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Skeletal filtrations of the fundamental group of a non-archimedean curve

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ABSTRACT

In this paper we study skeleta of residually tame coverings of a marked curve over a non-archimedean field. We first prove a simultaneous semistable reduction theorem for residually tame coverings, which we then use to construct a tropicalization functor from the category of residually tame coverings of a marked curve (X, D) to the category of tame coverings of a metrized complex Σ associated to (X, D). We enhance the latter category by adding a set of gluing data to every covering and we show that this yields an equivalence of categories. We use this skeletal interpretation to define the absolute decomposition and inertia group of a curve, which can be seen as the first subgroups in a ramification filtration of the fundamental group of the curve. We prove that the cyclic coverings that arise from the corresponding decomposition and inertia quotients coincide with the coverings that arise from the toric and connected parts of the analytic Jacobian of the curve.

1. Introduction

Let K be a complete, algebraically closed non-archimedean field with a non-trivial valuation. In this paper, we study tropicalizations of *residually tame* coverings of curves $\phi : X' \to X$, which are finite morphisms of smooth algebraic curves with an extra

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tameness condition on the induced morphism $\phi^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$ of Berkovich analytifications. Namely, for every $x' \in X'^{\mathrm{an}}$ mapping to $x \in X^{\mathrm{an}}$ we require the corresponding extension of completed residue fields $\mathcal{H}(x) \subset \mathcal{H}(x')$ to be tame. For these coverings, we prove the following simultaneous semistable reduction theorem, which can be seen as a non-discrete generalization of [22, Theorem 2.3].

Theorem 3.3. *Let $(X', D') \to (X, D)$ be a residually tame covering of marked curves and let Σ be a skeleton of (X, D). Then the inverse image of Σ under $X'^{\mathrm{an}} \to X^{\mathrm{an}}$ is a skeleton of (X', D').*

Here a skeleton of a marked curve (X, D) corresponds to a semistable model $\mathcal{X}$ of the marked curve (X, D) over the valuation ring R of K. In terms of this language, Theorem 3.3 says that we obtain an induced morphism of semistable models $\mathcal{X}' \to \mathcal{X}$ by taking the normalization of $\mathcal{X}$ in the function field of X'.

We now fix a skeleton Σ of a marked curve (X, D). Using Theorem 3.3, we can associate a finite harmonic morphism $\Sigma' \to \Sigma$ of metric graphs to every residually tame covering $(X', D') \to (X, D)$. This process is moreover functorial, so that we obtain a *tropicalization functor* $\mathcal{F}_\Sigma : \mathrm{Cov}_{\mathrm{Tame}}(X, D) \to \mathrm{Cov}(\Sigma)$ from the category of residually tame coverings of (X, D) to the category of finite harmonic coverings of Σ. One of the main goals in this paper is to investigate which properties of $\mathrm{Cov}_{\mathrm{Tame}}(X, D)$ are preserved under $\mathcal{F}_\Sigma$. For instance, the category $\mathrm{Cov}_{\mathrm{Tame}}(X, D)$ has the additional structure of a Galois category, which means that there is a profinite fundamental group $\pi_{\mathrm{Tame}}(X, D)$ that classifies residually tame coverings of (X, D) as in classical Galois theory. It is then natural to ask whether we can transfer this structure to $\mathrm{Cov}(\Sigma)$, or an extended category. The answer is yes, and we use the lifting results in [1] to do this.

Theorem 4.13. *Let Σ be a skeleton of a marked curve (X, D) and let $\mathcal{F}_\Sigma$ be the corresponding functor from the category of residually tame coverings of (X, D) to the category $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ of enhanced tame coverings of Σ. Then $\mathcal{F}_\Sigma$ induces an equivalence of categories*

$$\mathrm{Cov}_{\mathrm{Tame}}(X, D) \simeq \mathrm{Cov}_{\mathcal{G}}(\Sigma).$$

We refer the reader to Section 4.1 for the exact definition of an enhanced tame covering of Σ. Roughly speaking, it consists of a finite harmonic morphism $\Sigma' \to \Sigma$ of metric graphs, a set of tame algebraic coverings over the vertices of Σ that are compatible with the graph-theoretical data, and a set of gluing data over the edges of Σ to connect the various algebraic coverings. It follows from the results in [1] that we can lift an enhanced tame covering $\Sigma' \to \Sigma$ to a residually tame covering of marked curves $(X', D') \to (X, D)$. We extend this result here and show that morphisms can be uniquely lifted if we use a suitable notion of a morphism of enhanced tame coverings. From this, we directly obtain the equivalence of categories in Theorem 4.13.

Since $\mathrm{Cov}_{\mathrm{Tame}}(X, D)$ is a Galois category, we now find that $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ is also a Galois category. We write $\pi(\Sigma) \simeq \pi_{\mathrm{Tame}}(X, D)$ for the corresponding profinite fundamental group. We can define filtrations of this fundamental group $\pi(\Sigma)$ using the concepts of metrically unramified and completely split coverings. Here a covering ϕ is metrically unramified above an edge e of Σ if the dilation factors $d_{e'/e}(\phi)$ of the edges e' above e are all 1, and a covering is completely split above a vertex v of Σ if there are $\deg(\phi)$ vertices above v. For any subcomplex $\Sigma^0 \subset \Sigma$, we then consider all coverings of Σ that are metrically unramified or completely split over Σ^0. These coverings correspond to closed normal subgroups of $\pi(\Sigma)$ which we call the absolute inertia group $\mathfrak{I}(\Sigma^0)$ and the absolute decomposition group $\mathfrak{D}(\Sigma^0)$ respectively. Their quotients $\pi_{\mathfrak{I}}(\Sigma^0) := \pi(\Sigma)/\mathfrak{I}(\Sigma^0)$ and $\pi_{\mathfrak{D}}(\Sigma^0) := \pi(\Sigma)/\mathfrak{D}(\Sigma^0)$ classify the connected coverings of Σ that are unramified (resp. completely split) above Σ^0. If $D = \emptyset$ and $\Sigma^0 = \Sigma$, then we have the following theorem:

Theorem 5.14. *Let $\mathfrak{D}(\Sigma)$ be the decomposition group of Σ in $\pi(\Sigma)$. Then $\pi_{\mathfrak{D}}(\Sigma) := \pi(\Sigma)/\mathfrak{D}(\Sigma)$ is isomorphic to the profinite completion of the ordinary fundamental group of the underlying graph Γ of the metrized complex Σ.*

By algebraic topology, it follows that $\pi_{\mathfrak{D}}(\Sigma)$ is isomorphic to the profinite completion of the free group on $\beta(\Sigma)$ generators, where $\beta(\Sigma)$ is the first Betti number of Σ.

In Section 5.2, we turn to the abelianizations of the groups $\pi_{\mathfrak{D}}(\Sigma)$ and $\pi_{\mathfrak{I}}(\Sigma)$. For any n coprime to $\mathrm{char}(k)$, the cyclic étale coverings of degree n of an algebraic curve X are classified by the torsion points of the Jacobian $J := J(X)$ of X using the isomorphism $J[n] \simeq \mathrm{Hom}(\pi(\Sigma), \mathbb{Z}/n\mathbb{Z})$. The decomposition and inertia groups $\pi_{\mathfrak{D}}(\Sigma)$ and $\pi_{\mathfrak{I}}(\Sigma)$ naturally subdivide these cyclic coverings:

$$\mathrm{Hom}(\pi_{\mathfrak{D}}(\Sigma), \mathbb{Z}/n\mathbb{Z}) \subset \mathrm{Hom}(\pi_{\mathfrak{I}}(\Sigma), \mathbb{Z}/n\mathbb{Z}) \subset \mathrm{Hom}(\pi(\Sigma), \mathbb{Z}/n\mathbb{Z}). \tag{1}$$

On the other hand, we also have such a filtration in the Jacobian of X by the results in [10] and [8]. Indeed, let $\mathcal{X}$ be a semistable model of X with skeleton Σ and let $\mathrm{Jac}(\Sigma)$ be the tropical Jacobian or component group associated to Σ. We consider the kernel J^0 of the tropicalization map $\overline{\tau} : J^{\mathrm{an}} \to \mathrm{Jac}(\Sigma)$. The reduction $\overline{J}^0$ of J^0 fits in an exact sequence

$$1 \to \overline{T} \to \overline{J}^0 \to \overline{B} \to 1, \tag{2}$$

where $\overline{T}$ is a torus and $\overline{B}$ is an abelian variety over k. More explicitly, we have $\overline{B} = \prod_i \mathrm{Jac}(\Gamma_i)$, where the Γ_i are the components in the special fiber of a semistable model $\mathcal{X}$ for X. These are natural generalizations of concepts in the discretely valued case, where the role of the analytic Jacobian is played by the Néron model $\mathcal{J}/R$ of the Jacobian. This Néron model $\mathcal{J}$ has a fiberwise connected component $\mathcal{J}^0$ and the special fiber of $\mathcal{J}^0$ fits into an exact sequence similar to the one in Equation (2).

Returning to the analytic side, we now write $J^0[n] := \{P \in J[n] : \overline{\tau}(P) = 0\}$ and $T[n] := \{P \in J^0[n] : \pi(\overline{P}) = 0\}$, where $\overline{P}$ is the reduction of P and $\pi : \overline{J}^0 \to \overline{B}$ is the map from Equation (2). We then have the inclusions

$$T[n] \subset J^0[n] \subset J[n]. \tag{3}$$

We invite the reader to compare this with the material in [15, Exposé IX, §12]. We show here that the filtration of $J[n]$ in Equation (3) coincides with the filtration in Equation (1) under the isomorphism $J[n] \simeq \mathrm{Hom}(\pi(\Sigma), \mathbb{Z}/n\mathbb{Z})$.

Theorem 5.16. *Let $\pi_{\mathfrak{I}}(\Sigma)$ and $\pi_{\mathfrak{D}}(\Sigma)$ be the inertia and decomposition quotients of $\pi(\Sigma)$ respectively. Let n be an integer such that $\gcd(n, \mathrm{char}(k)) = 1$. Then the isomorphism $J[n] \simeq \mathrm{Hom}(\pi(\Sigma), \mathbb{Z}/n\mathbb{Z})$ induces isomorphisms*

$$J^0[n] \simeq \mathrm{Hom}(\pi_{\mathfrak{I}}(\Sigma), \mathbb{Z}/n\mathbb{Z})$$

and

$$T[n] \simeq \mathrm{Hom}(\pi_{\mathfrak{D}}(\Sigma), \mathbb{Z}/n\mathbb{Z}).$$

The main tool we use in the proof of this theorem is the analytic slope formula [9, Theorem 5.15], which says that the local reduced divisor of a function is determined by the slopes of its logarithm. This allows us to relate the various properties of n-torsion points in the Jacobian to the covering data on the level of metrized complexes. Overall, we interpret Theorem 5.16 as giving geometric interpretations for the various torsion points in the Jacobian. In line with this thought it now also seems natural to view the inertia and decomposition groups as non-abelian generalizations of the connected and toric parts of the Jacobian.[1]

The paper is organized as follows. We start in Section 2.1 by proving certain results on residually tame morphisms. We then review the results in [1] on lifting morphisms of metrized complexes in Section 2.2. In Section 3, we prove a generalized simultaneous semistable reduction theorem for residually tame coverings. We then introduce the notion of an enhanced covering of metrized complexes in Section 4.1 and prove the equivalence of categories in Theorem 4.13. In Section 4.2, we give an algebraic definition of enhanced coverings and show that this gives rise to an equivalent category. In Section 5.1, we introduce various filtrations of the fundamental group of the marked curve (X, D) using metrically unramified and completely split coverings. We then study the induced filtrations of the abelianization of $\pi_{\mathrm{Tame}}(X, D)$ in Section 5.2 and prove Theorem 5.16.

This paper uses a great deal of concepts and results from [1] and [10]. For the reader's convenience, we have included a short summary of the definitions and results we need from these papers in Sections 2.2 and 5.2. For the full version, we refer the reader to [1, Sections 2,4,6 and 7] and [10, Sections 4,5 and 6].

[1] The terminology for the toric part is borrowed from [15, Exposé IX, §12], the connected part is called the fixed part there.

1.1. Connections

We give an overview here of other similar results in the literature. An important object in the study of degenerations of coverings of curves is that of an admissible covering, which allows for certain mild singularities in the source and target. These were introduced by Harris and Mumford in [19] to compactify Hurwitz spaces of coverings of the projective line. This notion was consequently generalized to coverings of curves of arbitrary genus by Mochizuki in [24] in terms of log-admissible coverings. The corresponding stacks are often not normal, but if we take the normalization of such a stack then it becomes smooth over $\mathbb{Z}$, see [24, Section 3.23]. The points in this normalization can be given a moduli-theoretic interpretation as well using twisted stable maps, see [2].

The boundaries of these stacks can be studied using tropical geometry as in [12]. It was shown there that ordinary Hurwitz numbers can be calculated in terms of the tropical Hurwitz numbers associated to a *maximally degenerate* tropical curve Σ. Here Σ is maximally degenerate if its underlying stable graph is trivalent with vertices of weight zero, so that it belongs to a top-dimensional cone of the tropical moduli space M_g^{trop} of tropical curves of genus g. The formula in [12, Theorem 2] unfortunately does not extend to coverings of arbitrary tropical curves, as we will see in an explicit example in Remark 4.19. Theorem 4.13 explains the discrepancy in this example in terms of certain local twist factors, which slightly change the weight factors from [12]. It would be interesting to see if similar ideas involving Theorem 4.13 can be used to obtain more general explicit tropical formulae for Hurwitz numbers.

For more on fundamental groups in the context of non-archimedean spaces, we refer the reader to [14] and [4]. In [14], one starts with the ordinary fundamental group for algebraic coverings and then moves on to topological coverings (we call these completely split coverings). Our approach lies in between these two, since every topological covering is residually tame (see Proposition 5.13), but not every algebraic covering is residually tame (see Example 2.2). The statement of Theorem 5.14 appears in various guises throughout the literature, see for instance [14, Theorem 2.6]. Our results can be seen as a natural generalization where the category of coverings of a graph is replaced by the more general category of enhanced tame coverings of a metrized complex Σ.

A paper in this area that is also close to ours is [25]. Here one defines an auxiliary fundamental group using the intersection graph of a semistable model $\mathcal{X}/R$ defined over a discrete valuation ring R, and one then shows (under some restrictions) that its profinite completion is isomorphic to the étale fundamental group of an open subscheme of $\mathcal{X}$. Since the base ring is fixed, this excludes a large part of what we call *metrically ramified* morphisms, essentially by [25, Théoreme 3.7]. In particular, one does not consider all extensions that arise from the component group of the Néron model of the Jacobian of the curve. Our result is also stronger in another sense, namely that the isomorphism of profinite fundamental groups arises from an equivalence of Galois categories. This for instance implies that there is a unique Galois closure for an enhanced covering of metrized complexes.

