,Y)$ originates from an exact deformation of the round volume form on S^n. A necessary condition for the identification (3.32) to be possible is therefore that the S^n integral of the r.h.s. with measure $\sqrt{\mathring{g}}$ be independent of T and Y. Independence on Y is easily established by tuning the value of b_3',

$$b_3' = \frac{2(n-1)(D-n-2)}{a(D-2)(n+1)} \,. \tag{3.33}$$

This value guarantees that (3.24) agrees with our previous expressions (2.4), (2.7). In appendix B.2 we study when the integral of (3.32) becomes independent of T, as we vary the parameters $\tilde{k}$, $\frac{1}{2}ab_3 + b_3^2$. We identify two possibilities.

The first possibility is to set

$$\tilde{k} = 0 \,, \qquad \frac{1}{2}ab_3 + b_3^2 = 0 \,. \tag{3.34}$$

For these values of $\tilde{k}$, $\frac{1}{2}ab_3 + b_3^2$ the r.h.s. of (3.32) becomes the same as in the case without dilaton (3.14) with $k = 0$. We already know from the discussion around (3.16) that the integral is independent of T, for any symmetric matrix T, unimodular or not.

As observed above, in the cases without a dilaton the condition $k = 0$ fixes the allowed values of D, n. In the present context, we can solve the conditions (3.34) in terms of the parameters a and b_3. We find

$$a = -2b_3 \,, \qquad a^2 = 4 - \frac{2(n-1)(D-n-1)}{D-2} \,. \tag{3.35}$$

This is in agreement with (2.2): we have thus found a different argument that singles out the value of a^2 for which the truncation is possible [35]. Moreover, the relation $b_3 = -a/2$ ensure the compatibility of (3.24) with our previous formula (2.4).

Let us now turn to the second possible choice for $\tilde{k}$, $\frac{1}{2}ab_3 + b_3^2$ that renders the integral of (3.32) independent of T,

$$\tilde{k} = \frac{4(n+1)}{n+3}\,, \qquad \frac{1}{2}ab_3 + b_3^2 = -\frac{(n-1)(n+1)}{2(n+3)}\,. \tag{3.36}$$

This case is qualitatively different from (3.34): as discussed in appendix B.2, if we choose (3.36) the integral of (3.32) becomes a constant times $(\det T)^{-\frac{1}{2}}$, hence a constant for unimodular T, but not for a generic symmetric T. The values (3.36) are not compatible with the uplift formulae recorded earlier and checked against the literature. The interpretation of (3.36) seems more elusive and is left for future investigation.

4 Bott-Cattaneo formula and S^4 truncation

In the previous section we have studied a simplified setting in which the SO$(n+1)$ gauge fields are set to zero, and the T scalars to a constant. We have demonstrated that $\hat{\mathcal{F}}_n$ must contain an exact term given by the derivative of the $(n-1)$-form $(yTy)^{-1}\epsilon y(Ty)(dy)^{n-1}$, with the appropriate numerical prefactor. Let us now suppose we turn on the SO$(n+1)$ gauge fields, and we relax the assumption that T is constant. Gauge invariance requires the replacement $dy \to Dy$, leading to establish that the exact deformation of $\hat{\mathcal{F}}_n$ must contain the term $(yTy)^{-1}\epsilon y(Ty)(Dy)^{n-1}$. A priori, additional terms might be generated inside the exact deformation, proportional to F and/or DT. The explicit analysis of the uplift formulae in all examples collected in table 1, however, shows that this does not happen. While we cannot furnish a first-principle proof of this fact, we can point out an interesting connection with some results of Bott and Cattaneo [48] regarding the fiber integrals of global angular forms for even-dimensional spheres.

Reminder on the Bott-Cattaneo formula. Recall from section 2.1 that the canonical global angular form e_n for an n-sphere with n even is a closed form in the total space of an S^n fibration over a base space B. In this setting, we can consider the kth power of e_n and fiber-integrate it along the S^n directions to obtain a closed $n(k-1)$-form on the base space B. The result of this operation is the content of the Bott-Cattaneo formula [48],

$$\int_{S^{2m}} (e_{2m})^{2s+2} = 0\,, \qquad \int_{S^{2m}} (e_{2m})^{2s+1} = 2^{-2s}(p_m)^s\,, \qquad s = 0,1,2,\ldots \tag{4.1}$$

Here $n = 2m$ is the dimension of the sphere, and p_m denotes the mth Pontryagin *form* constructed with the curvature F^{IJ} of the SO$(n+1)$ gauge fields. This is a polynomial in F^{IJ} of degree m, a closed $4m$-form on the base space B with integral periods. Its cohomology class corresponds to an element of $H^{2m}(B;\mathbb{Z})$.

Application to the reduction of $D = 11$ supergravity on S^4. In this case, the flux $\hat{\mathcal{F}}_4$ that threads the S^4 is the $\hat{G}_4$ flux of 11d supergravity. The 11d action contains a Chern-Simons term $\hat{C}_3\hat{G}_4\hat{G}_4$. We are interested in analyzing the 7d couplings that are generated by fiber-integrating this Chern-Simons term along S^4. Following [43, 44], this is most easily performed by regarding 11d spacetime as the boundary of an auxiliary 12d space, which is an S^4 fibration over an 8d base B_8. The physical 7d spacetime is the boundary of B_8. We can consider the formal 12-form $\hat{G}_4^3$, do the fiber-integration to get an 8-form on B_8, and consider its restriction to the 7d boundary of B_8.

We know that $\hat{G}_4 \propto \hat{\mathcal{F}}_4$ takes the form

$$\hat{\mathcal{F}}_4 = e_4 + d\omega_3 \,, \tag{4.2}$$

where ω_3 is a globally-defined 3-form. We thus can write

$$\int_{S^4} \hat{\mathcal{F}}_4^3 = \int_{S^4} \left(e_4^3 + d\omega_{11}\right) = \int_{S^4} e_4^3 + d\int_{S^4} \omega_{11} = \tfrac{1}{4}p_2 + d\int_{S^4} \omega_{11} \,. \tag{4.3}$$

In the previous expressions we have defined the 11-form

$$\omega_{11} = 3e_4^2 \wedge \omega_3 + 3e_4 \wedge \omega_3 \wedge d\omega_3 + \omega_3 \wedge (d\omega_3)^2 \,, \tag{4.4}$$

and we have used the fact that fiber-integration and exterior derivative commute. In the final step of (4.3) we have applied the Bott-Cattaneo formula (4.1).

The r.h.s. of (4.3) is an 8-form that encodes topological couplings in the 7d action. The term with p_2 in the 8d bulk corresponds to a Chern-Simons form on the 7d boundary. Such coupling is indeed found in the action of 7d maximal SO(5) gauged supergravity [49].

The term $d\int_{S^4} \omega_{11}$ in (4.3) is an exact deviation from the Bott-Cattaneo formula. At the level of cohomology classes, the Bott-Cattaneo relation persists, but it is no longer valid at the level of differential forms. The term $d\int_{S^4} \omega_{11}$ in the 8d bulk can generate additional topological couplings $\int_{S^4} \omega_{11}$ on the 7d boundary.

We may now observe the following. We know that the consistent truncation selects the following form for ω_3,

$$\omega_3 = \frac{1}{\mathcal{V}_4 4!} \frac{4}{3} \frac{1}{yTy} \epsilon y(Ty)(Dy)^3 \,. \tag{4.5}$$

Plugging this ω_3 into (4.4), we verify that the 7-form $\int_{S^4} \omega_{11}$ vanishes identically.[10] Thus, the Bott-Cattaneo relation persist at the level of differential forms, and no additional topological couplings are generated in the 7d action.

