Superconformal blocks in diverse dimensions and BC symmetric functions

Francesco Aprile[1] **and Paul Heslop**[2]

[1] *Instituto de Fisica Teorica, UNESP, ICTP South American Institute for Fundamental Research Rua Dr Bento Teobaldo Ferraz 271, 01140-070, São Paulo, Brazil*

[2] *Mathematics Department, Durham University, Science Laboratories, South Rd, Durham DH1 3LE*

Abstract

We uncover a precise relation between superblocks for correlators of superconformal field theories (SCFTs) in various dimensions and symmetric functions related to the BC root system. The theories we consider are defined by two integers (m, n) together with a parameter θ and they include correlators of all half-BPS correlators in 4d theories with $\mathcal{N} = 2n$ supersymmetry, 6d theories with $(n, 0)$ supersymmetry and 3d theories with $\mathcal{N} = 4n$ supersymmetry, as well as all scalar correlators in any non SUSY theory in any dimension, and conjecturally various 5d, 2d and 1d superconformal theories. The superblocks are eigenfunctions of the super Casimir of the superconformal group whose action we find to be precisely that of the $BC_{m|n}$ Calogero-Moser-Sutherland Hamiltonian. When $m = 0$ the blocks are polynomials, and we show how these relate to BC_n Jacobi polynomials. However, differently from BC_n Jacobi polynomials, the $m = 0$ blocks possess a crucial stability property that has not been emphasised previously in the literature. This property allows for a novel supersymmetric uplift of the BC_n Jacobi polynomials, which in turn yields the $(m, n; \theta)$ superblocks. Superblocks defined in this way are related to Heckman-Opdam hypergeometrics and are non polynomial functions. A fruitful interaction between the mathematics of symmetric functions and SCFT follows, and we give a number of new results on both sides. One such example is a new Cauchy identity which naturally pairs our superconformal blocks with Sergeev-Veselov super Jacobi polynomials and yields the CPW decomposition of any free theory diagram in any dimension.

Contents

1 Introduction

In this paper we will obtain conformal and superconformal blocks for four-point functions of half-BPS scalar operators in diverse dimensions. Our construction is based on a single unified formalism, which uses analytic superspace [1–10] as its starting point, but includes various generalisations. In particular, a parameter θ which allows us to move across dimensions. By varying the value of θ, we will find superconformal blocks for theories in 1,2,3,4,6 and conjecturally 5 dimensions.

The superconformal blocks that we present have a beautiful interpretation in the theory of symmetric functions: they are in one-to-one correspondence with a natural supersymmetric extension of a class of stable polynomials that we call dual BC Jacobi polynomials. While the appearance of the BC root system in this context is not new,[1] the results here set these observations in a much more general context and apply them directly to the supersymmetric case as well as the bosonic CFT case.

In a quantum field theory, correlation functions can be decomposed by using the operator product expansion (OPE). If the theory is a Conformal Field Theory (CFT), it is well known that the OPE can be organised further by collecting the contribution of a primary operator with all its infinite descendants [18–21]. The object representing the common OPE between pairs of external operators in a four-point correlator is the conformal block. It can be represented as the solution of a Casimir operator (for the correlation function under consideration) with given boundary conditions [11, 22–25]. The conformal block encodes an infinite sum and is closely related to hypergeometric functions. We will show, more precisely, that the mathematics underlying the physics of superconformal blocks is that of the *multivariate* hypergeometric functions developed by Heckman-Opdam (applied to the BC root system).

The Heckman-Opdam (HO) hypergeometrics are rigorously defined as eigenfunctions of a system of differential equations [16, 17]. In the case of positive weights it was shown [26] that the A- and BC-type solutions reduced to Jack and Jacobi polynomials. These polynomials, on the other hand, were particularly well known in the literature because of their definition as orthogonal polynomials associated to root systems [27–29]. In fact, the study of orthogonal polynomials had its own independent trajectory, perhaps culminating with the introduction of the Koornwinder polynomials [30–32] and the proof of evaluation symmetry by Okounkov [33]. In this framework, supersymmetry makes its first appearance in [34–36], where Sergeev and Veselov used the $A_{m-1|n-1}$ root system to construct a supersymmetric version of Jack and Macdonald polynomials, i.e. super Jack and super Macdonald polynomials, which reduce to the bosonic family for the A_{n-1} root system.

The idea of expanding superconformal blocks in terms of super Jack polynomials was developed in [7–9, 11]. In particular the work of [9] showed that 4d superconformal blocks on the super Grassmannian $Gr(m|n, 2m|2n)$ admit a simple representation as a sum over super Schur polynomials [37], the latter being a particular (θ=1) case of super Jack polynomials. In this representation each super Schur polynomial contributes with a simple coefficient built out of gamma functions. Quite nicely, the whole series was shown to re-sum into a determinant of a matrix of Gauss hypergeometric functions.

[1] It was pointed out already in the foundational work [11], and more recently in [12–15], that bosonic conformal blocks solve BC_2-type differential equations.

An essential property of the construction in [9] is *stability*, which states that the (m,n) superconformal block reduces to the $(m-1,n)$ or $(m,n-1)$ superconformal block when one of its m x-variables, or n y-variables, is switched off. This property is crucial. It implies that the coefficients of a block expanded in super Schur polynomials are *independent* on m,n! Thus, they can be obtained by examining the $(m,0)$ bosonic conformal blocks, or even simpler, the $(0,n)$ compact analogues, which are just polynomials. This simple property then leads to a precise formula of all superblocks (in particular those of short or atypical representations) of half BPS correlators in 4d $\mathcal{N}=4,2$ supersymmetric theories.

As we will show here, it turns out that the $(0,n)$ polynomial of [9] is actually a rewriting of the BC_n Jacobi polynomial $J_{\underline{\lambda}}$ with the parameter $\theta=1$. A BC_n Jacobi polynomial is a polynomial in n variables defined by a Young diagram $\underline{\lambda}=[\lambda_1,..,\lambda_n]$ with row lengths $\lambda_i \geq \lambda_{i+1} \in \mathbb{Z}^{\geq 0}$. We will use the recent definition given by Koornwinder in [41]. Then, the precise relation between the $(0,n)$ block and a Jacobi polynomial is quite interesting as it involves taking a complementary Young diagram, $\beta^n \backslash \underline{\lambda} = [\beta-\lambda_n,..,\beta-\lambda_1]$ and inverting the original variables. Based on this observation we introduce a new class of polynomials, which we call the *dual* Jacobi polynomials. These are defined by

$$\tilde{J}_{\beta,\underline{\lambda}}(y_1,\ldots,y_n) \equiv (y_1\ldots y_n)^{\beta} J_{\beta^n\backslash\underline{\lambda}}(\tfrac{1}{y_1},\ldots,\tfrac{1}{y_n}) \tag{1.1}$$

where the new parameter β here is an arbitrary integer, sufficiently large to ensure the Jacobi in inverse variables is again polynomial. Remarkably, unlike the Jacobi polynomials themselves, the dual Jacobi polynomials are stable! This surprisingly simple fact is a key observation,[2] and opens up the way towards the more general definition of superconformal blocks in diverse dimensions that we give below. In fact, the above discussion was for $\theta=1$ but can be repeated for any value of θ by going from Schur to Jack polynomials.

We can now define a natural supersymmetric extension of $\tilde{J}_{\beta,\underline{\lambda}}$ using stability in a crucial way: We simply have to replace the expansion in Jack polynomials of $\tilde{J}_{\beta,\underline{\lambda}}$ with super Jack polynomials, keeping the same expansion coefficients. The sum over super Jack polynomials now is no longer cut off, and becomes infinite in the direction of the conformal subgroup, since the latter is non compact. We define in this way the $BC_{n|m}$ dual Jacobi functions.

The key claim then is that superconformal blocks $B_{\gamma,\underline{\lambda}}(\mathbf{x}|\mathbf{y})$ in any $(m,n;\theta)$ theory are given by these dual super Jacobi functions through a trivial redefinition, normalisation and transposition

$$B_{\gamma,\underline{\lambda}}(\mathbf{x}|\mathbf{y}) = \left(\frac{\prod_i x_i^{\theta}}{\prod_j y_j}\right)^{\frac{\gamma}{2}} (-1)^{|\underline{\lambda}'|}\Pi_{\underline{\lambda}'}(\tfrac{1}{\theta})\ \tilde{J}_{\beta,\underline{\lambda}'}(\mathbf{y}|\mathbf{x})\ . \tag{1.2}$$

The parameter θ will be related to the spacetime dimension in the four-point correlator, and β is given in terms of the parameter γ, related to the scaling dimension of the operator appearing in the block. Here and throughout the paper we use primed Young diagrams to denote transposed Young diagrams, i.e. the row lengths of $\underline{\lambda}'$ are the column lengths of $\underline{\lambda}$ and viceversa. The factor $\Pi_{\underline{\lambda}'}(\frac{1}{\theta})$ is an explicit known function of θ given later.

The dual Jacobi functions so constructed interpolate between the dual Jacobi polynomials for $m=0$, the (infinite series) bosonic conformal blocks of [11] for $n=0$ and arbitrary θ, and

[2] The importance of stability for the construction of BC symmetric functions from BC polynomials was discussed by E.Rains in [32].

reproduce the $\theta = 1$ (m, n) superconformal blocks in 4d. Quite remarkably, another class of supersymmetric Jacobi polynomials was constructed by Sergeev and Veselov in [38], however these are polynomial in both directions and are explicitly different from the super Jacobi functions we defined above.

The correlators in this formalism, which are defined entirely by specifying $(m, n; \theta)$ and charges for the external operators, include many cases of interest, for example 3d and 6d half BPS superconformal blocks which are notoriously difficult to study. Importantly, it will treat long and short representations on the same footing.

The use of symmetric polynomials, in relation with stability, has important payoffs for both the study of CFT and the theory of symmetric functions. The first one is that we will exhibit a formula for the expansion coefficients of our superconformal blocks, over the basis of super Jack polynomials, in terms of a (super) binomial coefficient [38], after a physically motivated redefinition of the external parameters. The second one, which is perhaps the most beautiful, is that we will able to prove, and indeed generalise, a superconformal Cauchy identity, which once properly interpreted yields the conformal partial wave expansion of any free theory propagator structures within the formalism. The objects paired in this Cauchy identity are, on the one side the superconformal blocks, on the other side the super Jacobi polynomials of Sergeev and Veselov [38], which we mentioned above. Another pay off is that we can read off and generalise explicit results derived for Heckman Opdam hypergeometrics with $\theta = 1$, allowing us to write down all higher order super Casimir operators for the superblocks. Finally we will find an interpretation of the coefficients occurring in the decomposition of superblocks into blocks as structure constants for dual Jacobi polynomials.

In the next section we will give a detailed outline of the whole paper before proceeding to the main body.

2 Overview

We would like to facilitate the reading of our paper, by presenting an extended overview of our results.

From a physics perspective, we will begin by presenting superconformal blocks in the formalism of analytic superspace as eigenfunctions of the superconformal Casimir operator which remarkably we will see is equivalent to a $BC_{m|n}$ CMS operator. This will anticipate the more detailed discussion in section 3. Then, we will outline a practical method to solve the differential equation associated to the $BC_{m|n}$ Casimir, by using a recursion relation. This recursion is very important in our story. Full details will be given in section 5, and an extended discussion about its analytic properties will be given in section 6.

From a more mathematically oriented point of view, we will motivate in section 2.3 our construction of dual BC_n Jacobi polynomials, and their supersymmetric extension to dual super Jacobi functions. We will show that superblocks are essentially dual super Jacobi functions. Properties of these functions, which parallel physical properties of the superconformal blocks, will be reviewed in sections 2.4 and 2.5, and proved later on in the corresponding sections. These have to do with the action of complementation, the Cauchy identity, and the structure constants for superconformal blocks.

2.1 Superconformal blocks

We will consider four-point functions of scalar operators living on certain coset spaces of appropriate superconformal groups with coordinates X. These operators include a number of cases of interest including half-BPS correlators in many superconformal field theories in $1, 2, 3, 4, 6$ and conjecturally 5 dimensions, as well as (non-supersymmetric) scalars on Minkowski space of any dimension, as well as scalars in non supersymmetric CFTs and analogous reps on purely internal (ie compact) spaces. These cases all belong to the more general family of theories indexed by three parameters $(m, n; \theta)$ with m, n non-negative integers. The above group theory interpretation exists only for certain values of $(m, n; \theta)$ and when there is such a group theory interpretation, the complexified coset space in question will be a maximal (possibly super and/or orthosymplectic) Grassmannian. They will all be given by a flag manifold which can be denoted by a (super) Dynkin diagram with a single marked node (see section 3 for more details, in particular section 3.1 gives a list of the physical CFTs to which the formalism applies). Scalar operators in this space have a weight, p, under a $\mathbb{C}^*$ subgroup.[3] We denote them by $\mathcal{O}_p$, and the four point functions by

$$\langle \mathcal{O}_{p_1}(X_1)\mathcal{O}_{p_2}(X_2)\mathcal{O}_{p_3}(X_3)\mathcal{O}_{p_4}(X_4)\rangle. \tag{2.1}$$

The operator $\mathcal{O}_1$ is the basic representation, and corresponds to a representation whose highest weight state is a scalar operator of scaling dimension $\theta = \frac{d-2}{2}$ and internal charge 1. Then the scaling dimension of $\mathcal{O}_p$ is $p\,\theta$ with internal charge p. In the supersymmetric cases $\mathcal{O}_p$ is a half BPS multiplet. Full details of all theories that fit into this (m, n, θ) classification can be found in section 3 and appendix A.

In a CFT we can bring two operators close to each other and replace their coincident limit with a sum over operators at a single point, as follows,

$$\mathcal{O}_{p_i}(X_1)\mathcal{O}_{p_j}(X_2) \;=\; \sum_{O^{\gamma,\underline{\lambda}}} C_{p_i p_j O^{\gamma,\underline{\lambda}}}\; g_{12}^{\frac{1}{2}(p_i+p_j-\gamma)}\; \mathbb{D}^{\gamma,\underline{\lambda}}(X_{12},\partial_2)\; O_{\gamma,\underline{\lambda}}(X_2)\;. \tag{2.2}$$

Formula (2.2) is known as the Operator Product Expansion or OPE. As an equality in group theory, the OPE corresponds to decomposing the tensor product of two (possibly infinite dimensional) representations into its irreducible primary representations, denoted above with $O_{\gamma,\underline{\lambda}}$, when $X_1 \to X_2$. In the $(m, n; \theta)$ theories that we consider here, the coordinates X_i can always be written as a square (super)-matrix, and the propagator, i.e. the two point function of two basic $\mathcal{O}_1$ operators, is[4]

$$g_{ij} = \langle \mathcal{O}_1(X_i)\mathcal{O}_1(X_j)\rangle = \mathrm{sdet}(X_i - X_j)^{-\#}\;. \tag{2.3}$$

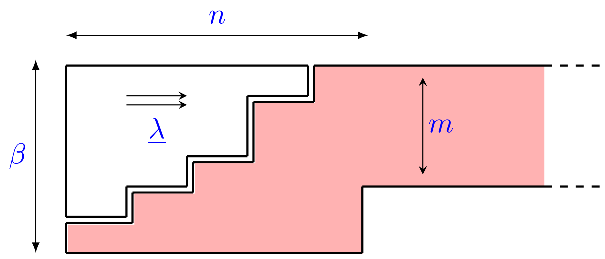
The operator $\mathbb{D}$ appearing in (2.2) generates descendants of the exchanged primary operators $O_{\gamma,\underline{\lambda}}$, and $C_{p_i p_j O_{\gamma,\underline{\lambda}}}$ is the OPE coefficient of $O_{\gamma,\underline{\lambda}}$ w.r.t. the external fields $\mathcal{O}_{p_i}(X_1)\mathcal{O}_{p_j}(X_2)$.

In the OPE of two scalars, as is the case here, the primaries exchanged can always be

[3]This is the complexification of either a $U(1)$ subgroup of the internal subgroup or the group of dilatations when there is no internal subgroup (i.e. $n = 0$) and it corresponds to the single marked Dynkin node.

[4]The power # here is defined so that $\mathcal{O}_1$ has dimension θ and/or internal charge 1. It depends on the theory in question: for $\theta = 1, 2$ we will have $\# = 1$ whereas for $\theta = \frac{1}{2}$ we will have $\# = \frac{1}{2}$. This precise value won't play any further role in the following.

specified by a weight under the afore-mentioned $\mathbb{C}^*$, which we denote by γ, together with the representation of the remaining isotropy group, $\underline{\lambda}$, so $O_{\gamma,\underline{\lambda}}$ (and also the external operators $\mathcal{O}$ can be specified in this way but with $\underline{\lambda}$ trivial). In a free theory,[5] γ varies over the range $\max(p_1{-}p_2, p_4{-}p_3) \leq \gamma \leq \min(p_1+p_2, p_3+p_4)$ in steps $\in 2\mathbb{Z}$ where we will assume $p_1 > p_2, p_4 > p_3$ without loss of generality. The representation of the isotropy group is specified via a Young diagram $\underline{\lambda}$, which will encode the various quantum numbers, for example spin and twist, of the operator exchanged. The Young diagram will be consistent with that describing an $SL(m|n)$ representation, meaning it will have at most m rows longer than n and at most n columns taller than m. Furthermore the overall height of the Young diagram must be smaller than $\beta \equiv \min\left(\frac{1}{2}(\gamma{-}p_{12}), \frac{1}{2}(\gamma{-}p_{43})\right)$. So a valid Young diagram $\underline{\lambda}$ must fit inside the red area here



(2.4)

Superconformal representations are conventionally specified via the dilation weight, Δ, various conformal quantum numbers or Dynkin labels (e.g. Lorentz spin), and quantum numbers or Dynkin labels describing the internal representation, whereas we specify the the representation by the Young diagram $\underline{\lambda}$ and the parameter γ. The dilation weight is then $\Delta = \sum_{i=1}^{m} \max(\lambda_i{-}n, 0) + m\theta\frac{\gamma}{2}$ and the other conformal quantum numbers are specified by differences of the first m row lengths $\lambda_i - \lambda_{i+1}$. The internal group quantum numbers are given as differences of Young diagram column heights $\lambda'_i - \lambda'_{i+1}$ as well as the special marked Dynkin label $b = \gamma - 2\lambda'_1$. We refer to section 3 for a more detailed explanation about how to read the Young diagram and relate it to conformal group representations. For now note that whilst all parameters are integers in the free theory, in an interacting quantum theory Δ alone can become non integer, with all other quantum numbers remaining integer. In particular, this means we can allow for anomalous dimensions by deforming the row lengths $\lambda_i \to \lambda_i + \tau$ for $i = 1,..,m$, arbitrary real τ and/ or $\lambda'_j \to \lambda'_j + \tau'$ for $j = 1,..,n$, arbitrary real τ' as long as we also have $\gamma \to \gamma + 2\tau'$ to ensure that $b = \gamma - 2\lambda'_1$ remains integer. Furthermore deforming thus, with $\tau = -\theta\tau'$ leaves the representation unchanged. We will return to this 'shift symmetry'.

The conformal structure of the theory and the use (twice) of the OPE, imply that we can decompose a four-point function as follows:

$$\langle \mathcal{O}_{p_1}(X_1)\mathcal{O}_{p_2}(X_2)\mathcal{O}_{p_3}(X_3)\mathcal{O}_{p_4}(X_4)\rangle = \\ g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{14}}{g_{24}}\right)^{\frac{p_{12}}{2}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{p_{43}}{2}} \sum_{\gamma,\underline{\lambda}} \left(\sum_{O} C^{O}_{p_1p_2} C_{O\,p_2p_3}\right) B^{(m,n)}_{\gamma,\underline{\lambda}}(X_i;\theta,p_{12},p_{43}) \qquad (2.5)$$

[5] We will focus initially on the case where all parameters are integers as for example occurs in a (generalised) free conformal field theory. However the analytic continuation to non integer values (for example to include anomalous dimensions) is considered in detail in section 6.

where $B_{\gamma,\underline{\lambda}}(X_i)$ is called the (super)conformal block.

The (super)conformal block $B_{\gamma,\underline{\lambda}}$ is the main subject of this paper. It represents the contribution of an exchanged primary operator $O_{\gamma,\underline{\lambda}}$ with its tail of descendants, which is common to the OPE of $\mathcal{O}_{p_1}\mathcal{O}_{p_2}$ and $\mathcal{O}_{p_3}\mathcal{O}_{p_4}$.Thus the block depends on the particular theory of interest $(m,n;\theta)$, as well as the external operators through the quantities

$$p_{12} \equiv p_1 - p_2 \qquad ; \qquad p_{43} \equiv p_4 - p_3. \tag{2.6}$$

Furthermore, because of conformal invariance, $B_{\gamma,\underline{\lambda}}(X_i)$ only depends on cross ratios built out of the four coordinates $X_{i=1,2,3,4}$.

The $X_{i=1,2,3,4}$ are square (super) matrices, and the cross ratios are the $m+n$ independent eigenvalues of the (super)-matrix $Z = X_{12}X_{24}^{-1}X_{43}X_{31}^{-1}$ where $X_{ij} = X_i - X_j$. These $m+n$ independent cross-ratios z_i split into two types, corresponding to whether they arise from the non-compact conformal group or the compact internal group, which we denote by x_i and y_j respectively, so

$$\begin{aligned} &B_{\gamma,\underline{\lambda}}(X_i) = B_{\gamma,\underline{\lambda}}(\mathbf{z}) \\ &\mathbf{z} = (z_1,\ldots,z_m|z_{m+1},\ldots,z_{m+n}), \qquad z_i = x_i, \qquad z_{j+m} = y_j \ . \end{aligned} \tag{2.7}$$

The details in all cases of physical interest will be given in section 3. For now note that

$$\mathrm{sdet}(Z)^{\#} = \frac{g_{24}g_{13}}{g_{12}g_{34}} = \frac{\prod_{i=1}^m x_i^\theta}{\prod_{j=1}^n y_j} \qquad \mathrm{sdet}(1-Z)^{\#} = \frac{g_{14}g_{23}}{g_{12}g_{34}} = \frac{\prod_{i=1}^m (1-x_i)^\theta}{\prod_{j=1}^n (1-y_j)} \ , \tag{2.8}$$

where $\#$ is the power in (2.3).

We can characterise $B_{\gamma,\underline{\lambda}}(\mathbf{z})$ group theoretically, as eigenfunctions of the quadratic Casimir acting at points 1 and 2 on the correlator. Pulling the Casimir through the prefactor of (2.5) to act on the block itself, it becomes a differential operator in $\mathbf{z}$. This differential operator, denoted hereafter by $\mathbf{C}$, has been computed explicitly in a number of cases[6] and we find, quite remarkably, that in all these cases it is equivalent to the $BC_{m|n}$ operator of the type of Calogero-Moser-Sutherland. We give the details of this identification in appendix B. We are led to conjecture that such an identification is true for all cases. Under this assumption we define superconformal blocks group theoretically through the eigenvalue equation:

$$\mathbf{C}^{(\theta,-\frac{1}{2}p_{12},-\frac{1}{2}p_{43},0)} B_{\gamma,\underline{\lambda}}(\mathbf{z}) = E_{\gamma,\underline{\lambda}}^{(m,n;\theta)} B_{\gamma,\underline{\lambda}}(\mathbf{z}) \ . \tag{2.9}$$

The Casimir $\mathbf{C}^{(\theta,a,b,c)}$ is a second order differential operator in the variables $\mathbf{z}$ given explicitly in (5.5). It depends on θ, a, b, c and implicitly on m, n. The eigenvalue is

$$E_{\gamma,\underline{\lambda}}^{(m,n;\theta)} = h_{\underline{\lambda}}^{(\theta)} + \theta\gamma|\underline{\lambda}| + \left[\gamma\theta|m^n| + h_{[e^m]}^{(\theta)} - \theta h_{[s^n]}^{(\frac{1}{\theta})}\right] \tag{2.10}$$

[6]The cases are: $(m,n) = (2,0)$ and $(0,2)$ in [11]. Then, $\theta = 1$, $m,n \in \mathbb{Z}^+$, in [9], by using a supermatrix formalism, and partially in the case of $\theta = 2$, $m,n \in \mathbb{Z}^+$ in [39]

where $e = +\frac{\theta\gamma}{2}$, $s = -\frac{\gamma}{2}$, and

$$h^{(\theta)}_{\underline{\lambda}} = \sum_i \lambda_i(\lambda_i - 2\theta(i-1)-1) \qquad ; \qquad |\underline{\lambda}| = \sum_i \lambda_i \tag{2.11}$$

with $\underline{\lambda} = [\lambda_1, \ldots]$ denoting the row lengths of the Young diagram, and

$$h^{(\theta)}_{[e^m]} = m\theta\tfrac{\gamma}{2}((\tfrac{\gamma}{2}+1-m)\theta - 1) \qquad ; \qquad \theta h^{(\frac{1}{\theta})}_{[s^n]} = n\tfrac{\gamma}{2}(\theta(\tfrac{\gamma}{2}+1) - 1 + n) \tag{2.12}$$

In particular, $[e^m]$ and $[s^n]$ are the diagrams with m rows of 'length' $+\frac{\theta\gamma}{2}$, and n columns of 'height' $-\frac{\gamma}{2}$.

What is important about (2.10) is that $h^{(\theta)}_{\underline{\lambda}} + \theta\gamma|\underline{\lambda}|$ does not depend on m, n but only depends on the Young diagram $\underline{\lambda}$, whereas the other terms clearly have explicit m, n dependence. We will revisit this key point shortly.

2.2 Solving the Casimir equation

From OPE considerations, it has been shown in a number of cases[7] that in the limit $\mathbf{z} \to 0$

$$B_{\gamma,\underline{\lambda}}(\mathbf{z}) = \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\frac{\gamma}{2}} \times \left(P_{\underline{\lambda}}(\mathbf{z};\theta) \;+\; \ldots\;\right) \tag{2.13}$$

where $P_{\underline{\lambda}}$ is the super Jack polynomial of Young diagram $\underline{\lambda}$. (Jack and super Jack polynomials are reviewed in appendix C). The physics behind (2.13) is quite simple to understand: When there is a group theory interpretation, the prefactor corresponds to the limit $z_i \to 0$ of a four-point propagator structure in which an operator of twist $\theta\gamma$ is the first operator exchanged, and $P_{\underline{\lambda}}$ accounts for the corresponding superprimary operator in $B_{\gamma,\underline{\lambda}}(\mathbf{z})$.

The eigenvalue equation (2.9), together with the asymptotics (2.13), gives a unique definition of the superblocks $B_{\gamma,\underline{\lambda}}(\mathbf{z})$.

For $\theta = 1, 2$ the whole superblock is known to be representable as a series over super Jack polynomials [7–9]. Thus for arbitrary θ we will seek a representation of $B_{\gamma,\underline{\lambda}}(\mathbf{z})$ as such a series, explicitly as,

$$B_{\gamma,\underline{\lambda}}(\mathbf{z}) = \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\frac{\gamma}{2}} F_{\gamma,\underline{\lambda}}(\mathbf{z}) \qquad ; \qquad F_{\gamma,\underline{\lambda}}(\mathbf{z};\theta,p_{12},p_{43}) = \sum_{\underline{\mu}\supseteq\underline{\lambda}} (T_{\gamma;\theta,p_{12},p_{43}})^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{z}) \;, \tag{2.14}$$

where the sum is over all Young diagrams $\underline{\mu}$ which contain $\underline{\lambda}$, i.e. $\underline{\lambda} \subseteq \underline{\mu}$. The expansion coefficients $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ will be sometimes referred to as the Jack→Block matrix. These coefficients depend on (θ, p_{12}, p_{43}) as well as γ and the two Young tableaux $\underline{\lambda}, \underline{\mu}$. In principle they should depend also on (m, n), but as we will see quite remarkably they in fact don't! The super Jacks themselves vanish if the Young diagram is not of $SL(m|n)$ shape (ie if the mth row

[7]For a direct derivation of this limit from the OPE see the discussion around eq. (20) in [9] for the $\theta = 1$ case and the discussion around (33)-(35) of [8] for $\theta = 2$. This limit is also implicit in earlier work for $\theta = 1$ [6] as well as in the bosonic case [11] for any θ.

is bigger than n, $\lambda_{m+1} > n$). Furthermore we will see that the coefficients T_γ vanish if the height of the Young digram is larger than β. So the non-vanishing contributions to the sum are only from Young diagrams $\underline{\mu}$ which fit inside the red area of (2.4).

From the defining Casimir for the blocks (2.9) we obtain a differential equation for $F_{\gamma,\underline{\lambda}}(\mathbf{z})$ by conjugating the original Casimir in (2.9) by the γ-prefactor in (2.14). It turns out that the result can be written in terms of a shifted version of the same differential operator $\mathbf{C}$, with a modified eigenvalue:

$$\mathbf{C}^{(\theta,\alpha,\beta,\gamma)} F_{\gamma,\underline{\lambda}}(\mathbf{z}) = (h^{(\theta)}_{\underline{\lambda}} + \gamma|\underline{\lambda}|) F_{\gamma,\underline{\lambda}}(\mathbf{z}) \ , \tag{2.15}$$

where

$$\alpha \equiv \max\left(\tfrac{1}{2}(\gamma - p_{12}), \tfrac{1}{2}(\gamma - p_{43})\right) , \quad \beta \equiv \min\left(\tfrac{1}{2}(\gamma - p_{12}), \tfrac{1}{2}(\gamma - p_{43})\right) \ . \tag{2.16}$$

A remarkable outcome of conjugating the Casimir is that the eigenvalue of $F_{\gamma,\underline{\lambda}}$ does not depend explicitly on (m,n), since the m,n dependent term of the original eigenvalue (2.10), the one in $[\ldots]$, is now missing. In other words, the eigenvalue of $F_{\gamma,\underline{\lambda}}$ depends only on $\underline{\lambda}$ and γ. This fact underlies a property known as 'stability' in the maths literature, which means that $F_{\gamma,\underline{\lambda}}$ depends only on the number of non-vanishing variables of each type. So the $F_{\gamma,\underline{\lambda}}$ of type $(m+1,n)$ in which one of the x's is set to zero reduces to the $F_{\gamma,\underline{\lambda}}$ of type (m,n). Similarly if one of the ys is set to zero. In formulae

$$F_{\gamma,\underline{\lambda}}(x_1,\ldots x_m,0|y_1,\ldots y_n) = F_{\gamma,\underline{\lambda}}(x_1,\ldots x_m|y_1,\ldots y_n,0) = F_{\gamma,\underline{\lambda}}(x_1,\ldots x_m|y_1,\ldots y_n) \ . \tag{2.17}$$

Stability is clear when there is a (super) matrix interpretation for the superconformal block (as for $\theta = \frac{1}{2}, 1, 2$), but the generalisation to arbitrary θ is non-trivial.

The next observation is that since super Jack polynomials are also stable,[8] the expansion coefficients $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ must be *independent* of (m,n)! Indeed this was the main insight of [9], which reduces the study of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ for m,n superconformal blocks to the study of the simpler generalised bosonic conformal blocks.

The representation of the super blocks $B_{\gamma,\underline{\lambda}}(\mathbf{z})$ as a sum over super Jack polynomials, (2.14), is not accidental. In fact, building on the observation that the Casimir is a differential operator for the $BC_{m,n}$ root system, as mentioned above, we are led to consider writing it in terms of operators $\mathbf{H}$ and $\sum_i z_i\partial_i$, the $A_{m-1,n-1}$ differential operators for which super Jack polynomials are eigenfunctions.[9] The corresponding decomposition takes the form,

$$\begin{aligned} \mathbf{C}^{(\theta,a,b,c)} = \mathbf{H}^{(\theta)} &+ \theta c \sum_{i=1}^{m+n} z_i\partial_i \\ &- \theta(a+b)\sum_{i=1}^{m+n} z_i^2\partial_i - \tfrac{1}{2}\Big[\mathbf{H}^{(\theta)}, \sum_{i=1}^{m+n} z_i^2\partial_i\Big] - \theta^2 ab \sum_{i=1}^{m+n} z_i(-\theta)^{-\pi_i} \ . \end{aligned} \tag{2.18}$$

The operators on the first line thus map Jack polynomials to themselves whereas those on the second line take a Jack polynomial to another Jack polynomial with an additional box in

[8] Notice also that the eigenvalue for $F_{\gamma,\underline{\lambda}}$ is the same as for $P_{\underline{\lambda}}$, since $\mathbf{H}P_{\underline{\lambda}} = h_{\underline{\lambda}}P_{\underline{\lambda}}$ and $\sum_i z_i\partial_i P_{\underline{\lambda}} = |\underline{\lambda}|P_{\underline{\lambda}}$.

[9] More precisely, $\mathbf{H}$ is the supersymmetric version of the Calogero-Moser-Sutherland (CMS) Hamiltonian found in [34,35]. A- and BC-type differential operators are reviewed in appendix B.

its Young diagram and will thus be interpreted as one-box raising operators. The action of the Casimir on the sum of super Jack polynomials, decomposed as in (2.18), thus turns the differential equation (2.15) into a recursion relation on the coefficients $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$. A particularly convenient representation of this recursion is

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_{i=1} (\mu_i - 1 - \theta(i-1-\alpha))\,(\mu_i - 1 - \theta(i-1-\beta))\,\mathbf{f}^{(i)}_{\underline{\mu}-\square_i}\,(T_\gamma)^{\underline{\mu}-\square_i}_{\underline{\lambda}}}{\left(h_{\underline{\mu}} - h_{\underline{\lambda}} + \theta\gamma\,(|\underline{\mu}| - |\underline{\lambda}|)\right)} . \tag{2.19}$$

Here $\underline{\mu}-\square_i$ represents a Young diagram obtained by removing the last box at row i from $\underline{\mu}$, when allowed, and $\mathbf{f}$ is a simple function given in (5.30). Let us emphasise that the recursion (2.19) is straightforward to implement on a computer, and very efficient up to high order.

The recursion (2.19) actually depends only on objects which can be defined combinatorially, and admits different equivalent representations, depending on whether we read the Young diagram just along the rows, or the columns, or we mix rows and columns as it is more appropriate in the supersymmetric theory. By the (m, n) independence, the value of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ does not depend on the chosen representation. This is particularly useful when constructing superconformal blocks for short (protected) representations, a case which is notoriously difficult to study with other methods.

As mentioned the above discussion is true for (generalised) free theory blocks where all quantum numbers are integers. In an interacting CFT however, the dilation weight can become non integer. Further it is interesting to consider analytic continuations of other quantum numbers such as spin. We then study possible analytic continuations of the recursion in the variables that describe the Young diagrams. We do so by promoting the external $\underline{\lambda}$ to complex values, and taking $\underline{\mu} = \underline{\lambda} + \vec{n}$ with $\vec{n} \in \mathbb{Z}^+$. It turns out that each one of the following representations of the recursion, either row type, column type, or supersymmetric one, now gives a distinct analytic continuation (thus analytic continuation breaks the m, n independence), which however coincide with the others when $\underline{\lambda}$ reduces to a valid Young diagram.

For long (non protected) representations we show in section 6 that a suitable (m, n) analytic continuation of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ exists, such that the following shift,

$$\begin{array}{ccccc} \lambda_i \to \lambda_i - \theta\tau' & & \lambda'_j \to \lambda'_j + \tau' & & \\ \mu_i \to \mu_i - \theta\tau' & ; & \mu'_j \to \mu'_j + \tau' & ; & \gamma \to \gamma + 2\tau' \,. \\ i = 1, \dots m & & j = 1, \dots n & & \end{array} \tag{2.20}$$

is a symmetry of the solution. For integer values this shift symmetry corresponds to the well understood equivalence of different Young diagrams describing $SL(m|n)$ reps for $\theta = 1$ and similarly for the other group theoretic cases (e.g. when $n = 0$ you can see it as the fact that full m columns correspond to the trivial $SL(m)$ rep and thus can be deleted). It can also be seen directly from the relation between the Young diagram and Dynkin labels (see (3.16)) that there is a redundancy in the Young diagram description of representations which is precisely this shift.