The simultaneous semistable reduction theorem we prove in Theorem 3.3 is a generalization of [22, Theorem 2.3]. There, the covering $X' \to X$ is assumed to be Galois and the Galois group G satisfies the tameness condition $p \nmid |G|$. In terms of this paper, we say that the covering is Galois-topologically tame, see Section 2.1. A possible proof for the Galois-topologically tame version of 3.3 was suggested in [13, Section 1.2.1] using results by Berkovich. We combine these with the lifting results in [1] to give a full proof in the more general residually tame case. We note that various other versions of Theorem 3.3 in the *discretely valued case* can be found in the literature. For instance, there is [24, Section 3.13], [3, Theorem 4.14 and Remark 4.15] and [18, Theorem 1.1]. Here, one can use purity of the branch locus and Abhyankar's Lemma, two results that are not available in the non-discrete case.

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2. Preliminaries

In this section we give a summary of algebraic and analytic results that we will need in the rest of the paper. We will moreover recall the notion of a residually tame morphism and some well-known facts concerning these. In Section 2.2 we review the lifting results in [1] and point out what adjustments have to be made to obtain a Galois category in Section 4.

2.1. Topological and residual tameness

We will use the following notation throughout this paper:

- K is a complete, algebraically closed non-archimedean field with non-trivial valuation $\mathrm{val} : K \to \mathbb{R} \cup \{\infty\}$ and value group $\Lambda := \mathrm{val}(K^*)$,
- R is its valuation ring,
- $\mathfrak{m}_R$ is its maximal ideal,
- $k = R/\mathfrak{m}_R$ is its residue field and
- ϖ is an element in K with $\mathrm{val}(\varpi) > 0$.

We use p to denote the characteristic of the residue field (which is allowed to be zero). We endow K with the absolute value $|f| = e^{-\mathrm{val}(f)}$, where e is Euler's constant.

Throughout the paper, we will impose tameness conditions of the form "$p \nmid n$" or "n is coprime to p", for n an integer. For $p = 0$ these are void conditions. A curve X over K is a smooth proper scheme of finite type over K of dimension 1. Note that we allow curves to be disconnected. A marked curve (X, D) is a curve X/K together with a finite set $D \subset X(K)$. A morphism of curves is a finite morphism of schemes $\phi : X' \to X$. We will moreover assume morphisms of curves to be separable, in the sense that they are étale at every generic point of X'. If a morphism of curves $\phi : X' \to X$ is dominant, then we say that it is a covering of X. If X is connected, then ϕ is automatically dominant since it is finite. A morphism of marked curves $(X', D') \to (X, D)$ is a morphism of curves that is étale on $X'\backslash D'$, where $\phi^{-1}(D) = D'$. We similarly define a covering of marked curves. A morphism between two coverings $(X', D') \to (X, D)$ and $(X'', D'') \to (X, D)$ is a commutative diagram

$$\begin{array}{ccc} (X', D') & \longrightarrow & (X'', D'') \\ & \searrow & \downarrow \\ & & (X, D) \end{array},$$

where $(X', D') \to (X'', D'')$ is a morphism of marked curves. A semistable model for a connected curve X/K is a triplet $(\mathcal{X}, \pi, \psi)$, where $\mathcal{X}$ is an integral scheme, $\pi : \mathcal{X} \to \mathrm{Spec}(R)$ is a flat, proper morphism such that the special fiber $\mathcal{X}_s$ is reduced and only has ordinary double singularities, and $\psi : \mathcal{X}_\eta \simeq X$ is an isomorphism (here $\mathcal{X}_\eta$ is the generic fiber of $\mathcal{X}$). We will denote a semistable model by $\mathcal{X}$. A strongly semistable model for X is a semistable model such that the irreducible components of the special fiber are smooth.

Let X be a scheme of finite type over K. The Berkovich analytification X^{an} of X is the set of pairs $x = (P, \mathrm{val}_P : K(P) \to \mathbb{R} \cup \{\infty\})$, where $P \in X$ and $\mathrm{val}_P(\cdot)$ is a valuation on the residue field of P that extends the valuation on K. Note that we can also identify points $x \in X^{\mathrm{an}}$ with equivalence classes of L-valued points of X, where L is a complete valued field extension of K. The completed residue field of a point $x = (P, \mathrm{val}_P : K(P) \to \mathbb{R} \cup \{\infty\})$ is the completion of $K(P)$ with respect to the induced valuation. We denote it by $\mathcal{H}(x)$. Its residue field is denoted by $\tilde{\mathcal{H}}(x)$. There is a natural topology on X^{an}, which for $X = \mathrm{Spec}(A)$ is the coarsest topology such that the maps $X^{\mathrm{an}} \to \mathbb{R} \cup \{\infty\}$ given by $(P, \mathrm{val}_P(\cdot)) \mapsto \mathrm{val}_P(f(P))$ for $f \in A$ are continuous. Here $\mathbb{R} \cup \{\infty\}$ is endowed with the order topology, and $f(P)$ is the image of f in $K(P)$. The topology on X^{an} for more general schemes is induced by gluing these for an affine open covering of X, see [17, Section 2]. The Berkovich analytification X^{an} moreover has the structure of an analytic space over K, see [6] and [28] for more details. We will only be needing good analytic spaces, which are analytic spaces such that every point has an affinoid neighborhood. For these spaces, we can freely use the material in both [5] and [6].

We now recall the definition of a semistable vertex set and a skeleton of a curve. A semistable vertex set V of a connected curve X is a finite set of points in X^{an} such that $X^{\mathrm{an}}\backslash V$ is a disjoint union of a set of open disks and finitely many open annuli, see [1, Definition 3.8]. A semistable vertex set of a marked connected curve (X, D) is a semistable vertex set of X such that the points in D are contained in distinct open disks of $X^{\mathrm{an}}\backslash V$. In particular, we have that $X^{\mathrm{an}}\backslash(V \cup D)$ is a disjoint union of a set of open disks and finitely many generalized open annuli. A triangulated marked curve $(X, V \cup D)$ is a marked curve (X, D) with a fixed semistable vertex set V of (X, D). The skeleton of $(X, V \cup D)$ is

$$\Sigma = \Sigma(X, V \cup D) = V \cup D \cup \bigcup \Sigma(A_i),$$

where the $\Sigma(A_i)$ are the skeleta of the finitely many generalized open annuli in $X^{\mathrm{an}}\backslash(V \cup D)$, see [1, Section 3.11] and [9, Section 2.3]. A semistable vertex set of a curve X corresponds to a unique semistable model $\mathcal{X}/R$ of X by [9, Theorem 4.11]. We say that V is strongly semistable if the irreducible components of the special fiber of $\mathcal{X}$ are smooth. For arbitrary marked curves, we define semistable vertex sets and skeleta by taking disjoint unions.

For any morphism $\phi : X' \to X$ of schemes of finite type over K, we obtain an induced morphism $\phi^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$ of analytic spaces. More explicitly, if we view a point $x' \in X'^{\mathrm{an}}$ as an L-valued point of X' for some complete valued field extension $L \supset K$, then this gives an induced L-valued point of X by composing $\mathrm{Spec}(L) \to X'$ with $\phi : X' \to X$. If x' is mapped to x by ϕ^{an}, then this induces a map of completed residue fields

$$i_{x'} : \mathcal{H}(x) \to \mathcal{H}(x').$$

We use this to define the notion of topological tameness and residual tameness for morphisms of curves $X' \to X$.

Definition 2.1. *(Topological and residual tameness)* Let X be a curve over K and let $\phi : X' \to X$ be a morphism of curves with analytification $\phi^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$. Let $x' \in X^{\mathrm{an}}$ with $\phi^{\mathrm{an}}(x') = x$ and consider the extension of completed residue fields

$$i_{x'} : \mathcal{H}(x) \to \mathcal{H}(x').$$

We say that

(1) ϕ^{an} is residually tame at x' if $i_{x'}$ is a tame extension of valued fields, see [6, Section 2.4].[2]
(2) ϕ^{an} is topologically tame at x' if $[\mathcal{H}(x') : \mathcal{H}(x)]$ is coprime to $p := \mathrm{char}(k)$.

[2] A tame extension of valued fields is called a moderately ramified extension in [6].

We say that ϕ or ϕ^{an} is topologically tame (resp. residually tame) if ϕ^{an} is topologically tame (resp. residually tame) at every point of X'^{an}. A morphism of marked curves $(X', D') \to (X, D)$ is topologically (resp. residually) tame if the underlying morphism $X' \to X$ has the corresponding property.

These concepts can also be found in [27], [13] and [6]. If ϕ^{an} is topologically tame at a point x', then it is also residually tame at that point, but the converse is not necessarily true, see Example 2.2. For any point x' at which ϕ^{an} is residually tame and étale, we have the following relation by [6, Proposition 2.4.7]:

$$[\mathcal{H}(x') : \mathcal{H}(x)] = [\tilde{\mathcal{H}}(x') : \tilde{\mathcal{H}}(x)][|\mathcal{H}(x')^*| : |\mathcal{H}(x)^*|] \tag{4}$$

The morphism ϕ^{an} is then topologically tame at x' if both of the indices in Equation (4) are not divisible by $\mathrm{char}(k)$.

Example 2.2. We give an example of a morphism of curves that is residually tame and étale, but not topologically tame. Consider an elliptic curve E over $\mathbb{C}_2$ with ordinary reduction, meaning that there is a model $\mathcal{E}/R$ with good reduction such that $\mathcal{E}[2](\overline{\mathbb{F}}_2) \simeq \mathbb{Z}/2\mathbb{Z}$. Here R is the valuation ring of $\mathbb{C}_2$. To be more explicit, consider the elliptic curve over $\mathbb{C}_2$ given by the equation

$$y^2 + xy = x^3 + 1.$$

The projective homogenization of this equation in $\mathbb{P}^2_R$ gives a model $\mathcal{E}/R$ with the desired reduction. The non-trivial 2-torsion point in the special fiber is given in local coordinates by $\overline{P} = (0, 1)$.

We denote the special fiber of $\mathcal{E}$ by $\overline{E}$. Since $\overline{E}$ has a non-trivial 2-torsion point, there is a unique étale morphism $\overline{E}' \to \overline{E}$ of degree two up to $\overline{E}$-isomorphism. More explicitly, it is the isogeny dual to the Frobenius morphism. Using [1, Theorem 7.4], we see that there is a lift of the morphism $\overline{E}' \to \overline{E}$ to a finite morphism of semistable models $\mathcal{E}' \to \mathcal{E}$. The generic points η' and η of the special fiber of $\mathcal{E}'$ and $\mathcal{E}$ correspond to type-2 points x' and x of the Berkovich spaces E'^{an} and E^{an}. The induced morphism $\tilde{\mathcal{H}}(x) \to \tilde{\mathcal{H}}(x')$ is just the map of function fields $\overline{\mathbb{F}}_2(E) \to \overline{\mathbb{F}}_2(E')$, which is separable by construction. We thus see that the morphism $E'^{\mathrm{an}} \to E^{\mathrm{an}}$ is residually tame over x. The morphism splits completely over the other points, so it defines a residually tame covering. Note that it is not topologically tame, since $[\mathcal{H}(x') : \mathcal{H}(x)] = 2$.

Remark 2.3. We point out one important difference between the usual notion of tameness for non-archimedean fields and the notion of topological tameness. For a finite separable extension of complete non-archimedean fields $K \subset L$, the tameness of L implies the tameness of the Galois closure $\overline{L}$ over K. The same is not necessarily true for topological tameness. To see this, we first note that the factor $[|\mathcal{H}(x')^*| : |\mathcal{H}(x)^*|]$ will not be divisible by $\mathrm{char}(k) = p$ after passing to the Galois closure, see [6, Section 2.4]. However, the degree $[\tilde{\mathcal{H}}(x') : \tilde{\mathcal{H}}(x)]$ of the extension of reductions can easily become divisible by p after passing to the Galois closure. In the proof of Theorem 3.3 we will show that in certain important cases the residual tameness does imply the topological tameness of the Galois closure.

Definition 2.4. Let $X' \to X$ be a morphism of connected curves with extension of function fields $K(X) \to K(X')$. Let $K(\overline{X})$ be the Galois closure of this extension and let $\overline{X} \to X$ be the corresponding covering of connected curves. We say that $X' \to X$ is Galois-topologically tame if $\overline{X} \to X$ is topologically tame.

Note that this definition was also used in [6, Section 6.3] to compare coverings of algebraic curves and coverings of Berkovich spaces. It also implicitly plays a role in [22, Theorem 2.3], where the coverings are Galois with Galois group G and $p \nmid |G|$.

Let $\overline{X}$ and X be two connected algebraic curves, and let $\phi : \overline{X} \to X$ be a Galois covering with Galois group G. Let $\overline{U}$ be the étale locus of ϕ and let $\overline{U} \to U$ be the étale morphism obtained by restricting ϕ to $\overline{U}$. The induced map of Berkovich analytifications $\overline{U}^{\mathrm{an}} \to U^{\mathrm{an}}$ is then also étale by [5, Proposition 3.4.6]. We have that $\overline{U}$ is a G-torsor, so that the induced map

$$\overline{U} \times_U G \to \overline{U} \times_U \overline{U}$$

is an isomorphism. For any complete valued field L extending K, we then obtain a bijection

$$\overline{U}(L) \times_{U(L)} G \to \overline{U}(L) \times_{U(L)} \overline{U}(L). \tag{5}$$

In other words, the action of G on the L-valued points of $\overline{U}$ is simply transitive. We use this to prove the following.

Proposition 2.5. *Let $\phi : X' \to X$ be a morphism of connected curves with Galois closure $\overline{X} \to X$ and consider a point $x \in X^{\mathrm{an}}$ over which $\overline{\phi}^{\mathrm{an}}$ is étale. Let x_i be the points in X'^{an} lying over x and let $\overline{x}$ be a point in $\overline{X}^{\mathrm{an}}$ lying over x. The extension $\mathcal{H}(\overline{x}) \supset \mathcal{H}(x)$ is Galois with Galois group $D_{\overline{x}/x} = \{\sigma \in G : \sigma(\overline{x}) = \overline{x}\}$. The composite of the $\mathcal{H}(x)$-embeddings $\mathcal{H}(x_i) \to \mathcal{H}(\overline{x})$ is $\mathcal{H}(\overline{x})$.*

Proof. We first note that the completed residue field $\mathcal{H}(\overline{x})$ of any $\overline{x}$ lying over x is a finite separable extension of $\mathcal{H}(x)$, so that we can view them as subextensions of the separable closure $L := \mathcal{H}(x)^{\mathrm{sep}}$. We write $x(L)$ for the L-valued point of U corresponding to $x \in U^{\mathrm{an}}$, Z for the fiber of $\overline{\phi}^{\mathrm{an}}$ over x and $Z(L)$ for the corresponding L-valued points of $\overline{U}$ that map to $x(L)$. By the bijectivity in Equation (5), the action of G on $Z(L)$ is simply transitive.