This remarkable property is generically lost if we consider a more general ω_3. For example, we can imagine adding a term with one field strength F, of the form

$$\omega_3 = \frac{1}{\mathcal{V}_4 4!} \left[\frac{4}{3} \frac{1}{yTy} \epsilon y(Ty)(Dy)^3 + \xi_2 \frac{1}{yTy} \epsilon y(Ty) F Dy \right] , \tag{4.6}$$

where ξ_2 is a constant coefficient. Notice that the new term goes to zero if we set $T = \mathbb{I}$. We can repeat the computation of $\int_{S^4} \omega_{11}$ with the new ω_3 in (4.6). The result is non-zero, of the schematic form

$$\int_{S^4} \omega_{11} \sim \mathcal{S}(T)_{I_1 \dots I_8} DT^{I_1 I_2} \wedge F^{I_3 I_4} \wedge F^{I_5 I_6} \wedge F^{I_7 I_8} \,. \tag{4.7}$$

[10] In fact, this is true as soon as ω_3 is proportional to $Dy^I \wedge Dy^J \wedge Dy^K$, since it is easy to verify that, in this case, no term in ω_{11} has four factors Dy, which is necessary to yield a non-zero result upon fiber-integrating over S^4.

Here $\mathcal{S}(T)_{I_1 \ldots I_8}$ stands for a tensor with 8 free SO(5) indices, constructed with T. A non-zero $\int_{S^4} \omega_{11}$ implies that the Bott-Cattaneo formula is no longer valid at the level of forms.

To summarize, the consistent truncation selects an exact deformation with precisely three Dy legs. Such a deformation is non-generic, in that it ensures that the Bott-Cattaneo relation persists at the level of differential forms. A deviation from a term with three Dy's generically induces topological terms in the 7d action of the form (4.7).[11]

5 Outlook

In this work we have made progress towards a unified approach to consistent truncations on spheres, centered around the classical geometric notion of global angular form. The latter is a mathematical object defined for an n-sphere bundle over a base space. In applications to consistent truncations, the base space is the spacetime of the lower-dimensional theory, and the sphere fiber is the internal space used in the compactification.

Our main results are equations (2.3)–(2.5). They describe universal features of the consistent Kaluza-Klein ansätze that are common to all the cases listed in table 1. In particular, our formulae capture the contributions of the $\mathrm{SO}(n+1)$ gauge fields, the $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ scalars describing the naive deformation space of S^n, and of the dilaton (when present). The ansatz (2.5) for the flux threading the sphere takes a particularly compact form, with the $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ scalars entering only via an exact shift. We have also shown how this exact shift is determined by computing the scalar potential of the lower-dimensional model by dimensional reduction, and imposing mild constraints on its functional dependence on the scalars.

An important problem that deserves further investigation is the analysis of the 'extra fields' that are required for consistency of the truncation in some cases (see last column of table 1). In particular, it is desirable to identify a way of detecting the necessity of extra fields based on the uplift formulae (2.3)–(2.5), without relying on the structure of supersymmetry multiplets in the lower-dimensional supergravity theory, and without performing a full explicit check that the lower-dimensional equations of motion imply the higher-dimensional ones. Progress in this direction would be particularly useful in light of applications to consistent truncations with less-than-maximal supersymmetry. In this context, matter supermultiplets can appear in the lower-dimensional model, making it more challenging to identify which modes should be retained for consistency of the reduction. It would also be beneficial to try to uncover general patterns in the way in which extra fields, when present, modify the uplift formulae (2.3)–(2.5), with the aim to learn lessons to be applied to more general setups with less supersymmetry.

In our approach, it is convenient to regard the construction of the consistent Kaluza-Klein ansatz for the flux as a two-step procedure. The first step is the identification of

[11]Such terms have four derivatives. This is more than the three derivatives in the supergravity action built by the Noether procedure from the Einstein-Hilbert term. (The term with three derivatives is the Chern-Simons term $d^{-1}p_2$.) In a maximal supergravity theory, the first higher-derivative corrections are expected to emerge at eight derivatives. These naive considerations suggest that terms such as (4.7) might be incompatible with supersymmetry.

the flux with the relevant (truncated) global angular form e'_n, see (1.1). At this stage we capture the $\mathrm{SO}(n+1)$ gauge fields related to the isometry of the sphere. As observed in the introduction, this stage is sufficient for the determination of the Chern-Simons terms in the lower-dimensional supergravity (for the AdS_7 and AdS_5 cases). The second step is the analysis of the effect of turning on scalar deformations.

The analog of the starting point (1.1) is available for more general setups, in which the internal space is not necessarily a sphere and the d-dimensional theory has less than maximal supersymmetry. In fact, we may consider replacing S^n with a compact internal space M_n, which is then fibered over a base space B_d, $M_n \hookrightarrow X_{d+n} \to B_d$. If M_n has a continuous isometry group G, we turn on G-connections on the bundle X_{d+n}. They correspond to gauge fields with gauge group G in the d-dimensional theory formulated on B_d. In this context, an important task is the identification of globally defined forms on the total space X_{d+n} which reduce to a given closed form on M_n if restricted to the fiber directions.[12] In [45–47] this problem has been studied systematically to compute anomalies in brane engineering and holography (see also [61]). The methods and results of [45–47] could also provide starting points for the construction of consistent Kaluza-Klein ansätze.[13]

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A Analysis of uplift formulae

A.1 Conventions for round spheres

The unit round S^n is defined by the equation

$$\delta_{IJ} y^I y^J = 1 \tag{A.1}$$

in $\mathbb{R}^{n+1}$ with Cartesian coordinates y^I, $I = 1, \dots, n+1$. Throughout this appendix we raise/lower I indices with δ. We use ξ^m, $m = 1, \dots, n$ for local coordinates on S^n, with $\partial_m = \partial / \partial \xi^m$. The round metric on S^n reads

$$\mathring{g}_{mn} = \delta_{IJ} \partial_m y^I \partial_n y^J \,. \tag{A.2}$$

[12] Mathematically, this is related to promoting an element of $H^\bullet(M_n;\mathbb{R})$ to a G-equivariant cohomology class in $H^\bullet_G(M_6;\mathbb{R})$. This process can be obstructed [57]. The physical interpretation of such obstructions involves the Stückelberg mechanism for p-form gauge fields [58] (see also [59, 60] for an analysis of this phenomenon in 4d QFTs engineered with M5-branes).

[13] For instance, M_6 could be an S^4 bundle over a Riemann surface. See [62, 63] for consistent truncations with such an internal space.

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It admits the Killing vectors

$$\mathcal{K}^m_{IJ} = \mathring{g}^{mn}(y_I \partial_n y_J - y_J \partial_n y_I)\,, \tag{A.3}$$

where $\mathring{g}^{mn}$ is the inverse of $\mathring{g}_{mn}$. A useful identity is

$$\mathring{g}^{mn}\partial_m y^I \partial_n y^J = \delta^{IJ} - y^I y^J\,. \tag{A.4}$$

The quantities y^I, regarded as a set of $n+1$ scalar functions on S^n, satisfy

$$\mathring{\nabla}_m \mathring{\nabla}_n y^I = -y^I \mathring{g}_{mn}\,, \tag{A.5}$$

where $\mathring{\nabla}_m$ denotes the Levi-Civita connection associated to $\mathring{g}_{mn}$. As a consequence of (A.5), the quantities y^I are eigenfunctions of the Laplacian constructed with the round metric,

$$\mathring{g}^{mn}\mathring{\nabla}_m \mathring{\nabla}_n y^I = -n y^I\,. \tag{A.6}$$

A.2 Formulae for Hodge stars

Let us consider a D-dimensional metric of the form

$$d\hat{s}^2_D = e^{2A} ds^2_d + g_{mn} D\xi^m D\xi^n\,, \tag{A.7}$$

where ξ^m, are local coordinates on the n-sphere with constrained coordinates y^I. The 1-forms $D\xi^m$ are related to the 1-forms Dy^I in (2.11) by

$$Dy^I = \partial_m y^I D\xi^m\,, \qquad D\xi^m = d\xi^m - \tfrac{1}{2} g A^{IJ} \mathcal{K}^m_{IJ}\,, \tag{A.8}$$

with $\mathcal{K}^m_{IJ}$ as in (A.3). The relation (A.4) is useful in verifying the compatibility between (A.8) and (2.11). We allow the quantities e^{2A} and for g_{mn} in (A.7) to depend both on the external coordinates and on the coordinates on S^n.