2.3 Superblocks as dual super Jacobi functions

In this section we start from scratch and motivate, from a purely mathematical point of view, the introduction of a certain supersymmetric generalisation of BC Jacobi polynomials [31,41]. More precisely, we will define a family of polynomials that we call *dual Jacobi polynomials*. These are closely related to the BC Jacobi polynomials of Koornwinder [41], but differently from those, they are stable, and thus allow a straightforward supersymmetric generalisation. The supersymmetric generalisation of a dual Jacobi polynomial is however not polynomial, in general. We will denote them as *dual super Jacobi functions*. It will turn out that superblocks are equivalent to these dual super Jacobi functions.

This section can be read largely independently of the previous sections (up to the point where we make the identification with blocks).

Dual Jacobi polynomials

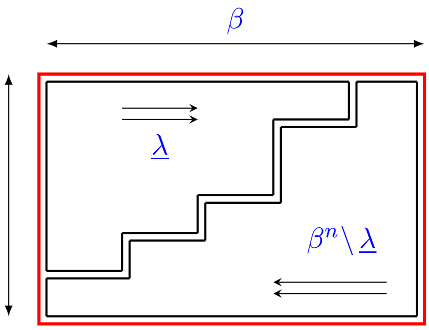
Consider the following operation: take a BC_n Jacobi polynomial $J_{\underline{\lambda}}()$ in inverse variables y_i^{-1}, and multiply by a sufficiently high power of $(y_1...y_n)^\beta$ for $\beta \in \mathbb{N}$ in order to ensure the result is polynomial again in the y_i. We will call this a *dual Jacobi polynomial*.

Note that when the above operation is performed on a *Jack* polynomial the result is again a Jack polynomial of the *complementary* Young diagram $\beta^n \backslash \underline{\lambda}$:

$$(y_1 \dots y_n)^\beta P_{\underline{\lambda}}(\tfrac{1}{y_1}, \dots, \tfrac{1}{y_n}) = P_{\beta^n \backslash \underline{\lambda}}(y_1, \dots, y_n) \ , \tag{2.21}$$

where the *complement* of $\underline{\lambda}$ in β^n, i.e. $\beta^n \backslash \underline{\lambda}$, is defined as the Young diagram with row lengths

$$(\beta^n \backslash \underline{\lambda})_j = \beta - (\underline{\lambda})_{n+1-j} \qquad ;$$



(2.22)

The integer β should be large enough to take the complement, i.e. $\beta \geq \lambda_1$.

The aforementioned operation maps a BC_n Jacobi polynomial, $J(\mathbf{y}; \theta, p_-, p_+)$ to a different polynomial, which we thus define as the *dual Jacobi polynomial*,

$$\tilde{J}_{\beta, \underline{\lambda}}(y_1, \dots, y_n) \equiv (y_1 \dots y_n)^\beta J_{\beta^n \backslash \underline{\lambda}}(\tfrac{1}{y_1}, \dots, \tfrac{1}{y_n}) \ . \tag{2.23}$$

Crucially, dual Jacobi polynomials turn out to be stable (meaning that switching off one of the variables reduces the polynomial to the polynomial with one variable fewer)

$$\tilde{J}_{\underline{\lambda}}(y_1, .., y_{n-1}, 0) = \tilde{J}_{\underline{\lambda}}(y_1, .., y_{n-1}) \ . \tag{2.24}$$

This is the same stability property possessed by Jack polynomials, $P_{\underline{\lambda}}(\mathbf{y},0) = P_{\underline{\lambda}}(\mathbf{y})$, but which is absent for the original Jacobi polynomials themselves: $J_{\underline{\lambda}}(\mathbf{y},0) \neq J_{\underline{\lambda}}(\mathbf{y})$.

This stability property is key to a direct supersymmetric uplift of the dual Jacobi polynomials to $BC_{n|m}$ functions. As we will see, this uplift lands precisely on our superconformal blocks!

The BC_n Jacobi polynomial has as an explicit expansion in Jack polynomials (just like the blocks)[10]

$$J_{\underline{\lambda}}(\mathbf{y};\theta,p^-,p^+) = \sum_{\underline{\mu}\subseteq\underline{\lambda}} (S^{(n)}_{\theta,p^-,p^+})^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{y};\theta) \ . \tag{2.25}$$

The coefficients, $(S^{(n)})^{\underline{\mu}}_{\underline{\lambda}}$ are not stable, meaning they have explicit n dependence and don't just depend on the Young diagrams $\underline{\lambda},\underline{\mu}$. Indeed, this is what prevents stability of the Jacobi polynomial J. Let us note that the $(S^{(n)})^{\underline{\mu}}_{\underline{\lambda}}$ can be computed quite explicitly, either through a recursion, investigated by Macdonald [28], or independently by using a binomial formula due to Okounkov [33,40,41]. In the latter case, $S^{(n)}$, is written in terms of BC_n interpolation polynomials (IPs). We will discuss this in much more detail in section 7.

We can understand the stability of the dual Jacobi polynomials by considering their defining differential equation. The original eigenvalue equation for the Jacobi polynomials, translated to the dual Jacobi polynomials, can be written in terms of the Casimir (2.18) as

$$\mathbf{C}^{(\frac{1}{\theta},a,b,c)}(|\mathbf{y})\ \tilde{J}_{\beta,\underline{\lambda}}(y_1,\dots y_n;\theta,p^-,p^+) = e^{(\theta)}_{\beta,\underline{\lambda}}\ \tilde{J}_{\beta,\underline{\lambda}}(y_1,\dots y_n;\theta,p^-,p^+) \tag{2.26}$$

$$a=\beta \qquad ; \qquad b=\beta+p^- \qquad ; \qquad c=2\beta+p^-+p^+ \ .$$

where the eigenvalue is

$$e^{(\theta)}_{\beta,\underline{\lambda}} = -\tfrac{1}{\theta}h^{(\theta)}_{\underline{\lambda}} + \tfrac{1}{\theta}(2\beta+p^-+p^+)|\underline{\lambda}| \tag{2.27}$$

Both $h^{(\theta)}_{\underline{\lambda}}$ and $|\underline{\lambda}|$, given already in (2.11), are functions of the Young diagram $\underline{\lambda}$ only, (unlike the corresponding eigenvalue for the Jacobi polynomial which has explicit n dependence) and stability (2.24) follows. In our conventions, the normalisation of $\tilde{J}$ will be chosen so that the leading term of $\tilde{J}_{\beta,\underline{\lambda}}$ is $P_{\underline{\lambda}}$.

Dual Jacobi polynomials depend on $\theta, p^{\pm}$, as the Jacobi polynomials do, and in addition depend on β, which sets the boundary for the Young diagram in (2.22). From the definition of $\tilde{J}_{\beta,\underline{\lambda}}$, in terms of the Jacobi polynomials (2.23), the expansion of Jacobi polynomials (2.25), and the relation (2.21), we obtain the following expansion of the dual Jacobi polynomials in Jack polynomials with coefficients given by binomial coefficients, S, in complemented Young diagrams

$$\tilde{J}_{\beta,\underline{\lambda}}(y_1,\dots,y_n) = \sum_{\underline{\mu}:\,\underline{\lambda}\subseteq\underline{\mu}} \big(\widetilde{S}_\beta\big)^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(y_1,\dots,y_n) \qquad ; \qquad \big(\widetilde{S}_\beta\big)^{\underline{\mu}}_{\underline{\lambda}} = \big(S^{(n)}\big)^{\beta^n\backslash\underline{\mu}}_{\beta^n\backslash\underline{\lambda}} \ . \tag{2.28}$$

Note that if the Young diagram $\underline{\mu}$ is wider than β (ie $\mu_1 > \beta$) the coefficient $\tilde{S}$ will vanish

[10] We have momentarily suppressed the parameters in $J_{\underline{\lambda}}(;\theta,p^-,p^+)$, since they do not play an immediate role. Note that $P_{\underline{\mu}}(y_1,\dots,y_n)$ here is $P^{(n,0)}_{\underline{\mu}}(y_1,\dots,y_n|;\theta)$ in supersymmetric notation, not to be confused with the $P^{(0,n)}_{\underline{\mu}}(|y_1,\dots,y_n;\theta)$ polynomials. See appendix C and C.5 for further details.

and so the sum is finite, $\underline{\mu} \subseteq \beta^n$ giving a polynomial.

Stability of dual Jacobi polynomials implies that the coefficients $(\widetilde{S}_\beta)^{\underline{\mu}}_{\underline{\lambda}}$ are independent of n. Looking at the way this is related to $S^{(n)}$ (2.28), we see that even though $S^{(n)}$ is n dependent, remarkably it becomes independent of n when specified by complemented Young diagrams as is the case for $(\widetilde{S}_\beta)^{\underline{\mu}}_{\underline{\lambda}}$.

Dual super Jacobi functions

The supersymmetric extension of the dual Jacobi polynomial which we call *dual super Jacobi functions* is now immediate from stability: replace the expansion over Jack polynomials with super Jack polynomials, keeping the same expansion coefficients $\widetilde{S}$. This operation defines the $n|m$ dual super Jacobi function

$$\tilde{J}_{\beta,\underline{\lambda}}(\mathbf{y}|\mathbf{x}) = \sum_{\underline{\mu}\supseteq\underline{\lambda}} (\widetilde{S}_\beta)^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{y}|\mathbf{x}) \ . \tag{2.29}$$

When $m > 0$, the sum is over all Young diagrams $\underline{\mu}$ such that $\underline{\lambda} \subseteq \underline{\mu}$. As before, if $\underline{\mu}$ is wider than β^n, the coefficient $\tilde{S}$ will vanish and so $\mu_1 \leq \beta$. But now $\underline{\mu}$ can have arbitrary height, unlike in the non supersymmetric case (2.28) where it was naturally cut-off by n since the Jack polynomials would vanish for $\mu'_1 > n$. This means that the sum is *infinite* in the vertical Young diagram direction, and this is why we call $\tilde{J}_{\beta,\underline{\lambda}}$ a function rather than a polynomial. As a result, the dual Jacobi functions $\tilde{J}_{\beta,\underline{\lambda}}$ are explicitly different from the super Jacobi polynomials introduced in [38]. The latter also provide a supersymmetric generalisations of Jacobi polynomials but the generalisation is polynomial in both x and y variables and not stable. We will have something to say about these other supersymmetric Jacobi polynomials in section 7.

Blocks as dual super Jacobi functions

The claim is then that the dual super Jacobi functions are precisely the superconformal blocks up to changes of conventions. Explicitly the relation is

$$F_{\gamma,\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta,p_{12},p_{43}) = (-1)^{|\underline{\lambda}'|}\Pi_{\underline{\lambda}'}(\tfrac{1}{\theta})\ \tilde{J}_{\beta,\underline{\lambda}'}(\mathbf{y}|\mathbf{x};\tfrac{1}{\theta},p^-,p^+) \tag{2.30}$$

where[11]

$$\gamma = 2\beta + p^+ + p^- \qquad ; \qquad p^\pm = \tfrac{1}{2}|p_{12} \pm p_{43}| \tag{2.31}$$

then $\Pi(\theta)$ is a numerical factor, given explicitly later on in (7.10), and p_{43} and p_{12} can be solved as linear combinations of p^+ and p^-. Specialising to the internal case $(0,n)$ we have a direct relation between internal blocks and Jacobi polynomials

$$\prod_{i=1}^{n} y_i^{\frac{1}{2}(p^++p^-)} \times B_{\gamma=2\beta+p^++p^-,\underline{\lambda}}(\mathbf{y};\theta,p_{12},p_{43}) \ = \ (-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}'}(\tfrac{1}{\theta}) \times J_{\beta^n-\underline{\lambda}'}\left(\tfrac{1}{\mathbf{y}};\tfrac{1}{\theta},p^-,p^+\right) \tag{2.32}$$

[11]In other words, $\beta = \min\left(\frac{1}{2}(\gamma - p_{12}), \frac{1}{2}(\gamma - p_{43})\right)$.

where the original Jacobi polynomial J appears on the RHS with complemented, transposed Young diagram.

Equation (2.32), is an equation purely between symmetric polynomials, and provides the cleanest way to prove the more general supersymmetric relation (2.30). Indeed it was the discovery of this relation which lead us to dual Jacobi polynomials and their supersymmetric generalisation. This polynomial identification (2.32) can be proved by simply showing they obey the same Casimir equation. Then the general supersymmetric relation (2.30) follows directly because both sides can be uplifted supersymmetrically in the same way, i.e. by expanding in Jack polynomials, and uplifting the bosonic polynomials to super Jack polynomials thanks to stability.

The relation between dual super Jacobi functions and superconformal blocks gives a way of seeing why there must be an infinite expansion in the $\mathbf{x}$ variable, at least in cases where there is a supergroup interpretation. This infinite sum is a consequence of the non compactness of the conformal subgroup.

To summarise this subsection then, we have a link between internal blocks and Jacobi polynomials, both of which, through stability, naturally uplift to superblocks and dual super Jacobi functions respectively:

(dual) BC_n Jacobi polynomial	$\sim$	internal $(0, n)$ block
$\downarrow$	(supersymmetric uplift)	$\downarrow$
dual $BC_{n\|m}$ super Jacobi function	$\sim$	(m, n) superblock

2.4 Binomial coefficient and Cauchy identities

The previous subsection outlined the relation between Jacobi polynomials and blocks. In this section we consider some consequences of this relation, summarising the results of section 7.

After computing the Jack→Block matrix $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ in many specific cases by solving the recursion (2.19) we find that, for generic θ, it is a rational functional with an increasingly complicated numerator as a function of γ. To understand this non-trial dependence on γ we use the relation to Jacobi polynomials of the $m = 0$ theory described in the previous section.

The relation between blocks and dual Jacobis (2.30) leads to a relation between the coefficients, $\tilde{S}$ and T_γ in their respective Jack expansions (2.14),(2.28), explicitly:[12]

$$(T_{\gamma;\theta,p_{12},p_{43}})^{\underline{\mu}}_{\underline{\lambda}} = (\widetilde{S}_{\beta;\frac{1}{\theta},p^-,p^+})^{\underline{\mu}'}_{\underline{\lambda}'} \times \frac{(-1)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-1)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \ . \tag{2.34}$$

Hence from (2.28) the coefficients T_γ are related to the binomial coefficients S (2.25)

$$(T_{\gamma;\theta,p_{12},p_{43}})^{\underline{\mu}}_{\underline{\lambda}} \ = \ \frac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \times (S^{(n)}_{\frac{1}{\theta};p^-,p^+})^{\beta^n\backslash\underline{\mu}'}_{\beta^n\backslash\underline{\lambda}'} \ , \tag{2.35}$$

where $\beta, p^\pm$ are read off γ, p_{12}, p_{43} through (2.31).

Now the BC_n Jacobi polynomials are very well studied objects. In particular, the coefficients in the expansion over Jack polynomials can be computed by Okounkov *binomial formula*. Okounkov's binomial formula for $S^{(n)}$ (see [32, 33, 40, 41]) is computed via objects called BC interpolation polynomials (IPs) evaluated on partitions. The combinatorics is thus completely different from the combinatorics induced by the recursion and so the above rewriting of T_γ (2.35) is quite non trivial.

In fact, a very non trivial feature of this relation (2.35) is the way the RHS depends on γ. In the binomial coefficient we have $\gamma = 2\beta + p^+ + p^-$ where β is an integer specifying the complemented Young diagram. On the other hand in the recursion, γ is a free parameter, and the solution of the recursion is a rational function of γ and thus straightforwardly analytically continued. This should also be the case for the binomial coefficient then. In the process of understanding how this works we find that the complicated γ dependence of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ is precisely captured by the interpolation polynomial. We also find that the inverse of the Jack→Block matrix (i.e. the Block→Jack matrix) T_γ^{-1}, has an even more direct characterisation in terms of the binomial coefficient.

Note that since $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ does not depend on (m, n), for integer quantum numbers, the above characterisation via the binomial coefficient, through the $(0, n)$ case, can be used to compute any superconformal block for arbitrary (m, n). However, from this insight we understand that the binomial coefficient can also be upgraded to a super binomial coefficient which uses the super interpolation polynomials introduced by Sergeev and Veselov [38]. We will investigate this whole story in detail in section 7, and then in appendix E we collect a number of explicit solutions for T_γ found by solving the recursion, and we show explicitly how the two sides of (2.35) match.

We will conclude the main body of the paper by giving two nice applications of the relation between blocks and Jacobi polynomials, together with stability. Firstly, in section 8 we see how upon re-interpreting a Cauchy identity for Jacobi polynomials, given by Mimachi in [42], we can obtain (essentially with no effort) a formula for the conformal partial wave expansion of any generalised free theory within the class analysed here, i.e. whose supergroup description fits (m, n, θ) for specific values. In the process of investigating this we discover a new

[12] To derive this relation one also needs to use the fact that super Jack polynomials are well behaved under transposition

$$P_{\underline{\lambda}'}(\mathbf{y}|\mathbf{x}; \tfrac{1}{\theta}) = (-1)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)P_{\underline{\lambda}}(\mathbf{x}|\mathbf{y}; \theta) \tag{2.33}$$

For more details see the discussion around (C.47).

non-trivial double uplift of Mimachi's Cauchy identity to a doubly supersymmetric Cauchy identity involving both dual super Jacobi functions and Sergeev and Veselov's super Jacobi polynomials. Secondly in section 5.1 we use results from the study of BC hypergeometric functions in the special $\theta = 1$ case of Shimeno [43] to give simple explicit formulae for all higher order super Casimirs.

2.5 Superblock to block decomposition: a conjecture

Our formalism for constructing superconformal blocks might be classified as *top-down*, because we start from a supergroup perspective in which superconformal symmetry is built-in, and we derive its consequences for the superconformal blocks. Instead, a *bottom-up* approach constructs superconformal blocks starting from an ansatz made of a sum of products of conformal and internal blocks, and afterwards imposes the constraints of superconformal symmetry, i.e. the superconformal Ward identity and the Casimir equation [44–51].

The top-down and the bottom-up approaches should obviously give the same final result. However, the two paths are quite different, and the crucial point is that decomposing a superconformal blocks, say for example on a basis of conformal and internal blocks, is quite hard. In fact, we will now show that all the nice properties about stability and the Jack→Block matrix, explained in the previous sections, become hidden in the details.

From the top-down approach we are able to provide an implicit formula for decomposing a (m, n) superconformal in subgroups. Mathematically, this formula is the equivalent of decomposing a super Jack polynomial $(m+m', n+n')$ into sum of products of two super Jack polynomials for (m, n) and (m', n'). This decomposition is achieved by using the structure constants for super Jack polynomials $(\mathcal{S})^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}$,

$$J^{(m+m',n+n')}_{\underline{\lambda}} = \sum_{\underline{\mu},\underline{\nu}} (\mathcal{S})^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} J^{(m,n)}_{\underline{\mu}} J^{(m',n')}_{\underline{\nu}} \tag{2.36}$$

which by stability are the same as those for bosonic Jack polynomials. For $\theta = 1$ these are the Littlewood-Richardson coefficients. Combining (2.36) with the expansion of superconformal blocks in super Jack polynomials we arrive at

$$F^{(m+m',n+n')}_{\gamma,\underline{\lambda}} = \sum_{\underline{\mu},\underline{\nu}} F^{(m,n)}_{\gamma,\underline{\mu}} \, (\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} \, F^{(m',n')}_{\gamma,\underline{\nu}} \ , \tag{2.37}$$

where the block structure constants $\mathcal{S}_\gamma$ depend on θ, p_{12}, p_{43} and are related to the Jack structure constants via the matrices T_γ and T_γ^{-1},

$$(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} = \sum_{\substack{\underline{\tilde\lambda}\supseteq\underline{\lambda} \\ \underline{\tilde\mu}\subseteq\underline{\mu} \\ \underline{\tilde\nu}\subseteq\underline{\nu}}} \mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}} \, (T_\gamma)^{\underline{\tilde\lambda}}_{\underline{\lambda}} (T_\gamma^{-1})^{\underline{\mu}}_{\underline{\tilde\mu}} (T_\gamma^{-1})^{\underline{\nu}}_{\underline{\tilde\nu}} \ . \tag{2.38}$$

Let us point out that [52–54] obtained a recursive formula for the Jack structure constants $\mathcal{S}$, but we are not aware of a more explicit formula, and to the best of our knowledge there is no closed formula expression for it, and therefore for $\mathcal{S}_\gamma$. The bottom-up construction of superblocks in terms of conformal and internal blocks mentioned above is a special case of the more general decomposition (2.37) when $m' = n = 0$, and the ignorance about $\mathcal{S}$ is what makes it complicated.

Not only is $\mathcal{S}$ not known explicitly, but a very non-obvious fact about (2.38) is the following: the Jack→Block matrix and its inverse are infinite size triangular matrices, however $(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}$ should truncate, because, for example, superconformal group theory for specific values of $\theta = \frac{1}{2}, 1, 2$ tells us that (2.37) is a finite sum. Consider again the case of conformal block $\times$ internal block decomposition, $m' = n = 0$. Whilst a truncation in the number of rows of $\underline{\nu}$ is expected, since $F_{\gamma,\underline{\lambda}}$ is polynomial in the $\mathbf{y}$, the truncation over finitely many conformal block $F_{\gamma,\underline{\mu}}$ is not at all manifest.

We conjecture that the subgroup decomposition in (2.37) is a finite sum. In section 9 we discuss in more detail various aspects of this conjecture, relating it to a more specific conjecture for a general decomposition and providing a number of examples. We have checked the conjecture with computer algebra, in many cases, and find that it holds for generic θ, even beyond the cases $\theta = \frac{1}{2}, 1, 2$ related to superconformal groups. Perhaps the connection with the binomial coefficient will provide a simple mechanism to prove it in the future.

That concludes our extended outline of the paper. The methods employed throughout the paper strongly rely on the use of Young diagrams and symmetric polynomials. To help the reader familiarise themself with the relevant mathematics, we have summarised in appendix C a number of results on the theory of symmetric polynomials.

3 Supergroups, physical theories, and beyond

In this section we detail the theories and the external operators for which we are computing superconformal blocks, as a function of $(m, n; \theta)$. We also explain the field theory interpretation of the labels assigned to the superconformal block $B_{\gamma,\underline{\lambda}}$, namely γ and $\underline{\lambda}$.

A unified description of all relevant theories can be achieved by using the formalism based on harmonic/analytic superspace, and building on previous literature. In particular, 4d superconformal theories with $\theta = 1$ discussed in [1–3, 6–10, 55, 56], 6d superconformal field theories with $\theta = 2$ discussed in [5, 8, 57], and 3d superconformal field theories with $\theta = \frac{1}{2}$ discussed in some detail in [4, 57]. This is explained in section 3.2, with supplementary material given in appendix A.

Let us emphasise that we will be able to construct $B_{\gamma,\underline{\lambda}}$ for any $(m, n; \theta)$ (for m, n positive integer), but only *some* values of $(m, n; \theta)$ appear to have a group theoretic meaning (and even fewer will correspond to a physical CFT). The spacetime dimension d and the parameter θ are identified as $\theta = \frac{d-2}{2}$, apart for special cases, that we will explicitly mention. A useful summary of the different cases is given here below.

3.1 List of theories and their superconformal blocks

The main examples to have in mind are three complete families of supergroups with $\theta = 1, 2, \frac{1}{2}$ and arbitrary m, n. These include the cases of four-point functions of half-BPS operators in 4d,6d,3d superconformal field theories respectively:

- $SU(m,m|2n)$: (m,n) arbitrary and $\theta=1$. For $(m,n)=(2,1)$ and $(2,2)$ we compute $\mathcal{N}=2,4$ superblocks, respectively, in four-dimensions. For $(m,n)=(1,1)$ we compute $\mathcal{N}=4$ superblocks in one-dimension.

- $OSp(4m^*|2n)$: (m,n) arbitrary and $\theta=2$. For $m=2$ we compute $\mathcal{N}=(n,0)$ superblocks in six-dimensions.

- $OSp(4n|2m)$: (m,n) arbitrary and $\theta=\frac{1}{2}$. For $m=2$ we compute $\mathcal{N}=4n$ superblocks in three-dimensions.

Alternatively, fixing $(m,n)=(2,0)$ or $(0,2)$ with arbitrary (half-integer) θ gives blocks in purely bosonic theories

- $SO(2,2\theta+2)$: $(m,n)=(2,0)$ with arbitrary θ: We compute standard conformal blocks for scalar correlators in dimensions $d=2\theta+2$.

- $SO(2/\theta+4)$: $(m,n)=(0,2)$ with arbitrary θ: We compute "compact blocks", i.e. blocks where the external operators are finite dimensional representations. These are the dual Jacobi polynomials.

Note the apparent duality

$$m\leftrightarrow n \qquad ; \qquad \theta\leftrightarrow\frac{1}{\theta}. \tag{3.1}$$

It means that whenever there is a group theory interpretation, there should be a corresponding interpretation for the dual case, although this exchanges non-compact groups with compact groups.

There are other theories, in addition to the ones listed above, which we would like to point out. We have not spelled out the details and therefore the discussion here will be in a sense speculative. Nevertheless we will provide evidence that these other cases appear as special cases in our formalism:[13]

- $m=1,n=1$, and arbitrary θ. There is a one-parameter family of one-dimensional $\mathcal{N}=4$ superconformal groups called $D(2,1;\alpha)$ [58]. For special values of α, $D(2,1;\alpha)$ is isomorphic (at the level of the algebra at least) to the following
  - $OSp(4^*|2)$, for $\alpha=-2,1$
  - $SU(1,1|2)\times SU(2)$, for $\alpha=-1,0,$
  - $OSp(4|2)$ for $\alpha=-\frac{1}{2}$

  Noting the relation with the supergroups discussed above for general m,n, we relate $\theta=-\alpha$ (or $\theta=\alpha+1$) and speculate that for $m=1,n=1$, arbitrary θ we obtain superblocks for $D(2,1;\alpha)$ superconformal theories.

[13] We thank Tadashi Okazaki for pointing out $D(2,1;\alpha)$ in the list.

- $m = 1, n = 2$, for various θ. There are four different possibilities for a one-dimensional $\mathcal{N} = 8$ CFT [58]:

$$OSp(8|2),\ SU(1,1|4),\ Osp(4^*|4) \qquad \text{and} \qquad F(4)\,. \tag{3.2}$$

  The first three correspond to supergroup theories discussed above for $\theta = \frac{1}{2}, 1, 2$ and $m = 1, n = 2$. To understand $F(4)$ note first that $F(4)$ has R-symmetry $SO(7)$. The blocks for this group belong to the family of $SO(2/\theta + 4)$ bosonic blocks mentioned above, and would correspond to $m = 0, n = 2, \theta = \frac{2}{3}$. Thus we conjecture that the $F(4)$ supergroup corresponds to $m = 1, n = 2, \theta = \frac{2}{3}$.

- Five-dimensional blocks. There is also an $F(4)$ superconformal group in five dimensions, and since the 5d bosonic blocks correspond to $m = 2, n = 0$ and $\theta = \frac{3}{2}$, we conjecture that the 5d superblocks correspond to $m = 2, n = 1$ and $\theta = \frac{3}{2}$. Note that, nicely, upon considering the duality $m \leftrightarrow n$ and $\theta \leftrightarrow \frac{1}{\theta}$, we obtain the other $F(4)$ group in (3.2).

- Two-dimensional blocks. In this case holomorphic/anti-holomorphic factorisation suggests that superconformal blocks are obtained from a basis constructed by first taking products of one-dimensional blocks, i.e. the ones that appeared at the previous points, and then by rearranging the product basis according to the relevant representations. An example of this has been worked out in [59].

3.2 Coset space formalism

In the previous section we introduced the groups corresponding to superconformal blocks with various values of $(m, n; \theta)$. In order to make our discussion self-contained we will now specify the space on which the superconformal reps lie, both the external and the exchanged states in the four-point function.

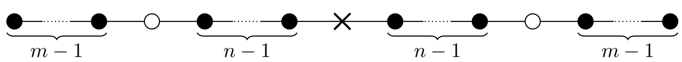
The external states we consider are scalar (super)fields (line bundles) on a certain coset space of the group. For example, it is well known that complexified Minkowski space in $d = 2\theta + 2$ dimensions, M_d, can be viewed as a coset of the complexified conformal group $SO(d + 2; \mathbb{C})$ divided by the subgroup consisting of Lorentz transformations, dilatations and special conformal transformations. This corresponds to the case $(m, n) = (2, 0)$ with arbitrary half integer θ. More generally, the coset space for arbitrary $(m, n; \theta)$ will always be a special type of (super) flag manifold, which can be specified by a (super) Dynkin diagram with a single marked node, also known as generalised (super) Grassmannian spaces.

A beautiful classification and description of flag manifolds, together with representations on them, can be found in [60]. A flag manifold and its field content, i.e. the irreducible representations of the group, can be read off from a marked Dynkin diagram and its associated Dynkin labels [60]. From a root system and a Dynkin diagram one can construct the corresponding Lie algebra. Similarly, from a root system and a *marked* Dynkin diagram one constructs a parabolic subalgebra by simply omitting the positive simple roots corresponding to the marked nodes and retaining all the negative roots. The coset space is then the group G divided by this parabolic subgroup H. Irreducible representations are defined by Dynkin labels giving the transformation properties of the highest weight state.

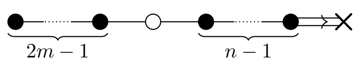
The generalisation of Dynkin diagrams to the supersymmetric case is well known (we recommend [61] for a nice introduction) and the generalisation of the flag manifold techniques of [60] to supergroups proceeds fairly straightforwardly and has been considered in the $\theta = 1, 2$ cases explicitly in [6–8, 56]. We should point out that, compared to the bosonic cases, when dealing with supergroups there is no longer a unique (super) Dynkin diagram.[14] A super Dynkin diagram will have even (black) and odd (white) nodes, and then the coset space of this superspace will be indicated by marked nodes (represented by crosses). Remarkably all unitary superconformal representations of the superconformal group are obtained as *unconstrained* analytic superfields on the the coset superspaces we use here. This is proven for 4d superconformal theories $\theta = 1, m = 2$ in [7] and is conjectured in other supersymmetric cases.[15]

Various details about the coset construction for the specific cases listed in section 3.1 are given in appendix A. We repeat the main points here.

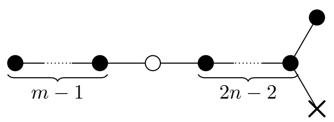
In the following will consider the *complexified*, e.g. $SU(m, m|2n) \xrightarrow{\mathbb{C}} SL(2m|2n; \mathbb{C})$, of the superconformal groups. One can return to the real case by choosing the appropriate real coordinates at the end. The marked super Dynkin diagrams specifying the relevant coset space for the three families $\theta = 1, 2, \frac{1}{2}$ are[16]



$SL(2m|2n)$



$OSp(4m|2n)$ (3.3)



$OSp(4n|2m)$

We consider a matrix representation for the (complexified) (super)group such that all positive roots correspond to upper triangular matrices. Crucially, in all cases the basis can be chosen

[14] An easy way to understand the existence of inequivalent Dynkin diagrams for the case of $SL(M|N)$ is to notice that a change of basis (in particular swapping even and odd basis elements) can not always be achieved via an $SL(M|N)$ transformation (unlike in the bosonic case). For each inequivalent change of basis there is a different Dynkin diagram. In the case of interest here, for $\theta = 1$ the standard basis for the complexified superconformal group $SL(2m|2n)$ would give a Dynkin diagram with $2m - 1$ even nodes then one odd node then $2n - 1$ even nodes. If instead we choose the basis $(m|2n|m)$ we end up with the diagram (3.3a) involving two odd nodes.

[15] These results suggest the existence of a supersymmetric generalisation of the Bott-Borel-Weil theorem, which would be interesting to explore further.

[16] Other cases will be discussed in appendix A.

such that (super)coset space $H\backslash G$ has the following block 2×2 structure

$$G \;=\; \left\{ \begin{pmatrix} \hat{a}^A{}_B & \hat{b}^{AB'} \\ \hat{c}_{A'B} & \hat{d}_{A'}^{\;B'} \end{pmatrix} \right\} \qquad\qquad H \;=\; \left\{ \begin{pmatrix} a^A{}_B & 0 \\ c_{A'B} & d_{A'}^{\;B'} \end{pmatrix} \right\} . \tag{3.4}$$

where $\hat{a},\hat{b},\hat{c},\hat{d},a,c,d$ are square (super-)matrices of equal dimensions (in general they will have some constraints). For $\theta=1$ they are $(m|n)\times(m|n)$ matrices; for $\theta=2$ they are $(2m|n)\times(2m|n)$ matrices; for $\theta=\frac{1}{2}$ they are $(m|2n)\times(m|2n)$ matrices.[17] For example, the special case $(m,n;\theta)=(2,0;1)$ corresponds to 4d Minkowski space viewed as the coset space of the complexified conformal group $SL(4)$ modded out by dilatations, Lorentz and special conformal transformations. Here the 2×2 matrices a,b give Lorentz and dilatations whereas c represents special conformal transformations.

Fields living on the coset space transform non-trivially under the block diagonal part of the parabolic subgroup H (known as the Levi subgroup, L). This subgroup consists of the matrices a,d (e.g. in the Minkowski space example these correspond to Lorentz and dilatations under which conformal operators transform). The Levi subgroup is read off from the same Dynkin diagrams (3.3) upon deleting the marked node (and replacing it with $\mathbb{C}^*$). In the above cases they are as follows,

θ	1	2	$\frac{1}{2}$
G	$SL(2m\|2n)$	$Osp(4m\|2n)$	$Osp(4n\|2m)$
L	$SL(m\|n)\otimes SL(m\|n)\otimes\mathbb{C}^*$	$SL(2m\|n)\otimes\mathbb{C}^*$	$SL(m\|2n)\otimes\mathbb{C}^*$

(3.5)

There is an overall constraint to take into account: for $h\in H$, $\mathrm{sdet}(h)=\mathrm{sdet}(a)\,\mathrm{sdet}(d)=1$. The $\mathbb{C}^*$ subgroup of the parabolic group is then identified with $\mathrm{sdet}(a)=1/(\mathrm{sdet}(d))$.

The coset space $H\backslash G$ is finally the collection of all the orbits under the equivalence $g\sim hg$ for $g\in G$ and $h\in H$. A representative for each orbit is[18]

$$g\sim s(X) \;=\; \begin{pmatrix} 1 & X^{AA'} \\ 0 & 1 \end{pmatrix} . \tag{3.6}$$

[17] More precisely, the second half of the basis is reversed compared to the first half. Taking $\theta=1$ as the illustrative example, then a is $(m|n)\times(m|n)$, X is $(m|n)\times(n|m)$, c is $(n|m)\times(m|n)$ and d is $(n|m)\times(n|m)$. However when referring to a,c,d,X we will consider them with the bases rearranged, so they all have the same dimensions $(m|n)\times(m|n)$.