By [6, Theorem 3.4.1], there is an equivalence of categories

$$\mathrm{Fét}(X, x) \to \mathrm{Fét}(\mathcal{H}(x)). \tag{6}$$

Consider the K-germ $(\overline{X}, Z)$, which forms an object of the first category in Equation (6). There is a natural map $G \to \mathrm{Aut}((\overline{X}, Z))$. By the aforementioned equivalence and the fact that the action of G on $Z(L)$ is simply transitive, we easily find that this map is bijective. The rest of the proposition now directly follows from standard methods in Galois theory; we leave the details to the reader. □

Corollary 2.6. Consider the set-up as in Proposition 2.5 and let $\overline{\mathcal{H}}(x_i)$ be a Galois closure of $\mathcal{H}(x_i)$ over $\mathcal{H}(x)$. Suppose that $p \nmid [\overline{\mathcal{H}}(x_i) : \mathcal{H}(x)]$ for every i. Then $p \nmid [\mathcal{H}(\overline{x}) : \mathcal{H}(x)]$.

Proof. This follows directly from Proposition 2.5 and the fact that the Galois group of a composite of Galois extensions is a quotient of the direct product of the individual Galois groups. □

Corollary 2.7. Consider the set-up as in Proposition 2.5 and suppose that ϕ is residually tame. Then the Galois closure $\overline{\phi}$ is residually tame.

Proof. By [6, Section 2.4], we find that if $\mathcal{H}(x_i) \supset \mathcal{H}(x)$ is tame, then the Galois closure is also tame and any composite of tame extensions is again tame. By Proposition 2.5, we then see that $\mathcal{H}(\overline{x})$ is also tame for any point $\overline{x}$ lying over x in the Galois closure $\overline{X}$ of X' over X. □

Lemma 2.8. *Suppose that $K(X) \subset K(X_1) \subset \overline{K(X)}$ and $K(X) \subset K(X_2) \subset \overline{K(X)}$ are function field extensions in a fixed algebraic closure $\overline{K(X)}$ of $K(X)$ corresponding to residually tame (resp. Galois-topologically tame) coverings of (X, D). Consider their composite $M = K(X_1) \cdot K(X_2)$. Then M is residually tame (resp. Galois-topologically tame).*

Proof. By Corollaries 2.6 and 2.7, the Galois closures of both $K(X_i)/K(X)$ are residually tame (resp. Galois-topologically tame). It suffices now to show that the composite of the Galois closures has the desired property. This composite is Galois, so we can use the local results from Proposition 2.5 again. Both cases easily follow from this. □

We now consider the composite M_{Tame} (resp. M_{GTop}) of all function fields $K(X')$ corresponding to residually tame (resp. Galois-topologically tame) coverings $(X', D') \to (X, D)$ of a fixed marked curve (X, D). Using Corollary 2.7 and Lemma 2.8, we easily see that M_{Tame} and M_{GTop} are Galois.

Lemma 2.9. *Let $K(X)$ be the function field of a curve X with a fixed set of marked points $D \subset X(K)$. Consider the composite M_{Tame} (resp. M_{GTop}) of all function field extensions $K(X') \supset K(X)$ arising from residually tame (resp. Galois-topologically tame) coverings $(X', D') \to (X, D)$. Then M_{Tame} and M_{GTop} are Galois.*

Proof. Let x be an element of M_{Tame} or M_{GTop} and let L be the field generated by x over $K(X)$. By Lemma 2.8, L is residually tame (resp. Galois-topologically tame) over $K(X)$. By Corollary 2.7, we then see that the Galois closure $\overline{L}$ over K is residually tame. For Galois-topologically tame coverings, $\overline{L}$ is automatically topologically tame. The extensions M_{Tame} and M_{GTop} are thus Galois. □

We denote the Galois groups of the extensions in Lemma 2.9 by $\pi_{\text{Tame}}(X, D)$ and $\pi_{\text{GTop}}(X, D)$. We view these as the fundamental groups of suitable Galois categories. For more background regarding the notion of a Galois category, we refer the reader to [26, Tag 0BMQ], [16], [20] and [11].

Definition 2.10. Let X/K be a connected curve with a set of marked points D. We write $\text{Cov}(X, D)$ for the category of all finite étale coverings of $X\backslash D$, or equivalently, the category of coverings of the marked curve (X, D). The full subcategories of $\text{Cov}(X, D)$ of all coverings that are residually (resp. Galois-topologically) tame are denoted by $\text{Cov}_{\text{Tame}}(X, D)$ and $\text{Cov}_{\text{GTop}}(X, D)$ respectively. By Lemma 2.9, these are Galois categories with profinite fundamental groups $\pi_{\text{Tame}}(X, D)$ and $\pi_{\text{GTop}}(X, D)$.

Remark 2.11. Throughout this paper, we suppress the base-points in the profinite fundamental groups $\pi(X, x)$.

2.2. A review of the lifting results in [1]

We now review the notion of a metrized complex, a tame covering of metrized complexes and the lifting results in [1] in the connected case. For disconnected metrized complexes, we refer the reader to Remark 2.18. Let Σ be a metric graph, as defined in [1, Section 2.1]. The essential vertices of Σ are the points of valence not equal to 2. Let Λ be the value group of K. A Λ-metric graph is a metric graph such that the distance between two essential vertices lies in Λ. A vertex set $V(\Sigma)$ of Σ is a finite subset of the Λ-points of Σ containing all essential vertices. This also gives rise to an edge set, which we denote by $E(\Sigma)$. To ease notation, we will from now on omit the value group Λ.

The analogue of a covering map for metric graphs is the notion of a *finite harmonic morphism*, see [1, Definitions 2.4 and 2.6]. We recall the harmonicity in this definition. Let $p' \in \Sigma'$ map to $p \in \Sigma$ under $\phi : \Sigma' \to \Sigma$. Let v be a tangent direction at p, which is an equivalence class of line segments starting at p. For small enough line segments e, we have that the map ϕ over e is affine linear with dilation factor $d_{v'/v} := d_{e'/e}$. We say that ϕ is harmonic at p' if the local degree

$$d_{p',v} := \sum_{v' \mapsto v} d_{v'/v} \tag{7}$$

is independent of the tangent direction v. Here the sum in Equation (7) ranges over all tangent directions starting at p' and mapping to v. The total degree of the harmonic morphism is the sum of the local degrees $d_{p',v}$ for p' mapping to p. This is independent of the chosen p. A morphism $\Sigma' \to \Sigma$ is said to be harmonic if it is surjective and harmonic at every $p' \in \Sigma'$.

If Σ is a subset of the Berkovich analytification X^{an} of a curve X/K, then every point $x \in \Sigma$ comes with additional algebraic data in the form of a residue field $\tilde{\mathcal{H}}(x)$. If x is a point of type two, then this is the function field of a unique smooth curve C_x/k, and the different tangent directions at x in X^{an} can be identified with the closed points of C_x. We can add this algebraic data to Σ as follows.

Definition 2.12. *(Metrized complexes)* A metrized complex of k-curves (Σ, C_x) consists of a metric graph Σ, a vertex set $V(\Sigma)$ of Σ, a set of smooth proper connected curves C_x/k for finite vertices $x \in V(\Sigma)$ and an identification of the edges e adjacent to x with closed points z_e of C_x. We will also simply call these metrized complexes.

Let $(\Sigma', C_{x'})$ and (Σ, C_x) be metrized complexes with vertex sets $V(\Sigma')$ and $V(\Sigma)$. A morphism $(\Sigma', C_{x'}) \to (\Sigma, C_x)$ of metrized complexes consists of a finite harmonic morphism of metric graphs $\phi : \Sigma' \to \Sigma$ represented by the vertex sets $V(\Sigma')$ and $V(\Sigma)$ (see [1, Definition 2.4]), together with a collection of finite morphisms of curves

$$\phi_{x'/x} : C_{x'} \to C_x$$

satisfying the following:

(1) If $e' \in E(\Sigma')$ maps to $e \in E(\Sigma)$ under ϕ, then $z_{e'}$ maps to z_e under $\phi_{x'/x}$.
(2) The morphism $\phi_{x'/x}$ is only ramified at the points $z_{e'}$ corresponding to the edges e' adjacent to x'.
(3) The ramification index of $z_{e'} \mapsto z_e$ is equal to the dilation factor $d_{e'/e}$.

We say that ϕ is tame if the maps $\phi_{x'/x}$ are all tame. This in particular implies that the $\phi_{x'/x}$ are separable.

Example 2.13. Let $(X, V \cup D)$ be a triangulated marked curve with skeleton Σ as in Section 2.1 or [1, Section 3.7]. This skeleton is a metric graph. The residue field $\tilde{\mathcal{H}}(x)$ of every $x \in V$ is isomorphic to the function field of a smooth proper connected curve C_x. We call this the *residue curve* of x. The tangent directions at x that belong to Σ can be identified with closed points on C_x, see [1, Section 3.20]. If e is an edge that represents a tangent direction, then we write z_e for this closed point. From this we obtain a natural metrized complex (Σ, C_x). We call this the metrized complex associated to the triangulated marked curve $(X, V \cup D)$.

Remark 2.14. We point out two differences between the definitions given here and the ones in [1]. In [1], metric graphs are augmented with a weight function, which gives the genera of the local curves C_x. We omitted this here, since the genus is implied by the residue curve. Secondly, we note that the second condition in Definition 2.12 is added so that the morphism of metrized complexes is generically étale in the terminology of [1]. This implies that the Riemann-Hurwitz formula holds at every finite vertex.

We now recall the notion of a star-shaped curve and the main lifting theorem from [1]. A star-shaped curve (U, x) consists of a smooth analytic space U of dimension 1 over K with a point $x \in U$ such that $U \backslash \{x\}$ is a disjoint union of open disks and finitely many open annuli, see [1, Definition 6.2]. A marked star-shaped curve (U, x, D) consists of a star-shaped curve (U, x) together with a set D of points of type 1 such that each $P \in D$ is contained in a distinct open disk in $U \backslash \{x\}$. We will also write U for the marked star-shaped curve if the central vertex x and the marked points D are clear from context. The space U comes with a natural retraction map $\tau : U \to \Sigma(U, \{x\} \cup D)$, where

$$\Sigma(U, \{x\} \cup D) = \{x\} \cup D \cup \bigcup \Sigma(A_i) \cup \bigcup \Sigma(B_i)$$

is the skeleton of the marked star-shaped curve, see [1, Section 6.3]. Here the A_i are the finitely many open annuli, the B_i are the punctured open disks corresponding to the points $P \in D$, and the $\Sigma(A_i)$ and $\Sigma(B_i)$ are the skeleta of the A_i and B_i as defined in [9, Section 2.3]. We can now state the main local lifting theorem in [1].

Theorem 2.15. *Let (U, x, D) be a marked star-shaped curve with residue curve C_x, and let $C' \to C_x$ be a tame covering that is only ramified over the closed points corresponding to tangent directions in $\Sigma(U, \{x\} \cup D)$. Then there is a unique lifting (up to a unique isomorphism) of $C' \to C_x$ over k to a tame covering of star-shaped curves over K.*

Proof. See [1, Theorem 6.18]. The notion of a tame covering of star-shaped curves can be found in [1, Definition 6.12]. □

This result gives us our local analytic models over K, which can be glued to give global coverings of algebraic curves. We explain this gluing in detail here, as it will be important in Section 4.

Let (Σ, C_x) be the metrized complex associated to a triangulated marked curve $(X, V \cup D)$. For every $x \in V$, there is an associated star-shaped curve $Y(x) \subset X^{\mathrm{an}}$. For a tame covering $(\Sigma', C_{x'}) \to (\Sigma, C_x)$ of metrized complexes, we then obtain a unique set of local coverings $Y(x') \to Y(x)$ of star-shaped curves by Theorem 2.15. Write $\overrightarrow{E}_f(\Sigma)$ for the set of *finite oriented edges* in Σ, and let $e = xy$ be a finite oriented edge with corresponding open edge e^o. We write τ_x for the retraction map associated to $Y(x)$, and τ_y for the retraction map associated to $Y(y)$. These are induced by the global retraction map τ coming from $\Sigma(X, V \cup D)$. In particular, we have $\tau_x^{-1}(e^o) = \tau_y^{-1}(e^o)$, which is an open annulus. We now consider an edge e' lying over e. We then have the sets $\tau_{x'}^{-1}(e'^o)$ and $\tau_{y'}^{-1}(e'^o)$, which are again open annuli. The induced maps $\tau_{x'}^{-1}(e'^o) \to \tau_x^{-1}(e^o)$ and $\tau_{y'}^{-1}(e'^o) \to \tau_y^{-1}(e^o)$ are *Kummer*, in the sense that they are obtained by extracting an

$d_{e'/e}$-th root of a parameter of the open annulus $\tau_x^{-1}(e^o) = \tau_y^{-1}(e^o)$. As such, there exists an isomorphism $\tau_{x'}^{-1}(e'^o) \to \tau_{y'}^{-1}(e'^o)$ that gives a commutative diagram

$$\begin{array}{ccc} \tau_{x'}^{-1}(e'^o) & \longrightarrow & \tau_{y'}^{-1}(e'^o) \\ \downarrow & & \downarrow \\ \tau_x^{-1}(e^o) & \longrightarrow & \tau_y^{-1}(e^o). \end{array} \tag{8}$$

These isomorphisms are not unique however, as we can compose any isomorphism with an element of $\mathrm{Aut}(\tau_{y'}^{-1}(e'^o)/\tau_y^{-1}(e^o)) \simeq \mathbb{Z}/d_{e'/e}\mathbb{Z}$.