Suppose α is a q-form with legs along the external spacetime ds^2_d only. The following identity holds,

$$\begin{aligned}\hat{*}(\alpha \wedge Dy^{I_1} \wedge \dots \wedge Dy^{I_p}) = (-)^{p(d-q)} e^{(d-2q)A} \frac{\sqrt{g}}{\sqrt{\mathring{g}}} (*\alpha) \wedge \\ \wedge \frac{1}{(n-p)!} \epsilon_{J_0 J_1 \dots J_p K_1 \dots K_{n-p}} y^{J_0} Q^{I_1 J_1} \dots Q^{I_p J_p} Dy^{K_1} \wedge \dots \wedge Dy^{K_{n-p}}\,.\end{aligned} \tag{A.9}$$

The symbol $\hat{*}$ on the l.h.s. denotes the Hodge star with respect to the metric (A.7), while $*$ on the r.h.s. is the Hodge star with respect to the external metric ds^2_d. The quantities g, $\mathring{g}$ are the determinants of g_{mn}, $\mathring{g}_{mn}$, respectively. The symmetric tensor Q^{IJ} is defined as

$$Q^{IJ} = g^{mn} \partial_m y^I \partial_n y^J\,, \tag{A.10}$$

where g^{mn} is the inverse of g_{mn} in (A.7).

The formula (A.9) takes a simpler form if specialized to a metric g_{mn} of the form

$$g_{mn} = e^{2A'} T^{-1}_{IJ} \partial_m y^I \partial_n y^J\,, \tag{A.11}$$

where A' is an arbitrary function of external and internal coordinates and T_{IJ} is a symmetric unimodular matrix, independent on the coordinates on S^n. In this case, the inverse g^{mn} of g_{mn} can be written in closed form in terms of the Killing vectors (A.3),

$$g^{mn} = e^{-2A'} \frac{1}{2(yTy)} T^{IJ} T^{KL} \mathcal{K}_{IK}^m \mathcal{K}_{JL}^m , \tag{A.12}$$

as it may be verified using (A.4). Plugging this expression into the Q tensor (A.10) and using again (A.4) we obtain

$$Q^{IJ} = e^{-2A'} \left[T^{IJ} - \frac{(Ty)^I (Ty)^J}{yTy} \right] . \tag{A.13}$$

The determinant of g_{mn} in (A.11) can also be written in closed form (see e.g. [36]),

$$\det g_{mn} = e^{2A'n} (yTy) \det \mathring{g}_{mn} , \qquad \frac{\sqrt{g}}{\sqrt{\mathring{g}}} = e^{A'n} (yTy)^{1/2} , \tag{A.14}$$

where we recalled that T is unimodular.

Let us close this section by sketching the derivation of (A.9). We can write the metric (A.7) in the form

$$d\hat{s}_D^2 = e^{2A} ds_d^2 + \delta_{\bar{m}\bar{n}} e^{\bar{m}} e^{\bar{n}} , \qquad e^{\bar{m}} = e^{\bar{m}}{}_p D\xi^p , \tag{A.15}$$

where a bar denotes a flat index of S^n and the vielbein components $e^{\bar{m}}{}_p$ satisfy $\delta_{\bar{m}\bar{n}} e^{\bar{m}}{}_p e^{\bar{n}}{}_q = g_{pq}$. Note that $e^{\bar{m}}{}_p$ depends in general on both internal and external coordinates. By using the inverse $e_{\bar{m}}{}^p$ of $e^{\bar{m}}{}_p$ we can write

$$\alpha \wedge Dy^{I_1} \wedge \ldots Dy^{I_p} = \partial_{m_1} y^{I_1} \ldots \partial_{m_p} y^{I_p} e_{\bar{n}_1}{}^{m_1} \ldots e_{\bar{n}_p}{}^{m_p} \alpha \wedge e^{\bar{n}_1} \wedge \cdots \wedge e^{\bar{n}_p} . \tag{A.16}$$

The Hodge star of $\alpha \wedge e^{\bar{n}_1} \wedge \cdots \wedge e^{\bar{n}_p}$ splits into the wedge product of $*\alpha$ and $*_g(e^{\bar{n}_1} \wedge \cdots \wedge e^{\bar{n}_p})$, where $*_g$ is the Hodge star with respect to g_{mn}. In the splitting, we generate a suitable power of the warp factor A, and the sign factor $(-)^{p(d-q)}$ from a reordering of indices.

We may now evaluate the quantity

$$\begin{aligned} &\partial_{m_1} y^{I_1} \ldots \partial_{m_p} y^{I_p} e_{\bar{n}_1}{}^{m_1} \ldots e_{\bar{n}_p}{}^{m_p} *_g (e^{\bar{n}_1} \wedge \cdots \wedge e^{\bar{n}_p}) = \\ &= \frac{1}{(n-p)!} \partial_{m_1} y^{I_1} \ldots \partial_{m_p} y^{I_p} e_{\bar{n}_1}{}^{m_1} \ldots e_{\bar{n}_p}{}^{m_p} \epsilon^{\bar{n}_1 \ldots \bar{n} \bar{r}_1 \ldots \bar{r}_{n-p}} e_{\bar{r}_1} \wedge \cdots \wedge e_{\bar{r}_{n-p}} . \end{aligned} \tag{A.17}$$

Here the epsilon symbol with flat indices takes values in the set $\{0, \pm 1\}$. Making use of $e^{\bar{m}} = e^{\bar{m}}{}_p D\xi^p$ we can also recast the same quantity in a way that does not make explicit reference to the vielbein,

$$\begin{aligned} &\partial_{m_1} y^{I_1} \ldots \partial_{m_p} y^{I_p} e_{\bar{n}_1}{}^{m_1} \ldots e_{\bar{n}_p}{}^{m_p} *_g (e^{\bar{n}_1} \wedge \cdots \wedge e^{\bar{n}_p}) = \\ &= \frac{1}{(n-p)!} \partial_{m_1} y^{I_1} \ldots \partial_{m_p} y^{I_p} g^{m_1 s_1} \ldots g^{m_p s_p} \epsilon_{s_1 \ldots s_p r_1 \ldots r_{n-p}} D\xi^{r_1} \wedge \cdots \wedge D\xi^{r_{n-p}} . \end{aligned} \tag{A.18}$$

Now the epsilon tensor with lower curved indices stands for the volume form of the metric g_{mn} and therefore takes values in the set $\{0, \pm\sqrt{g}\}$.

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The final step is to relate the epsilon tensor of the metric g_{mn} to that of the round metric $\mathring{g}_{mn}$, and write the latter in terms of the contrained coordinates y^I,

$$\epsilon_{m_1\ldots m_n} = \frac{\sqrt{g}}{\sqrt{\mathring{g}}}\mathring{\epsilon}_{m_1\ldots m_n} = \frac{\sqrt{g}}{\sqrt{\mathring{g}}}\epsilon_{I_0 I_1\ldots I_n} y^{I_0}\partial_{m_1}y^{I_1}\ldots\partial_{m_n}y^{I_n}\,. \tag{A.19}$$

Plugging this into (A.18) and making use of $\partial_m y^I D\xi^m = Dy^I$ and the definition (A.10) of the tensor Q^{IJ}, we arrive at the desired formula (A.9).