[18] A toy model for the coset construction, corresponding to $(m,n;\theta)=(0,1;1)$ is just the Riemann sphere, realised by taking $g=\left(\begin{smallmatrix}\hat{a}&\hat{b}\\ \hat{c}&\hat{d}\end{smallmatrix}\right)$ in $SL(2,\mathbb{C})$ and $h=\left(\begin{smallmatrix}a&0\\ c&d\end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix}1&x\\ 0&1\end{smallmatrix}\right)$ is the coset representative and $\left(\begin{smallmatrix}1&x\\ 0&1\end{smallmatrix}\right)g=h\left(\begin{smallmatrix}1&f(x)\\ 0&1\end{smallmatrix}\right)\sim\left(\begin{smallmatrix}1&f(x)\\ 0&1\end{smallmatrix}\right)$ where $f(x)=(\hat{d}x+\hat{b})/(\hat{c}x+\hat{a})$ and $a=\hat{a}+x\hat{c}, c=\hat{c}, d=\hat{d}-\hat{c}f(x)$. From f we recognise Möbius transformations acting on the Riemann sphere as the coset $H\backslash SL(2,\mathbb{C})$. By taking just the top row of the coset representative, $(1,x)$, we recognise this construction to be completely equivalent to the projective space P^1. Similarly, the general $(m,n;\theta)$ construction is equivalent to a Grassmannian space.

This supermatrix X are then coordinates for the coset space, which we write as

$$X = \left(\begin{array}{c|c} x & \rho \\ \hline \bar{\rho} & y \end{array} \right) . \tag{3.7}$$

For $\theta = 1$, x is $m \times m$, y is $n \times n$ and both are bosonic, whereas ρ is $m \times n$, $\bar{\rho}$ is $n \times m$ and they are fermionic. The other cases are similar except $m \to 2m$ when $\theta = 2$ and $n \to 2n$ when $\theta = \frac{1}{2}$ (and the supermatrix is generalised (anti-)symmetric in its indices, see appendix A for more details). For $\theta = 1$ the cosets are equivalent to super Grassmannians which can be seen by considering the upper half of the group matrix $(1, X) \sim a(1, X)$, which is indeed $Gr(m|n, 2m|2n)$, the space of $m|n$ planes in $2m|2n$ dimensions. Then the $\theta = 2, \frac{1}{2}$ cases are generalised Grassmannians. For $\theta = 1, 2, \frac{1}{2}$ note that $x^{\alpha\dot{\alpha}}$ represents coordinates for Minkowski space in dimensions $M_{d=4,6,3}$ respectively written in a (Weyl) spinor notation.

Now we can connect with the superblocks. The four-point superconformal invariant combinations of four coordinates X_1, X_2, X_3, X_2 can be viewed as the independent eigenvalues of the cross-ratio matrix:

$$Z = X_{12} X_{23}^{-1} X_{34} X_{41}^{-1} \qquad ; \qquad X_{ij} = X_i - X_j \; . \tag{3.8}$$

These independent eigenvalues then correspond to the arguments of the blocks, $\mathbf{z}$ (2.7). Further the building block two-point function g_{ij} in (2.3) is

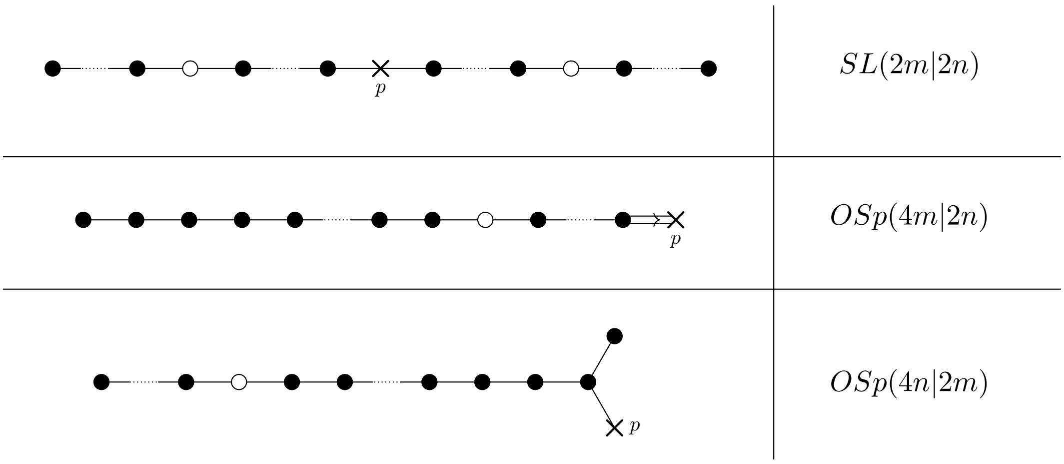
$$g_{ij} = \text{sdet}^{-\#}(X_i - X_j). \tag{3.9}$$

where *sdet* is the superdeterminant (Berezinian) and the exponent # is discussed in footnote 4.

The external representations appearing in the four-point function (2.5) are scalars, $\mathcal{O}_p(X)$, on this coset space, thus they transform non-trivially, with a certain weight p under $\mathbb{C}^*$ and are invariant under the rest of the parabolic group (3.5). In terms of the Dynkin labels, the label above a marked node gives the weight under the $\mathbb{C}^*$ subgroup of the parabolic subgroup associated with that node. The Dynkin labels next to the unmarked nodes give the representation under non-trivial subgroups of the parabolic subgroup.[19] The external

[19] Although we will be dealing with infinite dimensional representations, the representation of the parabolic subgroup itself will always be finite dimensional.

operators $\mathcal{O}_p$ are therefore written as,

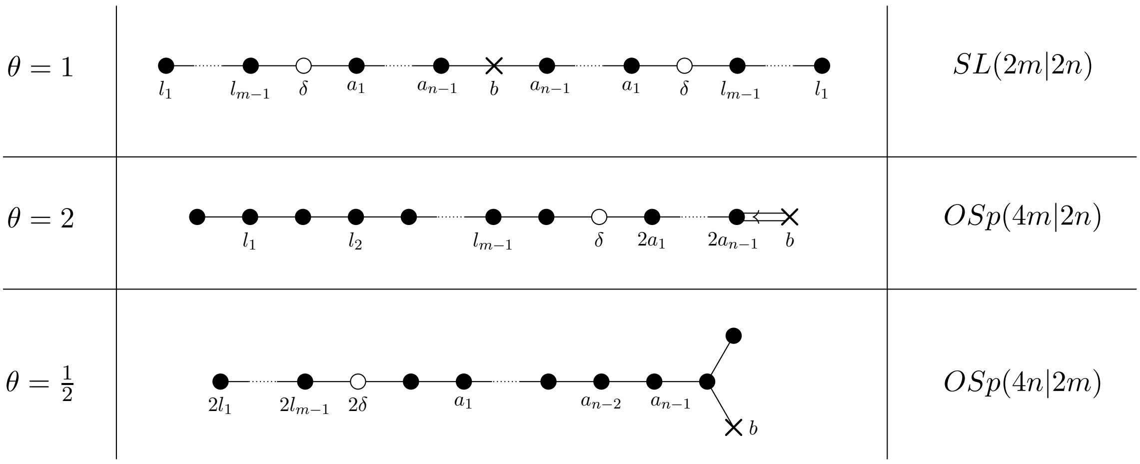


$$(3.10)$$

where all unlabelled nodes are understood to have the label 0. These fields transform under a general superconformal transformation g as follows:

$$s(X)\,g = h\,s(X') \qquad\qquad \mathcal{O}_p(X) \to \mathcal{O}'_p(X') = \mathrm{sdet}(a)^p \mathcal{O}_p(X)\ . \tag{3.11}$$

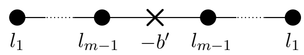
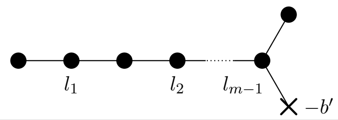
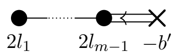
The superblock gives the contribution of an exchanged representation appearing in the common OPE (or tensor product) between the first two, $\mathcal{O}_1\mathcal{O}_2$, and last two, $\mathcal{O}_3\mathcal{O}_4$, external operators (see (2.2)). The representations which can appear in the OPE are summarised by the labelled Dynkin diagrams given below, for $n > 0$,



$$(3.12)$$

The Dynkin labels $l_{i=1,\ldots,m-1}$ and $a_{j=1,\ldots n-1}$, are all positive integers, and denote generalised spins of the conformal and compact subgroups respectively, of the superconformal groups. Note that for $OSp(4m|2n)$, the Dynkin labels for the $SO(2m)$ conformal subgroup are alternately vanishing. Similarly, for $OSp(4n|2m)$ the internal spins are alternately vanishing. All of the above representations are specified by the same set of labels l_i, δ, a_j, b even though they are describing representations of different groups.

The special cases of $m = 1, 0$ and $n = 1, 0$ require a small discussion. The Dynkin diagrams above indeed include the special cases: $m = 1$, which corresponds to having external nodes with l_i absent; $m = 0$, which corresponds to having external and odd nodes absent; $n = 1$, which corresponds to having internal uncrossed nodes labelled by a_j absent. However, the case $n = 0$, which corresponds to a bosonic scalar conformal field theory, can not be read off directly from the above diagrams, even though it follows directly from the same matrix construction. The relevant Dynkin diagrams for $n = 0$ is instead

$\theta = 1$		$SL(2m)$
$\theta = 2$		$OSp(4m)$
$\theta = \frac{1}{2}$		$Sp(2m)$

(3.13)

From all of the above Dynkin digrams we can immediately read off the corresponding field in the appropriate coset space. The weight under the $\mathbb{C}^*$ is the label on the marked node, and the non trivial transformation under the Levi subgroup, i.e. under the supermatrices a, d in (3.4) is that dictated by the rest of the Dynkin diagram. All of these reps can equivalently be given by a parameter γ and a Young diagram $\underline{\lambda}$ consistent with an $SL(m|n)$ rep. We will give the precise equations relating $\gamma, \underline{\lambda}$ to the above Dynkin labels shortly, but first we explain how the Young diagram arises as it slightly different in the three cases $\theta = 1, 2, \frac{1}{2}$.

First note that the representations appearing in the OPE of two scalars, $\mathcal{O}_{p_1}\mathcal{O}_{p_2}$, are not the most general possible (indeed these would have arbitrary Dynkin labels above every node in the diagrams above). Rather the representation of a must be identical to that of d. This statement can be understood most directly in the free theory, where the operators appearing in the OPE have the form

$$\mathcal{O}_{\underline{A}\underline{A'}} = \mathcal{O}_{p_1-w}(\partial^k)_{\underline{A}\underline{A'}}\mathcal{O}_{p_2-w}+.. \qquad ; \qquad (\partial^k)_{\underline{A}\underline{A'}} := (\delta_R)_{\underline{A}}^{A_1..A_k}(\delta_R)_{\underline{A'}}^{A'_1..A'_k}\prod_{i=1}^{k}\frac{\partial}{\partial X^{A_iA'_i}} \quad (3.14)$$

with $\underline{A}$ and $\underline{A'}$ multi-indices and A_i, A'_i (anti-)symmetrised into irreps R via the invariant tensors δ_R. Since partial derivatives $\partial_{AA'}$ commute with each other, the representation of $\underline{A}$ and $\underline{A'}$ must be the same as each other for this to be non-vanishing. Here the A indices carry the rep of a and A' indices carry the rep of d so we see these must be the same. Now for $\theta = 1$, a and d are independent and form the two $SL(m|n)$ subgroups of L (see (3.12)). Associating this common representation with its Young diagram, we define $\underline{\lambda}$. For $\theta = 2, \frac{1}{2}$, the matrices a and d are essentially equal to each other (see appendix A.2) and this means that the representation of $SL(2m|n)$ and $SL(m|2n)$ has a particular special form. For $\theta = 2$, the Young diagram of the $SL(2m|n)$ rep, $[r_1, \ldots]$, will have duplicated rows so that $r_{2i-1} = r_{2i}$.

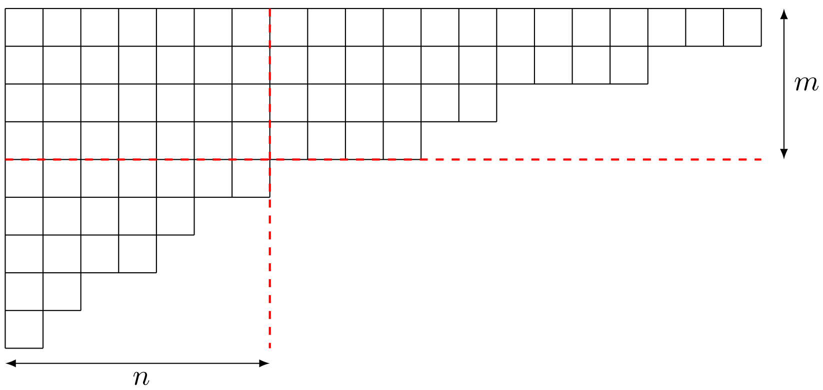
Thus, the only relevant information is given in a Young diagram $\underline{\lambda}$ defined by $\lambda_i = r_{2i}$, i.e. half the height, and this $\underline{\lambda}$ will label the blocks. Furthermore since the Young diagram $[r_1, \ldots]$ is consistent with $SL(2m|n)$ the resulting Young diagram $\underline{\lambda}$ will be consistent with $SL(m|n)$. For $\theta = \frac{1}{2}$ on the other hand, the resulting $SL(m|2n)$ Young diagram will always have duplicated columns so $r'_{2j-1} = r'_{2j}$. Thus,we will label the blocks with the Young diagram $\underline{\lambda}$ defined to have column lengths $\lambda'_j = r'_{2j}$.

Summarising: In all cases with a group theory interpretation we have specified a representation exchanged in the OPE of two scalars, through a Young diagram, $\underline{\lambda}$. Thus $\underline{\lambda}$, together with a parameter γ, is the data we will use to specify the superconformal blocks $B_{\gamma,\underline{\lambda}}$. It will turn out that this same data is very natural from the point of view of $BC_{m|n}$ functions!

A closing remark regarding the non-supersymmetric degeneration for either $n = 0$ or $m = 0$, which we discuss in appendix A. This has a group theory interpretation for any half integer (or inverse of a half integer) θ, and it is interesting to mention that the group theoretic interpretation we give here meshes closely with a previously understood interpretation of the BC_n Heckman Opdam hypergeometric functions as spherical functions on certain coset spaces [62]. The latter are Grassmannians in $SU(p,q), SO(p,q)$ and $Sp(p,q)$, for $\theta = 1, 2, \frac{1}{2}$ respectively. Upon setting $p = q = m$ (or n) they look very reminiscent of the ones that we are using and it would be interesting to pursue this connection further.

Young diagrams vs representation labels

A Young diagram $\underline{\lambda}$ with a $GL(m|n)$ structure has a hook shape in which the row lengths $\lambda_{i=1,2,\ldots}$ are such that $\lambda_{m+1} \leq n$. Equivalently, the column heights $\lambda'_{j=1,2,\ldots}$ are such that $\lambda'_{n+1} \leq m$. Graphically, the diagram fits inside the red dashed lines:[20]



(3.15)

We shall say that a box $\square \in \underline{\lambda}$ has integer coordinates (i, j), where i corresponds to the row index and j to the column index. For example, the rightmost boxes of $\underline{\lambda}$ from top to bottom have coordinates (i, λ_i) for $i = 1, 2, \ldots$. Equivalently, from bottom to top they have coordinates (λ'_j, j) for $j = 1, 2, \ldots$.

Young diagrams $\underline{\lambda}$ with a hook shape might or might not contain the rectangle n^m. In the supersymmetric case, a Young diagram $\underline{\lambda}$ which contains the box with coordinates (m, n), and therefore contains the full rectangle n^m, will correspond to a typical or long

[20] In this specific drawing $m = 4$, $n = 7$, then $\lambda_1 = 20$ and $\lambda_9 = 1$, while $\lambda'_1 = 9$ and $\lambda'_{20} = 1$.

representation. Otherwise the representation is atypical. We will discuss concrete examples related to physical theories later on (see discussion around (C.40) and (C.41) for examples and a fuller description).

The Dynkin labels given in (3.12) (and (3.13) for the case $n = 0$) translate to the data specifying a Young diagram $\underline{\lambda}$, and a parameter γ, for $\theta = 1, 2, \frac{1}{2}$ as follows

$$\begin{array}{llllll} l_{m-i} & = (\lambda_i - n)^+ - (\lambda_{i+1} - n)^+ & i = 1, \ldots, m-1 & ; & \delta = (\lambda_1 - n)^+ + \theta \lambda'_n \\ a_{n-i} & = (\lambda'_i - \lambda'_{i+1}) & i = 1, \ldots, n-1 & ; & b = \gamma - 2\lambda'_1 \\ b' & = \gamma + \frac{2}{\theta}\lambda_1 & & & \end{array} \tag{3.16}$$

where $(x)^+ = \max(x, 0)$. In physical applications it is useful to read off the dilation weight, especially as this is the only quantum number which can become non integer in an interacting CFT. The following equality gives the dilation weight in terms of the Dynkin labels, and then $\gamma, \underline{\lambda}$

$$\Delta = -\sum_{i=1}^{m-1} i l_i + m\left(\delta + \theta \sum_{j=1}^{n-1} a_j + \theta \frac{b}{2}\right) = \sum_{i=1}^{m} (\lambda_i - n)^+ + m\theta \frac{\gamma}{2} \ . \tag{3.17}$$

Note now that for long representations (those for which the box with coordinates (m, n) lies in the Young diagram $\underline{\lambda}$) the dictionary (3.16) is invariant under

$$\begin{array}{ccccc} \lambda_i \to \lambda_i - \theta\tau' & & \lambda'_j \to \lambda'_j + \tau' & & \\ & ; & & ; & \gamma \to \gamma + 2\tau' \ . \\ i = 1, \ldots, m & & j = 1, \ldots, n & & \end{array} \tag{3.18}$$

We will refer to this redundancy as the *shift invariance*.

The shift invariance (3.18) arises from the fact that $SL(m|n)$ reps are not uniquely specified by a Young diagram. This is very familiar in the bosonic case $n = 0$. In fact, for $SL(m)$ reps one can add arbitrarily many height m columns to a Young diagram without changing the rep. This leads directly to (3.18) for $n = 0$. In the same way, $SL(n)$ reps have a shift invariance, and therefore when $m = 0$ one can add arbitrarily many length n rows (since the Young diagram in $SL(0|n)$ is transposed w.r.t. $SL(n)$). For the supergroup $SL(m|n)$ what happens is that if there is a full width n row below the n^m rectangle in the Young diagram, this can be removed and replaced by a full height m column to the right of the rectangle without changing the representation. The presence of θ in the shift (3.18) reflects the fact that this shift applies to the reps of $SL(2m|n)$ or $SL(m|2n)$ with double numbers of rows or columns around (3.14). The implications of this shift for $\underline{\lambda}$ then give the θ dependence in (3.18).

Let us exemplify the relevant Young diagrams in some cases of physical interest. Blocks for maximally supersymmetric theories

$$\left.\begin{array}{c} \mathcal{N} = 4, 4d \text{ for } \theta = 1 \\ \mathcal{N} = (2,0), 6d \text{ for } \theta = 2 \\ \mathcal{N} = 8, 3d \text{ for } \theta = \frac{1}{2} \end{array}\right\} \quad \text{all correspond to} \quad m = 2, n = 2 \tag{3.19}$$

These blocks are labelled by γ and a Young diagram with at most two rows of length λ_1, λ_2

and two columns of length λ'_1, λ'_2. There are two quantum numbers for the conformal group, dilation weight $\Delta = (\lambda_1-2)^+ + (\lambda_2-2)^+ + \theta\gamma$ and spin $l_1 = (\lambda_1-2)^+ - (\lambda_2-2)^+$ (only symmetrised Lorentz indices appear in these four-point functions) and two analogous quantum numbers for the internal subgroup a_1, b.[21] When $\lambda_2 \geq 2$ the representation exchanged in the OPE (assuming integer dilation weight) is long or typical . If $\lambda_2 = 1, 0$ the diagram is a thin hook, with at most a single row and a single column, and the representation is short. When the Young diagram is empty it is half BPS. We emphasise that in our formalism there is no need to distinguish the different shortening conditions!

Another case of physical interest is given by blocks with $m = 2, n = 1$, which includes $\mathcal{N} = 2$ superconformal theories in 4d, the $\mathcal{N} = (0, 1)$ superconformal theories in 6d, and also $\mathcal{N} = 4$ in 3d. In this case the Young diagram has at most a single long column λ'_1 and therefore a single integer classifies the internal representations that can appear in the OPE. The reps are often defined by Δ, l and b. This time the atypical diagrams can only be the single row diagrams, since these do not contain the rectangle $[1^2]$. The rest of the allowed diagrams, $[\lambda_1, \lambda_2, 1^{\lambda'_1-2}]$, are necessarily long/typical.

4 Rank-one invitation

Before presenting the most general (m, n) superconformal block, we describe in some detail the bosonic $(1, 0)$ and $(0, 1)$ blocks, since these are simple enough to familiarise with the formalism based on super Jack polynomials.

4.1 Rank-one bosonic blocks

$(1, 0)$ blocks

Consider first the $(1, 0)$ blocks, i.e. a single x variable and no y variables. The only allowed Young diagrams have a single row shape $\underline{\lambda} = [\lambda]$. The Casimir (5.5) reduces to a one-variable Gauss hypergeometric equation, from which follows the solution (normalised according to (2.13)),

$$\begin{aligned} \mathbf{C} B_{\gamma,[\lambda]}(x|) &= (\lambda+\tfrac{\theta\gamma}{2})(\lambda+\tfrac{\theta\gamma}{2}-1) B_{\gamma,\underline{\lambda}}(x|) \\ B_{\gamma,[\lambda]}(x|) &= x^{\frac{\theta\gamma}{2}+\lambda}\; {}_2F_1\big(\lambda+\theta\alpha, \lambda+\theta\beta; 2\lambda+\theta\gamma; x\big) \end{aligned} \tag{4.1}$$

where recall $\alpha = \max\big(\tfrac{1}{2}(\gamma-p_{12}), \tfrac{1}{2}(\gamma-p_{43})\big)$, $\beta = \min\big(\tfrac{1}{2}(\gamma-p_{12}), \tfrac{1}{2}(\gamma-p_{43})\big)$ (2.16). The Casimir and the solution is symmetric in α and β. Furthermore, let us point out that out of the five parameters $\lambda, \gamma, \theta, p_{12}, p_{43}$ only the three quantities $\lambda + \frac{\theta\gamma}{2}$, $p_{12}\theta$ and $p_{43}\theta$ appear.[22]

Writing the $(1, 0)$ block as a series over superJack polynomials is very simple since the $(1, 0)$ Jack is simply $P_{[\lambda]}(x|) = x^\lambda$, therefore the Jack→Block matrix is just the coefficient in

[21] It might be instructive to compare with [45], which also treats all maximally supersymmetric cases together. They use internal labels $(a_{\texttt{there}}, b_{\texttt{there}})$ where $a_{there} = \frac{1}{2}\gamma-\lambda'_2$, $\quad b_{there} = \frac{1}{2}\gamma-\lambda'_1$.

[22] The combination $\lambda + \frac{\theta\gamma}{2}$ appears in the Dynkin diagram (A.14) for the $(1, 0)$ bosonic theory.

the Taylor expansion of the Gauss hypergeometric:

$$B_{\gamma,[\lambda]}(x|) = x^{\theta\frac{\gamma}{2}} \sum_{\mu=\lambda}^{\infty} (T_\gamma)_{[\lambda]}^{[\mu]} P_{[\mu]}(x|) \ , \qquad (T_\gamma)_{[\lambda]}^{[\mu]} = \frac{(\lambda+\theta\alpha)_{\mu-\lambda}(\lambda+\theta\beta)_{\mu-\lambda}}{(\mu-\lambda)!(2\lambda+\theta\gamma)_{\mu-\lambda}} \ . \tag{4.2}$$

$(0,1)$ **blocks**

Consider now the $(0,1)$ blocks, i.e. no x variables and a single y variable. Here the only allowed Young diagrams have a single column shape $\underline{\lambda} = [1^{\lambda'}]$. The Casimir reduces again to a hypergeometric equation, and is solved by:

$$\begin{aligned} \mathbf{C} B_{\gamma,[1^{\lambda'}]}(|y) &= -\theta(\lambda' - \tfrac{\gamma}{2})(\lambda' - \tfrac{\gamma}{2} - 1) B_{\gamma,[1^{\lambda'}]}(|y) \\ B_{\gamma,[1^{\lambda'}]}(|y) &= \tfrac{(-1)^{\lambda'} (\frac{1}{\theta})_{\lambda'}}{\lambda'!}\, y^{-\frac{\gamma}{2}+\lambda'} {}_2F_1\big(\lambda'-\alpha, \lambda'-\beta; 2\lambda'-\gamma; y\big) \ . \end{aligned} \tag{4.3}$$

Notice that the entries of the ${}_2F_1$ this time are independent of θ. Then, λ' and γ only enter through the combination $\frac{\gamma}{2} - \lambda'$.[23]

The normalisation of $B_{\gamma,[1^{\lambda'}]}(|y)$ has been chosen as for the the $(0,1)$ superJack polynomials (in agreement with (2.13)). In fact, the series expansion in superJack polynomials this time reads

$$B_{\gamma,[1^{\lambda'}]}(|y) = y^{-\frac{\gamma}{2}} \sum_{\mu'=\lambda'}^{\infty} (T_\gamma)_{[1^{\lambda'}]}^{[1^{\mu'}]} P_{[1^{\mu'}]}(|y) \ , \qquad P_{[1^{\mu'}]}(|y) = \frac{(\frac{1}{\theta})_{\mu'}}{\mu'!}(-y)^{\mu'} \tag{4.4}$$

where

$$(T_\gamma)_{[1^{\lambda'}]}^{[1^{\mu'}]} = \frac{(\lambda'-\alpha)_{\mu'-\lambda'}(\lambda'-\beta)_{\mu'-\lambda'}}{(\mu'-\lambda')!(2\lambda'-\gamma)_{\mu'-\lambda'}} \, \frac{\mu'!(\frac{1}{\theta})_{\lambda'}}{\lambda'!(\frac{1}{\theta})_{\mu'}} (-1)^{\mu'-\lambda'}, \tag{4.5}$$

An important observation to make at this point is that this series truncates when the arguments of the Pochhammers are negative. Thus it truncates for values of μ' greater than β.

Properties of T_γ

We can already with this information perform do a very rudimentary check that the coefficients T_γ are (m,n) independent i.e. they only depend on the Young diagrams, θ, and p_{12}, p_{43}. This implies that when the Young diagrams coincide, evaluating (4.2) and (4.4) should give the same result. In the $(1,0)$ case the Young diagram has a single row, and in $(0,1)$ case a single column, thus they can only be compared when the Young diagram is empty or consists of a single box i.e. the cases $(\underline{\lambda}, \underline{\mu}) = ([0],[0]), ([1],[1]), ([0],[1])$. We indeed see that for both $(1,0)$

[23] This combination is the value appearing in (A.15) sitting on crossed through Dynkin node.

and $(0,1)$ we obtain:

$$\begin{aligned} ([0],[0]) \quad &; \quad (T_\gamma)^{[0]}_{[0]} = 1 \\ ([1],[1]) \quad &; \quad (T_\gamma)^{[1]}_{[1]} = 1 \\ ([0],[1]) \quad &; \quad (T_\gamma)^{[1]}_{[0]} = \theta \tfrac{\alpha\beta}{\gamma} \ . \end{aligned} \tag{4.6}$$

The above computation is limited, but nevertheless it is nice to see that (4.2) and (4.4) do indeed give the same results where they overlap. This illustrates the more general (m,n) independence of T_γ which we will consider further in section 5.

4.2 Relation with the Heckman-Opdam hypergeometrics

Despite the fact that both $(1,0)$ and $(0,1)$ blocks are given by the same building block, a Gauss hypergeometric, there is an important difference, which we emphasised already: the $(0,1)$ block is a polynomial, while the $(1,0)$ block is an infinite series. However, since both external and internal blocks are based on the same ${}_2F_1$, comparing the two expressions (4.1) and (4.3), we read off the following formal relation between them:

$$B^{(1,0)}_{\gamma,\,[\lambda]}(y|;\theta,p_{12},p_{43}) = \frac{\lambda!}{(-1)^\lambda(\theta)_\lambda} B^{(0,1)}_{-\theta\gamma,\,[1^\lambda]}(|y;\tfrac{1}{\theta},-\theta p_{12},-\theta p_{43})\,. \tag{4.7}$$

So the $(1,0)$ and $(0,1)$ blocks can be viewed as analytic continuations of each other. In particular, if α,β,γ are assumed to be positive on the $(1,0)$ side, they become negative on the $(0,1)$ side, specifically, $(\alpha,\beta,\gamma) \to -\theta(\alpha,\beta,\gamma)$. This can be also understood from the general relation satisfied by the defining Casimir

$$\mathbf{C}^{(\theta,a,b,c)}(\mathbf{x}|\mathbf{y}) = -\theta\,\mathbf{C}^{(\frac{1}{\theta},-a\theta,-b\theta,-c\theta)}(\mathbf{y}|\mathbf{x}) \ . \tag{4.8}$$

Let us then try and understand this analytic continuation in the parameters and relate it to the Heckman Opdam hypergeometric function.

In section 2.3 we identified the internal $(0,n)$ block with the dual BC_n Jacobi polynomial, up to a normalisation, and we showed how the latter is obtained from the BC_n Jacobi polynomial (in particular see (2.32)). Now the BC_n Jacobi polynomials are also Heckman Opdam (HO) hypergeometric functions for positive weights [26], but the BC_n HO hypergeometric are defined more generally for arbitrary parameters. It is therefore interesting to compare these functions with our $(1,0)$ and $(0,1)$ blocks, and discuss in this context the analytic continuation w.r.t. the external parameters.

Both the BC_1 Jacobi polynomial and the dual Jacobi polynomial can be given explicitly in terms of ${}_2F_1$ Hypergeometrics as follows,[24]

$$J_{[\lambda]}(y;\theta,p^-,p^+) = \frac{(-)^\lambda(1+p_-)_\lambda}{(1+\lambda+p^++p^-)_\lambda}\, {}_2F_1(1+\lambda+p^++p^-,-\lambda;1+p^-;y) \qquad \lambda\in\mathbb{Z}^{\geq 0} \quad (4.9)$$

$$\begin{aligned}\tilde J_{\beta,[\lambda]}(y;\theta,p^-,p^+) &= y^\lambda\, {}_2F_1(\lambda-\beta-p^-,-(\beta-\lambda);2(\lambda-\beta)-p^+-p^-;y) \qquad \beta-\lambda\in\mathbb{Z}^{\geq 0}\\ &= y^\lambda {}_2F_1(\lambda-\alpha,\lambda-\beta;2\lambda-\gamma;y) \end{aligned} \quad (4.10)$$

where $\alpha=\beta+p^-$ and $\gamma=2\beta+p^++p^-$. The use of the Gauss hypergeometric to write the polynomials has the bonus that they give a natural analytic continuation to arbitrary values of the parameters, i.e. away from the polynomial restriction $\lambda\in\mathbb{Z}^+$ for J and $\beta-\lambda\in\mathbb{Z}^+$ for $\tilde J$. The key point here is that even though the Jacobi polynomial and the dual Jacobi polynomial are directly related to each other, the above expressions in terms of Gauss hypergeometrics yield two *inequivalent* analytic continuations of the parameters. The first is the HO hypergeometric, the second is the one relevant for blocks. In both cases the analytic continuation gives a function that is regular at $y=0$ with a branch cut between $y=1$ and $y=\infty$.

Let us explore how this works in more detail. The point is that when $\beta-\lambda\in\mathbb{Z}^+$, the hypergeometrics are polynomial and there is a hypergeometric identity [63]

$$\frac{(-1)^{\lambda-\beta}}{c(\alpha,\beta,\gamma,\lambda)}\, {}_2F_1(1-\lambda+\gamma-\beta,-(\beta-\lambda);1-\beta+\alpha;\tfrac{1}{y}) = y^{\lambda-\beta} {}_2F_1(\lambda-\alpha,-(\beta-\lambda);2\lambda-\gamma;y)\ , \quad (4.11)$$

where

$$c(\alpha,\beta,\gamma,\lambda) = \frac{\Gamma[1+\alpha-\beta]\Gamma\,[1+\gamma-2\lambda]}{\Gamma\,[1+\alpha-\lambda]\,\Gamma\,[1-\beta+\gamma-\lambda]}\ . \quad (4.12)$$

This is precisely the identity between the Jacobi and dual Jacobi polynomials $J_{[\beta-\lambda]}(1/y) = y^{-\beta}\tilde J_{\beta,[\lambda]}(y)$.

But as soon as we continue away from $\beta-\lambda\in\mathbb{Z}^+$, the above identity relaxes to a three term identity [64]:

$$\begin{aligned}{}_2F_1(1-\lambda+\gamma-\beta,\lambda-\beta;1-\beta+\alpha;\tfrac{1}{y}) = \quad &(-1)^{\lambda-\beta}c(\alpha,\beta,\gamma,\lambda)\times y^{\lambda-\beta} {}_2F_1\,(\lambda-\alpha,\lambda-\beta;2\lambda-\gamma;y)\\ &+(-1)^{\tilde\lambda-\tilde\beta}c(\tilde\alpha,\tilde\beta,\tilde\gamma,\tilde\lambda)\times y^{\tilde\lambda-\tilde\beta} {}_2F_1(\tilde\lambda-\tilde\alpha,\tilde\lambda-\tilde\beta;2\tilde\lambda-\tilde\gamma;y)\end{aligned} \quad (4.13)$$

where $\tilde\alpha=\alpha-\gamma, \tilde\beta=\beta-\gamma, \tilde\gamma=-\gamma, \tilde\lambda=1-\lambda$. This is a well know connection formula for the ${}_2F_1$. Notice in particular that $c(\tilde\alpha,\tilde\beta,\tilde\gamma,\tilde\lambda)\to 0$, as $\beta-\lambda\to n\in\mathbb{Z}^+$, therefore we recover the previous identity (4.11).

Now the combination on the RHS of (4.13) is precisely the expression for the BC_1 HO hypergeometric (in $1/y$) given as a specific sum of independent solutions of its defining equation (which is equivalent to the Casimir equation). In general the BC_n HO hypergeometric

[24] These can be obtained by solving their defining differential equations. In the dual case this is (2.26). For the Jacobi polynomial we refer to [28]. Alternatively, the 1d combinatorial formula in [41] straightforwardly gives the result. The normalisation of the Jacobi is taken such that $J_{[\lambda]}=y^\lambda+\ldots$, and this gives the prefactor w.r.t. to the ${}_2F_1$ series.

is defined as a sum over $W = S_n \ltimes (\mathbb{Z}_2)^n$ of building block functions called Harish Chandra functions [16, 17] each of which are independent solutions of the same defining equation. For BC_1 this sum is just a sum over $\mathbb{Z}_2$ and the two functions on the RHS of (4.13) are precisely these building block Harish Chandra functions (the normalisations are known as the Harish-Chandra c functions). More details for this BC_1 case can be found for example in [43].

In our context however on the RHS of (4.13), the first contribution is (up to a normalisation) the (1,0) block in (4.1) with parameters $B_{-\theta\gamma,[\lambda]}(y|;\frac{1}{\theta},-\theta p_{12},-\theta p_{43})$. The second contribution is $B_{-\theta\gamma;[1-\lambda]}$, and corresponds to an independent solution of the same Casimir equation (4.1), with different properties in the small y expansion. In the CFT context this is also an important object known as the shadow block.

So we see therefore that the block and the shadow block are both Harish-Chandra functions[25]. The blocks have different analyticity properties to the Heckman Opdam hypergeometrics, explaining why they are not the same even though they are trivially related for certain integer values of their parameters. In particular, in the variables we are using the blocks are regular at $y \to 0$, whereas instead the Heckman Opdam hypergeometrics are regular at $1/y \to 0$. For a more detailed analysis of the analytic structure of the standard bosonic conformal $m = 2, n = 0$ blocks and the relation to Harish Chandra functions see [12, 14].