Definition 2.16. *(Gluing data)* Let (Σ, C_x) be the metrized complex associated to a triangulated marked curve $(X, V \cup D)$. A set of gluing data for a pair of oriented edges $e' \in \overrightarrow{E}_f(\Sigma')$ and $e \in \overrightarrow{E}_f(\Sigma)$ with $\phi(e') = e$ is an isomorphism $\theta_{e'/e} : \tau_{x'}^{-1}(e'^o) \to \tau_{y'}^{-1}(e'^o)$ such that the diagram in Equation (8) commutes. A set of gluing data for a tame covering of metrized complexes consists of a set of gluing data for all pairs of finite oriented edges $(e', e) \in \overrightarrow{E}_f(\Sigma') \times \overrightarrow{E}_f(\Sigma)$ with $\phi(e') = e$. We impose the condition $\theta_{e'/e}^{-1} = \theta_{\overline{e}'/\overline{e}}$ for edges $\overline{e} = yx$ and $\overline{e}' = y'x'$ with the opposite orientation. A set of gluing data for a tame covering is denoted by $\mathfrak{g}$, and the set of all gluing data for a pair $(\Sigma' \to \Sigma, X)$ is denoted by $\mathcal{G}(\Sigma', X)$.

Note that there are only finitely many isomorphisms $\theta_{e'/e}$ for any given oriented edge. Indeed, the group $\mathrm{Aut}(\tau_{y'}^{-1}(e'^o)/\tau_y^{-1}(e^o)) \simeq \mathbb{Z}/d_{e'/e}\mathbb{Z}$ acts on these isomorphisms by postcomposition and this action is simply transitive. In other words, the set of $\theta_{e'/e}$ is a $\mathbb{Z}/d_{e'/e}\mathbb{Z}$-torsor. We can trivialize this torsor by fixing an initial isomorphism, which identifies the set of such isomorphisms $\theta_{e'/e}$ with $\mathbb{Z}/d_{e'/e}\mathbb{Z}$. We will see an algebraic version of this in Section 4.2.

Remark 2.17. In [1, Section 7], a set of gluing data is defined to be a set of isomorphisms

$$\theta_{e'/e} : \tau_{x'}^{-1}(e'^o) \to \tau_{y'}^{-1}(e'^o)$$

such that $\theta_{e'/e}^{-1} = \theta_{\overline{e}'/\overline{e}}$ for the edges $\overline{e}' = y'x'$ and $\overline{e} = yx$ with the opposite direction. This however gives us the possibility to compose any isomorphism with an arbitrary automorphism of the annulus in question. For instance, for any $u \in R^*$, we obtain a non-trivial automorphism by multiplying a parameter t by u. The elements in $\mathrm{Aut}(\tau_{y'}^{-1}(e'^o)/\tau_y^{-1}(e^o))$ can be seen as examples of this by taking u to be a primitive n-th root of unity. We now see that the annuli $\tau_{x'}^{-1}(e'^o)$ and $\tau_{y'}^{-1}(e'^o)$ afford infinitely many distinct automorphisms and this implies that the set of gluing data is infinite. If we however require the corresponding diagrams to be commutative as here, then the set of gluing data is finite. This seems to be implicit in the considerations in [1], as the sets of gluing data obtained in the examples there are finite and equal to the ones obtained here.

Remark 2.18. To obtain a Galois category of tropical coverings, we have to modify the definitions given in this section to allow for disconnected metrized complexes and morphisms between these. We leave it to the reader to write out their formal definitions. The most important modification here is that finite harmonic morphisms are not required to be surjective anymore. We however do require these morphisms to be surjective on the connected components, so that we retrieve the original definition from [1] in the connected case. If $\phi : \Sigma' \to \Sigma$ is surjective and tame, then we say that ϕ is a tame covering of Σ. This is analogous to our definition of coverings of curves, see Section 2.1.

3. A simultaneous semistable reduction theorem for tame coverings

In this section, we prove a simultaneous semistable reduction theorem for residually tame coverings of a marked algebraic curve (X, D). This theorem will be used in Section 4 to construct a functor from the category of residually tame coverings of (X, D) to the category of tame coverings of a metrized complex Σ associated to (X, D).

We first recall the notion of a Galois covering in the context of analytic spaces.

Definition 3.1. Let U, V be analytic spaces, let $\phi : U \to V$ be a finite étale morphism and let G be a finite group acting on U through V-automorphisms. We say that ϕ is a Galois if ϕ is a G-torsor, so that the map

$$U \times_V G \to U \times_V U$$

is an isomorphism. Note that since ϕ is finite étale, it is already locally trivial.

We now have the following important theorem by Berkovich.

Theorem 3.2. *(Topologically tame finite étale Galois coverings of open disks and annuli)*

(1) Let U be an open disk and let $\psi : V \to U$ be a topologically tame Galois covering. Then V is a disjoint union of open disks.
(2) Let U be an open annulus, let V be connected and let $V \to U$ be a topologically tame Galois covering. Then there exist isomorphisms $\mathbf{S}(a)_+ \simeq U$ and $V \simeq \mathbf{S}_+(a^{1/n})$ for $a \in K$ such that the composed map is given by $t \mapsto t^n$. Here $\mathbf{S}(a)_+$ and $\mathbf{S}_+(a^{1/n})$ are standard open annuli as in [1, Section 3.1].

Proof. This follows from [6, Theorems 6.3.2 and 6.3.5] by writing U as a union of closed disks or closed annuli. □

Theorem 3.3. *Let $(X', D') \to (X, D)$ be a residually tame covering of marked curves. Then the inverse image of any (strongly) semistable vertex set V of (X, D) is a (strongly) semistable vertex set for (X', D').*

Proof. It suffices to prove the connected case. Let $\phi : X' \to X$ be the morphism of curves and let $\overline{\phi} : \overline{X} \to X$ be its Galois closure with Galois group G. By basic ramification theory, we have that D is again the branch locus of $\overline{\phi}$. We write $\overline{D} = \overline{\phi}^{-1}(D)$ for the ramification locus. Throughout the proof, we will denote the analytified morphisms $X'^{\mathrm{an}} \to X^{\mathrm{an}}$ and $\overline{X}^{\mathrm{an}} \to X^{\mathrm{an}}$ by ϕ and $\overline{\phi}$ again to ease notation. Let U be a connected component of $X^{\mathrm{an}} \backslash V$ that is an open annulus or an open disk. We have that $\overline{\phi}^{-1}(U) \to U$ is a G-torsor, so that it is Galois. This also implies that the covering on any connected component of $\overline{\phi}^{-1}(U)$ is Galois, with Galois group the stabilizer of the component.

Suppose now that $\overline{\phi}^{-1}(U) \to U$ is topologically tame. We then conclude by Theorem 3.2 that $\overline{\phi}^{-1}(U)$ is a disjoint union of open annuli or open disks. On the annuli, the covering is Kummer and on the open disks the covering is trivial. This then also implies that the coverings $\phi^{-1}(U) \to U$ are of the above type. If we now consider the inverse image V' of the semistable vertex set V of X^{an} in X'^{an}, then we see that the complement $X'^{\mathrm{an}} \backslash V'$ is a disjoint union of open annuli and open disks. In other words, V' is a semistable vertex set for X'^{an}. The strongly semistable case also follows, since the induced morphism of metrized complexes from Proposition [1, Corollary 4.28] is finite, and thus Σ is loopless if and only if Σ' is loopless.

We now show that $\overline{\phi}$ is topologically tame over U. We first note that $\overline{\phi}$ and ϕ are topologically tame over any type-3 point. Indeed, this follows from the fact that for any point $x' \in X'^{\mathrm{an}}$ lying over x, we have that $\mathcal{H}(x) \subset \mathcal{H}(x')$ is residually tame if and only if $p \nmid [|\mathcal{H}(x')^{*}| : |\mathcal{H}(x)^{*}|]$. The latter in turn follows from $\tilde{\mathcal{H}}(x') = \tilde{\mathcal{H}}(x) = K$ and [6, Proposition 2.4.7]. We conclude using Lemma 2.6 that $\overline{\phi}$ is topologically tame at any type-3 point.

Suppose now for a contradiction that $\overline{\phi}$ is not topologically tame over some type-2 point $x \in U$ and let $\overline{x}$ be a point where $\overline{\phi}$ fails to be topologically tame. We will do the case where U is an open annulus, the open disk case is similar. We write x' for its image in X'^{an}. By the results in [1], we can find a pair of skeleta Σ'_1 and Σ_1 of X'^{an} and X^{an} respectively such that $x' \in \Sigma'_1$, $x \in \Sigma_1$ and $\phi^{-1}(\Sigma_1) = \Sigma'_1$. This in turn induces a finite morphism of triangulated marked curves $(X', V'_1 \cup D') \to (X, V_1 \cup D)$. From the residual tameness of ϕ, we conclude that the morphism of triangulated marked curves is a tame covering.

Consider the retraction morphism $\tau_{\Sigma} : X^{\mathrm{an}} \to \Sigma$ with $y := \tau_{\Sigma}(x)$. Note that the set of points in Σ_1 that retract to y under τ_{Σ} is a tree. For any type-2 point in the vertex set of Σ_1 that retracts to y (and not equal to y), we claim that the covering is trivial. Indeed, we can prove this inductively by starting at the leaf-vertices of Σ_1 retracting to y. Let z be such a leaf-vertex and choose a tangent direction outside Σ_1. Since ϕ is piecewise linear, we can find for any such tangent direction a geodesic $l' : [a, b] \to X'^{\mathrm{an}}$ such that the induced map $l : [am, bm] \to X^{\mathrm{an}}$ is a geodesic, see [1, Definitions 4.4 and 4.21]. Here m is the dilation factor. Suppose now that m is non-trivial in a direction outside Σ_1. Note that m is equal to the local degree $[\mathcal{H}(w') : \mathcal{H}(w')]$ for a pair of points in a geodesic representing the tangent direction, see [9, Propositions 2.2.1(2) and 2.5(1)]. A geodesic representing this direction contains points of type 2 and we obtain a contradiction using

[1, Proposition 4.35]. Since z is a leaf-vertex, there is only one direction in which the dilation factor can be non-trivial. But then the dilation factor has to be one, since this would otherwise give a tame covering of $\mathbb{P}^1$ ramified over only one point by [1, Theorem 4.23(2)], a contradiction. We conclude that ϕ is split over z. We then inductively continue this argument and see that ϕ is split over all type-2 points retracting to y and not equal to y. In particular, we must have $x = y = \tau_\Sigma(x)$, which implies that x is a type-2 point in an edge of Σ.

We now find that there are only two directions in which the dilation factor can be non-trivial. Indeed, by the previous argument the tangent directions belonging to Σ_1 are covered, and the covering is split over the other tangent directions. The tame covering of residue curves $C_{x'} \to C_x = \mathbb{P}^1$ is then ramified over exactly two points. This implies that $C_{x'} \to \mathbb{P}^1$ is a Kummer covering, of degree equal to the ramification degree in the two directions. This ramification degree is prime to p, since we can detect this using type-3 points. We conclude that $\mathcal{H}(x) \to \mathcal{H}(x')$ is Galois of degree coprime to p. By doing this for every point x' in X'^{an} lying over x, we find that all the corresponding morphisms $\mathcal{H}(x) \to \mathcal{H}(x')$ for x' lying over x are Galois of degree coprime to p. Using Lemma 2.5, we conclude that the degree of $\mathcal{H}(\overline{x})$ over $\mathcal{H}(x)$ is coprime to p for any $\overline{x}$ over x, a contradiction. We conclude that ϕ is topologically tame over any type-2 point in U. Since the map $\overline{\phi}$ is piecewise linear, we then also easily conclude that $\overline{\phi}$ is topologically tame at points of type 1 and 4 in U. □

Remark 3.4. For Galois-topologically tame coverings, the material in [1] on lifting morphisms to skeleta is not necessary, as we are done after the first paragraph of the proof. Since the theorems by Berkovich do not rely on any semistable reduction theorem, this gives a stand-alone proof of this theorem. The material in [1] on lifting morphisms however does rely on the semistable reduction theorem, so the second part does not give a stand-alone proof of this theorem for residually tame coverings. Finally, we note that the conclusion in the proof of Theorem 3.3 is similar to that of [27, Lemma 3.4.2]. Namely, if we have a residually tame morphism together with a skeleton of that morphism, then this morphism splits outside the skeleton.

Remark 3.5. Theorem 3.3 can be paraphrased in terms of ordinary algebraic geometry as follows. We first define a (strongly) semistable model for a marked curve (X, D) to be a strongly semistable model $\mathcal{X}$ such that D is mapped injectively to the smooth locus of $\mathcal{X}_s$ under the canonical reduction map $X(K) \to \mathcal{X}_s(k)$ assigned to $\mathcal{X}$. In terms of this language, Theorem 3.3 now says the following:

Let $\phi : X' \to X$ be a residually tame covering of smooth proper connected algebraic curves over K and let $\mathcal{X}$ be a (strongly) semistable model for (X, D), where D is the branch locus of ϕ. Let $\mathcal{X}'$ be the normalization of $\mathcal{X}$ in the function field $K(X')$. Then $\mathcal{X}'$ is (strongly) semistable and $\mathcal{X}' \to \mathcal{X}$ is a finite morphism of (strongly) semistable models over R.

This translated version quickly follows from the results in [1, Section 5] and the fact that semistable models are normal.

4. Enhanced tame coverings of metrized complexes

In this section we introduce the category of *enhanced* tame coverings of a metrized complex. These coverings consist of a tame covering of a metrized complex, together with a set of gluing data. In Section 4.1, we show that the category of enhanced tame coverings of a metrized complex is equivalent to the category of residually tame coverings of a marked curve. In Section 4.2, we show that we can give a purely algebraic definition of the category of enhanced tame coverings of a metrized complex. We conclude with several examples.

4.1. Enhanced tame coverings

In this section we define the category of enhanced coverings of a metrized complex and prove Theorem 4.13.

Remark 4.1. Throughout this section, we fix a triangulated marked curve $(X, V \cup D)$ whose underlying curve X is connected. We assume furthermore that V is strongly semistable, so that the associated metrized complex Σ is connected and loopless. We note that the metrized complexes Σ' for tame coverings $\Sigma' \to \Sigma$ are not necessarily connected.

We recall the set-up given before Definition 2.16. Let $(X, V \cup D)$ and Σ be as in Remark 4.1 and let $\phi : \Sigma' \to \Sigma$ be a tame covering of metrized complexes. We write $Y(x)$ and $Y(y)$ for the star-shaped curves corresponding to the endpoints of a finite oriented edge $e = xy$ in Σ with open edge e^o. By [1, Theorem 6.18], the morphisms of residue curves $C_{x'} \to C_x$ and $C_{y'} \to C_y$ lift to unique tame coverings of star-shaped curves $Y(x') \to Y(x)$ and $Y(y') \to Y(y)$. We denote the corresponding retraction maps of $Y(x')$ and $Y(y')$ by $\tau_{x'}$ and $\tau_{y'}$. A set of gluing data $\mathfrak{g}$ for ϕ consists of a set of $\tau^{-1}(e^o)$-isomorphisms $\theta_{e'/e} : \tau_{x'}^{-1}(e'^{\,o}) \to \tau_{y'}^{-1}(e'^{\,o})$ for all finite oriented edges $e' \mapsto e$, see Definition 2.16. These isomorphisms are moreover required to satisfy $\theta_{e'/e}^{-1} = \theta_{\overline{e}'/\overline{e}}$ for the edges $\overline{e}'$ and $\overline{e}$ with the opposite orientation. We emphasize here that for every pair $e' \mapsto e$, there are exactly $d_{e'/e}$ of these isomorphisms $\theta_{e'/e}$.