A.3 $D = 11$ supergravity on S^7

The complete uplift formulae for this consistent truncation are given in [18], including all scalars from the coset $E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$. The latter enter the uplift formulae via the symmetric matrices $\mathcal{M}_{MN}$, $\mathcal{I}_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma}$. Here $M, N = 1,\ldots,56$ are indices of the **56** of $E_{7(7)}$, $\Lambda,\Sigma = 1,\ldots,28$ are indices of the **28** of $\mathrm{SL}(8,\mathbb{R})$. A Λ index is equivalent to pair of antisymmetrized fundamental indices $[IJ]$ of $\mathrm{SL}(8,\mathbb{R})$. In terms of $\mathrm{SL}(8,\mathbb{R}) \subset E_{7(7)}$, a lower M index splits into a lower Λ index, and an upper Λ index. The matrix $\mathcal{M}_{MN}$ can be written in block form as

$$\mathcal{M}_{MN} = \begin{pmatrix} \mathcal{M}_{\Lambda\Sigma} & \mathcal{M}_{\Lambda}{}^{\Sigma} \\ \mathcal{M}^{\Lambda}{}_{\Sigma} & \mathcal{M}^{\Lambda\Sigma} \end{pmatrix}\,. \tag{A.20}$$

The four blocks of $\mathcal{M}_{MN}$ and the matrices $\mathcal{I}_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma}$ are related by (see e.g. [33])

$$\mathcal{M}_{\Lambda\Sigma} = -(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma}\,, \quad \mathcal{M}_{\Lambda}{}^{\Sigma} = \mathcal{M}^{\Sigma}{}_{\Lambda} = (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma}\,, \quad \mathcal{M}^{\Lambda\Sigma} = -(\mathcal{I}^{-1})^{\Lambda\Sigma}\,. \tag{A.21}$$

On the r.h.s. of the previous relations, Λ indices are contracted according to matrix multiplication.[14] The matrix $\mathcal{I}_{\Lambda\Sigma}$ is invertible and negative definite. If we set to zero the 35 pseudoscalars, and only retain the 35 proper scalars, we have

$$\mathcal{R}_{\Lambda\Sigma} = 0\,, \qquad \mathcal{I}_{\Lambda\Sigma} = \mathcal{I}_{[IJ][KL]} = -T^{-1}_{IK}T^{-1}_{JL} + T^{-1}_{IL}T^{-1}_{JK}\,. \tag{A.22}$$

The quantity T^{-1}_{IJ} is the inverse of the symmetric unimodular matrix parametrizing the 35 proper scalars of $\mathrm{SL}(8,\mathbb{R})/\mathrm{SO}(8)$.

The specialization (A.22) implies considerable simplifications in the general uplift formulae of [18]. The expressions for the internal metric and the warp factor are given in (24) and (26) of [18] and can be unpacked with the help of the expressions (C.3) in [33] for the generators of $\mathrm{SL}(8,\mathbb{R}) \subset E_{7(7)}$. The result is

$$d\hat{s}^2_{11} = (yTy)^{2/3}ds^2_4 + g^{-2}(yTy)^{-1/3}T^{-1}_{IJ}Dy^I Dy^J\,, \tag{A.23}$$

with (yTy) and Dy^I as in (2.3), (2.11).

The uplift formula for $\hat{G}_4$ is (27) in [18] (where the 4-form flux is denoted $\hat{F}_{(4)}$). Thanks to (A.22), the term dA drops away. The quantities $\mathcal{H}^{IJ}_{(4)}$, $\mathcal{H}_{(3)I}{}^{J}$, $\tilde{\mathcal{H}}_{(2)IJ}$ in (27) of [18] can be evaluated using (16) in [18]. The components of the X tensor that appears there can

[14] When we identify a Λ index with an antisymmetric pair $[IJ]$, we insert a factor 1/2 for each contracted pair. For instance, $(\mathcal{R}\mathcal{I}^{-1})_{[I_1 I_2]}{}^{[J_1 J_2]} = \frac{1}{2}\mathcal{R}_{[I_1 I_2][K_1 K_2]}(\mathcal{I}^{-1})^{[K_1 K_2][J_1 J_2]}$ and $(\mathcal{I}\mathcal{I}^{-1})_{[I_1 I_2]}{}^{[J_1 J_2]} = \frac{1}{2}\mathcal{I}_{[I_1 I_2][K_1 K_2]}(\mathcal{I}^{-1})^{[K_1 K_2][J_1 J_2]} = \mathbb{I}_{[I_1 I_2]}{}^{[J_1 J_2]} := 2\delta^{J_1}_{[I_1}\delta^{J_2}_{I_2]}$.

be found for instance in (C.9), (C.10) of [33] (for the SO(8) gauging, plug $\theta_{AB} = \delta_{AB}$ and $\xi^{AB} = 0$ in those equations). We arrive at the following expression for the 4-form flux (in our notation),

$$\hat{G}_4 = -\frac{1}{2g^2}(*F)^{JL} \wedge Dy^I \wedge Dy^K T^{-1}_{IJ} T^{-1}_{KL} - \frac{1}{g} T^{JK} y_K Dy^I \wedge *DT^{-1}_{IJ} + g\mathrm{vol}_4 \Big[(\mathrm{Tr}T)(yTy) - 2(yTTy) \Big] . \tag{A.24}$$

Here $yTTy = y_I T^{IJ} T_{JK} y^K$ and vol_4 is the volume form of the external metric ds_4^2. We may now compute the Hodge star of $\hat{G}_4$ with the help of the identities collected in section A.2 and verify that the resulting $\hat{G}_7$ takes the form (2.29) given in the main text.

A.4 Massive $D = 10$ type IIA supergravity on S^6

In this subsection, we use A, $B = 1, \ldots, 8$ for fundamental indices of $\mathrm{SL}(8, \mathbb{R})$, reserving I, $J = 1, \ldots, 7$ to fundamental indices of $\mathrm{SL}(7, \mathbb{R})$ or SO(7). The full set of scalar fields in four dimensions is again parametrized in terms of the matrices $\mathcal{M}_{MN}$, $\mathcal{I}_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma}$ as above. Now, we identify Λ indices with antisymmetrized pairs $[AB]$. In a first step, we freeze to zero all pseudoscalar modes, so that

$$\mathcal{R}_{\Lambda\Sigma} = 0 , \qquad \mathcal{I}_{\Lambda\Sigma} = \mathcal{I}_{[AB][CD]} = -\mathcal{T}^{-1}_{AC} \mathcal{T}^{-1}_{BD} + \mathcal{T}^{-1}_{AD} \mathcal{T}^{-1}_{BC} . \tag{A.25}$$

The quantity $\mathcal{T}$ is a symmetric unimodular 8×8 matrix. Notice that (A.25) is nothing but (A.22), written in a slightly different notation that is better suited to discuss the problem at hand.

Next, let us perform the index splitting $A \to (I, 8)$. We restrict the matrix $\mathcal{T}^{AB}$ to take the following block-diagonal form,

$$\mathcal{T}^{AB} = \begin{pmatrix} \mathcal{T}^{IJ} & \mathcal{T}^{I8} \\ \mathcal{T}^{8J} & \mathcal{T}^{88} \end{pmatrix} = \begin{pmatrix} Y^{1/7} T^{IJ} & 0 \\ 0 & Y^{-1} \end{pmatrix} . \tag{A.26}$$

Here Y is a positive real scalar and T^{IJ} is the symmetric unimodular matrix that parametrizes $\mathrm{SL}(7, \mathbb{R})/\mathrm{SO}(7)$.