Summarising. The BC_1 HO hypergeometric is the analytic continuation of the BC_1 Jacobi polynomial when $\beta - \lambda \in \mathbb{C}$ which is regular when its argument approaches zero. The dual Jacobi polynomials however have a different natural analytic continuation which is regular when its argument (which is the inverse of that of that of the Jacobi polynomial) approaches zero. This latter analytic continuation gives the Harish Chandra functions, the building blocks in the sum over $W = \pm 1$ giving the HO hypergeometric. Furthermore this latter analytic continuation interpolates between the (1,0) or (0,1) blocks as we vary the parameters.

This story would appear to have a natural generalisation to higher n. The BC_n Jacobi polynomials have a natural analytic continuation in their parameters as BC_n HO Hypergeometrics. The dual Jacobi polynomials (internal $(0,n)$ blocks) however will have a natural analytic continuation as $S_n \subset W = S_n \times (\mathbb{Z}_2)^n$ invariant combinations of Harish Chandra functions. This combination will then coincide with external $(n,0)$ blocks for certain values of their parameters. Indeed this latter story can be seen explicitly in the $\theta = 1$ case from the results of [9] where both external and internal blocks are given in terms of a determinant of Gauss Hypergeometric functions which indeed coincide on mapping the parameters appropriately. It is then interesting to consider the generalisation to the supersymmetric (m,n) case and if there is a relation between the dual super Jacobi functions and super Jacobi polynomials via analytic continuation. We consider (m,n) analytic continuations further in section 6

5 Superconformal blocks (I): recursion

Having considered in detail the (1,0) and (0,1) blocks in the previous section we return to the general case. In this section we explain how to solve the Casimir equation (2.9) for the

[25] In fact twisted Harish Chandra functions since we exclude the $(-1)^{\lambda-\beta}$ in the definition of the block.

superconformal blocks, defined via the expansion

$$B_{\gamma,\underline{\lambda}} = \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\frac{\gamma}{2}} F_{\gamma,\underline{\lambda}} \qquad ; \qquad F_{\gamma,\underline{\lambda}} = \sum_{\underline{\mu}\supseteq\underline{\lambda}} (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{z}) \qquad ; \tag{5.1}$$

for any values of the $m|n$ variables $\mathbf{z}$ as well as any θ, p_{12}, p_{43} and γ. We will use a recursion, as anticipated in section 2, and to find it we will first derive the representation of the Casimir (2.18) suited for acting on the basis of super Jack polynomials, $P_{\underline{\mu}}$ and then use this to turn the differential equation into a recurrence relation for the coefficients $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$.

We will write the recursion in a way which is manifestly independent of the dimension of the blocks, and the m, n labels for the super Jack polynomials, purely in terms of the Young diagram (5.16). Later this is made very explicit in a manner prepared for different possible analytic continuations, see for example (5.35), (5.36). Appendix C provides a concise summary of properties and definitions of Jack and super Jack polynomials which might serve as a helpful guide for the reader.

Various properties of the recursion, and therefore of its solution, the Jack→Block matrix T_γ, will then be discussed. First we will exemplify the special case of the half-BPS block $\underline{\lambda} = [\varnothing]$, which admits a simple hypergeometric solution. This is instructive because the details are still quite non trivial, despite the final simplicity of the solution. Then, we will consider the most general case. We do not obtain an explicit solution for the general case, however, it is very efficient to implement the recursion on a computer, and we use this to explore properties of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$, for example its dependence on θ, γ, p_{12} and p_{43}. The most intriguing is the dependence on γ, which becomes highly non-trivial and takes the form of a more and more complicated mostly non factorisable polynomial. In section 7 we will find a surprising interpretation for the γ dependence, by elaborating on the connection between superconformal blocks and Jacobi polynomials.

5.1 Derivation of the recursion and higher order Casimirs

Recall briefly our construction. Superconformal blocks as eigenfunctions of the Casimir operator introduced in (2.9)-(2.10). On $F_{\gamma,\underline{\lambda}}$, the eigenvalue problem reduces to

$$\mathbf{C}^{(\theta,\alpha,\beta,\gamma)} F_{\gamma,\underline{\lambda}} = (h^{(\theta)}_{\underline{\lambda}} + \theta\gamma|\underline{\lambda}|) F_{\gamma,\underline{\lambda}} \ , \tag{5.2}$$

where

$$h^{(\theta)}_{\underline{\lambda}} = \sum_i \lambda_i(\lambda_i - 2\theta(i-1) - 1) \tag{5.3}$$

and $|\underline{\lambda}| = \sum_i \lambda_i$, with $\underline{\lambda} = [\lambda_1, \ldots]$. As highlighted already, the eigenvalue for $F_{\gamma,\underline{\lambda}}$ does not depend explicitly on (m, n) unlike the original eigenvalue for $B_{\gamma,\underline{\lambda}}$ in (2.10). Recall also the definitions of α and β in terms of p_{12}, p_{43}, γ given in (2.16)

$$\alpha \equiv \max\left(\tfrac{1}{2}(\gamma - p_{12}), \tfrac{1}{2}(\gamma - p_{43})\right) , \quad \beta \equiv \min\left(\tfrac{1}{2}(\gamma - p_{12}), \tfrac{1}{2}(\gamma - p_{43})\right) \tag{5.4}$$

As discussed above (2.9), the Casimir can be nicely related to the $BC_{m|n}$ CMS operator, and we do so in appendix B. The explicit result in our notation takes the form,

$$\mathbf{C}^{(\theta,a,b,c)} = \sum_{I=1}^{m+n} (-\theta)^{\pi_I} \mathbf{D}_I^{(\theta,a,b,c)} + 2\theta \sum_{I=1}^{m+n} \sum_{J\neq I} \frac{z_I z_J}{z_I - z_J} (-\theta)^{-\pi_J} \mathbf{d}_I \tag{5.5}$$

where the two operators $\mathbf{d}_I$ and $\mathbf{D}_I$ are defined as

$$\mathbf{d}_I = (1 - z_I)\partial_I \tag{5.6}$$

$$\mathbf{D}_I^{(\theta,a,b,c)} = z_I^2 \partial_I (1 - z_I)\partial_I - (-\theta)^{1-\pi_I}(c - (a+b)z_I) z_I \partial_I - (-\theta)^{2-2\pi_I} ab z_I \; . \tag{5.7}$$

We use the vector $\mathbf{z} = (z_1, \ldots, z_m | z_{m+1}, \ldots, z_{m+n})$ to label the variables $z_i = x_{i=1,\ldots m}$ and $z_{m+j} = y_{j=1,\ldots n}$. The parity $\pi_i = 0$ is assigned to the non-compact direction $i = 1,..,m$, otherwise $\pi_j = 1$ is assigned to the compact direction $j = m+1,..,m+n$.

The identification of the Casimir with a $BC_{m|n}$ differential operator helps deducing the decomposition (2.18) of the Casimir into operators with well defined actions on $P_{\underline{\mu}}$:

$$\begin{aligned} \mathbf{C}^{(\theta,a,b,c)} =& \mathbf{H}^{(\theta)} + \theta c \sum_{i=1}^{m+n} z_I \partial_I \\ & - \theta(a+b) \sum_{I=1}^{m+n} z_I^2 \partial_I - \tfrac{1}{2}\Big[\mathbf{H}^{(\theta)}, \sum_{I=1}^{m+n} z_I^2 \partial_I\Big] - \theta^2 ab \sum_{I=1}^{m+n} z_I (-\theta)^{-\pi_I} \end{aligned} \tag{5.8}$$

where $\mathbf{H}^{(\theta)}$ is the $A_{m|n}$ CMS Hamiltonian (see appendix B)[26]

$$\mathbf{H}^{(\theta)} = \sum_{I=1}^{m+n} (-\theta)^{\pi_I} z_I^2 \partial_I^2 \; + \; 2\theta \sum_{I\neq J} \frac{z_I z_J}{z_I - z_J} (-\theta)^{-\pi_J} \partial_I \; . \tag{5.9}$$

The operator decomposition in (5.8) is perfectly suited to acting on the expansion $F_{\gamma,\underline{\lambda}} = \sum (T_\gamma)_{\underline{\lambda}}^{\underline{\mu}} P_{\underline{\mu}}$ because super Jack polynomials are eigenfunctions of both $\sum_I z_I \partial_I$ (reflecting the fact that they are homogeneous polynomials) and $\mathbf{H}^{(\theta)}$,[27] and moreover, the other two operators are themselves related to the simplest one-box super Jack polynomial, $P_\square$

$$\sum_I z_I (-\theta)^{-\pi_I} = P_\square(\mathbf{z}), \qquad \text{and} \qquad \sum_I z_I^2 \partial_I = \tfrac{1}{2}[\mathbf{H}^{(\theta)}, P_\square(\mathbf{z})] \; . \tag{5.10}$$

The action of all the operators $z_I\partial_I$, $\mathbf{H}$, $P_\square$ on super Jack polynomials yields a linear combination of Jack polynomials whose coefficients are independent of m, n. This will imply that the recursion for T_γ, which we derive from the Casimir, will be independent of m, n at every stage.

[26]Notice that $\mathbf{H}$ shows up directly in (5.8), since it is found by replacing $\mathbf{D}_I \to z_I^2 \partial_I^2$ and $\mathbf{d}_I \to \partial_I$. Terms of the Casimir in which $\mathbf{D}_I \to -z_I^3 \partial_I^2$ and $\mathbf{d}_I \to -z_I \partial_I$, are generated by the commutator.

[27]Indeed these are the first two Hamiltonians of a tower found in [34], which establishes the classical integrability of the system.

Let us begin with the simplest operator in (5.8), i.e. the momentum operator,

$$\sum_{I=1}^{m+n} z_I \partial_I \, P_{\underline{\mu}}(\mathbf{z}) = |\underline{\mu}| P_{\underline{\mu}}(\mathbf{z}) \ . \tag{5.11}$$

The eigenvalue here simply counts the number of boxes of the Young diagram $\underline{\mu}$ (which obviously does not depend on (m,n)). This operator just reflects the fact that super Jack polynomials are homogeneous functions of the z_I of degree $|\mu|$.

For the Hamiltonian, we have

$$\mathbf{H}^{(\theta)} \, P_{\underline{\mu}}(\mathbf{z}) \quad = \quad h^{(\theta)}_{\underline{\mu}} P_{\underline{\mu}}(\mathbf{z}) \ . \tag{5.12}$$

This equation is part of the standard definition of the super Jack polynomials and the eigenvalue $h^{(\theta)}_{\underline{\mu}}$ in (5.3) has no explicit dependence on (m,n).

Next, the two 'off-diagonal' operators in (5.10). The first of the two is simply the multiplicative operator acting with $P_{\Box}$. In general the product $P_{\underline{\mu}_1} P_{\underline{\mu}_2}$ gives a linear combination of super Jacks whose coefficients, $\mathcal{C}^{\underline{\mu}_3}_{\underline{\mu}_1 \underline{\mu}_2}(\theta)$, known as structure constants, are independent of m,n. When $\theta = 1$ Jack polynomials reduce to Schur polynomials and the structure constants become precisely the standard Littlewood-Richardson coefficients. Finding an explicit formula for the structure constants for general θ however is a difficult problem/task in modern mathematics [54]. Here however we only need the case in which one of the Young diagrams is a single box. This case reduces to the simplest case of the Pieri rule for Jack and super Jack polynomials, and $\mathcal{C}^{\underline{\nu}}_{\Box \, \underline{\mu}}$ *is* known combinatorially [65, 66]. It is perhaps less known that this combinatorial formula can be made fully explicit, as we will show shortly.

So the action of the first off-diagonal operator in (5.10) coincides with the product $P_{\Box} P_{\underline{\mu}}$ and returns

$$P_{\Box}(\mathbf{z}) \, P_{\underline{\mu}}(\mathbf{z}) = \sum_{\underline{\nu} \in \{\underline{\mu} + \Box\}} \mathcal{C}^{\underline{\nu}}_{\Box \, \underline{\mu}} P_{\underline{\nu}}(\mathbf{z}) \ . \tag{5.13}$$

where the RHS spans all polynomials labelled by the Young diagram $\underline{\nu}$ obtained by adding an extra box to the Young tableau $\underline{\mu}$, in all possible ways.

The action of the second off-diagonal operator in (5.10), $\sum_i z_i^2 \partial_i$, can then be obtained straightforwardly from its representation as a commutator together with (5.12),(5.13). We have

$$\sum_I z_I^2 \partial_I P_{\underline{\mu}}(\mathbf{z}) = \tfrac{1}{2} [\mathbf{H}^{(\theta)}, P_{\Box}(\mathbf{z})] P_{\underline{\mu}}(\mathbf{z}) = \sum_{\underline{\nu} \in \{\underline{\mu} + \Box\}} \tfrac{1}{2} (h_{\underline{\nu}} - h_{\underline{\mu}}) \mathcal{C}^{\underline{\nu}}_{\Box \, \underline{\mu}} P_{\underline{\mu} + \Box_i}(\mathbf{z}) \ . \tag{5.14}$$

Having understood the action of all the operators appearing in the operator decomposition of the Casimir, and having observed that all of them decompose into Jack polynomials with coefficients which only depend on the Young diagram, we can now write the representation of the Casimir itself in the basis of superJack polynomials. The result is quite neat and nicely generalises the structure noticed for $\theta = 1$ in [9],

$$\mathbf{C}^{(\theta,a,b,c)} P_{\underline{\mu}} = \left(h_{\underline{\mu}} + \theta c|\underline{\mu}|\right) P_{\underline{\mu}} \; - \sum_{\underline{\nu}\in\{\underline{\mu}+\square\}} \left(\tfrac{1}{2}(h_{\underline{\nu}}-h_{\underline{\mu}}) + \theta a\right) \left(\tfrac{1}{2}(h_{\underline{\nu}}-h_{\underline{\mu}}) + \theta b\right) \mathcal{C}^{\underline{\nu}}_{\square\,\underline{\mu}} \, P_{\underline{\nu}} \, . \tag{5.15}$$

From this and the definition $F_{\gamma,\underline{\lambda}} = \sum_{\underline{\mu}} (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}$, then the Casimir equation (5.2) gives the following recursion equation for the coefficients T_γ

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_{\underline{\nu}\in\{\underline{\mu}-\square\}} \left(\tfrac{1}{2}(h_{\underline{\mu}}-h_{\underline{\nu}}) + \theta\alpha\right) \left(\tfrac{1}{2}(h_{\underline{\mu}}-h_{\underline{\nu}}) + \theta\beta\right) \mathcal{C}^{\underline{\mu}}_{\square\,\underline{\nu}} \, (T_\gamma)^{\underline{\nu}}_{\underline{\lambda}}}{\left(h_{\underline{\mu}}-h_{\underline{\lambda}}+\theta\gamma\left(|\underline{\mu}|-|\underline{\lambda}|\right)\right)} \tag{5.16}$$

with α, β as defined earlier in terms of γ, p_{12}, p_{43} (5.4). Here we see that the recursion initiates from the Young diagram $\underline{\mu}$ (which contains $\underline{\lambda}$) and goes back to $\underline{\lambda}$ (for which we know $(T_\gamma)^{\underline{\lambda}}_{\underline{\lambda}} = 1$) recursively by subtracting boxes. Thus if $\underline{\mu}$ is compatible with the (m,n) structure (i.e. $\mu_{m+1} \leq n$) then $\underline{\nu} \in \{\underline{\mu} - \square\}$ is also compatible with it and the recursion always remains inside the (m,n) Young diagram. This means the recursion can be solved quite efficiently on a computer.

It is also interesting to consider the inverse of T_γ, i.e. the Block→Jack matrix, T_γ^{-1}. This gives Jack polynomials as a sum of blocks, via the inverse of (5.1),

$$P_{\underline{\lambda}} = \sum_{\underline{\mu}} (T_\gamma^{-1})^{\underline{\mu}}_{\underline{\lambda}} F_{\gamma,\underline{\mu}}. \tag{5.17}$$

Note that $F_{\gamma,\underline{\mu}}$, when $m > 0$ is an infinite series, and the sum, an infinite sum. Many non trivial cancellations take place in order to recover a polynomial. From (5.15) and (5.2) we obtain a recursion for T_γ^{-1} which is very similar looking to that of T_γ

$$(T_\gamma^{-1})^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_{\underline{\nu}\in\{\underline{\lambda}+\square\}} \left(\tfrac{1}{2}(h_{\underline{\nu}}-h_{\underline{\lambda}}) + \theta\alpha\right) \left(\tfrac{1}{2}(h_{\underline{\nu}}-h_{\underline{\lambda}} + \theta\beta\right) \mathcal{C}^{\underline{\nu}}_{\square\,\underline{\lambda}} \, (T_\gamma^{-1})^{\underline{\mu}}_{\underline{\nu}}}{\left(h_{\underline{\lambda}}-h_{\underline{\mu}}+\theta\gamma\left(|\underline{\lambda}|-|\underline{\mu}|\right)\right)} \tag{5.18}$$

The main difference is that here the recursion gives $(T_\gamma^{-1})^{\underline{\mu}}_{\underline{\lambda}}$ in terms of $(T_\gamma^{-1})^{\underline{\mu}}_{\underline{\lambda}+\square}$, obtained by *adding* boxes to $\underline{\lambda}$, whereas the recursion for $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ is in terms of $(T_\gamma)^{\underline{\mu}-\square}_{\underline{\lambda}}$, is obtained by *subtracting* boxes from $\underline{\mu}$.

We will return to the relation between T and T^{-1} later on in section 7.3.

Higher order Casimirs for $\theta = 1$

Before considering the recursion in more detail we would like to discuss higher order Casimirs in the CFT. These arise from the relation between the Casimir and CMS system with BC root system. In fact, higher order Casimirs correspond to the defining higher order differential operators of the Heckman Opdam BC hypergeometric functions. This is a general statement, but here we will specialise to $\theta = 1$, where formulae are both explicit and simple. Generalising the work of [43] (who focussed on the purely bosonic case $n = 0$) we now show that higher order Casimirs in the supersymmetric case are given by simply replacing the variables z_I in a super Schur with the corresponding second order operator $\mathbf{D}_I$ of (5.7) and conjugating with the Vandermonde (super-) determinant

$$\mathbf{C}_{\underline{\mu}}^{(\theta=1,a,b,c)} = V^{-1}(z_I)\, P_{\underline{\mu}}\Big(\mathbf{D}_I^{(\theta=1;a,b,c)}; \theta = 1\Big)\, V(z_I) \tag{5.19}$$

where the super van der Monde determinant is

$$V(z_I) = \frac{\prod_{1\leq i<j\leq m}(x_i^{-1} - x_j^{-1}) \prod_{1\leq i<j\leq n}(y_i^{-1} - y_j^{-1})}{\prod_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}(x_i^{-1} - y_j^{-1})} \ , \tag{5.20}$$

and where $\mathbf{D}_I$ is given in (5.7). The quadratic Casimir of (5.5) correspond to the above for the one-box super Schur, i.e. $\underline{\mu} = \square$. The latter is simply the sum $\sum_i x_i - \sum_j y_j$.

The superblocks $B_{\gamma,\underline{\lambda}}$ are eigenvalues of $\mathbf{C}_{\underline{\lambda}}^{(\theta,-\frac{1}{2}p_{12},-\frac{1}{2}p_{43},0)}$ for *all* Young diagrams $\underline{\mu}$

$$\mathbf{C}_{\underline{\mu}}{}^{(\theta,-\frac{1}{2}p_{12},-\frac{1}{2}p_{43},0)} B_{\gamma,\underline{\lambda}}(\mathbf{z}) = (E_{\underline{\mu}})_{\gamma,\underline{\lambda}}^{(m,n;\theta=1)} B_{\gamma,\underline{\lambda}}(\mathbf{z}) \ , \tag{5.21}$$

The corresponding eigenvalue is

$$(E_{\underline{\mu}})_{\gamma,\underline{\lambda}}^{(m,n;\theta=1)} = P_{\underline{\mu}}((E_I)_{\gamma,\underline{\lambda}}) - \tfrac{1}{3}(m-n-1)_3 \tag{5.22}$$

obtained by replacing z_I by E_I in the corresponding Schur polynomial $P_{\underline{\mu}}$ where

$$(E_I)_{\gamma,\underline{\lambda}}^{(m,n;\theta=1)} = \begin{cases} (\lambda_i - i + \frac{1}{2}\gamma + 1)(\lambda_i - i + \frac{1}{2}\gamma) & I = i = 1,..,m \\ (\lambda'_j - j - \frac{1}{2}\gamma + 1)(\lambda'_j + j - \frac{1}{2}\gamma) & I - m = j = 1,..,n \end{cases} \ . \tag{5.23}$$

The case $\underline{\mu} = \square$ is the one we discussed in our overview: one can check that putting the single box Young diagram, this agrees with (2.10) for $\theta = 1$.

Note that an eighth order super Casimir in $\mathcal{N} = 4$ SYM considered in [70] corresponds to (5.19) with $m{=}n{=}2$ and $\mu{=}[2,2]$ for which $P_{\underline{\mu}}(z_I){=}(x_1{-}y_1)(x_1{-}y_2)(x_2{-}y_1)(x_2{-}y_2)$. Higher order Casimirs both in 1d CFTs ($m{=}n{=}\theta{=}1$) and in $\mathcal{N}{=}4$ SYM ($m{=}n{=}2, \theta{=}1$) are considered in [73].

5.2 Explicit formulae for the recursion

So far the recursion (5.16) is entirely combinatoric, written in terms of Young diagrams, and our main concern has been to emphasize this property, that it depends only on objects that are (m, n) independent. We will now make the discussion more concrete by considering explicit formulae. Remarkably, there are different explicit forms that the recursion can take. The difference is simply a preference for rows over columns when reading the Young diagrams, or vice-versa, or even mixing rows and columns as is more natural in the supersymmetric (m, n) case.

Eigenvalue

Let us begin from $h_{\underline{\lambda}}^{(\theta)}$ which in (5.3) we wrote as

$$h_{\underline{\lambda}}^{(\theta)} = \sum_{i=1,2,..} \lambda_i(\lambda_i - 2\theta(i-1) - 1) \ . \tag{5.24}$$

It is simple to see that it admits an equivalent representation in which instead we take the sum to run over columns,

$$h_{\underline{\lambda}}^{(\theta)} = -\theta \sum_{j=1,2,..} \lambda'_j(\lambda'_j - 1 - \tfrac{2}{\theta}(j-1)) = -\theta h_{\underline{\lambda}'}^{(\frac{1}{\theta})} \ , \tag{5.25}$$

or if $\underline{\lambda}$ is a typical (m,n) Young diagram (ie it has $\lambda_m \geq n, \lambda_{n+1} \leq n$ see (C.40)) then it can be rewritten as a combination of these row and column formulae:

$$h_{\underline{\lambda}}^{(\theta)} = \sum_{i=1}^{m} \lambda_i(\lambda_i - 2\theta(i-1) - 1) - \theta \sum_{j=1}^{n} \lambda'_j(\lambda'_j - 1 - \tfrac{2}{\theta}(j-1)) - h_{n^m}^{(\theta)} \ . \tag{5.26}$$

where $h_{n^m}^{(\theta)} = nm((n-1) - \theta(m-1))$. All these forms for the eigenvalue can be easily derived from the following combinatoric formula

$$h_{\underline{\lambda}}^{(\theta)} = 2 \sum_{(i,j)\in\underline{\lambda}} \Big((j-1) - \theta(i-1)\Big) \ . \tag{5.27}$$

where the eigenvalue is given as a sum over boxes in the Young tableau where each box is understood to have row i column j. Then one obtains the equivalent formulae (5.3) or (5.25) by simply performing the first sum in each of the following ways of writing the sum over Young diagram boxes explicitly

$$\sum_{(i,j)\in\underline{\lambda}} = \sum_{i=1,2,..} \sum_{j=1}^{\lambda_i} = \sum_{j=1,2,..} \sum_{i=1}^{\lambda'_i} = \sum_{i=1}^{m} \sum_{j=1}^{\lambda_i} + \sum_{j=1}^{n} \sum_{i=1}^{\lambda'_i} - \sum_{i=1}^{m} \sum_{j=1}^{n} \ . \tag{5.28}$$

with the last formula for typical (long) (m,n) Young diagrams only.

In the numerator of the recursion (5.16) we have the expression

$$\left(\tfrac{1}{2}(h_{\underline{\mu}} - h_{\underline{\nu}}) + \theta\alpha\right)\left(\tfrac{1}{2}(h_{\underline{\mu}} - h_{\underline{\nu}}) + \theta\beta\right) \qquad \underline{\nu} \in \{\underline{\lambda} - \square\} \ . \tag{5.29}$$

Since $\underline{\nu}$ is related to $\underline{\mu}$ by subtracting a box in position (i,j), clearly from (5.27) we have $\frac{1}{2}(h_{\underline{\mu}} - h_{\underline{\nu}}) = (j-1) - \theta(i-1)$.

Structure constants

Let us come to $\mathcal{C}^{\underline{\mu}+\Box}_{\Box\,\underline{\mu}}$ in (5.16). The corresponding combinatorial formula is well known and can be found in [65]. It can be rearranged quite explicitly and we find[28]

$$\mathcal{C}^{\underline{\mu}+\Box_{ij}}_{\Box\,\underline{\mu}} = \mathbf{f}^{(i)}_{\underline{\mu}}(\theta) = \prod_{k=1}^{i-1} \frac{(\theta(i{-}k{+}1){-}1{+}\mu_k{-}\mu_i)\,(\theta(i{-}k{-}1){+}\mu_k{-}\mu_i)}{(\theta(i{-}k){-}1{+}\mu_k{-}\mu_i)\,(\theta(i{-}k){+}\mu_k{-}\mu_i)}\ . \tag{5.30}$$

where we are using the notation $\underline{\mu}+\Box_{ij}$ to mean that we add a box to row i and column j of the Young tableau so that we have $j=\mu_i+1$ and $i=\mu'_j+1$. The above formula is written in terms of Young diagram row lengths μ_i. Alternatively, we might prefer a rewriting of $\mathcal{C}^{\underline{\mu}+\Box_{ij}}_{\Box\,\underline{\mu}}$ over columns.

To do so we define first, $\mathbf{g}^{(i)}_{\underline{\mu}}$, obtained by multiplying $\mathbf{f}^{(i)}_{\underline{\mu}}$ by the following normalisation in terms of C symbols (defined more generally later in (7.9))

$$\mathbf{g}^{(i)}_{\underline{\mu}}(\theta) \equiv \frac{\Pi_{\underline{\mu}}(\theta)}{\Pi_{\underline{\mu}+\Box_{ij}}(\theta)}\ \mathbf{f}^{(i)}_{\underline{\mu}}(\theta) \tag{5.31}$$

$$\Pi_{\underline{\kappa}}(\theta) = \frac{C^-_{\underline{\kappa}}(\theta;\theta)}{C^-_{\underline{\kappa}}(1;\theta)} = \frac{\prod_{(ij)\in\underline{\kappa}}\Big(\kappa_i{-}j+\theta(\kappa'_j{-}i)+\theta\Big)}{\prod_{(ij)\in\underline{\lambda}}\Big(\kappa_i{-}j+\theta(\kappa'_j{-}i)+1\Big)}\ . \tag{5.32}$$

Note that this can be rewritten as an explicit function of the row lengths μ_i $(i=1,..,n)$ only[29]

$$\mathbf{g}^{(i)}_{\underline{\mu}}(\theta) = \frac{\mu_i-\mu_{i+1}+1}{\mu_i-\mu_{i+1}+\theta}\prod_{k=i+1}^{n}\frac{(\theta(k-i+1)+\mu_i-\mu_k)(\theta(k-i)+\mu_i-\mu_{k+1}+1)}{(\theta(k-i+1)+\mu_i-\mu_{k+1})(\theta(k-i)+\mu_i-\mu_k+1)} \tag{5.33}$$

(with μ_{n+1} defined to vanish).

The structure constant (5.30) then has an alternative writing in terms of column lengths using $\mathbf{g}^{(j)}_{\underline{\mu}'}$ as follows[30]

$$\mathcal{C}^{\underline{\mu}+\Box_{ij}}_{\Box\,\underline{\mu}} = \mathbf{f}^{(i)}_{\underline{\mu}}(\theta) = \tfrac{1}{\theta}\,\mathbf{g}^{(j)}_{\underline{\mu}'}(\tfrac{1}{\theta})\,,\qquad \mu'_j = i-1\,,\quad \mu_i = j-1\ . \tag{5.34}$$

[28]See also [66]. We actually first derived this formula using computed algebra, by seeking a generalisation of the formula for two-row polynomials, which can be derived with pencil and paper. Then, we confirmed its expression by rewriting the combinatorial formula coming from the Pieri rule [29].

[29]To do this we use explicit formulae for $C^-_{\underline{\kappa}}(w;\theta)$from [26],

$$C^-_{\underline{\mu}}(\theta;\theta) = \frac{\prod_{k=1}^{\mu'_1}\frac{(\mu_k+\theta(\mu'_1-k)+\theta-1)!}{(\theta-1)!}}{\prod_{1\le k_1<k_2\le\mu'_1}(\mu_{k_1}{-}\mu_{k_2}{+}\theta(k_2{-}k_1))_\theta}\qquad C^-_{\underline{\mu}}(1;\theta) = \frac{\prod_{k=1}^{\mu'_1}(\mu_k+\theta(\mu'_1-k))!}{\prod_{1\le k_1<k_2\le\mu'_1}(\mu_{k_1}{-}\mu_{k_2}{+}\theta(k_2{-}k_1)-(\theta{-}1))_\theta}\,.$$

after which the majority of contributions to $\Pi_{\underline{\mu}}(\theta)/\Pi_{\underline{\mu}+\Box_{ij}}(\theta)$ simplify in the ratio.

[30]A posteriori, $\mathbf{f}$ can also be written in terms of the A-type binomial coefficient, built out of the A-type interpolation polynomials of Okounkov [67], which can be seen as the limit $u=\infty$ in (C.30). The duality $\mathbf{f}\leftrightarrow\mathbf{g}$ can be proven using the identification with the A-type binomial coefficient.

where recall that (i,j) are the coordinates of $\Box_{ij}$.

Equivalent forms of the recursion

Choosing all row-type formulae is the most natural from a bosonic CFT point of view, since in a physical theory with conformal symmetry, we would be summing over $P_{\underline{\mu}}$ with a finite number of rows but an arbitrary number of columns such that $\underline{\lambda} \subseteq \underline{\mu}$, in order to construct $B_{\gamma,\underline{\lambda}}$. With this choice then we use (5.24) and (5.30) to write the recursion (5.16) in terms of row lengths explicitly as

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_i (\mu_i - 1 - \theta(i-1-\alpha))(\mu_i - 1 - \theta(i-1-\beta))\, \mathbf{f}^{(i)}_{\underline{\mu}-\Box_i}(\theta)\, (T_\gamma)^{\underline{\mu}-\Box_i}_{\underline{\lambda}}}{\sum_i (\mu_i - \lambda_i)(\lambda_i + \mu_i + \theta\gamma - 2\theta(i-1) - 1)} \tag{5.35}$$

where $\underline{\mu}-\Box_i := [\mu_1, .., \mu_{i-1}, \mu_i - 1, \mu_{i+1}, ...]$. Note that $\mathbf{f}$ imposes that $\mu_i > \mu_{i+1}$ automatically. In fact if it was the case that $\mu_i = \mu_{i-1}+1$ for some i, then second term in the denominator of $\mathbf{f}^{(i)}_{\underline{\mu}-\Box_i}$ in (5.30) would vanish for $k = i-1$.

If instead we choose all column-type explicit formulae then using (5.25) and (5.31) we write the recursion (5.16) in terms of column lengths explicitly as

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_j \left(\mu'_j - 1 - \alpha - \frac{j-1}{\theta}\right)\left(\mu'_j - 1 - \beta - \frac{j-1}{\theta}\right) \mathbf{g}^{(j)}_{\underline{\mu}'-\Box_j}(\tfrac{1}{\theta})(T_\gamma)^{(\underline{\mu}'-\Box_j)'}_{\underline{\lambda}}}{\sum_j (\lambda'_j - \mu'_j)(\lambda'_j + \mu'_j - \gamma - \frac{2}{\theta}(j-1) - 1)} \tag{5.36}$$

where $\underline{\mu}'-\Box_j := [\mu'_1, .., \mu'_{j-1}, \mu'_j - 1, \mu'_{j+1}, ...]$.

Finally we could also split the sum $\sum_{\underline{\nu}\in\{\underline{\mu}-\Box\}}$ in (5.16) into east and south part arbitrarily. The outcome for $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ will remain unchanged. This is most natural for an (m,n) supersymmetric theory and indeed is crucial if we wish to analytically continue consistently with the (m,n) structure. We will discuss this supersymmetric splitting in the section 6.

5.3 Special solutions: the half-BPS superconformal block

There is a special case in which we can solve the recursion for all coefficients explicitly with a particularly simple solution. It is the case corresponding to an empty diagram $\underline{\lambda} = [\varnothing]$, and therefore corresponding to a half-BPS superconformal block exchanged in a propagator structure with γ propagators going from $\mathcal{O}_{p_1}\mathcal{O}_{p_2}$ to $\mathcal{O}_{p_3}\mathcal{O}_{p_4}$. The solution is

$$(T_\gamma)^{\underline{\mu}}_{[\varnothing]} = \frac{C^0_{\underline{\mu}}(\theta\alpha;\theta)C^0_{\underline{\mu}}(\theta\beta;\theta)}{C^0_{\underline{\mu}}(\theta\gamma;\theta)} \frac{1}{C^-_{\underline{\mu}}(1;\theta)} \tag{5.37}$$

where the combinatorial symbols are

$$
\begin{aligned}
C^0_{\underline{\lambda}}(w;\theta) &= \prod_{(ij)\in\underline{\lambda}} (j-1-\theta(i-1)+w) \\
C^-_{\underline{\lambda}}(w;\theta) &= \prod_{(ij)\in\underline{\lambda}} \left(\lambda_i-j+\theta(\lambda'_j-i)+w\right) \ .
\end{aligned} \tag{5.38}
$$

Let us verify that this solution indeed satisfies the recursion using the row-type recursion (5.35). We will see that despite the final simplicity of the result, the recursion is solved in a very non trivial way. Afterwards we will present a more direct argument based on the fact that the Casimir annihilates the half-BPS superconformal block, therefore it can be split into simpler independent hypergeometric equations studied by Yan in [68].