Definition 4.2. *(Enhanced tame coverings)* An *enhanced tame covering* of a metrized complex Σ associated to a triangulated marked curve $(X, V \cup D)$ is a pair $\phi_{\mathfrak{g}} := (\phi, \mathfrak{g})$ consisting of:

(1) A tame covering $\phi : \Sigma' \to \Sigma$ of metrized complexes of k-curves,
(2) An element $\mathfrak{g}$ of the set of gluing data $\mathcal{G}(\Sigma', X)$, see Definition 2.16.

We will also refer to such a pair as an *enhanced covering* of Σ.

Definition 4.3. *(Morphisms of enhanced coverings)* Let $\phi_{1,\mathfrak{g}_1}$ and $\phi_{2,\mathfrak{g}_2}$ be two enhanced coverings of a metrized complex Σ. Write $\phi_i : \Sigma_i \to \Sigma$ for the morphisms of metrized complexes and $\theta_{e_i,i}$ for the isomorphisms arising from the gluing data. A morphism $\phi_{1,\mathfrak{g}_1} \to \phi_{2,\mathfrak{g}_2}$ is a morphism of metrized complexes $\psi : \Sigma_1 \to \Sigma_2$ satisfying the following two properties:

(1) The diagram

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\psi} & \Sigma_2 \\ \downarrow{\scriptstyle \phi_1} & \swarrow{\scriptstyle \phi_2} & \\ \Sigma & & \end{array}$$

is commutative.

(2) The induced diagram

$$\begin{array}{ccc} \tau_x^{-1}(e_1^o) & \xrightarrow{\theta_{e_1,1}} & \tau_y^{-1}(e_1^o) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \tau_{\psi(x)}^{-1}(e_2^o) & \xrightarrow{\theta_{e_2,2}} & \tau_{\psi(y)}^{-1}(e_2^o) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2} \\ \tau^{-1}(e^o) & \xrightarrow{\mathrm{id}} & \tau^{-1}(e^o) \end{array}$$

commutes for every edge $e_1 \in \Sigma_1$ with images $e_2 \in \Sigma_2$ and $e \in \Sigma$. Here we again write ψ and ϕ_2 for the induced unique maps (see [1, Theorem 6.18]) on the corresponding star-shaped curves (see [1, Definition 6.2]) and the restrictions of these maps to subannuli. Two morphisms of enhanced coverings are composed by composing the morphisms of metrized complexes and the maps of open annuli.

Remark 4.4. The composition of the maps $\psi : \tau_x^{-1}(e_1^o) \to \tau_{\psi(x)}^{-1}(e_2^o)$ and $\phi_2 : \tau_{\psi(x)}^{-1}(e_2^o) \to \tau^{-1}(e^o)$ is equal to $\phi_1 : \tau_x^{-1}(e_1^o) \to \tau^{-1}(e^o)$. Indeed, this follows from the commutativity in the first condition of Definition 4.3, together with the fact that the maps on the star-shaped curves are uniquely induced from the maps of metrized complexes.

Remark 4.5. We note that any morphism $\Sigma_1 \to \Sigma_2$ of enhanced tame coverings of Σ is automatically tame. Indeed, any subextension of a separable extension is separable (which gives tameness at the vertices) and ramification degrees are multiplicative in towers (which gives tameness at the edges).

Remark 4.6. We will occasionally denote the commutative diagram of metrized complexes in Definition 4.3 by $\Sigma_1 \to \Sigma_2 \to \Sigma$ to ease notation.

Definition 4.7. *(Category of enhanced tame coverings)* Let Σ be the metrized complex associated to a triangulated marked curve $(X, V \cup D)$. The *category* $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ *of enhanced tame coverings* of Σ is defined as follows:

(1) The objects of $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ are enhanced tame coverings of Σ, see Definition 4.2.
(2) The morphisms of $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ are morphisms of enhanced coverings, see Definition 4.3.

We will also call this the *category of enhanced coverings* of Σ.

Example 4.8. Let us examine [1, Example 7.8] from the viewpoint of enhanced coverings. The set-up is as follows. We assume that $\mathrm{char}(k) \neq 2$ and consider the Tate curve E, given by $K^*/\langle q \rangle$ for some $q \in K^*$ with $v(q) > 0$. Let $\Sigma' \to \Sigma$ be the degree 2 covering of metrized complexes as in [1, Example 7.8]. The gluing data $\mathcal{G}(\Sigma', E)$ of the covering $\Sigma' \to \Sigma$ consists of four elements, corresponding to a choice of an automorphism per edge. The automorphism group of $\Sigma' \to \Sigma$ in terms of metrized complexes has order four, but if we choose a set of gluing data $\mathfrak{g}$ and consider the corresponding enhanced automorphism group, then it has order two.

Our goal for the remainder of this section is to show that $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ is equivalent to $\mathrm{Cov}_{\mathrm{Tame}}(X, D)$. To that end, we first show that the category of enhanced coverings is equivalent to an intermediate category of tame coverings of triangulated marked curves, see [1, Definition 3.8]. We then show that this category is equivalent to the category of residually tame coverings of (X, D), which gives the final equivalence of categories in Theorem 4.13.

We start by creating a functor from the category of tame coverings of triangulated marked curves to the category of enhanced coverings. Let $\phi : (X', V' \cup D') \to (X, V \cup D)$ be a tame covering of triangulated marked curves. The inverse image $\Sigma' := (\phi^{\mathrm{an}})^{-1}(\Sigma)$ is then a skeleton of (X', D') and by [1, Corollary 4.28] we obtain a natural tame covering of metrized complexes $\Sigma' \to \Sigma$. In terms of that paper, we say that the morphism ϕ is a lifting of $\Sigma' \to \Sigma$. Let $x \in V$ and let $Y(x)$ be the canonical star-shaped neighborhood of x, as in [1, Sections 6 and 7]. We then consider the inverse image of $Y(x)$ in X'^{an}. This inverse image is a disjoint union of star-shaped curves $Y(x')$ for $x' \in X'^{\mathrm{an}}$ mapping to x. We similarly write y, $Y(y)$ and $Y(y')$ for a vertex y adjacent to x. For any two vertices x' and y' lying above x and y respectively, we consider the intersection $Y(x') \cap Y(y')$, which is a disjoint union of open edges: $Y(x') \cap Y(y') \simeq \coprod \tau'^{-1}(e'^{o})$. Here the disjoint union is over all edges e' that contain x' and y' and τ' is the retraction map on X'^{an}. Since the retraction maps $\tau_{x'}$ and $\tau_{y'}$ on the star-shaped curves $Y(x')$ and $Y(y')$ are the restrictions of the retraction map τ', we then have canonical $\tau^{-1}(e^{o})$-isomorphisms $\tau_{x'}^{-1}(e'^{o}) \simeq \tau^{-1}(e'^{o}) \simeq \tau_{y'}^{-1}(e'^{o})$, which gives a set of gluing data $\mathfrak{g}$ for $\Sigma' \to \Sigma$. We thus have an associated enhanced covering $\phi_{\mathfrak{g}}$ for ϕ. Furthermore, suppose that we have a morphism of tame coverings of a triangulated marked curve $(X, V \cup D)$, which is a commutative diagram

$$\begin{array}{ccc} (X_1, V_1 \cup D_1) & \xrightarrow{\psi} & (X_2, V_2 \cup D_2) \\ \downarrow{\scriptstyle \phi_1} & \swarrow{\scriptstyle \phi_2} & \\ (X, V \cup D) & & \end{array}.$$

This in turn comes from a commutative diagram of analytifications

$$\begin{array}{ccc} X_1^{\mathrm{an}} & \xrightarrow{\psi^{\mathrm{an}}} & X_2^{\mathrm{an}} \\ \downarrow{\scriptstyle \phi_1^{\mathrm{an}}} & \swarrow{\scriptstyle \phi_2^{\mathrm{an}}} & \\ X^{\mathrm{an}} & & \end{array}.$$

Using [1, Corollary 4.28], we see that this induces a commutative diagram of metrized complexes

$$\begin{array}{ccc} (\phi_1^{\mathrm{an}})^{-1}(\Sigma) & \xrightarrow{\psi} & (\phi_2^{\mathrm{an}})^{-1}(\Sigma) \\ \downarrow{\scriptstyle \phi_1^{\mathrm{an}}} & \swarrow{\scriptstyle \phi_2^{\mathrm{an}}} & \\ \Sigma & & \end{array}.$$

Since the diagram of analytifications is commutative, we directly obtain that the gluing data defined above commutes with ψ^{an} and the ϕ_i^{an} by considering the canonical opens $Y(x_i)$, $Y(y_i)$ and $Y(x_i) \cap Y(y_i)$ in X_i^{an}. We thus see that we have an induced morphism of enhanced tame coverings of Σ. In other words, we have a functor $\mathcal{F}_{\mathrm{tri}}$ from the category $\mathrm{Tame}(X, V \cup D)$ of tame coverings of a triangulated marked curve to the category $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ of enhanced tame coverings of Σ.

We now compare morphisms in these categories $\mathrm{Tame}(X, V \cup D)$ and $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$. To ease notation, we adopt the notation $X_i := (X_i, V_i \cup D_i)$ and $X := (X, V \cup D)$ for triangulated marked curves in this lemma.

Lemma 4.9. *Let $\phi_i : X_i \to X$ be tame coverings of a triangulated marked curve X with enhanced tame coverings $\Sigma_i \to \Sigma$ arising from the functor $\mathcal{F}_{\mathrm{tri}}$ constructed above. Then*

$$\mathrm{Hom}_X(X_1, X_2) \simeq \mathrm{Hom}_\Sigma(\Sigma_1, \Sigma_2).$$

Proof. We first show the injectivity of the induced map. Suppose that there are two coverings ψ_i that map to the same morphism of enhanced coverings. For every pair of vertices x_1 and x_2 with $\psi_i(x_1) = x_2$, we have a unique extension of the algebraic covering $C_{x_1} \to C_{x_2}$ to a covering of star-shaped curves $Y(x_1) \to Y(x_2)$ by [1, Theorem 6.18]. But these open neighborhoods cover X_1^{an} and X_2^{an}, so we conclude that $\psi_1 = \psi_2$.

Let $\psi : \Sigma_1 \to \Sigma_2 \to \Sigma$ be a morphism of enhanced tame coverings. We write $Y(x_i)$ for the star shaped curves corresponding to the vertices x_1 and x_2, where x_1 maps to x_2. Similarly, we write $Y(y_i)$ for the star-shaped curves corresponding to adjacent vertices. The commutativity of the diagram in the second condition of Definition 4.3 then implies that the local morphisms $Y(x_1) \to Y(x_2) \to Y(x)$ and $Y(y_1) \to Y(y_2) \to Y(y)$ lift to a well-defined global morphism on their union. But the union over all vertices is exactly X_i^{an}, so we conclude that ψ lifts to coverings $X_1^{\mathrm{an}} \to X_2^{\mathrm{an}} \to X^{\mathrm{an}}$. □

Corollary 4.10. Let $\mathrm{Tame}(X, V \cup D)$ be the category of tame coverings of a triangulated marked curve $(X, V \cup D)$ and let $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ be the category of enhanced tame coverings. Then $\mathcal{F}_{\mathrm{tri}}$ induces an equivalence of categories.

Proof. By Lemma 4.9, we see that the functor $\mathcal{F}_{\mathrm{tri}}$ is fully faithful. For any enhanced covering $(\Sigma' \to \Sigma, \mathfrak{g})$, we glue the local coverings on the star-shaped curves using $\mathfrak{g}$ to obtain a smooth proper analytic space X'^{an} with a covering $X'^{\mathrm{an}} \to X^{\mathrm{an}}$ as in [1, Theorem 7.4]. This comes from an algebraic covering $X' \to X$ and we easily verify that it has the correct properties. □

Consider the category $\mathrm{Cov}_{\mathrm{Tame}}(X, D)$ of residually tame coverings of the marked curve (X, D). Recall from Remark 4.1 that we have a fixed strongly semistable vertex set V for (X, D). Using Theorem 3.3, we see that the inverse image $V' = (\phi^{\mathrm{an}})^{-1}(V)$ for any residually tame covering $\phi : X' \to X$ is a strongly semistable vertex set of (X', D'), where D' is the inverse image of D. We moreover have the following:

Lemma 4.11. *Let V be a fixed strongly semistable vertex set for (X, D) and let $(X', D') \to (X, D)$ be a residually tame étale covering. Let $\phi_{\mathrm{tri}} : (X', V' \cup D') \to (X, V \cup D)$ be the finite morphism of triangulated marked curves induced from Theorem 3.3. Then ϕ_{tri} is a tame covering of triangulated marked curves. This induces a functor from the category of residually tame coverings of (X, D) to the category of tame coverings of $(X, V \cup D)$.*

Proof. Let x' be a type-2 point in X'^{an} with $\phi^{\mathrm{an}}(x') = x$. Then by [6, Proposition 2.4.7], the extension of residue fields $\tilde{\mathcal{H}}(x') \supset \tilde{\mathcal{H}}(x)$ is separable. As in the proof of Theorem 3.3, the expansion factor of an edge e is just the local degree $[\mathcal{H}(x') : \mathcal{H}(x)]$ for any point in e. We then take a point of type 3 and conclude that the degree is not divisible by p. This implies that ϕ_{tri} is a tame covering. We leave the functoriality to the reader. □

Definition 4.12. *(Tropicalization functor)* Let $\mathcal{F}_{\Sigma} : \mathrm{Cov}_{\mathrm{Tame}}(X, D) \to \mathrm{Cov}_{\mathcal{G}}(\Sigma)$ be the composite of the functors in Corollary 4.10 and Lemma 4.11. We call this the tropicalization functor associated to Σ.