The uplift formulae for the metric and the dilaton can be extracted from (3.14), (3.18), (3.23) in [34]. Since we only keep a subset of the scalars, the quantities denoted A_m, B_{mn}, A_{mnp} there are all zero. One may then readily reproduce the formulae (2.36), (2.38) given in the main text.

Next, we turn to the evaluation of the Ramond-Ramond 4-form field strength. It is denoted $\hat{F}_{(4)}$ in [34] and it is given by (A.4) therein in terms of the p-form potentials of type IIA. Using (3.12) in [34], keeping in mind that $A_m = B_{mn} = A_{mnp} = 0$ for us, we confirm that $\hat{F}_{(4)}$ takes the same form as in (3.27) in [34], with the terms implicit in the ellipses being zero in our simplified setting. This step can be checked making use of the relations (2.7), (2.8), (2.9) in [33]. The next task is then to evaluate the quantities $\mathcal{H}^{IJ}_{(4)}$, $\mathcal{H}_{(3)I}{}^J$, $\tilde{\mathcal{H}}_{(2)IJ}$ in (3.27) of [34]. This can be done using (2.23), (2.21), (2.19) in [33]. The latter can

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be unpacked using (C.3), (C.9) in the same reference, keeping in mind that for the dyonic gauging at hand the correct values of θ_{AB}, ξ^{AB} are given in (C.15) of [33]. We obtain

$$\mathcal{H}_{(3)I}{}^{J} = -T^{-1}_{IK} * DT^{KJ} + \frac{1}{7}\delta_I{}^J T^{-1}_{KL} * DT^{LK} \,, \tag{A.27}$$

$$\tilde{\mathcal{H}}_{(2)IJ} = -Y^{-\frac{2}{7}} T^{-1}_{IK} T^{-1}_{JL} * \mathcal{H}^{KL}_{(2)} \,, \tag{A.28}$$

$$\mathcal{H}^{IJ}_{(4)} = Y^{\frac{2}{7}} \Big[T^{IJ} \mathrm{Tr}\, T - 2(TT)^{IJ} \Big] \,. \tag{A.29}$$

Here $\mathcal{H}^{KL}_{(2)}$ is the notation of [33] for the field strengths F^{KL}. Interestingly, all Y and dY factors drop away from $\mathcal{H}_{(3)I}{}^{J}$.

Once $\hat{F}_{(4)}$ is computed, we may turn to $e^{\frac{1}{2}\hat{\phi}} \hat{*} \hat{F}_{(4)}$, which may be evaluated with the help of the identities of section A.2. Taking the Hodge star with respect to the metric (2.36) generates additional powers of Y. These conspire with the powers of Y in (A.27)–(A.29) to ensure that $\hat{*}\hat{F}_{(4)}$ has a single overall power of Y, which is precisely cancelled by the prefactor $e^{\frac{1}{2}\hat{\phi}}$. We thus confirm that $e^{\frac{1}{2}\hat{\phi}} \hat{*} \hat{F}_{(4)}$ does not contain Y. Finally, we verify that $e^{\frac{1}{2}\hat{\phi}} \hat{*} \hat{F}_{(4)}$ takes the form specified in (2.38), (2.39), up to an overall constant normalization factor.

B Scalar potential from reduction of the D-dimensional action

B.1 Reduction of the action: cases without dilaton

Reduction of the Einstein-Hilbert term. We find it convenient to write the metric ansatz (3.3) in the form

$$d\hat{s}^2_D = (yTy)^{b_1} d\bar{s}^2_D \,, \qquad d\bar{s}^2_D = ds^2_d + d\tilde{s}^2_n \,, \qquad d\tilde{s}^2_n = g^{-2} (yTy)^{b_2} \gamma_{mn} d\xi^m d\xi^n \,. \tag{B.1}$$

Here ξ^m are local coordinates on S^n, and we have introduced the notation

$$\gamma_{mn} = T^{-1}_{IJ} \partial_m y^I \partial_n y^J \,. \tag{B.2}$$

Geometrically, γ_{mn} describes an ellipsoid, whose principal axes are determined by the eigenvalues of the matrix T_{IJ}.

Our first task is the evaluation of the Ricci scalar of the metric $d\hat{s}^2_D$. In a first step, we express the Ricci scalar of $d\hat{s}^2_D$ in terms of that of $d\bar{s}^2_D$, making use of the fact that these two D-dimensional line elements are related by a Weyl rescaling by the factor $(yTy)^{b_1}$. Let $\hat{\mu}$, $\hat{\nu}$ be curved indices in D dimensions. The following formula is useful,

$$\begin{aligned} &\text{if } \hat{g}_{\hat{\mu}\hat{\nu}} = e^{2\varphi} \bar{g}_{\hat{\mu}\hat{\nu}} \,, \\ &R[\hat{g}_{\hat{\mu}\hat{\nu}}] = e^{-2\varphi} \Big[R[\bar{g}_{\hat{\mu}\hat{\nu}}] - 2(D-1)\, \bar{g}^{\hat{\mu}\hat{\nu}}\, \overline{\nabla}_{\hat{\mu}} \overline{\nabla}_{\hat{\nu}} \varphi - (D-1)(D-2)\, \bar{g}^{\hat{\mu}\hat{\nu}}\, \overline{\nabla}_{\hat{\mu}} \varphi\, \overline{\nabla}_{\hat{\nu}} \varphi \Big] \,. \end{aligned} \tag{B.3}$$

The symbol $\overline{\nabla}_{\hat{\mu}}$ denotes the Levi-Civita connection of the D-dimensional metric $d\bar{s}^2_D$. We apply (B.3) with $e^{2\varphi} = (yTy)^{b_1}$.

Next, we observe that the metric $d\bar{s}^2_D$ is a direct product between the d-dimensional metric $ds^2_d = g_{\mu\nu} dx^\mu dx^\nu$ and the n-dimensional metric $d\tilde{s}^2_n = \tilde{g}_{mn} d\xi^m d\xi^n$. (We have introduced μ, ν, which are curved indices in d dimensions.) This holds because we are working under the simplifying assumptions (3.2) of no gauge fields and constant T. As a result, we have the simple formula

$$R[\bar{g}_{\hat{\mu}\hat{\nu}}] = R[g_{\mu\nu}] + R[\tilde{g}_{mn}] \,. \tag{B.4}$$

Furthermore, the D-dimensional Levi-Civita connection $\overline{\nabla}_{\hat{\mu}}$ also splits in a trivial way,

$$\overline{\nabla}_{\hat{\mu}} = (\overline{\nabla}_{\mu}, \overline{\nabla}_m) = (\nabla_\mu, \tilde{\nabla}_m) \,. \tag{B.5}$$

Here ∇_μ is the Levi-Civita connection of $g_{\mu\nu}$ and $\tilde{\nabla}_m$ is that of $\tilde{g}_{mn}$. The conformal factor $e^{2\varphi} = (yTy)^{b_1}$ in (B.3) depends on the S^n coordinates only (because we are assuming T is constant). As a result, we can write

$$R[\hat{g}_{\hat{\mu}\hat{\nu}}] = e^{-2\varphi} \Big[R[g_{\mu\nu}] + R[\tilde{g}_{mn}] - 2(D-1)\, \tilde{g}^{mn}\, \tilde{\nabla}_m \tilde{\nabla}_n \varphi - (D-1)(D-2)\, \tilde{g}^{mn}\, \tilde{\nabla}_m \varphi\, \tilde{\nabla}_n \varphi \Big] \,. \tag{B.6}$$