Consider the recursion (5.35) with $\underline{\lambda}=[\varnothing]$

$$
(T_\gamma)^{\underline{\mu}}_{[\varnothing]} = \frac{\sum_i (\mu_i-1-\theta(i-1-\alpha))\,(\mu_i-1-\theta(i-1-\beta))\,\mathbf{f}^{(i)}_{\underline{\mu}-\square_i}(\theta)\,(T_\gamma)^{\underline{\mu}-\square_i}_{[\varnothing]}}{\left(h_{\underline{\mu}}+\theta\gamma\,|\underline{\mu}|\right)} \ . \tag{5.39}
$$

Inserting the solution (5.37) the terms $(\mu_i-1-\theta(i-1-\alpha))$ and $C^0_{\underline{\mu}-\square_i}(\theta\alpha;\theta)$ combine and simplify with the LHS $C^0_{\underline{\mu}}(\theta\alpha;\theta)$. Similarly for the terms with β. We are left with needing to prove the non trivial statement

$$
1 = \frac{1}{\left(h_{\underline{\mu}}+\theta\gamma\,|\underline{\mu}|\right)} \sum_i \left(\mathbf{f}^{(i)}_{\underline{\mu}-\square_i} \frac{C^-_{\underline{\mu}}(1;\theta)}{C^-_{\underline{\mu}-\square_i}(1;\theta)} \right) ((\mu_i-1)-\theta(i-1)+\theta\gamma) \ . \tag{5.40}
$$

We can deal with $C^-_{\underline{\mu}}$ by using the explicit formulae of footnote 29 and find that

$$
\mathbf{f}^{(i)}_{\underline{\mu}-\square_i} \frac{C^-_{\underline{\mu}}(1;\theta)}{C^-_{\underline{\mu}-\square_i}(1;\theta)} = (\mu_i+\theta(\mu'_1-i)) \prod_{k\neq i} \frac{\mu_i-\mu_k+\theta(k-i-1)}{\mu_i-\mu_k+\theta(k-i)} \ . \tag{5.41}
$$

Proving the recursion is now equivalent to proving that

$$
\begin{aligned}
&\sum_i (\mu_i+\theta(\mu'_1-i)) \prod_{k\neq i} \frac{\mu_i-\mu_k+\theta(k-i-1)}{\mu_i-\mu_k+\theta(k-i)} = |\underline{\mu}| \\
&\sum_i (\mu_i+\theta(\mu'_1-i)) \prod_{k\neq i} \frac{\mu_i-\mu_k+\theta(k-i-1)}{\mu_i-\mu_k+\theta(k-i)} (\mu_i-\theta i) = 2 \sum_{(ij)\in\underline{\mu}} (j-\tfrac{1}{2})-\theta(i-\tfrac{1}{2})
\end{aligned} \tag{5.42}
$$

where the RHS corresponds to a decomposition of $h_{\underline{\mu}}+\theta\gamma$. Note that even the first one of these relations is quite non trivial, since the common denominator in the LHS has to cancel out in the end, and this only happens after taking the sum. These two relations are proven in a beautiful way in [69] by using the Lagrange Lemma for alphabets.

The superconformal block $B_{\gamma,[\varnothing]}$ itself is now obtained by summing up $(T_\gamma)^{\underline{\mu}}_{[\varnothing]}$ with super

Jack polynomials. It takes the form of a multivariate ${}_2F_1$ series,

$$B_{\gamma,[\varnothing]} = \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\frac{\gamma}{2}} \sum_{\underline{\mu}} \frac{C^0_{\underline{\mu}}(\theta\alpha;\theta)C^0_{\underline{\mu}}(\theta\beta;\theta)}{C^0_{\underline{\mu}}(\theta\gamma;\theta)} \frac{P_{\underline{\mu}}(\mathbf{z};\theta)}{C^-_{\underline{\mu}}(1;\theta)}. \tag{5.43}$$

In the bosonic $(m,0)$ theory this is indeed equal to a well known multivariate hypergeometric function [68].[31] In particular, it is known to satisfy a set of m hypergeometric equations, which in our notation take the form

$$\left[z_I^{-1}\mathbf{D}_I^{(\theta,\alpha,\beta,\gamma)} - \theta(m-1)(1-z_I) + \theta\sum_{J\neq I} \frac{z_I(1-z_I)\partial_I - z_J(1-z_J)\partial_J}{z_I - z_J} \right] B_{\gamma,[\varnothing]} = 0 \quad ; \quad I = 1,\dots m \tag{5.44}$$

where $\mathbf{D}_I$ is the same operator introduced in (5.7). The trick to recover our Casimir $\mathbf{C}$ is to sum $\sum_I z_I \times (5.44)$ and put together $-(m-1) + \sum_{J\neq I} \frac{z_I}{z_I - z_J} = +\sum \frac{z_J}{z_I - z_J}$.

Since $h_\varnothing + \gamma|\varnothing| = 0$, the half-BPS block is annihilated by the Casimir, and therefore the system of equations (5.44) is equivalent to the original Casimir eigenvalue problem.

5.4 General features and non-trivial γ-dependence

The solution of the recursion for T_γ reveals many non trivial features. In order to systematise the discussion, let us first show that the α and β dependence of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ always takes the following form

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha;\theta)C^0_{\underline{\mu}/\underline{\lambda}}(\theta\beta;\theta) \times (T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} \tag{5.45}$$

with $(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}}$ then independent of α, β. Here by definition $C^0_{\underline{\mu}/\underline{\lambda}} \equiv C^0_{\underline{\mu}}/C^0_{\underline{\lambda}}$, thus[32]

$$C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha;\theta)C^0_{\underline{\mu}/\underline{\lambda}}(\theta\beta;\theta) = \prod_{(ij)\in\underline{\mu}/\underline{\lambda}} (j-1-\theta(i-1-\alpha))(j-1-\theta(i-1-\beta)). \tag{5.46}$$

The proof of (5.45) is immediate. In fact, the ratio of C^0 satisfies the relation

$$C^0_{\underline{\mu}/\underline{\lambda}}(\theta w;\theta) = (\mu_i - 1 - \theta(i-1-w))\, C^0_{\underline{\mu}-\square_i/\underline{\lambda}}(\theta w;\theta) \tag{5.47}$$

and therefore in the row-type formulae (5.35) for the recursion, the α and β dependence drops. This is indeed the same argument we saw at work in the half-BPS solution, around (5.39), with $\underline{\lambda} = [\varnothing]$. Then, it follows from (5.45) that $(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}}$ satisfies a simpler recursion,

$$(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_i \mathbf{f}^{(i)}_{\underline{\mu}-\square_i}(\theta)\, (T_{\underline{\lambda}}^{\texttt{rescaled}})^{\underline{\mu}-\square_i}_{\underline{\lambda}}}{\sum_i(\mu_i - \lambda_i)(\lambda_i + \mu_i + \theta\gamma - 2\theta(i-1) - 1)}, \tag{5.48}$$

and $(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}}$ is *only* a function of γ, θ and the Young diagrams $\underline{\mu}$ and $\underline{\lambda}$.

The recursion for $(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}}$ can be turned into column or mixed type formulae, exactly

[31] As discussed at the end of section 4.2 we believe this is also a certain S_n combination of Harish Chandra contributions to the corresponding Heckman-Opdam hypergeometric.

[32] The $C^0_{\underline{\mu}}(w;\theta)$ was defined in (5.38). Here we are giving the final result for $C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha;\theta)C^0_{\underline{\mu}/\underline{\lambda}}(\theta\beta;\theta)$.

as for $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ itself, as we did in section 5.2. The C^0 rescaling will change accordingly, see in particular section D.2.

Given the various explicit formulae provided in the previous section 5.2, it is not difficult to implement the recursion on a computer and solve it in many cases. Let us look at a couple of examples to illustrate the general features:

$$(T_\gamma^{\texttt{rescaled}})^{[5,3]}_{[3,1]} = \frac{162+222\theta+67\gamma\theta-168\theta^2+109\gamma\theta^2+7\gamma^2\theta^2+24\theta^3-36\gamma\theta^3+13\gamma^2\theta^3}{2(1+\theta)(3+\theta)_2(6+\gamma\theta)_2(2-2\theta+\gamma\theta)_2(4-\theta+\gamma\theta)_2} \tag{5.49}$$

The dependence on γ in the denominator as a product of linear factors in γ is simple to understand: it can only come from putting together all denominators encountered upon solving the recursion, and these are all of the form $(\theta\gamma-\theta\mathbb{N}+\mathbb{N})$ with integers coming from Young diagrams. The numerator on the other hand is fairly complicated, degree two in γ and degree three in θ. This degree depends on the Young diagrams. Another example shows this more clearly,

$$\begin{aligned}(T_\gamma^{\texttt{rescaled}})^{[6,5,3]}_{[3,2,1]} \sim\ & +21208+6(1861\gamma+7012)\theta+\left(1943\gamma^2+26524\gamma-56312\right)\theta^2 \\ & +\left(111\gamma^3+5312\gamma^2-17918\gamma+11112\right)\theta^3 \\ & +\left(339\gamma^3-965\gamma^2-796\gamma+2560\right)\theta^4 \\ & +\left(66\gamma^3-410\gamma^2+800\gamma-480\right)\theta^5\ .\end{aligned} \tag{5.50}$$

(we only display the complicated polynomial factor, and we omitted the various linear factors of the form $(\theta\gamma-\theta\mathbb{N}+\mathbb{N})$.)

The point is that guessing the γ dependence of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$, for arbitrary $\underline{\lambda}$, $\underline{\mu}$, and θ, would be quite challenging. However, as anticipated in the introduction, due to the relation with Jacobi polynomials we will find in section 7.2 an alternative description for this complicated factor as an interpolation polynomial.

6 Superconformal blocks (II): analytic continuation

We have understood in the previous section how to construct a superconformal block $B_{\gamma,\underline{\lambda}}$ given a Young diagram $\underline{\lambda}$ for which e.g. its row lengths $\lambda_i \in \mathbb{N}$ with $\lambda_i \geq \lambda_{i+1}$.

For physical applications in an interacting CFT however, non-protected operators exchanged in the OPE acquire an anomalous dimension and thus it is crucial to be able to understand non-integer quantum numbers, at least for the dilation weight. Even though the other quantum numbers remain integer in a unitary CFT, it is nevertheless also very useful to consider analytically continuing these too (for example the spin, see [70] for example).

For general θ, we do not have explicit solutions for either $B_{\gamma,\underline{\lambda}}$ or $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ to look at, in order to discuss analytic continuation beyond Young diagrams. In fact, only the case $\theta = 1$ has been solved for both $B_{\gamma,\underline{\lambda}}$ and $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ in [9]. We will then have to look at the recursion. The combinatorial form given in (5.16) is not immediately useful, since does not lend itself to analytic continuation. However, from the explicit forms of the recursion (5.35),(5.36) we can obtain analytic results for the coefficients T_γ as rational functions of the row lengths λ_i or column lengths λ'_i respectively.

The logic will be quite simple. For the row type recursion (5.35) we will think of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ as a function of $\underline{\lambda} \in \mathbb{C}$, with $\underline{\mu}$ given by $\underline{\mu} = \underline{\lambda} + \vec{n}$ with $n_i \in \mathbb{Z}^+$. The recursion takes $\underline{\mu}$ and goes back to $\underline{\lambda}$ by negative integer shifts, regardless of whether the λ_i are integer or not. In this framework, the various representations of the recursion, i.e. whether row-type or vertical-type or a mixed one, lead to different ways of analytically continuing the $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ in the variables describing the Young diagrams. Crucially we know that all these analytic continuations of T_γ must coincide in the cases when $\underline{\lambda}$ and $\underline{\mu}$ return to values such that they represent a valid Young diagrams.

6.1 Row-type representation on the east

The row-type representation of the recursion on the east, which we derived in the previous section using Young diagrams technology, is given in (5.35). This has the following $(m, 0)$ analytic continuation in the row lengths λ_i,

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_{i=1}^m (\mu_i - 1 - \theta(i-1-\alpha))\,(\mu_i - 1 - \theta(i-1-\beta))\,\mathbf{f}^{(i)}_{\underline{\mu}-\underline{e}_i}(\theta)\,(T_\gamma)^{\underline{\mu}-\underline{e}_i}_{\underline{\lambda}}}{\sum_{i=1}^m (\mu_i - \lambda_i)(\lambda_i + \mu_i + \theta\gamma - 2\theta(i-1) - 1)} \tag{6.1}$$

$$\lambda_i \in \mathbb{C}, \qquad \mu_i = \lambda_i + n_i, \qquad n_i \in \mathbb{Z}^+, \qquad (T_\gamma)^{\underline{\lambda}}_{\underline{\lambda}} = 1, \qquad (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = 0 \text{ if } n_i < 0$$

Here $\underline{\mu}$ and $\underline{\lambda}$ are parameters such that $\mu_i - \lambda_i = n_i$ is a positive integer, but $\lambda_i \in \mathbb{C}$ and $\underline{e}_i$ is the usual basis vector with a 1 in position i and zeroes elsewhere. As we discussed below (5.35) this recursion *automatically* gives $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = 0$ if $\mu_i = \mu_{i-1} + 1$. Thus if $\underline{\lambda}$ corresponds to a Young diagram, meaning that $\lambda_i - \lambda_{i-1} \in \mathbb{Z}^+$, then $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = 0$ unless $\underline{\mu}$ also corresponds to the row lengths of a Young diagram. This is a key point which allows the analytic continuation away from the Young diagram: if this was not the case, then the solution to the above recursion would not recover the correct solution when $\underline{\lambda}$ becomes a Young diagram. The expression for $\mathbf{f}^{(i)}_{\underline{\mu}}(\theta)$ is given in (5.30) and is rational in λ_i and θ. For fixed, integer $\mu_i - \lambda_i$, the recursion (6.1) has a solution which is manifestly rational in $\lambda_i, \theta, \alpha, \beta, \gamma$. We can thus view the solution of (6.1) as an analytic continuation of the solution of (5.35) which is valid only when λ_i are the row lengths of Young diagrams.

An important bonus of the solution of (6.1) is the manifest shift symmetry (2.20) (for any value of θ)

$$\lambda_i \to \lambda_i - \theta\tau', \qquad \mu_i \to \mu_i - \theta\tau', \qquad \gamma \to \gamma + 2\tau', \qquad i = 1, \dots m\ . \tag{6.2}$$

We can see this invariance immediately. First, the combinations $(\mu_i + \theta\alpha)$, $(\mu_i + \theta\beta)$, are manifestly invariant, since both α and β have the form $\frac{\gamma - p_{ij}}{2}$. Then, the combination of eigenvalues appearing in the denominator, i.e. $(\mu_i - \lambda_i)(\lambda_i + \mu_i + \theta\gamma - 2\theta(i-1) - 1)$, is also manifestly invariant. Finally, $\mathbf{f}_{\underline{\mu}-\underline{e}_i}$ only depends on differences $\mu_i - \mu_j$ and so is invariant.

The simplest analytic continuation we can think of is the $(1, 0)$ solution given in section 4.1, which we repeat here for convenience,

$$(T_\gamma)^{[\mu]}_{[\lambda]} = \frac{(\lambda + \theta\alpha)_{\mu-\lambda}(\lambda + \theta\beta)_{\mu-\lambda}}{(\mu-\lambda)!(2\lambda + \theta\gamma)_{\mu-\lambda}}\ . \tag{6.3}$$

We see that indeed, for fixed integer $\mu - \lambda$ this is a rational function of $\lambda, \theta, \alpha, \beta, \gamma$. (Of course, having an explicit solution we can now also see how to analytically continue in $\underline{\mu}$ if we desired, just by changing the Pochhammers to Gamma functions.)

Summarising, we have shown that defining $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ via (6.1) produces a rational (and hence) analytic function of $\underline{\lambda}, \theta, \alpha, \beta, \gamma$, which is also invariant under the shift symmetry (6.2).

6.2 Column-type representation on the south

An alternative analytic continuation of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ can be obtained by viewing the Young diagram as function of the column lengths λ'_j, rather than the row lengths λ_i. The column-type representation of the recursion on the south was derived in (5.36). It has the following $(0,n)$ analytic continuation,

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{\sum_{j=1}^n \left(\mu'_j - 1 - \alpha - \frac{j-1}{\theta}\right)\left(\mu'_j - 1 - \beta - \frac{j-1}{\theta}\right) \mathbf{g}^{(j)}_{\underline{\mu}'-\underline{e}_j}(\tfrac{1}{\theta})(T_\gamma)^{(\underline{\mu}'-\underline{e}_j)'}_{\underline{\lambda}}}{\sum_{j=1}^n (\lambda'_j - \mu'_j)(\lambda'_j + \mu'_j - \gamma - \frac{2}{\theta}(j-1) - 1)} \tag{6.4}$$

$$\lambda'_j \in \mathbb{C}, \qquad \mu'_j = \lambda'_j + n'_j, \qquad n'_j \in \mathbb{Z}^+, \qquad (T_\gamma)^{\underline{\lambda}}_{\underline{\lambda}} = 1, \qquad (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = 0 \text{ if } n'_j < 0$$

where the explicit formula for $\mathbf{g}$ was given already in (5.33) Similarly to the recursion on the east, for fixed integer $\mu'_j - \lambda'_j$, the solution is clearly a rational function of $\lambda'_j, \theta, \alpha, \beta, \gamma$.

The simplest solution of (6.4) we can look at is the $(0,1)$ solution studied in section 4.1. Namely

$$(T_\gamma)^{[1^{\mu'}]}_{[1^{\lambda'}]} = (-1)^{\mu'-\lambda'} \frac{(\lambda'-\alpha)_{\mu'-\lambda'}(\lambda'-\beta)_{\mu'-\lambda'}}{(\mu'-\lambda')!(2\lambda'-\gamma)_{\mu'-\lambda'}} \frac{(\lambda'+1)_{\mu'-\lambda'}}{(\lambda'+\frac{1}{\theta})_{\mu'-\lambda'}} \,. \tag{6.5}$$

Comparing with the row-type solution from (6.1) there is one small difference, which can appreciated also in (6.5). This $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ is not quite invariant under the $n=1$ case of the shift symmetry (6.2)

$$\lambda'_j \to \lambda'_j + \tau' \,, \qquad \mu'_j \to \mu'_j + \tau' \,, \qquad \gamma \to \gamma + 2\tau' \,, \qquad j = 1, \dots n \,. \tag{6.6}$$

The problem is a simple normalisation issue, due to the fact that we are expanding in $BC_{0,n}$ super Jack polynomials rather than directly in BC_n Jack polynomials. These two are essentially equal but have a different normalisation, (e.g see (C.47) with $m=0$)

$$P^{(n)}_{\underline{\lambda}'}(\mathbf{y}; \tfrac{1}{\theta}) = (-1)^{|\underline{\lambda}|} \Pi_{\underline{\lambda}}(\theta) P^{(0,n)}_{\lambda}(|\mathbf{y}; \theta) \,. \tag{6.7}$$

Thus the combination of superJack $\times$ coefficient T_γ does satisfy shift invariance.

6.3 Supersymmetric representation

We now come to a form of the recursion which gives analytic (indeed rational) results in the variables λ_i, λ'_j, the first m row lengths and first n column lengths, for long (or typical, the see discussion around (C.40) for these meanings) (m,n) Young diagrams $\underline{\lambda}$. This is suited for supersymmetric theory described by a (m,n) theory. It includes the previous cases as special cases by taking $m=0$ or $n=0$.

We first give a version of the recursion (5.16) for *long* reps $\underline{\lambda}$ in the (m,n) theory, which gives rational results in λ_i, λ'_j. To do this we split the RHS of the recursion into south and east components and define it as follows,

$$
\begin{aligned}
&\Big(h_{\underline{\mu}}-h_{\underline{\lambda}}+\theta\gamma\,(|\underline{\mu}|-|\underline{\lambda}|)\Big)(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \\
&\qquad \sum_{i=1}^{m}(\mu_i-1-\theta(i-1-\alpha))\,(\mu_i-1-\theta(i-1-\beta))\,\mathbf{f}^{(i)}_{\underline{\mu}-\underline{e}_i}(\theta)(T_\gamma)^{\underline{\mu}-\underline{e}_i}_{\underline{\lambda}}+ \\
&\qquad +\theta\sum_{j=1}^{n}\left(\mu'_j-1-\alpha-\tfrac{j-1}{\theta}\right)\left(\mu'_j-1-\beta-\tfrac{j-1}{\theta}\right)\mathbf{g}^{(j)}_{\underline{\mu}'-\underline{e}_j}(\tfrac{1}{\theta})(T_\gamma)^{(\underline{\mu}'-\underline{e}_j)'}_{\underline{\lambda}}
\end{aligned}
\tag{6.8}
$$

$$
\lambda_i,\lambda'_j\in\mathbb{C},\quad \mu'_j=\lambda'_j+n'_j, \mu_i=\lambda_i+n_i \quad n_i,n'_j\in\mathbb{Z}^+,\quad (T_\gamma)^{\underline{\lambda}}_{\underline{\lambda}}=1,\quad (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}=0 \text{ if } n_i,n'_j<0
$$

where (5.26) is used for $h_{\underline{\mu}}, h_{\underline{\lambda}}$ in the above. As in previous cases, for given integers n_i, n'_j this recursion yields a solution which is manifestly rational in $\lambda_i, \lambda'_j, \theta, \alpha, \beta, \gamma$ and therefore gives a supersymmetric-type analytic continuation in the variables λ_i, λ'_j of T_γ.

For example the explicit solution for the $(1,1)$ theory (relevant for 1d and 2d supersymmetric theories) is

$$
\begin{aligned}
(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = {} & \frac{(\mu'_1-1)!(\frac{1}{\theta})_{\lambda'_1-1}}{(\lambda'_1-1)!(\frac{1}{\theta})_{\mu'_1-1}}\frac{\theta^{2\lambda'_1-2\mu'_1}\,C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha;\theta)C^0_{\underline{\mu}/\underline{\lambda}}(\theta\beta;\theta)}{(\mu'_1-\lambda'_1)!(\mu_1-\lambda_1)!}\times \\
& \frac{(\lambda_1+\theta\lambda'_1-\theta)(\mu_1+\theta\mu'_1-1)+(1-\theta)(\lambda_1-\mu_1)\frac{(\theta\gamma+\lambda_1-\theta\mu'_1)}{(\theta\gamma+\lambda_1-\theta\lambda'_1)}}{(\lambda_1+\theta\lambda'_1-1)(\mu_1+\theta\mu'_1-\theta)(2\lambda_1+\theta\gamma)_{\mu_1-\lambda_1}(2\lambda'_1-\gamma)_{\mu'_1-\lambda'_1}(-1)^{\mu'_1-\lambda'_1}}
\end{aligned}
\tag{6.9}
$$

where $\underline{\mu}=[\mu_1,1^{\mu'_1-1}]$ and $\underline{\lambda}=[\lambda_1,1^{\lambda'_1-1}]$. Note that this $(1,1)$ coefficient reduces to the $(1,0)$ case for $\mu'_1=\lambda'_1=1$ and to the $(0,1)$ case for $\mu_1=\lambda_1=1$.

In appendix D we discuss in detail the shift symmetry of this form of the recursion relation and various other features.

We conclude with some comments.

Note that the recursion has been explicitly solved analytically for arbitrary λ_i and/or λ'_j in only a few cases: using both row and column variables, the determinantal formula of [9] for $\theta=1$, which we repeat in appendix E.3, and the $(1,1)$ solution presented above, with arbitrary θ. In this latter case the solution is precisely a formula depending on one row length and one column height.

In the next section 7 we will give a completely different description of the coefficients T_γ and T_γ^{-1}, closely related to the BC binomial formula of Okounkov [33] involving evaluating BC interpolation polynomials at partitions arising via the identification between $(0,n)$ blocks and BC_n Jacobi polynomials. It is tempting to speculate that this form should also have a supersymmetric representation, more suited for a (m,n) theory and indeed we will find that the super interpolation polynomials of Sergeev and Veselov [38] do the job, but only for T_γ^{-1} not T_γ itself.

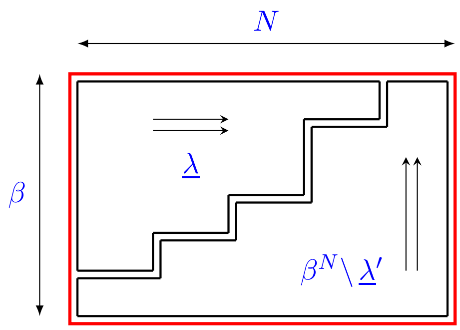
7 SCFT perspective on the binomial coefficient

In previous sections we defined superconformal blocks as an expansion in super Jack polynomials and solved for the coefficients of this expansion, T_γ, by setting up a recursion relation. As we pointed out in the overview, see section 2.3, the blocks can be viewed as supersymmetric generalisations of the BC Jacobi polynomials defined in [31, 41], which we called dual super Jacobi functions. Then, the claim about stability of superconformal blocks implies that the coefficients T_γ are directly related to the coefficients in the Jack expansion of BC_N Jacobi polynomials, $(S^{(N)}$. In our notation the explicit relation reads

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} \;=\; \frac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \times (S^{(N)}_{\frac{1}{\theta};p^{\pm}})^{\beta^N\backslash\underline{\lambda}'}_{\beta^N\backslash\underline{\mu}'} \,. \tag{7.1}$$

Specifically, the starting relation is the one between $(0, N)$ blocks and BC_N Jacobi polynomials (see (2.32)), and therefore (7.1) is valid for *any* integer $N \geq \mu_1$. Here we will use N rather than n to avoid confusion with the (m, n) labels of the superblock. Indeed, as emphasised in previous sections, we know that the $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ on the LHS of (7.1) do not depend on the particular (m, n) theory (when $\underline{\lambda}$ is a strict Young diagram), and therefore do not depend on the number of variables of the superblock. The crucial point about (7.1) is that the coefficients S can be given explicitly in terms of the BC_N interpolation polynomials P^* of Okounkov [33, 40].[33] evaluated at partitions. These polynomials are an important class of symmetric polynomials in N variables, whose definitions and properties are given explicitly below and in the appendices.

To begin appreciating the subtle aspects of (7.1) it is useful to picture how the Young diagrams involved change from LHS to RHS. On the LHS we have λ, μ and on the RHS we have their transposed complements in β^N



(7.2)

where $\beta^N\backslash\underline{\lambda}'$ is to be read along the columns, and bottom-up w.r.t. $\underline{\lambda}$.

At this point the relation (7.1) is quite intriguing because superficially the LHS appears to have many properties that are not manifest for the binomial coefficient on the RHS :

1. We know from the recursion that we can obtain the LHS as an explicit rational function of γ, whereas the RHS depends on γ only through the integer $\beta \equiv \min\left(\frac{1}{2}(\gamma - p_{12}), \frac{1}{2}(\gamma - p_{43})\right)$.

[33] In these refs, the more general Koornwinder polynomials are discussed, but the coefficients we are interested in are obtained by taking a limit on the Koornwinder polynomials as explained in [31, 41]

2. To obtain superblocks we sum over Young diagrams with arbitrarily large width μ_1. On the RHS this translates to arbitrarily large N and so we need interpolating polynomials with arbitrarily large number of variables which is very inefficient.

3. Moreover, the LHS depends mostly on the skew diagram $\underline{\mu}/\underline{\lambda}$ (indeed the recursion of the previous sections is solved along this skew diagram), the RHS instead depends on the increasing large (as N gets large, and also when β is large) Young diagrams $\beta^N \backslash \underline{\lambda}'$ and $\beta^N \backslash \underline{\mu}'$. Again this looks very inefficient.

From the above considerations we expect the binomial coefficient, $S^{(N)}$, to have various non-trivial hidden properties, in order to recover those of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$.

Our plan is thus the following: After giving a detailed definition of the binomial coefficient in terms of interpolating polynomials we will examine the above properties from this perspective. In the process we will obtain a rewriting of the binomial coefficient which deals with the first two of these points. Furthermore we will find that the inverse coefficient $(T_\gamma^{-1})^{\underline{\mu}}_{\underline{\lambda}}$ i.e. the coefficient in the expansion of a super Jack polynomial in terms of blocks, has a form which deals with all three points.

7.1 The binomial coefficient

Recall (2.25), we wrote the coefficients of a BC_N Jacobi polynomial, expanded in terms of Jack polynomials,

$$J_{\underline{\lambda}}(y_1,\ldots,y_N;\theta,p^-,p^+) = \sum_{\underline{\mu}\subseteq\underline{\lambda}} (S^{(N)}_{\theta,p^-,p^+})^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{y};\theta) \ . \tag{7.3}$$

Now these binomial coefficients $S^{(N)}$ have been well studied and are known to be given, up to normalisations, by BC_N interpolation polynomials $P^*_{\underline{\mu}}$ evaluated on $\underline{\lambda}$. The interpolation polynomials themselves are symmetric polynomials, depending on a single parameter u, and are uniquely defined [32, 33] by the property that they vanish when evaluated on a partition $\underline{\lambda}$ which is strictly contained in $\underline{\mu}$, i.e.

$$P^*_{\underline{\mu}}(\underline{\lambda};\theta,u) = 0 \qquad \text{if} \quad \underline{\lambda} \subset \underline{\mu} \ . \tag{7.4}$$

Quite remarkably, the interpolation polynomials admit a combinatorial definition which generalises the one of Jack polynomials in a simple way. See appendix C.3.

Now for some normalisation of the Jacobi polynomial, say $\hat{J}_{\underline{\lambda}}(\mathbf{y}) = J_{\underline{\lambda}}(\mathbf{y})/N_{\underline{\lambda}}$, and some normalisation of the Jack polynomials, say $\hat{P}_{\underline{\mu}}(\mathbf{y}) = P_{\underline{\mu}}(\mathbf{y})/M_{\underline{\mu}}$, the corresponding binomial coefficients in the expansion of $\hat{J}_{\underline{\lambda}}$ in the $\hat{P}_{\underline{\mu}}$ really are simply interpolating polynomials with the variables taking the values of the Young diagram $\underline{\lambda}$

$$\hat{J}_{\underline{\lambda}}(y_1,\ldots,y_N;\theta,A_1,A_2) = \sum_{\underline{\mu}\subseteq\underline{\lambda}} P^{*(N)}_{\underline{\mu}}(\underline{\lambda};\tfrac{A_1+A_2+1}{2})\,\hat{P}_{\underline{\mu}}(\mathbf{y};\theta) \ . \tag{7.5}$$

Returning to the standard choice of normalisations (such that the coefficient of $P_{\underline{\lambda}}$ in the expansion (2.25) is 1 and that $P_{\underline{\mu}}(\underline{0}) = \delta_{\underline{\mu},\varnothing}$) we can see that the normalisation of S will have the more complicated looking form:

$$(S^{(N)}_{\theta,\vec{A}})^{\underline{\mu}}_{\underline{\lambda}} = \frac{J_{\underline{\lambda}}(\underline{0};\theta,A_1,A_2)}{J_{\underline{\mu}}(\underline{0};\theta,A_1,A_2)} \frac{P^{*(N)}_{\underline{\mu}}(\underline{\lambda};\theta,u)}{P^{*(N)}_{\underline{\mu}}(\underline{\mu};\theta,u)}\bigg|_{u=\frac{A_1+A_2+1}{2}} . \tag{7.6}$$

However, the only non trivial part of (7.6) is still $P^{*(N)}_{\underline{\mu}}(\underline{\lambda};\theta,u)$! The remaining objects entering this formula are explicitly known formulae which are all products of linear factors in the parameters

$$P^{*(N)}_{\underline{\mu}}(\underline{\mu};\theta,u) = C^{+}_{\underline{\mu}}(2u{-}1{+}2\theta N;\theta)C^{-}_{\underline{\mu}}(1;\theta) \tag{7.7}$$

$$J^{(N)}_{\underline{\lambda}}(\underline{0};\theta,A_1,A_2) = \frac{(-)^{|\underline{\lambda}|}\ C^0_{\underline{\lambda}}(A_1{+}1{+}\theta(N-1);\theta)C^0_{\underline{\lambda}}(\theta N;\theta)}{P^{*}_{\underline{\lambda}}(\underline{\lambda};\theta,u)\Pi_{\underline{\lambda}}(\theta)}\bigg|_{u=\frac{A_1+A_2+1}{2}} , \tag{7.8}$$

where the combinatorial symbols $C^{0,\pm}$ are

$$\begin{aligned} C^0_{\underline{\lambda}}(w;\theta) &= \prod_{(ij)\in\underline{\lambda}} (j{-}1-\theta(i{-}1)+w) \\ C^-_{\underline{\lambda}}(w;\theta) &= \prod_{(ij)\in\underline{\lambda}} \big(\lambda_i{-}j+\theta(\lambda'_j{-}i)+w\big) \\ C^+_{\underline{\lambda}}(w;\theta) &= \prod_{(i,j)\in\underline{\lambda}} (\lambda_i+j-\theta(\lambda'_j+i)+w), \end{aligned} \tag{7.9}$$

and

$$\Pi_{\underline{\lambda}}(\theta) \equiv C^-_{\underline{\lambda}}(\theta;\theta)/C^-_{\underline{\lambda}}(1;\theta) \qquad ; \qquad \Pi_{\underline{\lambda}}(\tfrac{1}{\theta}) = 1/\Pi_{\underline{\lambda}'}(\theta). \tag{7.10}$$

Although these evaluation formulae may look complicated at first sight, they consist simply of products of linear factors. For later use we will sometimes use the notation $C^0_{\underline{\lambda}}(A,B;\theta) = C^0_{\underline{\lambda}}(A;\theta)C^0_{\underline{\lambda}}(B;\theta)$ and and $C^0_{\underline{\mu}/\underline{\lambda}} \equiv C^0_{\underline{\mu}}/C^0_{\underline{\lambda}}$ whenever $\underline{\lambda} \subseteq \underline{\mu}$.

Putting the above definitions in we arrive at the expression for the binomial coefficient

$$\big(S^{(N)}_{\theta,\vec{A}}\big)^{\underline{\mu}}_{\underline{\lambda}} = (-)^{|\underline{\lambda}|-|\underline{\mu}|}\frac{\Pi_{\underline{\mu}}(\theta)}{\Pi_{\underline{\lambda}}(\theta)}C^0_{\underline{\lambda}/\underline{\mu}}(A_1{+}1{+}\theta(N{-}1),\theta N;\theta) \times \frac{P^{*(N)}_{\underline{\mu}}(\underline{\lambda};\theta,u)}{P^{*(N)}_{\underline{\lambda}}(\underline{\lambda};\theta,u)}\bigg|_{u=\frac{A_1+A_2+1}{2}} . \tag{7.11}$$

Note that the P^* in the denominator now involves $P^*_{\underline{\lambda}}$.

7.2 Binomial representation of the Jack→Block matrix

From this expression (7.11) we can obtain an expression for T_γ in terms of interpolation polynomials via (7.1). Manipulating this with the help of some identities for interpolation polynomials of Rains [32] (the details will be given shortly), we obtain:

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = (\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}} \times \frac{P^{*(M)}_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\lambda};\theta,u)}{P^{*(M)}_{N^M\backslash\underline{\lambda}}(N^M\backslash\underline{\lambda};\theta,u)}\bigg|_{u=\frac{1}{2}-\theta\frac{\gamma}{2}-N} . \tag{7.12}$$

The normalisation is

$$(\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}} = C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha,\theta\beta;\theta) \times \frac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \, \frac{C^0_{\underline{\mu}/\underline{\lambda}}(1+\theta(M-1);\theta)}{C^0_{\underline{\mu}/\underline{\lambda}}(M\theta;\theta)} \tag{7.13}$$

where α, β are as defined in (D.2), and the symbols Π and C^0 are given in (7.9).

Crucially the whole expression is independent of M, N as long as the rectangle N^M contains the Young diagram $\underline{\mu}$ (which in turn contains $\underline{\lambda}$). For practical computations we would always choose the minimal case $M = \mu'_1$ and $N = \mu_1$.

Note that the manipulations we performed have produced a formula which solves the first two problems listed on page 47. Firstly, the non factorisable γ dependence of T_γ now appears entirely in the parameter of $P^*_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\lambda};\theta,u)$, and can thus be continued to arbitrary non-integer values. This polynomial is precisely the complicated polynomial in γ in $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ discussed in section 5.4. Secondly, when the Young diagrams $\underline{\lambda}$ and $\underline{\mu}$ are arbitrarily wide with limited height, i.e. less than or equal to β for superblocks, the dimension of the interpolating polynomials M is limited (since we can choose $M = \beta$ for example) and does not grow indefinitely as for the original formula.

Finally, note that the bosonic $(m, 0)$ case of the shift symmetry (D.1) is manifest in this formula: choosing $N = \mu_1$, we see that γ only appears through the combination $\mu_1 + \theta\frac{\gamma}{2}$ and the row lengths λ_i, μ_i only appear as differences with μ_1 via $\mu_1^M\backslash\underline{\lambda}$ and $\mu_1^M\backslash\underline{\mu}$ all of which leave the shift invariant.