Theorem 4.13. *Let $\mathcal{F}_{\Sigma}$ be the tropicalization functor from the category of residually tame coverings of a marked curve (X, D) to the category of enhanced tame coverings of Σ.*

Then $\mathcal{F}_\Sigma$ induces an equivalence of categories

$$\mathrm{Cov}_{\mathrm{Tame}}(X,D) \simeq \mathrm{Cov}_{\mathcal{G}}(\Sigma).$$

Proof. Using Lemma 4.10, we see that we only have to show that the functor from $\mathrm{Cov}_{\mathrm{Tame}}(X,D)$ to $\mathrm{Tame}(X, V \cup D)$ defines an equivalence. To do this, it suffices to show that any tame covering of triangulated marked curves is also residually tame. By [1, Proposition 4.35] any tame covering is an isomorphism outside the skeleton Σ'. At the vertices of Σ', it defines a separable covering, so these give tame extensions. For an edge e' in Σ', the covering is piecewise linear, with dilation factor coprime to p. As in the proof of Theorem 3.3, this dilation factor is the degree $[\mathcal{H}(x') : \mathcal{H}(x)]$ for the points x' in e'. This proves that ϕ is residually tame. □

4.2. An algebraic definition of enhanced coverings

In this section we give an algebraic definition of enhanced coverings, without any reference to Berkovich spaces. Consider a metrized complex of k-curves Σ. For every vertex $x \in V(\Sigma)$, the residue curve C_x over k is smooth and proper. Every adjacent edge $e = xy$ then corresponds to a pair of closed points $z_{e,x}$ and $z_{e,y}$ of C_x and C_y respectively. We will think of e as an oriented edge, and we will write $\overline{e} = yx$ for the edge with the reverse orientation. Since C_x is smooth over k, we have

$$\hat{\mathcal{O}}_{C_x,z_{e,x}} \simeq k[[u]]$$

and similarly for $z_{e,y}$ and C_y. In particular we find that the completed local rings $\hat{\mathcal{O}}_{C_x,z_{e,x}}$ and $\hat{\mathcal{O}}_{C_y,z_{e,y}}$ are isomorphic. There is no canonical isomorphism however, so we are led to the following definition.

Definition 4.14. *(Algebraic gluing sets)* Let $\overrightarrow{E}_f(\Sigma)$ be the set of finite oriented edges of Σ. An algebraic set of gluing data for Σ is a set of isomorphisms

$$\psi_{\Sigma,e} : \hat{\mathcal{O}}_{C_x,z_{e,x}} \to \hat{\mathcal{O}}_{C_y,z_{e,y}}$$

for $e \in \overrightarrow{E}_f(\Sigma)$. We impose the condition $\psi_{\Sigma,e} = \psi^{-1}_{\Sigma,\overline{e}}$ for the edge $\overline{e} = yx$ with the reverse orientation. A pair $(\Sigma, \psi_{\Sigma,e})$ is called an *algebraically glued metrized complex*. If Σ is clear from context, then we denote $\psi_{\Sigma,e}$ by ψ_e.

Definition 4.15. *(Algebraically enhanced coverings)* Let $(\Sigma', \psi_{e'})$ and (Σ, ψ_e) be two algebraically glued metrized complexes. An algebraically enhanced covering $\phi : (\Sigma', \psi_{e'}) \to (\Sigma, \psi_e)$ is a tame covering of metrized complexes $\Sigma' \to \Sigma$ such that for every oriented edge $e' = x'y'$ mapping to the oriented edge $e = xy$ the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{O}}_{C_{x'},z_{e',x'}} & \longrightarrow & \hat{\mathcal{O}}_{C_{y'},z_{e',y'}} \\ \uparrow & & \uparrow \\ \hat{\mathcal{O}}_{C_x,z_{e,x}} & \longrightarrow & \hat{\mathcal{O}}_{C_y,z_{e,y}} \end{array}.$$

Here the vertical maps $\hat{\mathcal{O}}_{C_x,z_{e,x}} \to \hat{\mathcal{O}}_{C_{x'},z_{e',x'}}$ and $\hat{\mathcal{O}}_{C_y,z_{e,y}} \to \hat{\mathcal{O}}_{C_{y'},z_{e',y'}}$ are induced from the morphism of metrized complexes, and the horizontal maps are induced by ψ_e and $\psi_{e'}$. The category of algebraically enhanced coverings of (Σ, ψ_e) has as its objects the algebraically enhanced coverings of (Σ, ψ_e). Morphisms between algebraically enhanced coverings are defined as in Definition 4.3. For a given tame covering of metrized complexes $\Sigma' \to \Sigma$ and a fixed algebraic gluing set ψ_e for Σ, the set of algebraic gluing data for the pair $(\Sigma' \to \Sigma,\ \psi_e)$ is the set of all gluing data $\psi_{e'}$ for Σ' that induce an algebraically enhanced covering of (Σ, ψ_e). This set is denoted by $\mathcal{G}_a(\Sigma', \psi_e)$.

Remark 4.16. If the above diagram commutes for a given pair of oriented edges e' and e, then the corresponding diagram for $\overline{e}'$ and $\overline{e}$ also automatically commutes by the condition on the inverses in Definition 4.14.

We now argue that the set of algebraic gluing data $\mathcal{G}_a(\Sigma', \psi_e)$ for a given morphism $\Sigma' \to \Sigma$ of metrized complexes is finite. In fact, consider the finite set $\prod_{e' \in E(\Sigma')} \mathbb{Z}/d_{e'/e}(\phi)\mathbb{Z}$. We can identify $\mathcal{G}_a(\Sigma', \psi_e)$ with this set by noting that any two isomorphisms for a given oriented edge e' are related by an automorphism of $\hat{\mathcal{O}}_{C_{x'},z_{e,x'}}$ over $\hat{\mathcal{O}}_{C_x,z_{e,x}}$. This automorphism group is just $\mathbb{Z}/d_{e'/e}(\phi)\mathbb{Z}$, so we only have to note that these isomorphisms in fact exist. This follows from the fact that there is exactly one tamely ramified extension of $\hat{\mathcal{O}}_{C_x,z_{e,x}} \simeq k[[u]]$ of any given degree coprime to char(k). More abstractly, we have that the set of isomorphisms in Definition 4.15 for a given pair of edges e' and e is a $\mathbb{Z}/d_{e'/e}(\phi)\mathbb{Z}$-torsor, and we can trivialize this torsor by choosing a specific isomorphism.

Suppose that we are given a triangulated marked curve $(X, V \cup D)$ with metrized complex Σ and a tame covering of metrized complexes $\Sigma' \to \Sigma$. As with the algebraic gluing data, the analytic gluing data in Definition 2.16 can be trivialized by choosing initial isomorphisms. Moreover, the resulting analytic trivializations are isomorphic to the algebraic trivializations by inspection. We conclude that the corresponding torsors in the algebraic and analytic context are isomorphic. We note however that this isomorphism is not unique, as we for instance can twist by automorphisms of the $\mathbb{Z}/d_{e'/e}(\phi)\mathbb{Z}$. We can nonetheless fix a set of isomorphisms for all tame coverings $\Sigma' \to \Sigma$, and all edges of the Σ'. By the way that morphisms were defined in Definitions 4.3 and 4.15, we then directly find an *equivalence of categories*. The category of enhanced coverings can thus be defined without any reference to Berkovich spaces.

Example 4.17. We determine all connected degree two enhanced coverings of the metrized complex in Fig. 1, where char$(k) \neq 2$ and the vertex of genus one corresponds to an

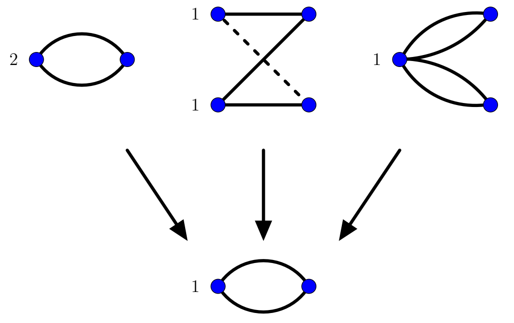

Fig. 1. The enhanced connected coverings of degree two of a metrized complex of genus 2 in Example 4.17. The multiplicities from left to right are 8, 1 and 6. These give a total of 15 different coverings of the base complex Σ.

elliptic curve $\overline{E}/k$. Note that the genus of the complex is 2, so a degree-2 covering has genus 3 by the Riemann-Hurwitz formula. The dilation factor $d_{e'/e}(\phi)$ over the edges is either 1 or 2. If one of them is 2, then there is graph-theoretically only one option, see Fig. 1. On the vertex of genus 1, this induces an étale degree two covering of $\overline{E}\backslash\{p_1, p_2\}$. Using Grothendieck's results on the tame fundamental group of a curve over a field [16, Corollaire 2.12], one then sees that 4 of these coverings $C \to \overline{E}\backslash\{p_1, p_2\}$ are ramified over at least one p_i. Each of these maps is ramified over the edges and the corresponding gluing data can be identified with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. These give rise to two distinct isomorphism classes of enhanced coverings, as in the case of an elliptic curve with multiplicative reduction. For each of the four coverings above we thus have two liftings, giving 8 coverings in total that are ramified over at least one edge. We call this the *multiplicity* of the combinatorial type on the left in Fig. 1.

If the dilation factor is 1 on both edges, then there are graph-theoretically two options for a degree two covering. It is either split over the vertex of genus one or it is not. If it is not split, then it is defined by a non-trivial étale covering of degree two of $\overline{E}$ and there are exactly 3 such coverings. We still have some freedom in identifying the closed points corresponding to the edges however. This gives a total of four options per covering of $\overline{E}$. Using the non-trivial automorphism of $\overline{E}'$ over $\overline{E}$, we find that out of the four options, only two are non-isomorphic. In total, we thus obtain 6 of these coverings. Lastly, suppose that the covering is completely split over both the edges and the vertices. There is then only one option, namely the graph-theoretical covering of degree two of the circle. In total, we find 15 different non-trivial coverings of degree two.

Example 4.18. Consider the covering $\Sigma' \to \Sigma$ depicted in Fig. 2, where the five edges all have dilation factor 3. We can turn this into a tame covering of metrized complexes by adding appropriate $\mathbb{Z}/3\mathbb{Z}$-coverings of the projective line. We thus have a covering of

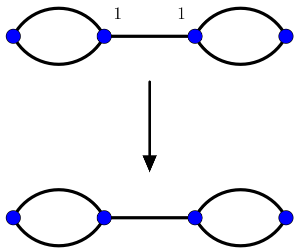

Fig. 2. A covering of degree three that is locally a $\mathbb{Z}/3\mathbb{Z}$-covering, but not globally.

degree three that is locally a $\mathbb{Z}/3\mathbb{Z}$-covering. It does *not* lift to a global $\mathbb{Z}/3\mathbb{Z}$-covering however. Indeed, consider a collection of automorphisms over the vertices. Note that the induced automorphisms of $\hat{\mathcal{O}}_{\mathbb{P}^1,x_i}$ over the two closed points x_i on the left are inverse to each other, whereas the automorphisms on the $\hat{\mathcal{O}}_{E,y_i}$ for the two corresponding points y_i are not (since otherwise the morphism $E \to \mathbb{P}^1$ would be étale at the third point corresponding to the bridge). In other words, we do not obtain a commutative diagram as in the definition of gluing data and we thus do not have an automorphism of order three. Note that the induced condition on the gluing data for a loop is completely analogous to the balancing condition found in the theory of admissible coverings, see [2, Definition 4.3.1].

Another way to see the non-realizability as a $\mathbb{Z}/3\mathbb{Z}$-covering is using the material in Section 5.2. Namely, this covering would correspond to a 3-torsion point D in the tropical Jacobian of Σ whose associated piecewise linear function ϕ with $3D = \text{div}(\phi)$ has slope not divisible by 3 on all the edges. This does not exist, so this covering is not realizable as a $\mathbb{Z}/3\mathbb{Z}$-covering. We now find that this covering has a non-trivial Galois closure $\overline{\Sigma} \to \Sigma$ with Galois group S_3. Moreover, the intermediate degree-two morphism $\Sigma_Q \to \Sigma$ is a topological covering of Σ by the imposed conditions on $\Sigma' \to \Sigma$, and the induced covering $\overline{\Sigma} \to \Sigma_Q$ corresponds to a non-trivial torsion point of order three in the tropical Jacobian of Σ_Q by the considerations in Section 5.2.

Remark 4.19. We illustrate the difference between algebraically enhanced coverings, admissible coverings (see [19], [24] and [12]), and twisted stable maps (see [2]). We first note that the different notions of an admissible covering in the papers above are equivalent, essentially by [24, Section 3.11]. We write $\overline{\mathcal{H}}_{g'\to g,d}$ for the corresponding stack of unramified admissible coverings of degree d between curves of genera g' and g. Here by unramified, we mean that the coverings are unramified over the smooth locus, so that the ramification pattern μ in the terminology of [12] is trivial. The stack $\overline{\mathcal{H}}_{g'\to g,d}$ is usually not normal at the boundary strata, but its normalization is smooth over $\mathbb{Z}$, see [19, Proof of Theorem 4] and [24, Section 3.23]. In [2] the limit objects in the normalization were given a modular interpretation using twisted stable maps.

To see the difference between our notion of an algebraically enhanced covering and the notion of an admissible covering, consider the Hurwitz stack $\overline{\mathcal{H}}_{3\to 2,2}$ of unramified admissible $\mathbb{Z}/2\mathbb{Z}$-coverings of curves of genus two. Since we are dealing with G-coverings for a finite group G, the notions above all give the same stack, see [2, Theorem 4.3.2]. Note that specific strata of this stack were also considered in Example 4.17. Let $\overline{\mathcal{H}}_{3\to 2,2} \to \overline{\mathcal{M}}_2$ be the target map, sending a covering $\mathcal{X}' \to \mathcal{X}$ to $\mathcal{X}$. We now fix a k-valued point of $\overline{\mathcal{M}}_2$, corresponding to a stable genus-two curve X over k. If X is smooth, then there are 15 different coverings in $\overline{\mathcal{H}}_{3\to 2,2}$ lying over the point corresponding to X. In the boundary strata, this number is often lower. For instance, the 8 coverings on the left in Fig. 1 only give 4 different admissible coverings.

This discrepancy is due to the fact that we are only considering set-theoretic fibers here. To address this problem, various weights for the coverings were introduced in [12, Definition 22] (we called these multiplicities in Example 4.17). By attaching this weight $\varpi(\Theta)$ to every combinatorial type Θ, one obtains the right Hurwitz number for *maximally degenerate tropical coverings*,[3] see [12, Theorem 2]. In the proof of that theorem, the authors work with the completed local ring rather than the fiber to calculate the right multiplicities. Our set-up is similar, in that algebraically enhanced coverings allow us to work over an infinitesimal neighborhood rather than the fiber.