We may now proceed by writing quantities associated to $\tilde{g}_{mn}$ in terms of quantities associated to γ_{mn}. The two metrics are related as $\tilde{g}_{mn} = e^{2\varphi'} \gamma_{mn}$ with $e^{2\varphi'} = g^{-2} (yTy)^{b_2}$. For the Ricci scalar $R[\tilde{g}_{mn}]$ we can use a formula completely analogous to (B.3). We also have immediately $\tilde{\nabla}_m \varphi = \partial_m \varphi$. The Laplacian term $\tilde{g}^{mn}\, \tilde{\nabla}_m \tilde{\nabla}_n \varphi$ can be addressed with the help of the identity

$$\text{if } \; \tilde{g}_{mn} = e^{2\varphi'} \gamma_{mn} \,, \; \tilde{g}^{mn}\, \tilde{\nabla}_m \tilde{\nabla}_n \varphi = e^{-2\varphi'} \Big[\gamma^{mn} \nabla^{(\gamma)}_m \nabla^{(\gamma)}_n \varphi + (n-2) \gamma^{mn} \partial_m \varphi \partial_n \varphi' \Big] \,. \tag{B.7}$$

We have introduced the notation $\nabla^{(\gamma)}_m$ for the Levi-Civita connection of γ_{mn}.

Retracing all the steps outlined above, we arrive at the following formula for the Ricci scalar of the D-dimensional metric $d\hat{s}^2_D$,

$$\begin{aligned} R[\hat{g}_{\hat{\mu}\hat{\nu}}] = (yTy)^{-b_1}\, R[g_{\mu\nu}] + g^2\, (yTy)^{-b_1 - b_2} \Big[& R[\gamma_{mn}] + \mathcal{K}_{(\mathrm{L})}\, \gamma^{mn}\, \nabla^{(\gamma)}_m \nabla^{(\gamma)}_n \log(yTy) \\ & + \mathcal{K}_{(\mathrm{G})}\, \gamma^{mn}\, \partial_m \log(yTy) \partial_n \log(yTy) \Big] \,. \end{aligned} \tag{B.8}$$

We have introduced the constants

$$\begin{aligned} \mathcal{K}_{(\mathrm{L})} &= b_1 - b_1 D + b_2 - b_2 n \,, \\ \mathcal{K}_{(\mathrm{G})} &= -\frac{b_1^2}{4} (D-1)(D-2) - \frac{b_2^2}{4} (n-1)(n-2) - \frac{b_1 b_2}{2}\, (D-1)(n-2) \,. \end{aligned} \tag{B.9}$$

The volume form of the D-dimensional metric $d\hat{s}^2_D$ is written in terms of the d-dimensional metric $g_{\mu\nu}$ and the n-dimensional metric γ_{mn} as

$$\sqrt{-\hat{g}_D} = g^{-n}\, (yTy)^{b_1 D/2 + b_2 n/2}\, \sqrt{-g_d}\, \sqrt{\gamma} \,. \tag{B.10}$$

The D-dimensional Einstein-Hilbert Lagrangian $\sqrt{-\hat{g}_D} R[\hat{g}_{\hat{\mu}\hat{\nu}}]$ is readily written by combining (B.8) and (B.10).

We are free to add to $\sqrt{-\hat{g}_D}R[\hat{g}_{\hat{\mu}\hat{\nu}}]$ any total divergence on S^n, of the form $\sqrt{\gamma}\nabla^{(\gamma)}_m(\dots)^m$, because we are interested in the integral of the D-dimensional action over S^n. If we add a term of the form

$$\sqrt{\gamma}\nabla^{(\gamma)}_m\left(\sqrt{-g_d}\gamma^{mn}\nabla^{(\gamma)}_n(yTy)^{\frac{b_1D}{2}-b_1+\frac{b_2n}{2}-b_2}\right) \tag{B.11}$$

with the appropriate constant prefactor, we eliminate the Laplacian term $\gamma^{mn}\nabla^{(\gamma)}_m\nabla^{(\gamma)}_n\log(yTy)$. We thus get the simpler expression

$$\begin{aligned}\sqrt{-\hat{g}_D}R[\hat{g}_{\hat{\mu}\hat{\nu}}] &= g^{-n}(yTy)^{-b_1+b_1D/2+b_2n/2}\sqrt{-g_d}\,\sqrt{\gamma}R[g_{\mu\nu}] \\ &+ g^{-n+2}\,(yTy)^{-b_1-b_2+b_1D/2+b_2n/2}\sqrt{-g_d}\,\sqrt{\gamma}\Big[R[\gamma_{mn}]+\mathcal{K}'_{(\mathrm{G})}\,\gamma^{mn}\,\partial_m\log(yTy)\partial_n\log(yTy)\Big]\,,\end{aligned} \tag{B.12}$$

where the new constant $\mathcal{K}'_{(\mathrm{G})}$ is given by

$$\mathcal{K}'_{(\mathrm{G})}=\mathcal{K}_{(\mathrm{G})}-\mathcal{K}_{(\mathrm{L})}\left(\frac{b_1D}{2}-b_1+\frac{b_2n}{2}-b_2\right)\,. \tag{B.13}$$

In a final step, we rewrite the quantities expressed in terms of the ellipsoid metric γ_{mn} in terms of the round metric $\mathring{g}_{mn}$ defined in (A.2). The following identities are useful,

$$\sqrt{\gamma}=(yTy)^{1/2}\sqrt{\mathring{g}}\,,\qquad \gamma^{mn}\,\partial_m(yTy)\partial_n(yTy)=4\,(yT^3y)-4\,\frac{(yT^2y)^2}{yTy}\,. \tag{B.14}$$

We also need the expression of the Ricci scalar of the ellipsoid metric [42],

$$R[\gamma_{mn}]=\frac{(\mathrm{Tr}\,T)^2}{yTy}-\frac{\mathrm{Tr}\,T^2}{yTy}-\frac{2\,(\mathrm{Tr}\,T)\,(yT^2y)}{(yTy)^2}+\frac{2\,yT^3y}{(yTy)^2}\,. \tag{B.15}$$

Combining all the above ingredients, we finally arrive at the result quoted in (3.4), (3.5).

As explained in the main text, the values of the parameters b_1, b_2 are fixed according to (3.10). After that, we find it convenient to add another total divergence on S^n. More precisely, we add a constant multiple of

$$\begin{aligned}&\sqrt{\gamma}\nabla^{(\gamma)}_m\left(\sqrt{-g_d}\gamma^{mn}\nabla^{(\gamma)}_n(yTy)^{1/2}\right)= \\ &\quad=2\sqrt{-g_d}\sqrt{\mathring{g}}\left[\frac{(yT^2y)^2}{(yTy)^2}-\frac{(yT^3y)}{yTy}+\frac{1}{2}\,\mathrm{Tr}(T^2)-\frac{(\mathrm{Tr}\,T)\,(yT^2y)}{2\,(yTy)}\right].\end{aligned} \tag{B.16}$$

Adding this total divergence allows us to cancel the terms with yT^3y and complete a perfect square, thus arriving at the result (3.11) quoted in the main text.