Proof of (7.12)

To show that (7.1) is equivalent to (7.12) we will need two formulae which can both be found in [32], by taking the appropriate classical limits. The first relates BC_N interpolation polynomials with a Young diagram $\underline{\kappa}$ to one with a Young diagram with additional ν full (ie height N) columns

$$P^{*(N)}_{\underline{\kappa}+\nu^N}(w_1\dots w_N;\theta,s) = \prod_{i=1}^{N}\prod_{j=1}^{\nu}(w_i^2-(s+j-1)^2)\ P^{*(N)}_{\underline{\kappa}}(w_1\dots w_N;\theta,s+\nu)\ . \tag{7.14}$$

The second relation is an identity between a BC_N interpolation polynomial with Young diagram $\underline{\mu}$ evaluated on a Young diagram $\underline{\kappa}$ and a BC_M interpolation polynomial with transposed Young diagram $\underline{\mu}'$ evaluated on the transposed Young diagram $\underline{\kappa}'$. Here M, N can be arbitrary (this freedom to choose M, N in fact arises from the previous relation) as long as they are big enough to contain the relevant Young diagrams, so $M \geq \mu'_1$ and $N \geq \mu_1$. It reads

$$\begin{aligned} \frac{P^{*(N)}_{\underline{\mu}}(\underline{\kappa};\theta,u)}{P^{*(N)}_{\underline{\mu}}(\underline{\mu};\theta,u)} &= \frac{P^{*(M)}_{\underline{\mu}'}(\underline{\kappa}';\frac{1}{\theta},u')}{P^{*(M)}_{\underline{\mu}'}(\underline{\mu}';\frac{1}{\theta},u')} \\ \text{where}\qquad \theta u' + (M-\tfrac{1}{2}) &= -u-\theta(N-\tfrac{1}{2})\ . \end{aligned} \tag{7.15}$$

We can now use (7.14) to remove the explicit β dependence in the Young diagram of the non-trivial P^* term in (7.1) with (7.11). Taking $\nu = M - \beta$ positive we find

$$P^{*(N)}_{\beta^N\backslash\underline{\mu}'}(\beta^N\backslash\underline{\lambda}';\theta^{-1},\tfrac{1}{2}(p_+ + p_- + 1)) = P^{*(N)}_{M^N\backslash\underline{\mu}'}(M^N\backslash\underline{\lambda}';\theta^{-1},\tfrac{1}{2}(\gamma+1)-M)\times\mathcal{N}_1 \tag{7.16}$$

where we recalled the relation between $\beta, \gamma, p_\pm$ in (2.31) to get this. Note however that this equation is true for both ν positive or negative and indeed is valid for any $M \geq \mu_1$. The normalisation $\mathcal{N}_1$ is given in terms of the known evaluation formula (7.7) for $P^*_\mu(\mu)$ as

$$\mathcal{N}_1 = \frac{P^{*(N)}_{\beta^N\backslash\underline{\lambda}'}(\beta^N\backslash\underline{\lambda}';\frac{1}{\theta},\frac{1}{2}(p_+ + p_- + 1))}{P^{*(N)}_{M^N\backslash\underline{\lambda}'}(M^N\backslash\underline{\lambda}';\frac{1}{\theta},\frac{1}{2}(\gamma+1)-M)} \tag{7.17}$$

Next we use the relation (7.15) to transpose the Young diagrams. Thus obtaining

$$P^{*(N)}_{M^N\backslash\underline{\mu}'}(M^N\backslash\underline{\lambda}';\,\tfrac{1}{\theta},\tfrac{1}{2}(\gamma+1)-M) = P^{*(M)}_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\lambda};\,\theta,\tfrac{1}{2}(1-\gamma\theta)-N)\times\mathcal{N}_2\ , \tag{7.18}$$

with the normalisation $\mathcal{N}_2$ also given in terms of explicitly known formulae as

$$\mathcal{N}_2 = \frac{P^{*(N)}_{M^N\backslash\underline{\mu}'}(M^N\backslash\underline{\mu}';\,\frac{1}{\theta},\frac{1}{2}(\gamma+1)-M)}{P^{*(M)}_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\mu};\,\theta,\frac{1}{2}(1-\gamma\theta)-N)}\ . \tag{7.19}$$

So inserting these into the formula for T_γ (7.1) using the definition of the binomial coefficient (7.6) we obtain (7.12) with the normalisation:

$$(\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}} = \frac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)}\times\frac{J_{\beta^N\backslash\underline{\lambda}'}(\underline{0};\frac{1}{\theta},p^-,p^+)}{J_{\beta^N\backslash\underline{\mu}'}(\underline{0};\frac{1}{\theta},p^-,p^+)}\times\mathcal{N}_1\mathcal{N}_2\times\frac{P^{*(M)}_{N^M\backslash\underline{\lambda}}(N^M\backslash\underline{\lambda};\theta,\frac{1}{2}(1-\gamma\theta)-N)}{P^{*(N)}_{\beta^N\backslash\underline{\mu}'}(\beta^N\backslash\underline{\mu}';\frac{1}{\theta},\frac{1}{2}(p_+ + p_- + 1)))}\ . \tag{7.20}$$

This normalisation, despite its complicated appearance, is just made of known linear factors. Moreover, it dramatically simplifies!! We know this in advance because it is clear from the recursion for T_γ that only the γ dependence is non factorisable and complicated. In particular all factors of C^+ occurring in the evaluation formulae for (7.7) appear in ratios with transposed Young diagrams and appropriate parameters such that they cancel using the relations

$$\begin{aligned} C^0_{\underline{\mu}}(x;\theta) &= (-\theta)^{|\underline{\mu}|}\ C^0_{\underline{\mu}'}(-\tfrac{x}{\theta};\tfrac{1}{\theta}) \\ C^-_{\underline{\mu}}(x;\theta) &= (+\theta)^{|\underline{\mu}|}\ C^-_{\underline{\mu}'}(+\tfrac{x}{\theta};\tfrac{1}{\theta}) \\ C^+_{\underline{\mu}}(x;\theta) &= (-\theta)^{|\mu|}\ C^+_{\underline{\mu}'}(-\tfrac{x}{\theta};\tfrac{1}{\theta})\ , \end{aligned} \tag{7.21}$$

together with the identities

$$\frac{C^0_{N^M\backslash\underline{\mu}}(x;\theta)}{C^0_{N^M}(x;\theta)} = \frac{(-)^{|\underline{\mu}|}}{C^0_{\underline{\mu}}(1-N+\theta(M-1)-x;\theta)}\ , \tag{7.22}$$

and

$$\frac{\Pi_{N^M\backslash\underline{\mu}}(\theta)\Pi_{N^\beta\backslash\underline{\lambda}}(\theta)}{\Pi_{N^M\backslash\underline{\lambda}}(\theta)\Pi_{N^\beta\backslash\underline{\mu}}(\theta)} = \frac{C^0_{\underline{\mu}/\underline{\lambda}}(\beta\theta;\theta)}{C^0_{\underline{\mu}/\underline{\lambda}}(M\theta;\theta)} \times \frac{C^0_{\underline{\mu}/\underline{\lambda}}(1{+}\theta(M-1);\theta)}{C^0_{\underline{\mu}/\underline{\lambda}}(1{+}\theta(\beta-1);\theta)} \tag{7.23}$$

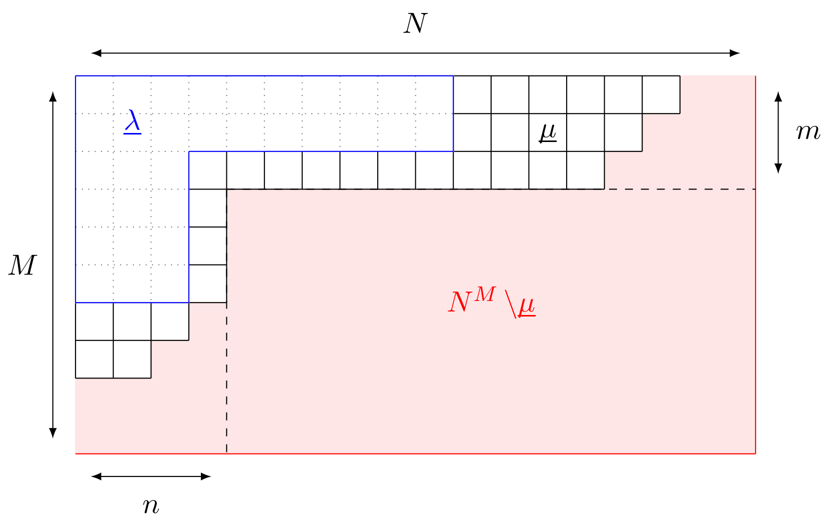
The remaining products of linear factors give the much simpler expression quoted in (7.13).

Comparing the recursion formula with the binomial coefficient

The binomial formula for T_γ in (7.12) can be tested, on a computer, against the recursion of section 5, very much exhaustively. It can also be tested analytically in all known cases in which the recursion has been solved explicitly. These are: rank-one $(1,0)$ and $(0,1)$ cases for any θ; rank-two $(2,0)$ for any θ and finally (m,n) for $\theta=1$. We explicitly check all of them in appendix E, where we refer the reader for a detailed discussion. The computation is instructive since, as we pointed out in various occasions, the interpolation polynomials and the recursion are based on two very different combinatorics, therefore there will be many non trivial operations involved in order for the two to agree.

7.3 Inverse Jack→Block matrix and complementation

We now have two completely different methods for computing the Jack→Block matrix T_γ, via the binomial formula (7.12) in terms of interpolation polynomials or via the recursion of section 5. The recursion has the advantage that in order to compute $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$, we only run over Young diagrams in between $\underline{\mu}/\underline{\lambda}$, and therefore we use the minimum number of steps. Moreover, in the super case the steps respect the (m,n) hook structure of the Young diagrams. This structure is broken in the binomial formula (7.12) because we take the complement Young diagrams $N^M\backslash\underline{\lambda}$. For physical applications, where m,n are small, this is not efficient because the diagrams generically will have a slim hook structure, but in order to compute the P^* we would need to consider the red diagram in the following picture,



(7.24)

which can become quite ‘fat’, compared to the skew shape $\underline{\mu}/\underline{\lambda}$.

In a number of applications, (T_γ^{-1}), the inverse of the Jack→Block matrix, shows up and plays an important role. This matrix is such that

$$P_{\underline{\lambda}}(\mathbf{z}) = \sum_{\underline{\mu}:\underline{\lambda}\subseteq\underline{\mu}} (T_\gamma^{-1})_{\underline{\lambda}}^{\underline{\mu}}\, F_{\gamma,\underline{\mu}}(\mathbf{z}) \tag{7.25}$$

where as usual we understand the dependence on θ, p_{12}, p_{43} in $F_{\gamma,\underline{\mu}}$. Note that when $m > 0$ the Jack polynomials is represented as an infinite sum. We will now see that the inverse of the Jack→Block matrix, the Block→Jack matrix, has a very nice interpretation in terms of interpolation polynomials, and avoids the above mentioned problems with complementation.

An illuminating way of obtaining the Block→Jack matrix is to apply the Casimir (5.15) and obtain a recursion very similar to the recursions for T_γ, see (5.18). By carefully examining and comparing the two recursions (5.16) and (5.18), in particular the denominator term in each, one can deduce that there is a direct relation between T_γ^{-1} and $T_{\tilde{\gamma}}$ with complemented Young diagrams. This complementation then accounts for the difference that in (5.16) we sum over Young diagrams with one subtracted box, whereas in (5.18) we sum over Young diagrams with one added box. The precise relation is then

$$(T_\gamma^{-1})_{\underline{\lambda}}^{\underline{\mu}}(\theta, p_{12}, p_{43}) = \frac{C^0_{(N^M\backslash\underline{\lambda})/(N^M\backslash\underline{\mu})}(M\theta;\theta)}{C^0_{(N^M\backslash\underline{\lambda})/(N^M\backslash\underline{\mu})}(1+\theta(M-1);\theta)} \times \frac{\Pi_{N^M\backslash\underline{\mu}}(\theta)}{\Pi_{N^M\backslash\underline{\lambda}}(\theta)}\,(T_{\tilde{\gamma}})_{N^M\backslash\underline{\lambda}}^{N^M\backslash\underline{\mu}}(\theta, -p_{12}, -p_{43}) \tag{7.26}$$

where

$$\tilde{\gamma} = -\gamma + 2(M-1) - \tfrac{2}{\theta}(N-1)\ . \tag{7.27}$$

This identity is valid for any γ, p_{12}, p_{43} and $N \geq \mu_1$, $M \geq \mu_1'$. The intuition behind (7.26) is that $\underline{\mu}/\underline{\lambda}$ and $(N^M\backslash\underline{\lambda})/(N^M\backslash\underline{\mu})$ represent the same skew diagram, seen from different orientations. Then, the map $\gamma \to \tilde{\gamma}$ is the only real non trivial transformation in going from left to right in (7.26), since the γ dependence is the non factorisable one. Analyticity in γ of T_γ is also crucial, since γ might be an integer but $\tilde{\gamma}$ is certainly not. The factors of C^0 can be eventually rewritten/simplified by using (7.22)-(7.23).[34] The form given above is useful for the computation that follows.

Now notice that the binomial formula for the Jack→Block matrix, T_γ (7.12) involves complemented Young diagrams, as does the RHS formula (7.26) for T_γ^{-1} in terms of T_γ. We thus conclude that the *inverse* matrix T_γ^{-1} is very naturally computed by a binomial coefficient, given by interpolation polynomials directly in $\underline{\mu}$ and $\underline{\lambda}$! In fact, by using (7.12) on the RHS of (7.26), a number of simplifications take place,[35] and the final formula is very

[34] For example, the RHS can be transposed by using

$$(T_{-\theta\gamma})_{\underline{\lambda}'}^{\underline{\mu}'}(\tfrac{1}{\theta}, -\theta p_{12}, -\theta p_{43}) = \frac{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)}{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}(T_\gamma)_{\underline{\lambda}}^{\underline{\mu}}(\theta, p_{12}, p_{43})$$

which follows from $\mathbf{C}^{(\theta,a,b,c)}(\mathbf{x}|\mathbf{y}) = -\theta\, \mathbf{C}^{(\frac{1}{\theta},-a\theta,-b\theta,-c\theta)}(\mathbf{y}|\mathbf{x})$, and properties of the (super) Jack polynomials under transposition.

[35] Upon writing $(T_{\tilde{\gamma}})_{N^M\backslash\underline{\lambda}}^{N^M\backslash\underline{\mu}}(\theta, -p_{12}, -p_{43})$ note that $C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha, \theta\beta; \theta)$ is recovered by using (7.22).

clean and reads:

$$(T_\gamma^{-1})_{\underline{\lambda}}^{\underline{\mu}} = \frac{(-)^{|\underline{\mu}|}}{(-)^{|\underline{\lambda}|}} C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha, \theta\beta; \theta) \left. \frac{P^{*(M)}_{\underline{\lambda}}(\underline{\mu}; \theta, v)}{P^{*(M)}_{\underline{\mu}}(\underline{\mu}; \theta, v)} \right|_{v=-\frac{1}{2}+\theta\frac{\gamma}{2}-\theta(M-1)} \tag{7.28}$$

where $P^*_{\underline{\lambda}}$ are BC_M interpolation polynomials with any $M \geq \mu'_1$ and α, β, as always, given by (D.2). Note, the formula above is independent of M with the minimal choice $M = \mu'_1$. One can also deduce this relation from formulae in Rains [32], together with the identification (7.12), where it is seen that both the binomial coefficient and its inverse are given by interpolation polynomials with complemented Young diagrams (see [32] (4.1),(4.2),(4.22),(4.23)). Observe that this expression for T_γ^{-1} is very similar to the expression for the binomial coefficients S written in (7.11). We will make use of this observation when examining the Cauchy identity in the next section.

The nice feature of (7.28) is that the Young diagrams labelling the interpolation polynomials are now $\underline{\lambda}$ and $\underline{\mu}$ rather than their complements. As a result, computing T_γ^{-1} by using the binomial coefficient is also quite efficient. Finally, the evaluation of the interpolation polynomial in the numerator, $P^*_{\underline{\lambda}}(\underline{\mu}; \theta, v)$, is again the only term not involving products of linear factors, and depends non trivially on γ.

Even though (7.28) has nice properties, it still has the inconvenience that the number of variables in the interpolation polynomials grows with M. Therefore, if we are considering blocks with large β, since $\mu_1 \leq \beta$ and $M \geq \mu'_1$, with $M = \mu'_1$ being the minimal choice, we will will have to consider interpolation polynomials with fixed but large number of variables. In fact, (7.28) is still a bosonic formula, which reads the Young diagram only by rows, whereas in the super block the typical Young diagram is a (m, n) hook. We can then ask if there is an interpolation polynomial type formula which respects the (m, n) structure. Nicely enough, this problem has a simple solution which is suggested by the independence of the dimension of the interpolation polynomials, M, of the expression T^{-1} (7.28). This stability property immediately suggests that one might be able to replace the bosonic interpolation polynomials with *super*-interpolation polynomials, where the latter have a natural hook-type parametrisation of their variables.

Sergeev and Veselov introduced the $BC_{M|N}$ super interpolation polynomials in [38] that we need. They first modified the free parameter on which the interpolation polynomials depend, to $h = v + \theta M$ and defined the modified interpolation polynomial $\tilde{P}^{*(M)}_{\underline{\lambda}}$ as

$$\tilde{P}^{*(M)}_{\underline{\lambda}}(\mathbf{x}; \theta, h) := P^{*(M)}_{\underline{\lambda}}(\mathbf{x}; \theta, h-\theta M) \ . \tag{7.29}$$

They then showed that these modified IPs have a natural supersymmetric generalisation, denoted by $\tilde{P}^{*(M|N)}_{\underline{\lambda}}(\mathbf{x}|\mathbf{y}; \theta, h)$, where taking $N = 0$ returns the bosonic interpolation polynomial

$$\tilde{P}^{*(M)}_{\underline{\lambda}}(\mathbf{x}; \theta, h) = \tilde{P}^{*(M|0)}_{\underline{\lambda}}(\mathbf{x}|\ ; \theta, h) \ . \tag{7.30}$$

We spell out the conventions [38] in appendix C.7.

In [38] these super polynomials were used to obtain super binomial coefficients, i.e. the coefficients of the expansion of super Jacobi polynomials in terms of super Jacks. We will return to this point in the next section. Here we notice that stability of T_γ^{-1} in (7.28) gives further insight into these objects. Indeed, we understand from stability that the ratio of interpolation polynomials in (7.28) is independent of M,

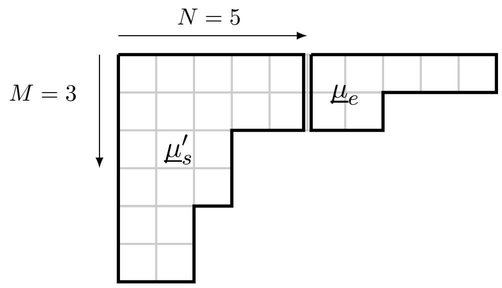
$$\frac{\tilde{P}^{*(M)}_{\underline{\lambda}}(\underline{\mu};\theta,h)}{\tilde{P}^{*(M)}_{\underline{\mu}}(\underline{\mu};\theta,h)} = \frac{\tilde{P}^{*(\mathsf{M})}_{\underline{\lambda}}(\underline{\mu};\theta,h)}{\tilde{P}^{*(\mathsf{M})}_{\underline{\mu}}(\underline{\mu};\theta,h)} \tag{7.31}$$

for any $\mathsf{M}, M \geq \mu'_1$. This in turn suggests that one can further replace this ratio with the appropriate ratio of *super* interpolation polynomials evaluated on Young diagrams. Namely,

$$\frac{\tilde{P}^{*(\mathsf{M})}_{\underline{\lambda}}(\underline{\mu};\theta,h)}{\tilde{P}^{*(\mathsf{M})}_{\underline{\mu}}(\underline{\mu};\theta,h)} = \frac{\tilde{P}^{*(M|N)}_{\underline{\lambda}}(\underline{\mu}_e|\underline{\mu}_s;\theta,h)}{\tilde{P}^{*(M|N)}_{\underline{\mu}}(\underline{\mu}_e|\underline{\mu}_s;\theta,h)}\ . \tag{7.32}$$

where $\underline{\mu}_s = [\mu'_1, .., \mu'_N]$ and $\underline{\mu}_e \equiv \underline{\mu}/\underline{\mu}'_s = [(\mu_1 - N)_+, .., (\mu_M - N)_+]$ with $(x)_+ \equiv \max(x,0)$.

An example illustrating the decomposition of $\underline{\mu}$ into $\underline{\mu}_s$ and $\underline{\mu}_e$ relevant here is the following,



$$\begin{gathered}\underline{\mu} = [10,7,3,3,2,2]\\ \underline{\mu}_s = [6,6,4,2,2]\\ \underline{\mu}_e = [5,2,0]\end{gathered} \tag{7.33}$$

We have checked extensively that (7.32) equality is true.

Putting the above reasoning together, we conclude that the inverse block coefficients can be written in terms of super-interpolation polynomials as

$$(T^{-1}_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{(-)^{|\underline{\mu}|}}{(-)^{|\underline{\lambda}|}} C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha,\theta\beta;\theta)\, \frac{\tilde{P}^{*(M|N)}_{\underline{\lambda}}(\underline{\mu}_e|\underline{\mu}_s;\theta,v)}{\tilde{P}^{*(M|N)}_{\underline{\mu}}(\underline{\mu}_e|\underline{\mu}_s;\theta,v)}\Bigg|_{v=-\frac{1}{2}+\theta\frac{\gamma}{2}+\theta}\ , \tag{7.34}$$

and this expression is independent of M, N.

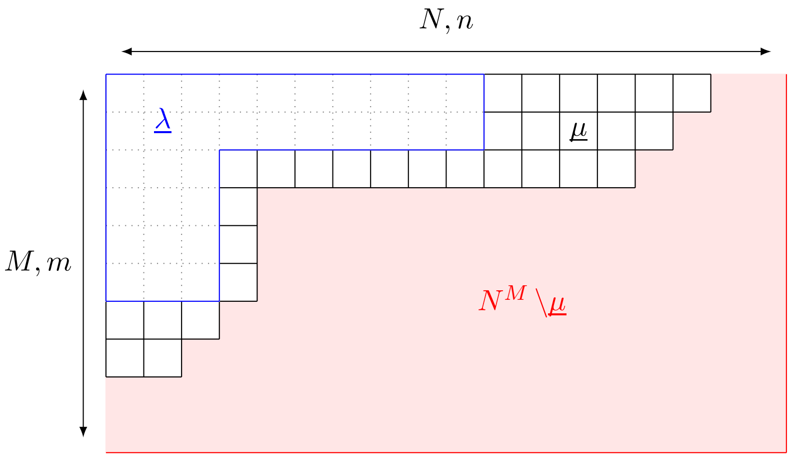
Thus in the expansion of an (m,n) superblock we are free to take $M = m$ and $N = n$, and then we will only ever need to evaluate super interpolation polynomials in a fixed number $(m+n)$ of variables.

An interesting application of (7.34) for the inverse matrix will involve a transformation of the dual Cauchy identity. This is explained in the next section, and will lead to a formula for decomposing *any* free theory diagram into superconformal blocks for any θ, therefore any dimension.

8 Generalised free theory from a Cauchy identity

The most straightforward application of the identification of superblocks with dual super Jacobi functions consists in borrowing from the maths literature a Cauchy identity for Jacobi polynomials, found by Mimachi [42], and re-interpreting it to yield a formula for *every* coefficient in the superblock expansion of *any* generalised free theory in *any* theory of the type described by this formalism (summarised in section 3.1). Inspired by this we also conjecture the uplift of this Cauchy identity to a new dually supersymmetric Cauchy identity.

In the following we will have in mind the following picture,



(8.1)

as in the previous section.

8.1 Super Cauchy identity

Let us consider eq. (4.2) of Mimachi [42], which gives a Cauchy identity involving BC_n and BC_M Jacobi polynomials. Looking at that identity, we immediately see that out of the two Jacobi polynomials entering the formula, one depends on a complemented Young diagram $M^n \backslash \underline{\lambda}'$ and is more naturally written in terms of our *dual* Jacobi polynomial (2.23). Mimachi's Cauchy formula then reads

$$\prod_i^M \prod_j^n (1 - y_i y'_j) = \sum_{\underline{\lambda}} (-1)^{|\underline{\lambda}|} J^{(M)}_{\underline{\lambda}}(\mathbf{y}; \theta, \tilde{p}^-, \tilde{p}^+) \tilde{J}^{(n)}_{M,\underline{\lambda}'}(\mathbf{y}'; \tfrac{1}{\theta}, p^-, p^+) \tag{8.2}$$

$$\tilde{p}^- = \theta p^- + \theta - 1 \qquad ; \qquad \tilde{p}^+ = \theta p^+ + \theta - 1 \ . \tag{8.3}$$

Note that the sum on the RHS is automatically cut off, so that $\underline{\lambda} \subseteq n^M$. Indeed, the BC_M Jacobi polynomial, $J_{\underline{\lambda}}$ vanishes if $\underline{\lambda}$ has more than M rows and the BC_n dual Jacobi $\tilde{J}_{M,\underline{\lambda}'}$ vanishes if $\underline{\lambda}'$ has more than n rows. Also note that the LHS is manifestly stable in both M and n: setting $y_M = 0$ or $y_n = 0$ yields the same formula with $M \to M-1$ or $n \to n-1$. The RHS is also manifestly stable in n since, as we know, the dual Jacobi's $\tilde{J}$ are stable. However the RHS is not manifestly stable in M. It depends on M both in the Jacobi $J^{(M)}$, which is not stable, and in $\tilde{J}_{M,\underline{\lambda}'}$. Remarkably these two M dependencies must precisely cancel each other out! To understand how this is possible, in the next subsection we will re-derive (8.2) starting from the simpler and well known Cauchy identity involving Jack polynomials. The derivation will then explain the stability property in N and also explain the appearance of the modified parameters $\tilde{p}^{\pm}$.

But first let us motivate various supersymmetric generalisations of this Jacobi Cauchy identity which we will explicitly prove in the next section. Through stability, we know that dual Jacobi polynomials have a natural uplift to dual super Jacobi functions. Thus we can uplift $\tilde{J}^{(n)}_{M,\underline{\lambda}'}(\mathbf{y}')$ on the RHS of the previous formula (8.2) to (n,m) variables. Regarding the LHS of (8.2), the eigenvalue interpretation of the variables in the super matrix formalism for $\theta = 1, 2, \frac{1}{2}$ (see appendix A) provides intuition for the corresponding uplift. Namely, we interpret the LHS as the product of determinants $\prod_i \det(1 - y_i Z)$ where Z is the diagonal matrix with eigenvalues y'_j, and therefore we simply lift the matrix Z to the supermatrix (see (A.1),(A.6),(A.11)).

This then leads to propose the following supersymmetric Jacobi Cauchy identity

$$\prod_{i}^{M} \frac{\prod_j^n (1 - y_i y'_j)}{\prod_k^m (1 - y_i x'_k)^\theta} = \sum_{\underline{\lambda}} (-1)^{|\underline{\lambda}|} J^{(M)}_{\underline{\lambda}}(\mathbf{y}; \theta, \tilde{p}^-, \tilde{p}^+) \tilde{J}^{(n|m)}_{M,\underline{\lambda}'}(\mathbf{y}'|\mathbf{x}'; \tfrac{1}{\theta}, p^-, p^+) \;, \tag{8.4}$$

involving BC_N Jacobi polynomials and $BC_{n|m}$ super dual Jacobi functions.

Now, the dual super Jacobi functions on the RHS of (8.4) are simply superblocks (2.30). Thus, redefining variables this becomes

$$\prod_{r=1}^{\beta} \frac{\prod_{j=1}^n (1 - s_r y_j)}{\prod_{i=1}^m (1 - s_r x_i)^\theta} = \sum_{\underline{\lambda}} J^{(\beta)}_{\underline{\lambda}}(\mathbf{s}; \theta, \tilde{p}^-, \tilde{p}^+)\; \Pi_{\underline{\lambda}}(\theta)\; F_{\gamma,\underline{\lambda}}(\mathbf{z}; \theta, p_{12}, p_{43}) \;, \tag{8.5}$$

where $M = \beta$ and therefore $\gamma = 2\beta + p^+ + p^-$, as usual. Upon setting $s_{r=1,\ldots,\beta}$ to the values 0 or 1, the LHS of this formula can be interpreted as a diagram contributing to a generalised free theory correlator in a (m,n,θ) SCFT as we show in section 8.3. The RHS then gives the expansion in superconformal blocks of this diagram and we see that the coefficients in a superblock decomposition of any generalised free theory diagram are simply bosonic Jacobi polynomials $J^{(\beta)}_{\underline{\lambda}}(\mathbf{s}; \theta, \tilde{p}^-, \tilde{p}^+)$ evaluated on 0s and 1s, (up to multiplication by an explicitly known factor $\Pi_{\underline{\lambda}}(\theta)$).

Indeed, if we now specialise back to $m = 0$, the coefficients of this expansion don't change and we get

$$\prod_{r=1}^{\beta} \prod_{j=1}^{n} (1 - s_r y_j) = \sum_{\underline{\lambda}} J^{(\beta)}_{\underline{\lambda}}(\mathbf{s}; \theta, \tilde{p}^-, \tilde{p}^+)\; \Pi_{\underline{\lambda}}(\theta)\; F_{\gamma,\underline{\lambda}}(|\mathbf{y}; \theta, p_{12}, p_{43}) \;. \tag{8.6}$$

which is just Mimachi's formula (8.2). The above is a formula involving only polynomials! Nevertheless it gives the same information, the superblock coefficients i.e. $J^{(\beta)}_{\underline{\lambda}}(\mathbf{s}; \theta, \tilde{p}^-, \tilde{p}^+)\Pi_{\underline{\lambda}}$, as (8.5), which instead contains the full (infinite series) superconformal blocks.

But before discussing this generalised free CFT interpretation of the Cauchy formula let us try to first generalise the formula further. Having lifted the dual Jacobi polynomials to supersymmetric functions, it is natural to also lift the Jacobi polynomials! We will do so in two steps. By looking at (8.4), we first consider that BC_M Jacobi polynomials have an uplift to $BC_{M|N}$ super Jacobi polynomials, $J^{(M|N)}_{\underline{\lambda}}(\mathbf{y}|\mathbf{x})$ discussed below (7.29). Once we disentangle the various pieces, we arrive at the doubly supersymmetric Cauchy identity:

$$\frac{\prod_{i,j}^{M,n}(1-y_iy'_j)\cdot\prod_{l,k}^{N,m}(1-x_lx'_k)}{\prod_{i,k}^{M,m}(1-y_ix'_k)^{\theta}\cdot\prod_{l,j}^{N,n}(1-x_ly'_j)^{\frac{1}{\theta}}} = \sum_{\underline{\lambda}}(-1)^{|\underline{\lambda}|}J^{(M|N)}_{\underline{\lambda}}(\mathbf{y}|\mathbf{x};\theta,\tilde{p}^-,\tilde{p}^+)\tilde{J}^{(n|m)}_{M-\frac{1}{\theta}N,\underline{\lambda}'}(\mathbf{y}'|\mathbf{x}';\tfrac{1}{\theta},p^-,p^+) \tag{8.7}$$

which is therefore the master formula for all Cauchy identities previously discussed.

In the next section we will prove the supersymmetric Cauchy identities (8.4) and (8.7) and in the process, illuminate the appearance of the parameters $\tilde{p}^{\pm}$ as well as explaining how stability manifests in all parameters.

8.2 Proof of the super Cauchy identity

The idea of the proof is to start from analogous super Cauchy identities involving superJack polynomials [71] and by using properties of the Jack→Block matrix and its inverse, derive from them the super Jacobi Cauchy identities (8.4) and (8.7). We will first use formulae involving interpolation polynomials, in order to have a direct connection with mathematics. Then we will give a perspective based on our Jack→Block matrix T and its inverse T^{-1}.

It is useful to begin with the internal case, namely derive (8.2), the Jacobi Cauchy identity in Mimachi [42], from the Cauchy identity for Jack polynomials,

$$\prod_{i=1}^{M}\prod_{j=1}^{n}(1-y_iy'_j) = \sum_{\underline{\lambda}}(-)^{|\underline{\lambda}|}P^{(M)}_{\underline{\lambda}}(\mathbf{y};\theta)P^{(n)}_{\underline{\lambda}'}(\mathbf{y}';\tfrac{1}{\theta}) \ . \tag{8.8}$$

The rough idea is to change basis from Jacks to Jacobis and dual Jacobis, noting that the change of basis matrices are inverses of each other.

So to convert this into (8.2) recall first the relevant definitions for $\tilde{S}, S$ (2.28),

$$J^{(M)}_{\underline{\mu}} = \sum_{\underline{\lambda}\subseteq\underline{\mu}}(S^{(M)}_{\theta,p^-,p^+})^{\underline{\lambda}}_{\underline{\mu}}P_{\underline{\lambda}}(y_1,\ldots,y_M;\theta) \tag{8.9}$$

$$\tilde{J}^{(n)}_{\beta,\underline{\lambda}} = \sum_{\underline{\lambda}\subseteq\underline{\mu}}\left(\widetilde{S}_{\beta;\theta,p^-,p^+}\right)^{\underline{\mu}}_{\underline{\lambda}}P_{\underline{\mu}}(y_1,\ldots,y_n;\theta) \qquad ; \qquad \left(\widetilde{S}_{\beta}\right)^{\underline{\mu}}_{\underline{\lambda}} = \left(S^{(n)}\right)^{\beta^n\backslash\underline{\mu}}_{\beta^n\backslash\underline{\lambda}} \ . \tag{8.10}$$

We will proceed by using the observation that the inverse of the dual Jacobi coefficient $\tilde{S}^{-1}_{\beta}$, once written via interpolation polynomials is very similar to the binomial coefficient S for Jacobi polynomials. Given the precise relation $S\sim\tilde{S}^{-1}$, we will use it in (8.8) to obtain the desired identity (8.2).

Let us go into details. Continuing from (7.28), with the $\tilde{P}$ given in (7.29)-(7.31), and using (2.34), we find

$$(\tilde{S}^{-1}_{\beta;\frac{1}{\theta},p^-,p^+})^{\underline{\mu}'}_{\underline{\lambda}'} = \frac{\Pi_{\underline{\lambda}}(\theta)}{\Pi_{\underline{\mu}}(\theta)} C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha,\theta\beta;\theta)\, \frac{\tilde{P}^{*(\mathsf{M})}_{\underline{\lambda}}(\underline{\mu};\theta,v)}{\tilde{P}^{*(\mathsf{M})}_{\underline{\mu}}(\underline{\mu};\theta,v)}\Bigg|_{v=-\frac{1}{2}+\theta(\frac{p^++p^-}{2}+1+\beta)} \tag{8.11}$$

where on the RHS we are assuming the usual relations

$$\alpha = \max\left(\tfrac{1}{2}(\gamma-p_{12}),\tfrac{1}{2}(\gamma-p_{43})\right)\,,\quad \beta = \min\left(\tfrac{1}{2}(\gamma-p_{12}),\tfrac{1}{2}(\gamma-p_{43})\right)\,,\quad p^{\pm} = \tfrac{1}{2}|p_{12}\pm p_{43}|,$$

and therefore $\gamma = 2\beta + p^+ + p^-$ and $\alpha = \beta + p^-$.