A difference between the two approaches is that algebraically enhanced coverings give the right multiplicities for *all* types of curves in the boundary (and not just maximally degenerate curves). In fact, if we use the weight factors from [12] on Example 4.17, then we obtain 21/4, rather than 15/2. Here the two coverings on the left in Fig. 1 have the correct weights, but the covering on the right does not, as one obtains 3/4 rather than 6/2. To adjust the formula in this case, first note that both the graph-theoretical automorphism of the covering and the algebraic automorphism of the vertex give a factor of 1/2 with the formula in [12], even though only one should be taken into account. Moreover, the different twists arising from edge identifications are not accounted for, so that the weight is missing a factor 2. All in all, we have to multiply the weight by 4 to obtain the correct Hurwitz number. It seems likely that one can write down similar formulas for the weights of other types of tropical coverings using the equivalence in Theorem 4.13.

5. Fundamental groups for metrized complexes

In this section, we use the equivalence of categories in Theorem 4.13 to endow the category of tame coverings of a metrized complex with the structure of a Galois category,

[3] Here maximally degenerate means that the target tropical curve is in a top-dimensional cone of the tropical moduli space of marked tropical curves of genus g. That is, the corresponding stable graph is trivalent with vertices of weight zero. Note that any algebraic curve whose skeleton is in a top-dimensional cone is automatically Mumford, but not conversely.

giving a natural notion of a fundamental group for a metrized complex. We furthermore define the notions of unramified coverings and completely split coverings above a subcomplex $\Sigma^0 \subset \Sigma$. These correspond to closed subgroups of $\pi(\Sigma)$, which we call the absolute inertia group $\mathfrak{I}(\Sigma^0)$ and decomposition group $\mathfrak{D}(\Sigma^0)$ of Σ^0. For unmarked curves, we prove that the quotient $\pi_{\mathfrak{D}}(\Sigma(X)) := \pi(\Sigma)/\mathfrak{D}(\Sigma)$ is isomorphic to the profinite completion of the ordinary fundamental group of the underlying graph of Σ, see Theorem 5.14. Furthermore, we show that the coverings that come from the abelianizations of the quotients $\pi_{\mathfrak{D}}(\Sigma)$ and $\pi_{\mathfrak{I}}(\Sigma)$ correspond to the extensions that come from the toric and connected parts in the analytic Jacobian of the curve, see Theorem 5.16.

5.1. Fundamental, inertia and decomposition groups

As in Section 4, we fix a semistable vertex V for a marked curve (X, D) with skeleton $\Sigma := \Sigma(X, V \cup D)$. Here we again assume that X is connected. By Theorem 4.13, the tropicalization functor $\mathcal{F}_\Sigma$ gives an equivalence of categories

$$\mathcal{F}_\Sigma : \mathrm{Cov}_{\mathrm{Tame}}(X, D) \to \mathrm{Cov}_{\mathcal{G}}(\Sigma).$$

Since $\mathrm{Cov}_{\mathrm{Tame}}(X, D)$ is a Galois category, we obtain an induced Galois category structure on $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$.

Definition 5.1. *(Fundamental group of a metrized complex)* Let $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$ be the category of enhanced tame coverings of Σ. We endow it with the Galois category structure induced by Theorem 4.13. The corresponding fundamental group is denoted by $\pi(\Sigma) = \pi_{\mathrm{Tame}}(X, D)$. We call this group the tame profinite tropical fundamental group of Σ, or just fundamental group of Σ.

Let us state some facts about the interaction between the profinite group $\pi(\Sigma)$ and the category $\mathrm{Cov}_{\mathcal{G}}(\Sigma)$. All of these results directly follow from the theory of Galois categories.

Proposition 5.2. *Let Σ be a fixed metrized complex associated to the marked curve (X, D) and consider the category of enhanced tame coverings of Σ. Then the following are true.*

(1) There is an equivalence of categories

$$\mathrm{Cov}_{\mathcal{G}}(\Sigma) \to (\mathrm{Finite}\,\pi(\Sigma)\text{-Sets}).$$

(2) The closed subgroups of finite index in $\pi(\Sigma)$ correspond bijectively to connected enhanced tame coverings $\Sigma' \to \Sigma$.

Example 5.3. Consider an elliptic curve E/K with $\mathrm{char}(K) = 0$. The ordinary profinite fundamental group of E is $\hat{\mathbb{Z}}^2$. For $\mathrm{char}(k) = 0$, we have that every such covering is automatically residually tame, so $\pi(\Sigma) \simeq \hat{\mathbb{Z}}^2$. If $\mathrm{char}(k) = p$, then the group depends on the reduction type of E. If E has good reduction, then the minimal skeleton consists of a single vertex of genus 1 and we write Σ for this skeleton. We then have $\pi(\Sigma) \simeq \pi(\overline{E})$, where $\pi(\overline{E})$ is the étale fundamental group of $\overline{E}$. Suppose that the reduced curve $\overline{E}$ has supersingular reduction. We then have $\pi(\Sigma) \simeq \hat{\mathbb{Z}}'^2$, where the prime means that the inverse limit in the definition of $\hat{\mathbb{Z}}$ runs over all subgroups of index coprime to p. The étale coverings of degree p^n are thus not residually tame. If the reduced curve $\overline{E}$ has ordinary reduction, then there are coverings of degree p^n that are residually tame, see Example 2.2. We have $\pi(\Sigma) \simeq \hat{\mathbb{Z}}'^2 \times \mathbb{Z}_p$ in this case. Suppose now that Σ has Betti number one and let E/K be an elliptic curve with skeleton Σ. Then $\pi(\Sigma) \simeq \hat{\mathbb{Z}}'^2 \times \mathbb{Z}_p$. Indeed, using the Tate uniformization $E^{\mathrm{an}} \simeq \mathbb{G}^{\mathrm{an}}_{m,K}/\langle q \rangle$, one easily obtains a description of all finite étale coverings. The induced coverings on the skeleton are either topological coverings or coverings with some expansion factor d on the cycle. The topological coverings are residually tame (see Proposition 5.13), but the coverings with dilation factor p^n on both edges are not.

Remark 5.4. Every covering $X' \to X$ of connected smooth curves comes with a natural extension of function fields $K(X) \to K(X')$ and this is a bijective correspondence. The closed subgroups of $\pi_{\mathrm{Tame}}(X, D) = \pi(\Sigma)$ then correspond to (possibly infinite) field extensions, which are composites of finite function field extensions $K(X') \supset K(X)$ where $X' \to X$ is residually tame. We will create closed subgroups of $\pi(\Sigma)$ using this correspondence.

Definition 5.5. *(Subcomplexes)* A (strict) finite subcomplex Σ^0 of Σ consists of subsets $V(\Sigma^0) \subset V_f(\Sigma)$ and $E(\Sigma^0) \subset E_f(\Sigma)$. Here $V_f(\Sigma)$ and $E_f(\Sigma)$ denote the sets of finite vertices and edges of Σ respectively.

Definition 5.6. *(Unramified and completely split coverings)* Let $\phi : \Sigma' \to \Sigma$ be a connected enhanced covering of metrized complexes and let $\Sigma^0 \subseteq \Sigma$ be a subcomplex. We say that

(1) ϕ is *metrically unramified* (or *unramified*) above Σ^0 if for every edge $e' \in E(\Sigma')$ mapping to $e \in E(\Sigma^0)$, we have $d_{e'/e}(\phi) = 1$,
(2) ϕ is *completely split* above Σ^0 if ϕ is unramified above Σ^0 and for every vertex $v \in V(\Sigma^0)$, we have that there are $\deg(\phi)$ vertices $v' \in V(\Sigma')$ such that $\phi(v') = v$.

We then say that an enhanced covering of metrized complexes has any of the above properties if the connected components have these properties. The enhanced covering $\Sigma' \to \Sigma$ is unramified (resp. completely split) if it is unramified (resp. completely split) over the maximal finite subcomplex Σ^0 in Σ. We say that a residually tame covering $\phi : (X', D') \to (X, D)$ is unramified (resp. completely split) above a subcomplex $\Sigma^0 \subseteq \Sigma$ if $\mathcal{F}_\Sigma(\phi)$ is unramified (resp. completely split) above Σ^0 (and similarly for the maximal finite subcomplex).

Remark 5.7. Our definition of ϕ being completely split above an edge or vertex corresponds to ϕ being a topological covering above that edge or vertex. The definition of being metrically unramified is not related to the Berkovich definition of being unramified or étale. Indeed, any morphism $X'^{\mathrm{an}} \to X^{\mathrm{an}}$ will automatically be étale outside the branch points. However, the morphism $\mathcal{X}' \to \mathcal{X}$ of strongly semistable models corresponding to $\phi : \Sigma' \to \Sigma$ is not étale at the closed points corresponding to edges if and only if $d_{e'/e}(\phi) \neq 1$. This can be seen by considering the corresponding morphism of completed rings for the ordinary double points. It is for this reason that we call these morphisms metrically unramified at an edge if $d_{e'/e}(\phi) = 1$.

Lemma 5.8. *The notions of being unramified and completely split above a subcomplex Σ^0 are stable under taking composites.*

Proof. We first show that these are stable under taking Galois closures. For the notion of being unramified, we can take a type-3 point x in an edge of Σ^0. If $[\mathcal{H}(x') : \mathcal{H}(x)] = 1$ for every point x' lying above x, then by Proposition 2.5 we conclude that $[\mathcal{H}(\overline{x}) : \mathcal{H}(x)] = 1$. We can also apply the above reasoning to a point of type 2 that is completely split to obtain the desired statement for completely split morphisms. For the composite $K(\overline{X})$ of two fields $K(X_i)$, we argue as in 2.8: we reduce to the case of two Galois extensions and then find that the composite of $\mathcal{H}(x_1)$ and $\mathcal{H}(x_2)$ for two points $x_i \in X_i^{\mathrm{an}}$ lying above x is $\mathcal{H}(\overline{x})$, where $\overline{x} \in \overline{X}^{\mathrm{an}}$ lies above the x_i. If $[\mathcal{H}(x_i) : \mathcal{H}(x)] = 1$ for both i, then $[\mathcal{H}(\overline{x}) : \mathcal{H}(x)] = 1$, which quickly gives the desired statement for both unramified and completely split morphisms. □

We now consider the closed subgroups in $\pi(\Sigma)$ corresponding to the coverings that are unramified and completely split.

Definition 5.9. *(Inertia and decomposition groups)* We define $\mathfrak{I}(\Sigma^0)$ and $\mathfrak{D}(\Sigma^0)$ to be the closed subgroups of $\pi(\Sigma)$ corresponding to the coverings that are unramified and completely split above Σ^0 respectively. We refer to them as the (absolute) inertia and decomposition groups of Σ^0 respectively. If Σ^0 consists of all finite edges and vertices of Σ, then we denote these groups by $\mathfrak{I}(\Sigma)$ and $\mathfrak{D}(\Sigma)$.

Proposition 5.10. *The subgroups $\mathfrak{I}(\Sigma^0)$ and $\mathfrak{D}(\Sigma^0)$ are normal subgroups of $\pi(\Sigma)$.*

Proof. Let $x \in M$, where M is the field corresponding to either $\mathfrak{I}(\Sigma^0)$ or $\mathfrak{D}(\Sigma^0)$ and write $L = K(X)(x)$ for the field generated by x over $K(X)$. By Lemma, 5.8, the covering $X' \to X$ corresponding to L is unramified or completely split above Σ^0, and the Galois closure then also has the same property. Thus the conjugates of x are contained in M, which proves that it is Galois. □

Definition 5.11. *(Inertia and decomposition quotients)* Let $\Sigma^0 \subseteq \Sigma$ be a subcomplex and let $\mathfrak{I}(\Sigma^0)$ and $\mathfrak{D}(\Sigma^0)$ be the absolute inertia and decomposition groups of Σ^0. We define

$\pi_{\mathfrak{I}}(\Sigma^0) := \pi(\Sigma)/\mathfrak{I}(\Sigma^0)$ and $\pi_{\mathfrak{D}}(\Sigma^0) := \pi(\Sigma)/\mathfrak{D}(\Sigma^0)$. If Σ^0 consists of all finite vertices and edges of Σ, then we write $\pi_{\mathfrak{I}}(\Sigma)$ and $\pi_{\mathfrak{D}}(\Sigma)$ for these groups.

Remark 5.12. We will see in Section 5.2 that there is a natural connection between the cyclic abelian extensions coming from $\pi_{\mathfrak{D}}(\Sigma)$ and $\pi_{\mathfrak{I}}(\Sigma)$ and the cyclic abelian extensions coming from the toric and connected parts of the analytic Jacobian of X. It is in this sense that we think of the groups $\pi_{\mathfrak{D}}(\Sigma)$ and $\pi_{\mathfrak{I}}(\Sigma)$ as natural *non-abelian* generalizations of the extensions coming from the toric and connected parts of the Jacobian in the tame case.

We now connect the group $\pi_{\mathfrak{D}}(\Sigma)$ to the profinite completion of the ordinary fundamental group of the graph underlying Σ. Let Γ be the finite connected graph underlying Σ. We denote the category of finite coverings by $\mathrm{Cov}(\Gamma)$, its profinite fundamental group by $\hat{\pi}(\Gamma)$ and its ordinary fundamental group by $\pi(\Gamma)$ (this is the only time we will use this notation for a nonprofinite group). The normal subgroup $\mathfrak{D}(\Sigma)$ gives rise to a Galois subcategory of $\mathrm{Cov}(\Sigma)$, which we denote by $\mathrm{Cov}_{\mathfrak{D}}(\Sigma)$. The corresponding profinite fundamental group is $\pi_{\mathfrak{D}}(\Sigma)$. We now have the following

Proposition 5.13. *Let* $\mathrm{Cov}_{\mathfrak{D}}(X)$ *be as above and consider the forgetful functor* $\mathrm{Cov}_{\mathfrak{D}}(\Sigma) \to \mathrm{Cov}(\Gamma)$*. This induces an equivalence of categories*

$$\mathrm{Cov}_{\mathfrak{D}}(\Sigma) \to \mathrm{Cov}(\Gamma).$$

Proof. Let $\Gamma' \to \Gamma$ be a finite covering of graphs. By assigning the same length function on Γ' as on Γ (induced by Σ) and by assigning to every vertex of Γ' a projective line with the right identifications of the edges, we easily obtain a finite tame covering of metrized complexes $\Sigma' \to \Sigma$. There is no non-trivial gluing data (indeed, $\mathrm{Aut}_{\tau_x^{-1}(e^0)}(\tau_{x'}^{-1}(e'^0)) = (1)$ for every finite vertex x' with image x and adjacent edges e' and e), so this also gives a canonical enhanced morphism. This shows that the functor is essentially surjective. For any two completely split coverings $\Sigma_i \to \Sigma$, giving an enhanced Σ-covering $\Sigma_1 \to \Sigma_2$ of metrized complexes is no different from giving a covering of graphs (since there are no non-trivial coverings of algebraic curves and no non-trivial dilation factors). This gives an isomorphism of Hom-sets and we conclude that the categories are equivalent. □

Theorem 5.14. *Let* $\mathfrak{D}(\Sigma)$ *be the decomposition group of* Σ *in* $\pi(\Sigma)$*. Then* $\pi_{\mathfrak{D}}(\Sigma) := \pi(\Sigma)/\mathfrak{D}(\Sigma)$ *is isomorphic to the profinite completion of the ordinary fundamental group of the underlying graph* Γ *of a metrized complex* Σ *corresponding to* X*.*

Proof. Note that the category $\mathrm{Cov}(\Gamma)$ is a Galois category with profinite fundamental group equal to $\hat{\pi}(\Gamma)$ by algebraic topology. The theorem then follows from Proposition 5.13 and [26, Lemma 0BMV]. □

5.2. Filtrations of the abelianization of $\pi(\Sigma)$

In this section, we study the abelianizations of the profinite groups $\pi(\Sigma)$, $\pi_{\mathfrak{I}}(\Sigma)$ and $\pi_{\mathfrak{D}}(\Sigma)$ introduced in Section 5.1. We start by reviewing some material on divisors on metric graphs and skeleta of Jacobians. For more details, we refer the reader to [10].