Reduction of kinetic term for the flux. The kinetic term for $\hat{\mathcal{F}}_n$ is proportional to the quantity

$$\sqrt{-\hat{g}_D}|\hat{\mathcal{F}}_n|^2=\frac{1}{n!}\sqrt{-\hat{g}_D}\hat{\mathcal{F}}_{\hat{\mu}_1\dots\hat{\mu}_n}\hat{\mathcal{F}}_{\hat{\nu}_1\dots\hat{\nu}_n}\hat{g}^{\hat{\mu}_1\hat{\nu}_1}\dots\hat{g}^{\hat{\mu}_n\hat{\nu}_n}\,. \tag{B.17}$$

The ansatz (3.7) for $\hat{\mathcal{F}}_n$ implies that the only non-zero components of $\hat{\mathcal{F}}_n$ are

$$\hat{\mathcal{F}}_{p_1\dots p_n}=f(y,T)\mathring{\epsilon}_{p_1\dots p_n}\,, \tag{B.18}$$

where $\mathring{\epsilon}$ is the volume form of the round metric $\mathring{g}_{mn}$ on S^n. In a first step, we can rewrite (B.17) in terms of the D-dimensional metric $d\bar{s}_D^2$, generating an overall yTy prefactor,

$$\sqrt{-\hat{g}_D}|\hat{\mathcal{F}}_n|^2 = \frac{1}{n!}(yTy)^{b_1 D/2 - b_1 n}\sqrt{-\bar{g}_D}\hat{\mathcal{F}}_{\hat{\mu}_1\dots\hat{\mu}_n}\hat{\mathcal{F}}_{\hat{\nu}_1\dots\hat{\nu}_n}\bar{g}^{\hat{\mu}_1\hat{\nu}_1}\dots\bar{g}^{\hat{\mu}_n\hat{\nu}_n}\,. \tag{B.19}$$

Next, we use the fact that $d\bar{s}_D^2$ is a direct product metric, $d\bar{s}_D^2 = ds_d^2 + d\tilde{s}_n^2$, and that $\hat{\mathcal{F}}_n$ has only internal legs,

$$\sqrt{-\hat{g}_D}|\hat{\mathcal{F}}_n|^2 = \frac{1}{n!}(yTy)^{b_1 D/2 - b_1 n}\sqrt{-g_d}\sqrt{\tilde{g}}\hat{\mathcal{F}}_{p_1\dots p_n}\hat{\mathcal{F}}_{q_1\dots q_n}\tilde{g}^{p_1 q_1}\dots\tilde{g}^{p_n q_n}\,. \tag{B.20}$$

We proceed by trading the metric $\tilde{g}_{mn}$ with the ellipsoid metric γ_{mn} and using (B.18),

$$\sqrt{-\hat{g}_D}|\hat{\mathcal{F}}_n|^2 = \frac{1}{n!}g^n(yTy)^{b_1 D/2 - b_1 n - b_2 n/2}\sqrt{-g_d}\sqrt{\gamma}f^2\mathring{\epsilon}_{p_1\dots p_n}\mathring{\epsilon}_{q_1\dots q_n}\gamma^{p_1 q_1}\dots\gamma^{p_n q_n}\,. \tag{B.21}$$

To compute the contraction of two *round* volume forms $\mathring{\epsilon}$ with the *ellipsoid* inverse metric, we use $\det\gamma_{mn} = (yTy)\det\mathring{g}_{mn}$ (valid for unimodular T) to write

$$\mathring{\epsilon}_{p_1\dots p_n}\mathring{\epsilon}_{q_1\dots q_n}\gamma^{p_1 q_1}\dots\gamma^{p_n q_n} = (yTy)^{-1}\epsilon^{(\gamma)}_{p_1\dots p_n}\epsilon^{(\gamma)}_{q_1\dots q_n}\gamma^{p_1 q_1}\dots\gamma^{p_n q_n} = n!(yTy)^{-1}\,, \tag{B.22}$$

where $\epsilon^{(\gamma)}$ is the volume form of the ellipsoid metric γ_{mn}. We thus reproduce the result (3.13) quoted in the main text.

B.2 Integrals on S^n and independence on T

Our goal is to determine under which conditions the following quantity has an integral over S^n that is independent of T (provided $\det T = 1$),

$$h(T,y) = (yTy)^{-k/2+u}\left[\frac{yT^2y}{(yTy)^2} - \frac{4-k}{8}\frac{\mathrm{Tr}\,T}{yTy}\right]. \tag{B.23}$$

We have included two independent constant parameters k, u, in order to be able to apply the present argument to the case with a dilaton.

We consider a T matrix of the form $T_{IJ} = \exp(t)_{IJ}$, with $\mathrm{Tr}\,t = 0$, so that $\det T = 1$. We can expand $h(e^t, y)$ in a power series around $t = 0$. The integral over S^n can be performed order by order with the help of identities

$$\int_{S^n} d^n\xi\sqrt{\mathring{g}}y^{I_1}\dots y^{I_{2p}} = \frac{(2p-1)!!\,\delta^{(I_1 I_2}\delta^{I_3 I_4}\dots\delta^{I_{2p-1}I_{2p})}}{(n+1)(n+3)\dots(n+2p-1)}\,, \qquad p = 1,2,3,\dots \tag{B.24}$$

For simplicity, let us first freeze $u = 0$, keeping k as only free parameter. This is enough for applications to the cases without dilaton. We find

$$\begin{aligned}\frac{1}{\mathcal{V}_n}\int_{S^n} d^n\xi\sqrt{\mathring{g}}h(e^t,y) &= \frac{1}{8}((k-4)n+k+4) \\ &+ \frac{k\left(k^2(n+1)-kn(n+2)+7k+4(n-3)(n+1)\right)}{32(n+1)(n+3)}\mathrm{Tr}(t^2) \\ &- \frac{k(k-n-1)\left(2k^2(n+1)+k(19-(n-2)n)+4(n-7)(n+1)\right)}{96(n+1)(n+3)(n+5)}\mathrm{Tr}(t^3) + \mathcal{O}(t^4)\,.\end{aligned} \tag{B.25}$$

We see that the only way to ensure that the result is independent of t, without imposing any restriction on t (except tracelessness), is to select

$$k = 0\,. \tag{B.26}$$

We know that (B.23) with $(k,u) = (0,0)$ is related to a total derivative, thanks to the identity (3.16), repeated here for convenience,

$$\epsilon y(dy)^n + d\left[\frac{n}{n-1}\frac{1}{yTy}\epsilon y(Ty)(dy)^{n-1}\right] = -\frac{2}{n-1}\left[\frac{yT^2y}{(yTy)^2} - \frac{1}{2}\frac{\operatorname{Tr}T}{yTy}\right]\epsilon y(dy)^n\,. \tag{B.27}$$

This relation demonstrates that

$$(k,u) = (0,0) \quad \Rightarrow \quad \int_{S^n} d^n\xi \sqrt{\mathring{g}}h(y,T) = -\frac{n-1}{2}\mathcal{V}_n\,, \tag{B.28}$$

which is indeed non-zero for $n \geq 2$ and independent of T.

Let us now repeat the analysis keeping both k and u as independent parameters, as required for applications to the cases with a dilaton. We have studied the integral of (B.23) up to fourth order in t. By collecting the coefficients of the $\operatorname{Tr}(t^2)$, $\operatorname{Tr}(t^3)$, $\operatorname{Tr}(t^4)$, and $[\operatorname{Tr}(t^2)]^2$ structures, and setting them to zero, we obtain a set of relations among k, u, n, which admit the following solutions,

$$(k,u) = (0,0)\,, \quad (k,u) = \left(\tfrac{4(n-1)}{n+1}, -\tfrac{(n-3)(n-1)}{2(n+1)}\right)\,, \quad (k,u) = \left(\tfrac{4(n+1)}{n+3}, -\tfrac{(n-1)(n+1)}{2(n+3)}\right)\,. \tag{B.29}$$

These three options are the only possible values of (k,u) for which the integral of (B.23) has a chance of being independent of T, for unimodular T. We now study each case and prove that the integral has indeed this property.

We have already encountered the case $(k,u) = 0$, and we have already demonstrated in (B.28) that for these values of k, u the integral of h is indeed independent of T.