Recall that M is arbitrary in (8.11) as long as $\mathsf{M} \geq \mu'_1$. Now compare this with the binomial coefficient for the Jacobi polynomial (7.11)

$$(S^{(M)}_{\theta,\vec{A}})^{\underline{\lambda}}_{\underline{\mu}} = (-)^{|\underline{\mu}|-|\underline{\lambda}|}\frac{\Pi_{\underline{\lambda}}(\theta)}{\Pi_{\underline{\mu}}(\theta)} C^0_{\underline{\mu}/\underline{\lambda}}(A_1+1+\theta(M-1),\theta M;\theta) \times \frac{\tilde{P}^{*(\mathsf{M})}_{\underline{\lambda}}(\underline{\mu};\theta,u)}{\tilde{P}^{*(\mathsf{M})}_{\underline{\mu}}(\underline{\mu};\theta,u)}\Bigg|_{u=\frac{A_1+A_2+1}{2}+\theta M} \tag{8.12}$$

where the M-dimensional $\tilde{P}$ are introduced to display the same M independence as (8.11). Comparing (8.12) with (8.11) we see that if we match the parameters as

$$\beta \leftrightarrow M \qquad ; \qquad A_1 \leftrightarrow \tilde{p}^- \qquad ; \qquad A_2 \leftrightarrow \tilde{p}^+ \tag{8.13}$$

as defined in (8.3), i.e. $\tilde{p}^{\pm} = \theta p^{\pm} + \theta - 1$, there is a precise matching (up to a sign) between $S^{(M)}$ and $\tilde{S}^{-1}_M$, namely

$$(S^{(M)}_{\theta,\tilde{p}^-,\tilde{p}^+})^{\underline{\lambda}}_{\underline{\mu}} = (-)^{|\underline{\mu}|-|\underline{\lambda}|}(\tilde{S}^{-1}_{M;\frac{1}{\theta},p^-,p^+})^{\underline{\mu}'}_{\underline{\lambda}'}\,. \tag{8.14}$$

If we now expand the Jack polynomial $P^{(n)}_{\underline{\lambda}'}(\mathbf{y},\frac{1}{\theta})$ on the RHS of the Cauchy identity (8.8) in terms of dual Jacobi polynomials, and then use the relation (8.14), we will recognize a Jacobi polynomial (7.3), for specific values of the parameters. Let us explain the reasoning step by step: First

$$\begin{aligned}\prod_{i=1}^{M}\prod_{n=1}^{n}(1-y_iy'_j) &= \sum_{\underline{\lambda}}(-)^{|\underline{\lambda}|}P^{(M)}_{\underline{\lambda}}(\mathbf{y};\theta)P^{(n)}_{\underline{\lambda}'}(\mathbf{y}',\tfrac{1}{\theta})\\ &= \sum_{\underline{\lambda}}(-)^{|\underline{\lambda}|}P^{(M)}_{\underline{\lambda}}(\mathbf{y};\theta)\sum_{\underline{\mu}}(\tilde{S}^{-1}_{\beta;\frac{1}{\theta},p^-,p^+})^{\underline{\mu}'}_{\underline{\lambda}'}\,\tilde{J}^{(n)}_{\beta,\underline{\mu}'}(\mathbf{y}';\tfrac{1}{\theta},p^-,p^+)\end{aligned} \tag{8.15}$$

where β is still arbitrary, since it was not entering the Cauchy identity for Jack polynomials we started with. Then, we fine tune $\beta = M$ and $A_{1,2}$ as in (8.13), so to use (8.14)

$$\begin{aligned}\prod_{i=1}^{M}\prod_{n=1}^{n}(1-y_iy'_j) &= \sum_{\underline{\mu}}\sum_{\underline{\lambda}}(-)^{|\underline{\mu}|}(S^{(M)}_{\theta,\tilde{p}^-,\tilde{p}^+})^{\underline{\lambda}}_{\underline{\mu}}P^{(M)}_{\underline{\lambda}}(\mathbf{y};\theta)\,\tilde{J}^{(n)}_{M,\underline{\mu}'}(\mathbf{y}';\theta^{-1},p^-,p^+)\\ &= \sum_{\underline{\mu}}(-)^{|\underline{\mu}|}J^{(M)}_{\underline{\mu}}(\mathbf{y};\theta,\tilde{p}^-,\tilde{p}^+)\,\tilde{J}^{(n)}_{M,\underline{\mu}'}(\mathbf{y}';\tfrac{1}{\theta},p^-,p^+)\,.\end{aligned} \tag{8.16}$$

To obtain the second line we recognised that the sum over $\underline{\lambda}$ gives the Jacobi polynomial.[36]

Our proof of (8.2) is thus concluded. Our derivation here explains both the appearance of the parameter combinations $\tilde{p}^{\pm}$ in the Cauchy identity, needed to obtain the relation (8.14), as well as its stability in M. The latter follows because the Cauchy identity for Jack polynomials we started with is manifestly stable in M.

A bonus of our discussion is that following the logic that leads to (8.16) we can prove other supersymmetric generalisations of the Cauchy identity for Jacobi and dual Jacobi polynomials, starting from known Cauchy identity involving supersymmetric Jack polynomials [71].

There is an immediate supersymmetrisation of $P^{(n)}(\mathbf{y}') \to P^{(n|m)}(\mathbf{y}'|\mathbf{x}')$ that we can perform and will lead to (8.4). The known Cauchy identity for this case is [71]

$$\prod_i^M \frac{\prod_j^n (1-y_i y'_j)}{\prod_k^m (1-y_i x'_k)^\theta} = \sum_{\underline{\lambda}} (-1)^{|\underline{\lambda}|} P^{(M)}_{\underline{\lambda}}(\mathbf{y};\theta) P^{(n|m)}_{\underline{\lambda}'}(\mathbf{y}'|\mathbf{x}';\tfrac{1}{\theta}) \ , \tag{8.17}$$

The key point is that the super expansion of $P^{(n|m)}_{\underline{\lambda}'}$ in terms of $\tilde{J}^{(n|m)}_{M,\underline{\lambda}'}$ depends exactly on the same coefficients $(\tilde{S}^{-1}_M)^{\underline{\mu}'}_{\underline{\lambda}'}$ as in the bosonic case, because $\tilde{J}$ is stable. Thus, by the very same manipulations as in (8.16), we arrive at (8.4),

$$\prod_i^M \frac{\prod_j^n (1-y_i y'_j)}{\prod_k^m (1-y_i x'_k)^\theta} = \sum_{\underline{\mu}} (-1)^{|\underline{\mu}|} J^{(M)}_{\underline{\mu}}(\mathbf{y};\theta,\tilde{p}^-,\tilde{p}^+) \tilde{J}^{(n|m)}_{M,\underline{\mu}'}(\mathbf{y}'|\mathbf{x}';\tfrac{1}{\theta},p^-,p^+) \ . \tag{8.18}$$

For completeness we point out that the sum over $\underline{\mu}$ is now unbounded on the east, since it only truncates to M rows due to $J^{(M)}$. The $J^{(n|m)}$ is instead an infinite series in small $\mathbf{x}'$, as it is the LHS.

Next, we want to supersymmetrise $P^{(M)}(\mathbf{y}) \to P^{(M|N)}(\mathbf{y}|\mathbf{x})$ and obtain the doubly supersymmetric Cauchy identity (8.7). As before, the starting point is the doubly supersymmetric Cauchy identity for super Jack polynomials [71]

$$\frac{\prod_{i,j}^{M,n}(1-y_i y'_j)\cdot \prod_{l,k}^{N,m}(1-x_l x'_k)}{\prod_{i,k}^{M,m}(1-y_i x'_k)^\theta \cdot \prod_{l,j}^{N,n}(1-x_l y'_j)^{\frac{1}{\theta}}} = \sum_{\underline{\lambda}} (-1)^{|\underline{\lambda}|} P^{(M|N)}_{\underline{\lambda}}(\mathbf{y}|\mathbf{x};\theta) P^{(n|m)}_{\underline{\lambda}'}(\mathbf{y}'|\mathbf{x}';\tfrac{1}{\theta}) \ . \tag{8.19}$$

and the initial step is the same as (8.15), for arbitrary β. But this time we will need to carefully understand how to fine tune β with both M and N.

In [38] super Jacobi polynomials were defined and shown to have an expansion in terms of super Jacks with coefficients given in terms of super interpolation polynomials:

$$J^{(M|N)}_{\underline{\mu}}(\mathbf{y}|\mathbf{x};\theta,p^-,p^+) = \sum_{\underline{\lambda}\subseteq\underline{\mu}} (S^{(M|N)}_{\theta,p^-,p^+})^{\underline{\lambda}}_{\underline{\mu}} P^{(M|N)}_{\underline{\lambda}}(\mathbf{y}|\mathbf{x};\theta) \ , \tag{8.20}$$

[36] The sum over $\underline{\lambda}$ in (8.15) we kept unbounded (since it automatically truncates to M^n) will now also automatically truncate since $(S^{(M)})^{\underline{\lambda}}_{\underline{\mu}} = 0$ when $\underline{\mu} \subset \underline{\lambda}$. The result is therefore a polynomial, i.e. the Jacobi polynomial.

where

$$(S^{(M|N)}_{\theta,A_1,A_2})^{\underline{\lambda}}_{\underline{\mu}} = (-)^{|\underline{\mu}|-|\underline{\lambda}|}\frac{\Pi_{\underline{\lambda}}(\theta)}{\Pi_{\underline{\mu}}(\theta)}C^0_{\underline{\mu}/\underline{\lambda}}(a,b;\theta) \times \frac{\tilde{P}^{*(M|N)}_{\underline{\lambda}}(\underline{\mu}_e|\underline{\mu}_s;\theta,u)}{\tilde{P}^{*(M|N)}_{\underline{\mu}}(\underline{\mu}_e|\underline{\mu}_s;\theta,u)}$$

$$a = A_1{+}1{+}\theta(M{-}1){-}N, \quad b = \theta M{-}N, \quad u = \tfrac{A_1+A_2+1}{2}{+}\theta M{-}N\ , \tag{8.21}$$

with $\underline{\mu}_s, \underline{\mu}_e$ defined below (7.32).

In the above formula the dimensions of the interpolation polynomials $(M|N)$ is tied directly to the dimension of the super Jacobi. However in (7.32) we saw that the dimensions of the interpolation polynomials can be arbitrary (as long as they are big enough to contain the corresponding Young diagram). So we can write the more general expression:

$$(S^{(M|N)}_{\theta,A_1,A_2})^{\underline{\lambda}}_{\underline{\mu}} = (-)^{|\underline{\mu}|-|\underline{\lambda}|}\frac{\Pi_{\underline{\lambda}}(\theta)}{\Pi_{\underline{\mu}}(\theta)}C^0_{\underline{\mu}/\underline{\lambda}}(a,b;\theta) \times \left.\frac{\tilde{P}^{*(\mathsf{M|N})}_{\underline{\lambda}}(\underline{\mu}_e|\underline{\mu}_s;\theta,u)}{\tilde{P}^{*(\mathsf{M|N})}_{\underline{\mu}}(\underline{\mu}_e|\underline{\mu}_s;\theta,u)}\right|_{u=\frac{A_1+A_2+1}{2}+\theta M-N} \tag{8.22}$$

with a, b as above, any $\mathsf{M|N}$. (In particular, we could choose $\mathsf{N} = 0$ and obtain the super Jacobi coefficients in terms of non supersymmetric interpolation polynomials).

Compare (8.22) with $\tilde{S}^{-1}_{\beta}$, in which we use the analogue of the supersymmetric formula (7.34) for T^{-1}_{γ}, namely,

$$(\tilde{S}^{-1}_{\beta;\frac{1}{\theta},p^-,p^+})^{\underline{\mu}'}_{\underline{\lambda}'} = \frac{\Pi_{\underline{\lambda}}(\theta)}{\Pi_{\underline{\mu}}(\theta)}C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha,\theta\beta;\theta)\left.\frac{\tilde{P}^{*(\mathsf{M|N})}_{\underline{\lambda}}(\underline{\mu}_s|\underline{\mu}_e;\theta,v)}{\tilde{P}^{*(\mathsf{M|N})}_{\underline{\mu}}(\underline{\mu}_s|\underline{\mu}_e;\theta,v)}\right|_{v=-\frac{1}{2}+\theta(\frac{p^++p^-}{2}+1+\beta)} \tag{8.23}$$

we see that

$$(S^{(M|N)}_{\theta,\tilde{p}^-,\tilde{p}^+})^{\underline{\lambda}}_{\underline{\mu}} = (-)^{|\underline{\mu}|-|\underline{\lambda}|}(\tilde{S}^{-1}_{M-\frac{1}{\theta}N;\frac{1}{\theta},p^-,p^+})^{\underline{\mu}'}_{\underline{\lambda}'}\ . \tag{8.24}$$

We can now apply manipulations similar to the ones performed in (8.16) to derive the doubly supersymmetric Jacobi Cauchy (8.7) from the doubly supersymmetric Jack Cauchy (8.19).

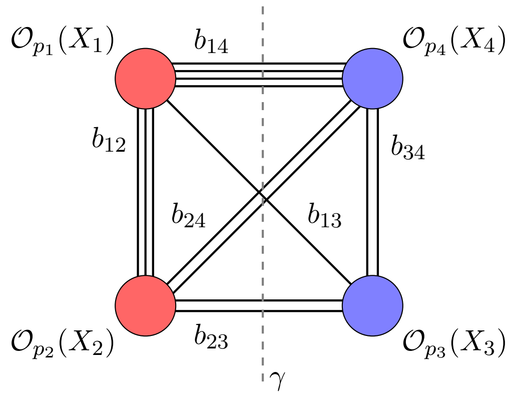
8.3 Free theory block coefficients

We now turn to our main interest in the Cauchy formulae, namely, we will compute conformal partial wave (CPW) coefficients, in generalised free theories, with no effort, just using Cauchy identities and nothing else.

Four-point functions of generalised free theories are given simply as sums of products of propagators with the correct weights:

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{free}} = \sum_{\{b_{ij}\}} a_{\{b_{ij}\}} \prod_{1\le i<j\le 4} g_{ij}^{b_{ij}}\,, \qquad \sum_i b_{ij} = p_j\ . \tag{8.25}$$

Diagrammatically each term in the sum looks as



(8.26)

Here there are six b_{ij} connecting operators between insertion points, and four constraints $\sum_i b_{ij} = p_j$ leaving two degrees of freedom. We parametrise these with $\gamma := p_1 + p_2 - 2b_{12}$ and $k := b_{23}$.[37] The free four-point correlator can then be re-written as

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{free}} = g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac{p_{21}}{2}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{p_{43}}{2}} \sum_{\gamma,k} a_{\gamma,k} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{\gamma}{2}} \left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{k} \tag{8.27}$$

where $\gamma \in \{\gamma_m, \gamma_m + 2, \ldots, \gamma_M\}$ with $\gamma_m = \max(|p_{12}|, |p_{43}|)$, $\gamma_M = \min(p_1 + p_2, p_3 + p_4)$ and $k = 0, 1, .., \frac{1}{2}(\gamma - \gamma_m)$. So the propagator structures are classified firstly by the total number of bridges γ going from the pair of operators $\mathcal{O}_{p_1}\mathcal{O}_{p_2}$ to the pair $\mathcal{O}_{p_3}\mathcal{O}_{p_4}$, and secondly by the the number of bridges k connecting $\mathcal{O}_{p_2}$ with $\mathcal{O}_{p_3}$. The parameters $a_{\gamma,k}$ are arbitrary coefficients, determined by the particular microscopic theory in question.

The form of (8.27), has exactly the same prefactor as the superconformal blocks (2.5). Moreover, changing from $B_{\gamma,\underline{\lambda}}$ to $F_{\gamma,\underline{\lambda}}$, as in (2.14), we can drop the first γ-dependent prefactor. Thus the superconformal block decomposition of the above propagator structure (8.26) reduces to

$$\left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{k} = \sum_{\underline{\lambda}} A_{\gamma,k,\underline{\lambda}} F_{\gamma,\underline{\lambda}} \tag{8.28}$$

where the expansion coefficients $A_{\gamma,k,\underline{\lambda}}$ also depend on $\theta; p_{12}, p_{43}$, as $B_{\gamma,\underline{\lambda}}$ does.

[37]This then gives

$$b_{12} = \tfrac{1}{2}(p_1 + p_2 - \gamma) \quad ; \quad b_{34} = \tfrac{1}{2}(p_3 + p_4 - \gamma) \quad ; \quad b_{13} = \tfrac{1}{2}(p_3 - p_4 + \gamma) - k$$
$$b_{24} = \tfrac{1}{2}(p_2 - p_1 + \gamma) - k \quad ; \quad b_{23} = k \quad ; \quad b_{14} = \tfrac{1}{2}(p_1 - p_2 - p_3 + p_4) + k\ .$$

An equivalent parametrisations, $\gamma_{ij} = p_i - p_j - 2b_{ij}$, counts the number of bridges going from $\mathcal{O}_{p_i}\mathcal{O}_{p_j}$ to the opposite pair. $\gamma_{12} + \gamma_{13} + \gamma_{23} = \sum_i p_i$, is a Mandelstam-type constraint for free theory four-point correlators.

The LHS of (8.28) is simple to understand, using from (2.8) we have that:

$$\left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^k = \mathrm{sdet}(1-Z)^{-k\#} = \prod_{r=1}^{\beta} \frac{\prod_{j=1}^{n}(1-s_r y_j)}{\prod_{i=1}^{m}(1-s_r x_i)^\theta} \qquad ; \qquad \begin{matrix} s_{i=1,\dots,k}=1 \\ s_{i=k+1,\dots,\beta}=0 \end{matrix} \tag{8.29}$$

where $\beta = \frac{1}{2}\min(\gamma - p_{43}, \gamma - p_{12})$ as usual. We recognise here the LHS of the superconformal Cauchy identity (8.5). Thus comparing (8.28) with this Cauchy identity (8.5) we can immediately identify the CPW coefficients $A_{\gamma,k,\underline{\lambda}}$ as an evaluation formula for a Jacobi polynomial:

$$A_{\gamma,k,\underline{\lambda};\theta,p_{12},p_{43}} = \Pi_{\underline{\lambda}}(\theta)\, J^{(\beta)}_{\underline{\lambda}}(\mathbf{s};\theta,\tilde{p}^-,\tilde{p}^+)|_{\mathbf{s}=(1^k,0^{\beta-k})} \tag{8.30}$$

with $\tilde{p}^{\pm}$ defined in the same way, i.e. $\tilde{p}^{\pm} = \theta p^{\pm} + \theta - 1$ and $p^{\pm} = \frac{1}{2}|p_{12} \pm p_{43}|$.

Finally then, through Okounkov's binomial formula (7.3), the free theory CPW coefficients, $A_{\gamma,k,\underline{\lambda}}$, can be written as a sum over interpolation polynomials evaluated at partitions multiplied by Jack polynomials evaluated at 1 and 0, $P_{\underline{\mu}}(1^k, 0^{\beta-k};\theta)$. Note that such a Jack polynomial, $P_{\underline{\mu}}(1^k, 0^{\beta-k};\theta)$, truncates to a Young diagram with just k rows out of β, $P_{[\mu_1,\dots,\mu_k]}(1^k;\theta)$, from stability. The latter is a known explicit evaluation formula due to Stanley [65]. Putting this together then we arrive at an explicit formula for the CPW coefficients as a sum of binomial coefficients,

$$A_{\gamma,k,\underline{\lambda};\theta,p_{12},p_{43}} = \Pi_{\underline{\lambda}}(\theta) \sum_{\underline{\mu}\subseteq\underline{\lambda}} (S^{(\beta)}_{\theta,\tilde{p}^-,\tilde{p}^+})^{\underline{\mu}}_{\underline{\lambda}} \times \prod_{1\le i<j\le k} \frac{(\theta(j-i+1))_{\mu_i-\mu_j}}{(\theta(j-i))_{\mu_i-\mu_j}}\,. \tag{8.31}$$

This formula provides the decomposition of any free theory diagram in superconformal blocks. It depends on γ, k, then θ and the external charges p_{12} and p_{43}. It does not of course depend on (m,n) by stability of the super Cauchy identity. Indeed, because of stability, the derivation of (8.31) doesn't rely on any supersymmetry at all, and can be derived directly from the original bosonic Cauchy identity of Mimachi (8.2).

We will investigate this formula for free theory CPW coefficients further in a separate publication [72] giving even simpler and completely explicit formulae for the $\theta = 1$ case.

9 Decomposing superblocks

The superconformal block (or equivalently dual super Jacobi functions) $B_{\gamma,\underline{\lambda}}$ has been so far defined as the multivariate series

$$B_{\gamma,\underline{\lambda}} = \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\frac{\gamma}{2}} F_{\gamma,\underline{\lambda}}\,, \qquad F_{\gamma,\underline{\lambda}} = \sum_{\underline{\mu}\supseteq\underline{\lambda}} (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{z};\theta) \tag{9.1}$$

with $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ computed either from the recursion of section 5 or by the binomial coefficient of section 7.

In this section we want to relate this to a more common approach where instead $B_{\gamma,\underline{\lambda}}$ is decomposed into its constituent bosonic blocks. It is useful to view this in a more general context of decomposing superblocks into smaller *super*blocks,

$$(m+m'|n+n') \to (m|n) \otimes (m'|n') \ . \tag{9.2}$$

Then we will specialise to the case of decomposing into bosonic blocks an internal blocks by considering $m' = n = 0$.

9.1 General decomposition of blocks

The analogous decomposition (9.2) for superJack polynomials is given in (C.59), and it involves the structure constants $\mathcal{S}$. Note that these structure constants are independent of the dimensions m, n, m', n', because of stability! Combining this with the expansion of Blocks into Jacks (9.1) we arrive at the decomposition of superblocks as

$$F^{(m+m',n+n')}_{\gamma,\underline{\lambda}} = \sum_{\underline{\mu},\underline{\nu}} F^{(m,n)}_{\gamma,\underline{\mu}} \, (\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} \, F^{(m',n')}_{\gamma,\underline{\nu}} \ , \tag{9.3}$$

where the block structure constants $\mathcal{S}_\gamma$ depend on θ, p_{12}, p_{43} and are related to the Jack structure constants via the (inverse) block to Jack matrices T_γ as

$$(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} = \sum_{\substack{\underline{\tilde\lambda}\supseteq\underline{\lambda}\\ \underline{\tilde\mu}\subseteq\underline{\mu}\\ \underline{\tilde\nu}\subseteq\underline{\nu}}} \mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}} \, (T_\gamma)^{\underline{\lambda}}_{\underline{\tilde\lambda}} (T^{-1}_\gamma)^{\underline{\mu}}_{\underline{\tilde\mu}} (T^{-1}_\gamma)^{\underline{\nu}}_{\underline{\tilde\nu}} \ . \tag{9.4}$$

Note we distinguish the Jack structure constants $\mathcal{S}$ from the block structure constants $\mathcal{S}_\gamma$ purely by the presence of the subscript for the latte.r Since we know T_γ infinite triangular matrices, the issue in this formula is to understand when and how the $\mathcal{S}_\gamma$ are non vanishing.

For the structure constants of *Jack* polynomials, since these are homogeneous, we clearly require $|\underline{\tilde\lambda}| = |\underline{\tilde\mu}|+|\underline{\tilde\nu}|$ for $\mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}} \neq 0$. A stronger statement conjectured by Stanley (conjecture 8.4 in [65]) is that the Jack structure constants are non vanishing if and only if the corresponding Schur structure constants (given by the Littlewood-Richardson rule) are non vanishing

$$\mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}}(\theta) \neq 0 \qquad \Leftrightarrow \qquad \mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}}(1) \neq 0 \ . \tag{9.5}$$

In any case, clearly for a given $\underline{\tilde\lambda}$ there are only a finite number of non vanishing Jack structure constants $\mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}}$ and thus a finite number of terms in the decomposition of $P_{\underline{\tilde\lambda}}$ into smaller Jacks.

What about the block structure constants? Since the Jack→Block matrix T_γ is triangular we see that (9.4) gives the $(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}$ in terms of $\mathcal{S}^{\underline{\tilde\mu}\underline{\tilde\nu}}_{\underline{\tilde\lambda}}$ with $\underline{\tilde\lambda} \supseteq \underline{\lambda}$. Thus $\underline{\tilde\lambda}$ can become arbitrarily large and the decomposition of a superblock into smaller superblocks might involve an infinite number of terms. However let us consider more carefully the structure of the Young diagrams which can survive.

First note that the decomposition (9.3) has a vertical cut-off: the height of the Young diagram $\tilde\lambda$ can be no larger than $\beta = \min\left(\frac{1}{2}(\gamma - p_{12}), \frac{1}{2}(\gamma - p_{43})\right)$, since T_γ has such a vertical cut-off (see discussion above (D.20)).

Now we claim that if $\underline{\mu}$ is much wider than $\underline{\lambda}$ (i.e. $\mu_1 >> \lambda_1$) then $\underline{\nu}$ will also have to be very wide in order for $(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} \neq 0$. More precisely we make the following conjecture:

$$(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} \neq 0 \qquad \Rightarrow \qquad \exists\, \underline{\tilde{\mu}}, \underline{\tilde{\nu}}, \underline{\kappa}, \text{ such that } \mathcal{S}^{\underline{\tilde{\mu}}\underline{\tilde{\nu}}}_{\underline{\lambda}} \neq 0,\ \mathcal{S}^{\underline{\tilde{\mu}}\underline{\kappa}}_{\underline{\mu}} \neq 0,\ \mathcal{S}^{\underline{\tilde{\nu}}\underline{\kappa}}_{\underline{\nu}} \neq 0\ . \tag{9.6}$$

To get a feel for this claim, consider the case $\underline{\lambda} = \varnothing$ (corresponding to half BPS operators in a superconformal theory). Then the above conditions will require $\underline{\tilde{\mu}} = \underline{\tilde{\nu}} = \varnothing$ and so $\underline{\mu} = \underline{\nu} (= \underline{\kappa})$. Therefore the conjecture implies the following in this case

$$(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\varnothing} \neq 0 \qquad \Rightarrow \qquad \underline{\mu} = \underline{\nu}\ , \tag{9.7}$$

so the decomposition of a superblock with a trivial Young diagram is diagonal. This is not obvious from (9.4) and requires non trivial relations between the Jack structure constants $\mathcal{S}$ and the block to Jack matrix T_γ.

More generally, the conjecture (9.6) can be proven in the case $\theta = 1$ where there is a group theory interpretation. In the $\theta = 1$ case the decomposition of the superblock $F^{(m+m'|n+n')}_{\gamma,\underline{\lambda}}$ in (9.3) has the group theoretic interpretation of decomposing reps of $U(2m{+}2m'|2n{+}2n') \to U(2m|2n) \otimes U(2m'|2n')$ and taking only the states which are diagonal under the decomposition $U(2m{+}2m'|2n{+}2n') \to U(m+m'|n+n') \otimes U(m+m'|n+n')$ (equivalently taking only those reps whose Dynkin labels are symmetric as only such states contribute to the block).[38] Let us perform this decomposition $U(2m+2m'|2n+2n') \to U(2m|2n) \otimes U(2m'|2n')$ for the induced $U(2m+2m'|2n+2n')$ representations $\mathcal{O}_{\gamma,\underline{\lambda}}(X^{AA'})$ (described in section 3.2. We must first decompose the $U(m{+}m'|n{+}n')$ rep $\underline{\lambda}$ into $U(m|n) \otimes U(m'|n')$ reps. In fact we do this decomposition twice but then take the same reps for both copies of $U(m{+}m'|n{+}n')$. Under the decomposition let the $U(m{+}m'|n{+}n')$ index A split into the $U(m|n)$ indices (α, a) and similarly $A' \to (\dot{\alpha}, a')$. So

$$\mathcal{O}_{\gamma,\underline{\lambda}(A)\underline{\lambda}(A')}(X^{AA'}) \to \bigcup_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}} \mathcal{O}_{\gamma,\underline{\tilde{\mu}}(\alpha)\underline{\tilde{\nu}}(a)\underline{\tilde{\mu}}(\dot{\alpha})\underline{\tilde{\nu}}(a')}(X^{AA'})$$

(this decomposition is dictated by $\mathcal{S}^{\underline{\tilde{\mu}}\underline{\tilde{\nu}}}_{\underline{\lambda}}$ and explains the appearance of this in (9.6)). We must now also consider the derivatives $\partial_{AA'}$ acting on these. In particular the off-diagonal derivatives $\partial_{\alpha a'}$ and $\partial_{a\dot{\alpha}}$ will yield new induced representations of the subgroup $U(2m|2n) \otimes U(2m'|2n')$. Since the derivatives commute (and we again project onto the diagonal rep) we arrive at

$$\mathcal{O}_{\gamma,\underline{\tilde{\mu}}(\alpha)\underline{\tilde{\mu}}(\dot{\alpha})\underline{\nu}(a)\underline{\nu}(a')}(X^{AA'}) \to \bigcup_{\underline{\kappa}} \partial^{|\underline{\kappa}|}_{\underline{\kappa}(\alpha)\underline{\kappa}(\dot{\alpha})\underline{\kappa}(a)\underline{\kappa}(a')} \mathcal{O}_{\gamma,\underline{\tilde{\mu}}(\alpha)\underline{\tilde{\mu}}(\dot{\alpha})\underline{\nu}(a)\underline{\nu}(a')}(X^{\alpha\dot{\alpha}}, X^{aa'})\ ,$$

and arrive at the tensor products $\underline{\kappa} \otimes \underline{\tilde{\mu}}$ and $\underline{\kappa} \otimes \underline{\tilde{\nu}}$ which explains the structure constants $\mathcal{S}^{\underline{\tilde{\mu}}\underline{\kappa}}_{\underline{\mu}}, \mathcal{S}^{\underline{\tilde{\nu}}\underline{\kappa}}_{\underline{\nu}}$ appearing in (9.6).

We believe a similar argument for (9.6) can be made for the other group theoretic cases $\theta = 2, \frac{1}{2}$ although we haven't gone carefully through the details.

[38] Note the difference with the corresponding group theory interpretation of the Jack decomposition (C.59) which is instead equivalent to decomposing reps of $U(m+m'|n+n') \to U(m|n) \otimes U(m'|n')$ in the $\theta = 1$ case.

9.2 Superblock to block decomposition

We now focus on the main interest, the case most relevant for physics application, namely that of decomposing a superblock into its corresponding bosonic blocks. This corresponds to taking $m' = n = 0$ in the general case described in the previous section. Specialising to this case the decomposition (9.3) becomes

$$F^{(m,n)}_{\gamma,\underline{\lambda}}(\mathbf{x}|\mathbf{y}) = \sum_{\underline{\mu},\underline{\nu}} F^{(m,0)}_{\gamma,\underline{\mu}}(\mathbf{x})\,(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}\,F^{(0,n)}_{\gamma,\underline{\nu}}(\mathbf{y})\ . \tag{9.8}$$

Now note that the conjecture (9.6) together with Stanley's conjecture (9.5) implies that the sum on the rhs is finite in this case. First, as already noted, the Young diagrams $\underline{\mu},\underline{\nu}$ can not extend vertically below row β. Since $\underline{\nu}$ is a $(0,n)$ rep it can not be wider than n, $\nu_1 \leq n$ (and cannot have no more than n columns, since this is a $U(n)$ rep transposed). Thus $\underline{\nu}$ is restricted to sit inside the rectangle $[n^\beta]$. But what about $\underline{\mu}$? If this was much wider than $\underline{\lambda}$ it would be impossible to satisfy the conjecture (9.6). To see this, first note that the Littlewood-Richardson rule (and Stanley's conjecture (9.5)) means that $\mathcal{S}^{\tilde{\underline{\mu}}\tilde{\underline{\nu}}}_{\underline{\lambda}} \neq 0$ implies $\tilde{\mu}_1, \tilde{\nu}_1 \leq \lambda_1 \leq \tilde{\mu}_1 + \tilde{\nu}_1$. Then this together with conjecture (9.6) means that $(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}} \neq 0$ only if $\mu_1 - \lambda_1 \leq \mu_1 - \tilde{\mu}_1 \leq \kappa_1 \leq \nu_1 \leq n$. So in particular the width μ_1 of $\underline{\mu}$ can not be bigger than $\lambda_1 + n$, i.e. the width of $\underline{\lambda} + n$. There are tighter bounds but this is already enough to show that there are a finite number of terms. Note that specialising to $m = n = 2, \theta = 1$ the decomposition should reproduce the diagonal component reps in the decomposition of $\mathcal{N} = 4$ superconformal multiplets which were written out in great detail in appendix B of [44].

9.3 Examples of the superblock to block decomposition

To ground the discussion about the conformal and internal block decomposition of a superconformal block, we give some explicit examples here.

Let us fix the external data to be $\frac{\gamma}{2} = \alpha = \beta = 5$, $\theta = \frac{1}{2}$, and $\underline{\lambda} = [3,1]$, and consider different values of (m,n). This also illustrates the m,n independence.

In the $(1,1)$ theory, the decomposition of $F_{\gamma=10,[3,1]}(x|y;\frac{1}{2})$ gives

$(\mathcal{S}_{\gamma=10})^{\underline{\nu}\underline{\mu}}_{[3,1]}$: (9.9)

$\underline{\nu}'$ \ $\underline{\mu}$	[1]	[2]	[3]	[4]
[0]	0	0	0	0
[1]	0	0	1	0
[2]	0	$\frac{7}{9}$	0	$\frac{1573}{40320}$
[3]	0	0	$\frac{39}{1120}$	0
[4]	0	0	0	0
[5]	0	0	0	0

Now let us look at the same external data but in the $(m,n) = (2,1)$ theory, thus considering the decomposition of $F_{\gamma=10,[3,1]}(x_1, x_2|y; \frac{1}{2})$. We find

$(\mathcal{S}_{\gamma=10})^{\mu\nu}_{[3,1]}$:

$\underline{\nu}'$ \ $\underline{\mu}$	[1]	[2]	[3]	[4]	[1,1]	[2,1]	[3,1]	[4,1]	[2,2]	[3,2]	[4,2]	[3,3]	[4,3]	[4,4]
[0]	0	0	0	0	0	0	1	0	0	0	0	0	0	0
[1]	0	0	1	0	0	$\frac{28}{45}$	0	$\frac{1573}{53550}$	0	$\frac{4096}{133875}$	0	0	0	0
[2]	0	$\frac{7}{9}$	0	$\frac{1573}{40320}$	0	0	$\frac{93428}{1461915}$	0	$\frac{64}{2025}$	0	$\frac{37752}{33632375}$	0	0	0
[3]	0	0	$\frac{39}{1120}$	0	0	$\frac{8}{225}$	0	$\frac{1573}{937125}$	0	$\frac{1664}{1561875}$	0	0	0	0
[4]	0	0	0	0	0	0	$\frac{13}{7875}$	0	0	0	0	0	0	0
[5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(9.10)

Note that the red diamond is the same as in the $(1, 1)$ theory. The other contributions correspond to conformal blocks with two full rows, which therefore did not exist in the previous case with $m = 1$.