Let X be a connected curve over K as in Section 2.1 and let $J := J(X)$ be its Jacobian. We write $\mathcal{X}$ for a fixed strongly semistable model of X with skeleton Σ and retraction map $\tau : X^{\mathrm{an}} \to \Sigma$. By linearly extending τ, we then obtain a map

$$\tau_* : \mathrm{Div}(X) \to \mathrm{Div}_\Lambda(\Sigma).$$

On divisors of degree zero, this leads to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Prin}(X) & \longrightarrow & \mathrm{Div}^0(X) & \longrightarrow & J(K) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Prin}_\Lambda(\Sigma) & \longrightarrow & \mathrm{Div}^0_\Lambda(\Sigma) & \longrightarrow & \mathrm{Jac}_\Lambda(\Sigma) & \longrightarrow & 0 \end{array}$$

Here $\mathrm{Jac}_\Lambda(\Sigma)$ is the Λ-valued tropical Jacobian of Σ. We will view this as a subset of the real tropical Jacobian $\mathrm{Jac}(\Sigma) = \mathrm{Jac}_{\mathbb{R}}(\Sigma)$. By the results in [10], this real tropical Jacobian arises naturally as the skeleton of J^{an}. Furthermore, under this identification the retraction map

$$\overline{\tau} : J^{\mathrm{an}} \to \mathrm{Jac}(\Sigma)$$

is given on type-1 points by the earlier maps on divisor groups. We now consider the kernel of $\overline{\tau}$, which is the unique compact analytic domain $J^0 \subset J^{\mathrm{an}}$ that is also a formal K-analytic subgroup, see [10, Corollary 6.8]. We then have an exact sequence

$$(1) \to T^0 \to J^0 \to B^{\mathrm{an}} \to (1)$$

where $T^0 \subset T^{\mathrm{an}}$ is an affinoid torus and B is an abelian variety with good reduction. Note that if $T^{\mathrm{an}} = (\mathbb{G}^{\mathrm{an}}_{m,K})^n$, then the canonical reduction $\overline{T}$ of T^0 is isomorphic to $(\mathbb{G}_{m,k})^n$. As in [8, Theorem 5.1.c] and [10, Section 7.2], the reduction of J^0 is equal to the Jacobian of the special fiber $\mathcal{X}_s$, which gives the exact sequence

$$(1) \to \overline{T} \to \overline{J^0} \to \prod_{i=1}^{n} \mathrm{Jac}(\Gamma_i) = \overline{B} \to (1). \tag{9}$$

Here the Γ_i are the irreducible components of $\mathcal{X}_s$ and $\overline{T} \simeq (k^*)^t$. This toric rank t is equal to the first Betti number of the intersection graph of $\mathcal{X}_s$ by [21, Chapter 7, Lemma 5.18]. We write $a = \sum_{i=1}^n g(C_i)$ for the abelian rank of J^0 and π for the map $\overline{J}^0 \to \prod_{i=1}^n \mathrm{Jac}(\Gamma_i) = \overline{B}$ in Equation (9).

The group scheme $J[n]$ is étale over K by our assumptions on n, so we can identify it with its K-rational points. These K-rational points give a set of type-1 points of J^{an} and we again denote these by $J[n]$. We then define

$$J^0[n] = \{P \in J[n] : \overline{\tau}(P) = 0\},$$
$$T[n] = \{P \in J^0[n] : \pi(\overline{P}) = 0\}.$$

Here $\overline{P}$ is the image of P in $\overline{J}^0$ under the reduction map.

Proposition 5.15. *Let n be any integer that is coprime to the residue characteristic of K. Then*

$$J^0[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{t+2a},$$
$$T[n] \simeq (\mathbb{Z}/n\mathbb{Z})^t.$$

Proof. This follows from [21, Chapter 7, Corollary 4.41], the exact sequence in Equation (9) and the fact that the reduction map restricted to the n-torsion has no kernel for n coprime to $\mathrm{char}(k)$. □

We can now characterize the torsion points in J using the groups $\pi_{\mathfrak{D}}(\Sigma)$ and $\pi_{\mathfrak{I}}(\Sigma)$. We assume throughout that n is coprime to the residue characteristic. We first note that we have an isomorphism

$$J[n] \simeq \mathrm{Hom}(\pi(\Sigma), \mathbb{Z}/n\mathbb{Z})$$

by [23, Chapter III, Lemma 9.2]. That is, cyclic étale coverings of X are given by torsion points of J. To be more explicit, let $D \in J[n]$ and suppose that D has order n. Then $nD = \mathrm{div}(f)$ for some $f \in K(X)$ and we can consider the covering on the level of function fields defined by

$$z^n = f.$$

The condition on the order of D ensures that this equation is irreducible, and the corresponding covering is étale since f is étale-locally an n-th power. Conversely, every cyclic covering of X is given on the level of function fields by a covering of the form $z^n = f$ by Kummer theory. In order for this covering to be étale, the valuation of f at every closed point must be divisible by n. This then quickly gives an n-torsion point in J.

Theorem 5.16. *Let $\mathfrak{I}(\Sigma)$ and $\mathfrak{D}(\Sigma)$ be the inertia and decomposition group of Σ in $\pi(\Sigma)$ and let $\pi_{\mathfrak{I}}(\Sigma)$ and $\pi_{\mathfrak{D}}(\Sigma)$ be their corresponding quotients in $\pi(\Sigma)$. Let n be an integer such that $\gcd(n, \mathrm{char}(k)) = 1$. Then the isomorphism $J[n] \simeq \mathrm{Hom}(\pi(\Sigma), \mathbb{Z}/n\mathbb{Z})$ induces isomorphisms*

$$J^0[n] \simeq \mathrm{Hom}(\pi_{\mathfrak{J}}(\Sigma), \mathbb{Z}/n\mathbb{Z})$$

and

$$T[n] \simeq \mathrm{Hom}(\pi_{\mathfrak{D}}(\Sigma), \mathbb{Z}/n\mathbb{Z}).$$

Proof. Let $\mathcal{X}$ be a strongly semistable model for X. On the level of function fields, any cyclic étale covering $X' \to X$ is given by $K(X') = K(X)(\alpha)$, where $\alpha^n = f$ and $f \in K(X)$. This defines a Galois-topologically tame covering of X and we denote the induced morphism of metrized complexes by $\psi : \Sigma' \to \Sigma$ and the morphism of semistable models by $\mathcal{X}' \to \mathcal{X}$. For any vertex v of $\Sigma(\mathcal{X})$, we denote the corresponding generic point of the special fiber $\mathcal{X}_s$ by η_v. The local ring $\mathcal{O}_{\mathcal{X},\eta_v}$ is a valuation ring of rank 1, being locally generated by $\mathfrak{m}_R \subset R$. We can directly describe the normalization of $\mathcal{X}$ in $K(X')$ above this valuation ring as follows. Write $f = \omega^n f_v$ for some f_v with $v_{\eta_v}(f_v) = 0$. The element $\alpha' = \alpha/\omega$ is then integral over $\mathcal{O}_{\mathcal{X},\eta}$ as it satisfies $\alpha'^n = f_v$. Since $\gcd(n, \mathrm{char}(k)) = 1$, we find that this gives an étale extension, which thus describes the normalization above η_v. That is, the points of $\mathcal{X}'$ that lie above η_v are described by the étale $\mathcal{O}_{\mathcal{X},\eta_v}$-scheme

$$Z = \mathrm{Spec}(\mathcal{O}_{\mathcal{X},\eta_v}[z]/(z^n - f_v))$$

or equivalently by the base change of Z over $\mathrm{Spec}(k(\eta_v))$. The closed points of this scheme are as follows. Write $\overline{f}_v = f_1^d$ in $k(\eta_v)$, where d is the largest divisor of n such that $\overline{f}_v$ is a d-th power. Let ζ be a primitive d-th root of unity. The factors $z^{n/d} - \zeta^i f_1$ of $z^n - \overline{f}_v$ are then irreducible over $k(\eta_v)$ for every $i \in 0, 1, ..., d-1$ by our assumption on d. We thus find that the polynomials $z^{n/d} - \zeta^i f_1$ define the extensions of residue fields $k(\eta_v) \subset k(\eta_{v_i})$. These are all isomorphic over $k(\eta_v)$, so it suffices to consider the one determined by $z^{n/d} - f_1$.

We now start with the correspondence for J^0. Let $D \in J^0[n]$. Then $\overline{\tau}(D) = 0$, so there is a piecewise linear function ϕ_D such that $\tau_*(D) = \Delta(\phi_D)$. Here $\Delta(\cdot)$ is the Laplace operator on piecewise linear functions. Let $f \in K(C)$ be such that $\mathrm{div}(f) = nD$ and let ϕ_f be a piecewise linear function on Σ such that $\Delta(\phi_f) = \tau_*(\mathrm{div}(f))$. Since the divisors of $n \cdot \phi_D$ and ϕ_f are the same, we can use [7, Theorem 3] and translate these functions so that $n \cdot \phi_D = \phi_f$. The slope of ϕ_f on every edge is then divisible by n. Let v be a vertex in Σ and scale f as in the previous paragraph so that $v_{\eta_v}(f_v) = 0$. By the *slope formula* (see [9, Theorem 5.15(3)]) we find that the order of the reduction of f_v at the closed point w_e in C_v corresponding to an edge e is divisible by n. Write $\overline{f}_v = f_1^d$ as in the first paragraph and consider a factor $z^{n/d} - f_1$ defining the extension of residue curves. Since the order of $\overline{f}_v$ at the closed point w_e is divisible by n, we find that the order of f_1 at w_e is divisible by n/d. A computation similar to the one in the previous paragraph then shows that the morphisms $C_{v_i} \to C_v$ are completely split above w_e (divide or multiply by a suitable power of a uniformizer at w_e). We then apply [1, Theorem 4.23] and see that $d_{e'/e}(\psi) = 1$ for every edge e' lying above e, which shows that ψ is unramified above e.

Conversely, suppose that ψ is unramified above every e. Let v be a vertex and write $z^{n/d} - f_1$ for the polynomial defining an irreducible component above v. We claim that the order of f_1 at any closed point corresponding to an edge is divisible by n/d. Indeed, otherwise the Newton polygon of $z^{n/d} - f_1$ with respect to the discrete valuation corresponding to the closed point would contain a non-rational slope and thus the extension would be ramified, a contradiction. We conclude by the slope formula that the slope of the piecewise linear function ϕ_f on every edge $e \in E(\Sigma)$ is divisible by n. We can then write $\phi_f = n \cdot \phi'$ for some piecewise linear function ϕ'. But then $n\Delta(\phi') = \Delta(\phi_f) = n\tau_*(D)$ and thus $\tau_*(D) = \Delta(\phi')$. We conclude that $D \in J^0[n]$, as desired.

Suppose now that $D \in T[n]$. We have to check that the induced covering of metrized complexes splits completely. Since $T[n] \subseteq J^0[n]$, we already know that the covering splits on the edges. We thus only have to show that the covering splits on the vertices. By the exact sequence in Equation (9), we know that the reduction of the divisor D at every component is principal, that is $\text{red}(D, \Gamma_i) = (h_i)$ for an $h_i \in k(\eta_v)$. Furthermore, we have $u_i \cdot h_i^n = \overline{f}_v$ for some $u_i \in k^*$ by the condition $nD = \text{div}(f)$. It now follows from the explicit description of the normalization given in the first paragraph of the proof that the covering is completely split above every v.

Conversely, suppose that the extension induced by $D \in J[n]$ splits above every vertex. Then the reduction of $z^n - f_v$ splits completely at every v. This means that $\overline{f}_v$ is an n-th power in $k(\eta_v)$. We then see that $\text{red}(D, \Gamma_i) = (h_i)$ for some $h_i \in k(\eta_v)$ and thus the image of D in $\text{Jac}(\mathcal{X}_s)$ maps to zero in the Jacobian of every component Γ_i. In other words, $D \in T[n]$, as desired. □

Example 5.17. Consider a genus two curve X with skeleton as in Fig. 1. To be more explicit, consider the smooth proper curve X defined locally by

$$y^2 = x(x - \varpi)f(x)$$

for a polynomial $f(x) \in R[x]$ of degree 3 and an element $\varpi \in R$ with $\text{val}(\varpi) > 0$. Here we assume that $\text{char}(k) \neq 2$, $f(0) \neq 0$, $f(\varpi) \neq 0$, $\deg(\overline{f}(x)) = 3$, $\gcd(\overline{f}(x), \overline{f}'(x)) = 1$ and $\overline{f}(0) \neq 0$. A simple computation shows that X has a skeleton of the desired type. For any n coprime to $\text{char}(k)$, we have

$$J[n] = (\mathbb{Z}/n\mathbb{Z})^4,$$
$$J^0[n] = (\mathbb{Z}/n\mathbb{Z})^3,$$
$$T[n] = (\mathbb{Z}/n\mathbb{Z}).$$

By Theorem 5.16, the coverings that come from $T[n]$ correspond to the topological abelian coverings of Σ, and the coverings that come from $J^0[n]$ correspond to the metrically unramified abelian coverings of Σ. We invite the reader to compare this for $n = 2$ with Example 4.17.

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