For the case $(k,u) = \left(\tfrac{4(n-1)}{n+1}, -\tfrac{(n-3)(n-1)}{2(n+1)}\right)$ we use the relation

$$d\left[(yTy)^{-\frac{n+1}{2}}\epsilon y(Ty)(dy)^{n-1}\right] = -\frac{n+1}{n}(yTy)^{1-\frac{n+1}{2}}\left[\frac{yT^2y}{(yTy)^2} - \frac{1}{n+1}\frac{\operatorname{Tr}T}{yTy}\right]\epsilon y(dy)^n\,. \tag{B.30}$$

We conclude that

$$(k,u) = \left(\tfrac{4(n-1)}{n+1}, -\tfrac{(n-3)(n-1)}{2(n+1)}\right) \quad \Rightarrow \quad \int_{S^n} d^n\xi \sqrt{\mathring{g}}h(y,T) \equiv 0\,. \tag{B.31}$$

While indeed independent of T, a flux that is identically zero is not acceptable, because it does not reproduce the non-zero flux of the case $T = \mathbb{I}$.

The third option in (B.29) requires a more careful analysis. We start from

$$\sqrt{\mathring{g}}\mathring{\nabla}^m\mathring{\nabla}_m(yTy)^\lambda = \sqrt{\mathring{g}}\left[2\,\lambda\,(yTy)^{\lambda-1}\operatorname{Tr}T - 2\,\lambda\,(n+1)\,(yTy)^\lambda\right. \\ \left. + 4\,\lambda\,(\lambda-1)\,(yTy)^{\lambda-2}\,(yT^2y) - 4\,\lambda\,(\lambda-1)\,(yTy)^\lambda\right]\,, \tag{B.32}$$

where $\mathring{\nabla}^m\mathring{\nabla}_m$ is the scalar Laplace operator constructed with the round metric on S^n and λ is a real constant. If we specialize to $\lambda = -(n+1)/2$, we obtain

$$\sqrt{\mathring{g}}\mathring{\nabla}^m\mathring{\nabla}_m(yTy)^\lambda = (n+1)(n+3)\sqrt{\mathring{g}}(yTy)^{-\frac{n+1}{2}}\left[\frac{yT^2y}{(yTy)^2} - \frac{1}{n+3}\frac{\operatorname{Tr}T}{yTy} - \frac{2}{n+3}\right]. \quad \text{(B.33)}$$

Since the l.h.s. is a total divergence, we conclude that

$$(k,u) = \left(\tfrac{4(n+1)}{n+3}, -\tfrac{(n-1)(n+1)}{2(n+3)}\right) \quad \Rightarrow \quad \int_{S^n}\sqrt{\mathring{g}}h(y,T) = \frac{2}{n+3}\int_{S^n} d^n\xi\sqrt{\mathring{g}}(yTy)^{-\frac{n+1}{2}}. \quad \text{(B.34)}$$

The integral on the r.h.s. can be evaluated as follows. We start from the standard Gaussian integral

$$\int_{\mathbb{R}^{n+1}} d^{n+1}x e^{-\frac{1}{2}xTx} = (2\pi)^{\frac{n+1}{2}}(\det T)^{-\frac{1}{2}}, \quad \text{(B.35)}$$

where x^I are Cartesian coordinates on $\mathbb{R}^{n+1}$. We introduce polar coordinates $x^I = ry^I$, with $y^I y_I = 1$. We can then write

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} d^{n+1}x e^{-\frac{1}{2}xTx} &= \int_{S^n} d^n\xi\sqrt{\mathring{g}}\int_0^\infty dr r^n e^{-\frac{1}{2}r^2(yTy)} \\ &= \int_{S^n} d^n\xi\sqrt{\mathring{g}}2^{\frac{n-1}{2}}(yTy)^{-\frac{n+1}{2}}\Gamma(\tfrac{n+1}{2}). \end{aligned} \quad \text{(B.36)}$$

Comparing (B.35) and (B.36) and recalling (2.8), we find

$$\int_{S^n} d^n\xi\sqrt{\mathring{g}}(yTy)^{-\frac{n+1}{2}} = \mathcal{V}_n(\det T)^{-\frac{1}{2}}. \quad \text{(B.37)}$$

Using (B.34) we conclude

$$(k,u) = \left(\tfrac{4(n+1)}{n+3}, -\tfrac{(n-1)(n+1)}{2(n+3)}\right) \quad \Rightarrow \quad \int_{S^n}\sqrt{\mathring{g}}h(y,T) = \frac{2}{n+3}\mathcal{V}_n(\det T)^{-\frac{1}{2}}. \quad \text{(B.38)}$$

The result is not identically zero and depends on T. The dependence on T, however, drops out after imposing the unimodularity constraint.

To summarize: the integral of h in (B.23) is independent of T for unimodular T if and only if one of the following three cases is realized:

- $(k,u) = (0,0)$: the integral of h is non-zero and independent of T without using the condition $\det T = 1$;
- $(k,u) = (\frac{4(n-1)}{n+1}, -\frac{(n-3)(n-1)}{2(n+1)})$: the integral of h is identically zero for any symmetric T without using the condition $\det T = 1$;
- $(k,u) = (\frac{4(n+1)}{n+3}, -\frac{(n-1)(n+1)}{2(n+3)})$: the integral of T is a non-zero constant times $(\det T)^{-\frac{1}{2}}$, and thus independent of T using $\det T = 1$.

B.3 Features of non-trivial integrals over S^n

The reduction of the Einstein-Hilbert term yields the integrand $\mathcal{I}$ in (3.11). The terms with a non-trivial y dependence are collected in the perfect square $[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T]^2$. This is a rational function of y, T, homogeneous of degree 2 under a formal rescaling $T_{IJ} \to \lambda T_{IJ}$. (In the present discussion, we relax the unimodularity constraint on T and study integrals of expressions such as $\frac{yT^2y}{yTy}$ as functions of an arbitrary symmetric T.) If we integrate $[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T]^2$ over S^n with measure $\sqrt{\mathring{g}}$, we get a function of T_{IJ} which is manifestly homogeneous of degree 2. It is not, however, a quadratic function, nor a rational function of the entries of T.

This can be verified explicitly in the example $n = 2$, choosing for simplicity a diagonal T of the form

$$T_{IJ} = \operatorname{diag}(t, t, s)\,, \tag{B.39}$$

where t, s are positive parameters. The constrained coordinates y^I may be parametrized in terms of an interval coordinate $-1 \leq \mu \leq 1$ and an angle ϕ with period 2π,

$$y^1 = \sqrt{1-\mu^2}\cos\phi\,, \qquad y^2 = \sqrt{1-\mu^2}\sin\phi\,, \qquad y^3 = \mu\,. \tag{B.40}$$

The round metric on S^2 in these coordinates reads $d\mathring{s}^2 = \frac{d\mu^2}{1-\mu^2} + (1-\mu^2)d\phi^2$ and therefore $\sqrt{\mathring{g}} = 1$. We also check that the quantity $[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T]^2$ is independent of ϕ, due to the fact that the first two diagonal entries of T are equal. We are thus left to compute an integration in the variable μ only. Considering for definiteness the case $s > t$, we find[15]

$$\frac{1}{2\pi}\int_{S^2}\sqrt{\mathring{g}}\left[\frac{yT^2y}{yTy} - \frac{4-k}{8}\operatorname{Tr} T\right]^2 = st + \frac{((k+4)s+2kt)^2}{32} - \frac{s((k+2)s+2kt)\arctan\sqrt{\frac{s}{t}-1}}{2\sqrt{\frac{s}{t}-1}}\,. \tag{B.41}$$

The r.h.s. is homogeneous of degree 2 under a simultaneous rescaling $(t, s) \to (\lambda t, \lambda s)$, but it is not a rational function of t, s.

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