Finally let us look in the $(m, n) = (3, 1)$ theory at the decomposition of $F_{\gamma=10,[3,1]}(x_1, x_2, x_3|y; \frac{1}{2})$. We find $(\mathcal{S}_{\gamma=10})^{\mu\nu}_{[3,1]} =$

$\underline{\nu}'$ \ $\underline{\mu}$	[1]	[2]	[3]	[4]	[1,1]	[2,1]	[3,1]	[4,1]	[2,2]	[3,2]	[4,2]	[3,3]	[1,1,1]	[2,1,1]	[3,1,1]	[4,1,1]	[2,2,1]	[3,2,1]	[4,2,1]	[3,3,1]
[0]	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[1]	0	0	1	0	0	$\frac{28}{45}$	0	$\frac{1573}{53550}$	0	$\frac{4096}{133875}$	0	0	0	0	$\frac{169}{4320}$	0	0	0	0	0
[2]	0	$\frac{7}{9}$	0	$\frac{1573}{40320}$	0	0	$\frac{93428}{1461915}$	0	$\frac{64}{2025}$	0	$\frac{37752}{33632375}$	0	0	$\frac{8}{243}$	0	$\frac{1573}{1049580}$	0	$\frac{1352}{1136025}$	0	0
[3]	0	0	$\frac{39}{1120}$	0	0	$\frac{8}{225}$	0	$\frac{1573}{937125}$	0	$\frac{1664}{1561875}$	0	0	0	0	$\frac{13}{4080}$	0	$\frac{5}{3564}$	0	$\frac{11583}{241076864}$	0
[4]	0	0	0	0	0	0	$\frac{13}{7875}$	0	0	0	0	0	0	$\frac{7}{4050}$	0	$\frac{1573}{19992000}$	0	$\frac{3328}{66268125}$	0	0
[5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{13}{129600}$	0	0	0	0	0

(9.11)

The $m \otimes n$ decomposition generically will have a diamond/rhomboid structure w.r.t. to the products basis of conformal and internal blocks. However we should say that in order to see such a diamond structure, one has to considered a generic Young diagram $\underline{\lambda}$, and a generic γ, otherwise parts of the diamond might be reflected on the boundaries.

One interesting aspect that the above example illustrates is the appearance of the coefficients '1' at the top of each diamond. In the cases $\theta = \frac{1}{2}, 1, 2$, where there is a group theory interpretation, this '1' is easily explained by considering the corresponding decomposition in Minkowski superspace.[39] There the decomposition simply corresponds to the standard super-

[39] Minkowski superpsace corresponds to the coset space obtained by putting cross through all odd (white)

space expansion of the corresponding superfield over Grassmann odd variables and the first term in the expansion (the term obtained by switching off all Grassmann odd variables) has corresponding coefficient '1'. We can view this superspace expansion for different values of (m,n), thus giving different '1' coefficients. So for example in the $(1,1)$ theory this gives the red '1' in (9.10) and in the $(2,1)$ theory it gives the blue '1'. But of course stability means the red '1' remains there even for the $(2,1)$ theory.

It is interesting to see how to recover the "1 at the top of the diamond", i.e. the leading conformal block in the decomposition of a superblock from the formula (9.4) and show that it is true for all θ regardless of whether there is a group theory interpretation. This follows directly from the observation that the Jack structure constants are '1' whenever the large Young diagram is built from the smaller ones put on top of each other (C.27). A similar result then follows for the block structure constants

$$(\mathcal{S}_\gamma)^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta) = 1 \qquad \text{if} \qquad \underline{\lambda} = [\underline{\mu}, \underline{\nu}] \ . \tag{9.12}$$

since $\mathcal{S}$ and $\mathcal{S}_\gamma$ are simply related via a triangular change of (infinite) basis with 1's on the diagonal (9.4) (recall $(T_\gamma)^{\underline{\lambda}}_{\underline{\lambda}} = 1$).

Testing the conjecture (9.6) in general turns into a sophisticated computation, because the Jack structure constants $\mathcal{S}$ themselves are non-trivial and explicit formulae are not known. The works [52–54] obtain a recursive formulation which can be used to carry out the computation, but we are not aware of a more explicit formula. A naive approach we used, for generic number of rows, which nevertheless gives the desired result, is to start from the definitions, and use computer algebra. But we point out that the remarkable consequence of the conjecture (9.6) is that even though $\mathcal{S}_\gamma$ involves infinite matrices T_γ, the result always has only finitely many non zero values. In particular if we keep α, β, γ arbitrary in the recursion relation for T_γ this truncation does not occur, but only takes place when β (or α) is an integer larger than the height of $\underline{\lambda}$. To help clarify these points we now give some explicit results for the two row case.

Explicit formulae for two-rows

For Young diagrams with two rows we can give alternative explicit (i.e. non combinatorial) formulae for the building blocks of the block structure constants (9.4). Indeed, the Jack structure constants for two-row diagrams are known thanks to a result quoted in [100],

$$\mathcal{C}^{[\mu_1,\mu_2]}_{[\kappa_1,\kappa_2]\ [\omega_1,\omega_2]} = \frac{(\theta, 2\theta+\mu_-, \omega_2+\kappa_1-\mu_2+1, \omega_1+\kappa_2-\mu_2+1)_{\mu_2-\kappa_2-\omega_2}}{(1, 1+\theta+\mu_-, \omega_2+\kappa_1-\mu_2+\theta, \omega_1+\kappa_2-\mu_2+\theta)_{\mu_2-\kappa_2-\omega_2}} \tag{9.13}$$

and

$$\mathcal{S}^{\underline{\kappa}\,\underline{\omega}}_{\underline{\mu}} = \frac{\Pi_{\underline{\kappa}}(\theta)\Pi_{\underline{\omega}}(\theta)}{\Pi_{\underline{\mu}}(\theta)} \mathcal{C}^{\underline{\mu}}_{\underline{\kappa}\,\underline{\omega}} \ . \tag{9.14}$$

(Where $(a,b,c)_n$ represents a product of Pochhammers $a_n b_n c_n$.) Note this formula is manifestly symmetric in $\underline{\kappa} \leftrightarrow \underline{\omega}$. It is also clear that for $\theta = 1$ the value of $\mathcal{C}$ collapse to unity, when it is non vanishing.

nodes in the relevant super Dynkin diagram rather than the crossed through nodes of analytic superspace (3.3).

Now, we assemble the block structure constants $\mathcal{S}_\gamma$ in (9.4). The two-row Jack→Block matrix T_γ can be taken from (E.18), and the inverse T_γ^{-1} is given by the formula (7.28), in terms of the interpolation polynomials in (E.22). Putting them together then gives the block structure constants $\mathcal{S}_\gamma$.

In the recursion we may leave all variables α, β, γ unspecified in T_γ and T_γ^{-1}. We find that in this case there is actually no truncation and there are infinitely many non-zero structure constants $(\mathcal{S}_\gamma)_{\underline{\lambda}}^{\underline{\mu}\underline{\nu}}(\theta)$ for given $\underline{\lambda}$. However one finds that for arbitrary α, β, γ, for $\underline{\nu}$ large enough they always appear with a factor of $(\alpha - 2)(\beta - 2)$. So for any $\underline{\lambda}, \underline{\mu}$ exists a $\underline{\nu}^*$ such that

$$(\mathcal{S}_\gamma)_{\underline{\lambda}}^{\underline{\mu}\underline{\nu}}(\theta) = (\alpha - 2)(\beta - 2)\Big[\dots\Big] \qquad \forall\, \underline{\nu} \supset \underline{\nu}^* \tag{9.15}$$

where the terms understood in $\Big[\dots\Big]$ might be cumbersome, but their knowledge is not needed to understand the truncation. It is then clear from (9.15) that upon specifying either β or α to be the value of the maximal number of rows of the Young diagram, in the case of this section $\beta = 2$, the block structure constants $\mathcal{S}_\gamma$ will truncate. We checked explicitly that the very same mechanism generalises to any number of rows.

The $m \otimes n$ decomposition then can be carried out in practice for arbitrary Young diagrams and theories $(m, n; \theta)$. It depends on the Young diagrams and θ through rational functions. For example, consider the $(2, 2)$ theory and $B_{\gamma=4,[\lambda_1,\lambda_2]}$. Then, we know that

$$B_{\gamma=4,[\lambda_1,\lambda_2]} = \left(\frac{\prod_{i=1}^2 x_i^\theta}{\prod_{j=1}^2 y_j} \right)^{\frac{\gamma}{2}} \left(F_{\gamma,[\lambda_1,\lambda_2]}^{(2,0)}(\mathbf{x}) \times F_{\gamma,\varnothing}^{(0,2)}(\mathbf{y}) + \dots \right) . \tag{9.16}$$

The coefficients of the next terms beyond $F_{\gamma=4,[\lambda_1,\lambda_2]} \times F_{\gamma=4,\varnothing}$, with $p_{12} = p_{43} = 0$, are

$$\begin{aligned}
(\mathcal{S}_{\gamma=4})_{[\lambda_1,\lambda_2]}^{[\lambda_1,\lambda_2-1],\square} &= \tfrac{\theta\lambda_2}{\lambda_2+\theta-1} \\
(\mathcal{S}_{\gamma=4})_{[\lambda_1,\lambda_2]}^{[\lambda_1-1,\lambda_2],\square} &= \tfrac{\theta(\lambda_1+\theta)(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+2\theta-1)}{(\lambda_1+2\theta-1)(\lambda_1-\lambda_2+\theta-1)(\lambda_1-\lambda_2+\theta)} \\
(\mathcal{S}_{\gamma=4})_{[\lambda_1,\lambda_2]}^{[\lambda_1+1,\lambda_2],\square} &= \tfrac{\theta(\lambda_1+2\theta)(\lambda_1+3\theta-1)(\lambda_1+\lambda_2+2\theta)(\lambda_1+\lambda_2+4\theta-1)}{4(2\lambda_1+4\theta-1)(2\lambda_1+4\theta+1)(\lambda_1+\lambda_2+3\theta-1)(\lambda_1+\lambda_2+3\theta)} \\
(\mathcal{S}_{\gamma=4})_{[\lambda_1,\lambda_2]}^{[\lambda_1,\lambda_2+1],\square} &= \tfrac{\theta(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+2\theta-1)(\lambda_2+\theta)(\lambda_2+2\theta-1)(\lambda_1+\lambda_2+2\theta)(\lambda_1+\lambda_2+4\theta-1)}{4(\lambda_1-\lambda_2+\theta-1)(\lambda_1-\lambda_2+\theta)(\lambda_1+\lambda_2+3\theta-1)(3\theta+\lambda_1+\lambda_2)(2\lambda_2+2\theta-1)(2\lambda_2+2\theta+1)}
\end{aligned} \tag{9.17}$$

The complexity of these rational functions comes both from the complexity of the structure constants and the solution of the recursion, equivalently the complexity of interpolation polynomials evaluated at partitions.

10 Conclusions and outlook

Let us summarise our findings and conclude with an outlook.

In this paper we have uncovered a non trivial connection between the theory of conformal and superconformal blocks for four-point correlators of scalar fields, and the theory of symmetric functions (as well as Heckman Opdam hypergeometric functions, Calogero Sutherland Moser wave functions and supersymmetric generalisations of all of these). This connection generalises previous work done in [9] for theories defined on a super Grassmannian $Gr(m|n, 2m|2n)$, corresponding to the case $(m, n; \theta = 1)$, where the connection between symmetric functions and superconformal blocks passed through the use of super Schur polynomials [37]. It also generalises the connection between bosonic external/internal blocks and BC_2 hypergeometric functions/Jacobi polynomials uncovered in [11–15]. The starting point has been to consider four-point functions of scalar operators on certain special generalised (m, n) analytic superspaces in 3,4 and 6 dimensions following [1–10]. We then looked at the very concrete problem of solving a Casimir eigenvalue differential equation, and we have found that:

• The Casimir operator for these theories coincides with the $BC_{m|n}$ CMS operator for special values of a parameter θ ($\theta = \frac{1}{2}, 1, 2$). This then immediately suggests to define $(m, n; \theta)$ analytic superspaces for more general θ as discussed in section 3.1.

• The eigenfunctions of the $BC_{m|n}$ Casimir admit a representation as a multivariate series over super Jack polynomials,

$$B_{\gamma,\underline{\lambda}}(\mathbf{z}) = \left(\frac{\prod_i x_i^\theta}{\prod_j y_j} \right)^{\frac{\gamma}{2}} F_{\gamma,\underline{\lambda}}(\mathbf{z}) \ , \qquad\qquad F_{\gamma,\underline{\lambda}}(\mathbf{z}) = \sum_{\underline{\mu} \supseteq \underline{\lambda}} (T_\gamma)_{\underline{\lambda}}^{\underline{\mu}} P_{\underline{\mu}}(\mathbf{z}) \ , \tag{10.1}$$

with expansion coefficients T_γ enjoying special properties.

• The expansion coefficients $(T_\gamma)_{\underline{\lambda}}^{\underline{\mu}}$ do not depend on m, n as long as $\underline{\lambda}, \underline{\mu}$ are Young diagrams. Since super Jack polynomials are themselves stable, by construction, our superconformal blocks are stable!

• Although the coefficients T_γ do not depend on (m, n) when $\underline{\lambda}, \underline{\mu}$ are Young diagrams, the recursion (and its solution) can non-the-less be represented in various forms, in terms of the row lengths of the Young diagram, the column lengths, or a mixture of the two. All representations give the same result, as shown in section 5.2, when the parameters coincide with those of a Young diagram. However, the different choices yield inequivalent analytic continuations (section 6). A mixed representation consistent with the (m, n) Young diagram structure is then most appropriate for superblocks in an $(m, n; \theta)$ theory and will give the correct superblocks for operators with non-integer (anomalous) dimensions. Such an (m, n) representation satisfies the shift symmetry (2.20) for arbitrary θ (which is a manifest symmetry for the physical cases $\theta = \frac{1}{2}, 1, 2$).

• We exemplified this (m, n) analytic continuation in the simplest $(1, 1)$ theory, for arbitrary θ. This shows, in a concrete case, that the recursion can be solved by using a mixed, or supersymmetric, representation where the Young diagrams are read both along rows and columns.

The above results find immediate application for superconformal theories not only in dimensions 3,4,6, but also in 1,2 and possibly also 5 dimensions, as well as ordinary bosonic theories in any dimension, as explained in section 3.1. The superconformal block understanding is particularly important for theories that have a gravity dual through the AdS/CFT

correspondence, where we expect the SCFT to provide important insights towards the quest for a theory of quantum gravity. The simplest examples which have made full use of this $(m, n; \theta)$ formalism have been so far $AdS_5 \times S^5$ [74–76]. There has been some use of the formalism also for $AdS_7 \times S^4$ but restricted to the lowest charge correlators [8,78].[40] It would be interesting to extend this further and consider other $AdS \times S$ backgrounds. An interesting case that has not been attacked yet is $AdS_3 \times S^3 \times S^3$, which realises a one-parameter family of two-dimensional superconformal theories with $D(2, 1; \alpha) \times D(2, 1; \alpha)$ symmetry.[41]

The connection between (super)conformal blocks and symmetric functions, that we have uncovered, yielded in turn a number of mathematical results, providing further inspiration on the mathematics of $BC_{n|m}$ symmetric functions, as well as further insights for the superconformal blocks themselves. In this context, we found that:

- The reduction our blocks from $BC_{m|n}$ to BC_n produces the dual Jacobi polynomials $\tilde{J}_{\beta,\underline{\lambda}}(; \theta, p^{\pm})$ in n variables, which are stable, and were not noticed before. From the knowledge of these polynomials (in particular their stability), we showed that they admit a natural supersymmetric extension which give superconformal blocks.

- Jacobi polynomials $J_{\underline{\lambda}}(; \theta, p^{\pm})$ have a binomial formula due to Okounkov [32,33,41], and so do the dual Jacobi polynomials. The difference is the way the Young diagrams enter the interpolation polynomials P^*. For the dual Jacobi polynomials, the Young diagrams are complemented and transposed w.r.t. a given rectangle R. This manipulation stabilises the binomial formula for the Jacobi polynomials.

- The dependence on the variables of the interpolation polynomials $P^*_{N^M \backslash \underline{\mu}}(N^M \backslash \underline{\lambda}; \theta, u = \frac{1}{2} - \theta\frac{\gamma}{2} - N)$, provides the highly non-trivial γ dependence of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$, which was quite challenging to understand from the point of view of the recursion.

- Elaborating on the way complementation works on the Jack→Block matrix, we showed that its inverse, the Block→Jack matrix, T_γ, has a natural representation in terms of interpolation polynomials, which we upgraded to super interpolation polynomials of Sergeev and Veselov [36]. Thus, whenever an identity pairs a Young diagram with its complement, we can use the Jack→Block and the Block→Jack matrix to uplift that identity to an identity for superconformal blocks.

[40] More recent work has considered superblocks for higher charge correlators in the $AdS_7 \times S^4$ case using the Ward identity approach discussed at the start of section 2.5 [51,79]

[41] In the context of integrability this has been studied in, e.g. [80].

• From the (m,n) Cauchy identity for superJack polynomials we proved a superconformal Cauchy identity,

$$\prod_{r=1}^{\beta} \frac{\prod_{j=1}^{n}(1-s_r y_j)}{\prod_{i=1}^{m}(1-s_r x_i)^{\theta}} = \sum_{\underline{\lambda}} J_{\underline{\lambda}}(\mathbf{s};\theta,\tilde{p}^-,\tilde{p}^+)\, \Pi_{\underline{\lambda}}(\theta)\, F_{\gamma,\underline{\lambda}}(\mathbf{z};\theta,p_{12},p_{43}) \tag{10.2}$$

Then we showed that the LHS becomes a propagator structure when $\mathbf{s}=(1^k,0^{\beta-k})$, up to the overall prefactor $(\prod_{i,j} x_i^{\theta}/y_j)^{\frac{\gamma}{2}}$, which distinguishes $B_{\gamma,\underline{\lambda}}$ from $F_{\gamma,\underline{\lambda}}$. It follows that, with no effort, the above Cauchy identity provides the decomposition of any (generalised) free theory diagram within our formalism.

• The physics understanding of the (m,n) super Cauchy identity allowed us to prove a new doubly supersymmetric Cauchy identity (8.7) involving superconformal blocks (dual super Jacobi functions) and the super Jacobi polynomial of Sergeev and Veselov [36].

• Finally we could read off known results on higher order operators for Heckman Opdam hypergeometrics to obtain all higher order Casimirs in $\theta=1$ theories (in particular $\mathcal{N}=4$ SYM).

Overall, we showed in section 7 and 8 that the connection between superconformal blocks and symmetric functions comes with a non trivial exchange of information, which is fruitful for both sides. The superconformal Cauchy identity is a beautiful example of this exchange. On one side the understanding of stability clarifies the proofs greatly, on the other side, the Cauchy formula allows us to read off, with no effort, the superconformal block decomposition of any free theory diagram.

There are a number of topics one could investigate further. There is still more to be understood about the analytic properties of the superblocks. In [12, 14] bosonic blocks were identified with twisted versions of the Harish Chandra functions giving Heckman Opdam hypergeometric functions. We give here two new viewpoints on this. Both viewpoints start with the much simpler BC_n Jacobi polynomials or more accurately the dual BC_n Jacobi polynomials which are stable. This stability then leads immediately to a family of (n,m) supersymmetric dual Jacobi functions, and then specialising to $n=0$ we obtain the conformal blocks (which are certain S_n symmetric combinations of Harish Chandra functions). But the rank one considerations suggest another viewpoint, namely that the analytic continuation, regular at the origin, of the dual Jacobi polynomials to negative row lengths also yield the conformal blocks. We then expect similarly that analytic continuation of the super dual Jacobi functions to negative row lengths for the infinite non compact direction will give the super Jacobi polynomials of [36].

Besides the immediate relevance of our work for theories that have a generalised free limit, the superconformal block expansion has a non perturbative nature, and recent work in [81–85] has shown that a fully non perturbative reconstruction of a four-point correlator can be given within the framework of the inversion formula [70]. A prominent role is played by the Polyakov-Regge block.We think that this discussion should have a counterpart in the theory of symmetric functions worth investigating.[42]

[42]For similar reasons, it would be interesting to investigate the diagonal limit [86] of the superconformal

It would be interesting to try to adapt our story beyond scalar representations, as well as for higher points, possibly by adapting the techniques of [88–91].

10.1 Outlook on the q-deformed superblocks

Finally, it would be fascinating to find a place for a possible q-deformation of the superblocks, since the various group theoretic objects that we have been discussing, i.e. Jack, Jacobi, interpolation polynomials etc., have well known q-deformed generalisations. In particular, the expansion coefficients (7.12) of the superblocks

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = (\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}} \times \left. \frac{P^{*(M)}_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\lambda};\theta,u)}{P^{*(M)}_{N^M\backslash\underline{\lambda}}(N^M\backslash\underline{\lambda};\theta,u)} \right|_{u=\frac{1}{2}-\theta\frac{\gamma}{2}-N} . \tag{10.3}$$

written as binomial coefficient has such a q-deformation. A simple question to ask is whether the q-deformed binomial coefficient plays a role in physical theories.[43] For example, is there a notion of q-deformed CFTs in any dimension? In these theories, how do q-deformed conformal blocks compare to our multivariate series? For applications within the AdS/CFT correspondence, is it possible to investigate the field theory dual to the q-deformed world-sheet σ-model on $AdS_5 \times S^5$ [96–98].

In a little more detail, the q-deformed binomial coefficient can be used to define the Koornwinder polynomials $K_{\underline{\lambda}}(\mathbf{z}, q, t; a_{i=1,2,3,4})$, as shown by Okounkov [33]. The latter are BC orthogonal (Laurent) polynomials in the bosonic variables $\mathbf{z}$, and for a choice of parameters $\vec{a}$, the $q \to 1$ defines the BC Jacobi polynomials (see e.g. [41]):

$$\lim_{q\to 1} K_{\underline{\lambda}}(\mathbf{z}) = (-4)^{|\underline{\lambda}|} J_{\underline{\lambda}}\left(-\tfrac{1}{4}(\sqrt{z_i} - \tfrac{1}{\sqrt{z_i}})^2\right) . \tag{10.4}$$

An obvious proposal to build the q-deformed generalisation of the $(0, n)$ internal block would be to start from the Koornwinder polynomials. Koornwinder polynomials have a binomial expansion, but the polynomials involved are themselves q-deformed BC interpolation polynomials.[44] One might have expected the generalisation of the internal block would rather correspond to a series over Macdonald polynomials, instead of BC interpolation polynomials. After all the super Jack polynomials, which provided the relevant basis for the (m, n) superconformal blocks, admit a natural generalisation to the super Macdonald operators of Sergeev and Veselov [36]. On the other hand, there is a fundamental reason why Koornwinder polynomials sum over BC interpolation polynomials, and this is evaluation symmetry, i.e. the property that evaluating $K_{\underline{\mu}}$ at partition $\underline{\lambda}$ is invariant under $\underline{\mu} \leftrightarrow \underline{\lambda}$. This property is manifest in the binomial coefficient formalism [33], precisely because the expansion coefficients on the basis of BC interpolation polynomials are again BC interpolation polynomial evaluated at a partition, and therefore evaluation symmetry "swaps" the two.

A proposal for the q-deformed generalisation of the $(0, n)$ internal block can be read from

blocks, as well as the change of variables to the radial coordinate proposed in [87].

[43] In this paper, we used the limit $q \to 1$ and $t \to q^\theta$, in the notation of [32].

[44] The limit to the Jacobi polynomials is designed to degenerate the BC interpolation polynomials to Jack polynomials and the q-deformed binomial coefficient to its classical counterpart [41].

E.Rains' results in [32].[45] It is the virtual Koornwinder "polynomial" $\hat{K}_{Q,\underline{\lambda}}$ (see Definition 7 and Theorem 7.13 in [32]). This is a stable S_n (rather than BC_n) *function* which provides a 1-parameter, Q, generalisation of Koornwinder polynomials $K^{(n)}_{\underline{\lambda}}$ in the following sense,

$$\hat{K}_{Q,\underline{\lambda}}(\ldots,z_n;q,t;\vec{a})\Big|_{Q=q^m} = \prod_i z_i^m\, K^{(n)}_{m^n\backslash\underline{\lambda}}(\mathbf{z};q,t,\vec{a}) \ . \tag{10.5}$$

In more detail, $\hat{K}_{Q,\underline{\lambda}}$ is defined by a binomial formula that sums over Young diagrams $\underline{\mu}$ such that $\underline{\lambda}\subseteq\mu$, and uses the equivalent of the q-deformed Jack$\to$Block matrix, as in our dual Jacobi function $\tilde{J}_{\beta,\underline{\lambda}}(\mathbf{y}|\,)$ in (2.29), analytically continued in β (see the discussion in section 6 and appendix D.2.) The similarity is even more precise because when $Q=q^m$ the sum truncates to the polynomial on the r.h.s. of (10.5), which indeed has the same characteristics as dual Jacobi polynomials:

$$\tilde{J}_{\beta,\underline{\lambda}}(\mathbf{y}|\,)\Big|_{\beta=m} \equiv \prod_i y_i^m J_{m^n\backslash\underline{\lambda}}(\ldots,\tfrac{1}{y_n}) \ . \tag{10.6}$$

We can see now that by taking the limit from Koornwinder polynomials to Jacobi polynomials in (10.5), and re-expressing the result in terms of our dual Jacobi polynomials we find that

$$\lim_{q\to1}\hat{K}_{Q,\underline{\lambda}}(\mathbf{z})\Big|_{Q=q^m} = \prod_i(1-z_i)^{2m}\tilde{J}_{m,\underline{\lambda}}\left(\frac{-4z_i}{(1-z_i)^2}\right) \tag{10.7}$$

The argument of $\tilde{J}$ might be surprising at first, but $Z(z)=-4z/(1-z)^2$ is just a conformal mapping which sends the unit disk in the z plane to $Z\in\mathbb{C}\backslash(1,+\infty)$. Stability of the r.h.s. of (10.7) then follows from that of $\hat{K}_{Q,\underline{\lambda}}$, and coincides with the stability of $\tilde{J}_{\beta,\underline{\lambda}}$ which we discussed in this paper (since that of $\prod_i(1-z_i)^{2m}$ is obvious). We thus conclude that $\hat{K}_{Q,\underline{\lambda}}$ is the q-generalisation of our $\tilde{J}_{\beta,\underline{\lambda}}(\mathbf{y}|\,)$ where $Q=q^\beta$.

From stability, following the same logic that led us from dual Jacobi polynomials to super Jacobi function $\tilde{J}_\beta(\mathbf{y}|\mathbf{x})$ in section 2.3, we infer that there is a supersymmetric uplift of the virtual $\hat{K}$ to a "super virtual Koornwinder" $\hat{K}_Q(\mathbf{y}|\mathbf{x})$ obtained by supersymmetrising the expansion polynomials that define $\hat{K}$, called virtual P^* for reference, but keeping the same expansion coefficients, i.e. the q-deformed Jack$\to$Block matrix. It is crucial here that such virtual P^* are again S_n stable polynomials, the "super virtual" P^* will then generalise the super Macdonald polynomials of [36]. The super virtual Koornwinder $\hat{K}_Q(\mathbf{y}|\mathbf{x})$ would then give the q-generalisation of our superconformal blocks. Following our general motivations, it would be fascinating to study in more detail the properties of the super $\hat{K}_Q(\mathbf{y}|\mathbf{x})$ and understand their role in the context of q-deformed CFTs.

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[45] We thank Ole Warnaar for pointing out this to us.

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A Super/conformal/compact groups of interest

Here we give details about the supercoset construction of section 3, and in particular show how the general formalism ties in with the more standard approach in the case of non-supersymmetric conformal blocks.

A.1 $\theta = 1$: $SU(m,m|2n)$

The $SU(m,m|2n)$ is the simplest family of theories with a supergroup interpretation for any value of m, n positive integers. This corresponds to the case $\theta = 1$.

The supercoset here is the most straightforward of the three general classes: Firstly view the complexified group $SU(m,m|2n) = SL(2m|2n;\mathbb{C})$ as the set of $(m|2n|m) \times (m|2n|m)$ matrices (this is a straightforward change of basis of the more standard $(2m|2n) \times (2m|2n)$ matrices). Then the supercoset space we consider has the 2×2 block structure of (3.4) with each block being a $(m|n) \times (m|n)$ matrix (or rearrangement thereof - see footnote 17.) These blocks are unconstrained beyond the overall unit superdeterminant condition: $\mathrm{sdet}(G) = \mathrm{sdet}(H) = 1$. So in particular the coordinates $X^{AA'}$ of (3.6) are unconstrained $(m|n) \times (m|n)$ matrices. Here A and A' are both superindices carrying the fundamental representation of two independent $SL(m|n)$ subgroups. This supercoset corresponds to the super Grassmannian $Gr(m|n, 2m|2n)$. For more details see [2, 3, 6, 7, 56].

Then a four point function (and in particular the superconformal blocks) can be written in terms of a function of the four points X_1, X_2, X_3, X_4 invariant under the action of G. This in turn boils down to a function of the $(m|n) \times (m|n)$ matrix Z invariant under conjugation (see [7] for more details)

$$Z = X_{12}X_{23}^{-1}X_{34}X_{41}^{-1} \sim \mathrm{diag}(x_1, .., x_m|y_1, .., y_n) \ . \tag{A.1}$$

The $m + n$ eigenvalues x_i, y_i then yield the $m + n$ arguments of the superblocks $B_{\gamma,\underline{\lambda}}$.

Thus any function of Z invariant under conjugation will automatically solve the superconformal Ward identities associated with the four point function. As first pointed out in this context in [7] the simplest way to construct a basis of all such polynomials associated with a Young diagram $\underline{\lambda}$ naturally yields (super) Schur polynomials (which are just the Jack polynomials with $\theta = 1$). The construction is as follows. Take $|\underline{\lambda}|$ copies of Z_A^B and symmetrise the upper (B) indices according to the Young symmetriser of $\underline{\lambda}$. Then contract all upper and lower indices. For example

$$\begin{array}{llcl}
\underline{\lambda} = \square & \rightarrow & Z^A_A = \mathrm{Tr}(Z) & = \; P_{\underline{\lambda}}(\mathbf{z};\theta{=}1) \\
\underline{\lambda} = \square\square & \rightarrow & Z^{(A_1}_{A_1} Z^{A_2)}_{A_2} = \frac{1}{2}\left(\mathrm{Tr}(Z)^2 + \mathrm{Tr}(Z^2)\right) & = \; P_{\underline{\lambda}}(\mathbf{z};\theta{=}1) \\
\underline{\lambda} = \begin{smallmatrix}\square\\ \square\end{smallmatrix} & \rightarrow & Z^{[A_1}_{A_1} Z^{A_2]}_{A_2} = \frac{1}{2}\left(\mathrm{Tr}(Z)^2 - \mathrm{Tr}(Z^2)\right) & = \; P_{\underline{\lambda}}(\mathbf{z};\theta{=}1)
\end{array} \tag{A.2}$$

Inputting the eigenvalues of Z (A.1) into these we obtain (up to an overall normalisation) precisely the corresponding Schur polynomials i.e. Jack polynomials with $\theta = 1$, $P_{\underline{\lambda}}(\mathbf{z};\theta = 1)$. This correspondence works for any Young diagram $\underline{\lambda}$.

A.2 $\theta = 2 : OSp(4m|2n)$

Details of this supercoset construction can be found (specialised to the $m = 2$ case) in [5,8]. We summarise here.

When $\theta = 2$ the relevant (complexified) supergroup is $OSp(4m|2n)$ which has bosonic subgroup $SO(4m) \times Sp(2n)$ with $SO(4m)$ non-compact and $Sp(2n)$ compact. Of physical interest is the case $m = 2$ in which $SO(8)$ is the complexification of the 6d conformal group $SO(2,6)$ and $Sp(2n)$ is the internal symmetry group for $(0,n)$ superconformal field theories.

The supercoset space is defined by the marked Dynkin diagram (3.12)b (or (3.13)b if $n = 0$) and can be realised in the block 2×2 matrix form of (3.4). To see this one needs to realise the group $osp(4m|2n)$ as the set of supermatrices orthogonal wrt the metric J:

$$osp(4m|2n) = \left\{ M \in \mathbb{C}^{(2m|2n|2m)\times(2m|2n|2m)} : MJM^T = J = \left(\begin{array}{c|cc|c} & & & 1_{2m} \\ \hline & & 1_n & \\ & -1_n & & \\ \hline 1_{2m} & & & \end{array}\right) \right\} . \tag{A.3}$$

Inputting $M = H$ in the block form of (3.4) into this we find that a is related to d but is itself unconstrained. Thus the Levi subgroup (under which the operators transform explicitly) is isomorphic to $GL(2m|n)$. Similarly, inputting $M = s(X)$ in the form of (3.6), we find that the coordinates are $(2m|n) \times (2m|n)$ antisymmetric supermatrices

$$X = -X^T . \tag{A.4}$$

Note that here and elsewhere X^T denotes the supertranspose $(X^{AB})^T = X^{BA}(-1)^{AB}$. Four-point functions are written in terms of a function of four such Xs, X_1, X_2, X_3, X_4 invariant under $OSp(4m|2n)$. We can use the symmetry to set $X_3 \rightarrow 0$ and then $X_2^{-1} \rightarrow 0$. Then we can use the remaining symmetry to set

$$X_1 \rightarrow K = \left(\begin{array}{cc|c} & 1_m & \\ -1_m & & \\ \hline & & 1_n \end{array}\right) . \tag{A.5}$$

Finally we end up with the problem of finding a function of X_4 which is invariant under $X_4 \rightarrow AX_4A^T$ where $AKA^T = K$. The group of matrices satisfying $AKA^T = K$ is $OSp(n|2m)$. So letting $Z = X_4K$ we seek functions of Z invariant under conjugation under $OSp(n|2m) \subset$

$GL(2m|n)$. Ultimately, an invariant function of X_1, X_2, X_3, X_4 will be a function of the eigenvalues of this matrix Z. The eigenvalues of the $m \times m$ piece of the matrix Z (3.8) are repeated

$$Z = X_{12} X_{23}^{-1} X_{34} X_{41}^{-1} \sim \text{diag}(x_1, x_1, x_2, x_2, .., x_m, x_m | y_1, .., y_n) \ . \tag{A.6}$$

As always the independent ones yield the $m|n$ arguments of the superblocks (2.7).

This construction again provides a completely manifest way of solving the superconformal Ward identities. Fascinatingly the simplest way to construct a basis of functions solving these Ward identities yields the super Jack polynomials! To construct a basis of such functions in 1-1 correspondence with Young diagrams $\underline{\lambda}$ proceed as follows. Take $|\underline{\lambda}|$ copies of $W = X_4$, and symmetrise all indices according to the Young symmetriser of $\hat{\underline{\lambda}}$ the Young diagram obtained from $\underline{\lambda}$ by duplicating all rows. Then contract all indices with $|\underline{\lambda}|$ copies of K.

Let us illustrate this with the simplest examples:

$$\begin{aligned}
&\underline{\lambda} = \square &&\Rightarrow \quad \hat{\underline{\lambda}} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array} &&\rightarrow \qquad W^{AB} K_{AB} = \text{Tr}(Z) = P_{\underline{\lambda}}(\mathbf{z}; \theta{=}2) \\
&\underline{\lambda} = \square\square &&\Rightarrow \quad \hat{\underline{\lambda}} = \begin{array}{|c|c|}\hline A & C \\ \hline B & D \\ \hline \end{array} &&\rightarrow \quad -W^{B(A} W^{C)D} K_{AB} K_{CD} = \tfrac{1}{2} \left(\text{Tr}(Z)^2 + \text{Tr}(Z^2) \right) = P_{\underline{\lambda}}(\mathbf{z}; \theta{=}2) \\
&\underline{\lambda} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} &&\Rightarrow \quad \hat{\underline{\lambda}} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline D \\ \hline \end{array} &&\rightarrow \quad W^{[AB} W^{CD]} K_{AB} K_{CD} = \tfrac{1}{3} \left(\text{Tr}(Z)^2 - 2\,\text{Tr}(Z^2) \right) = P_{\underline{\lambda}}(\mathbf{z}; \theta{=}2)
\end{aligned} \tag{A.7}$$

Inputting the eigenvalues of Z (A.6) into these we obtain (up to an overall normalisation) precisely the corresponding Jack polynomial with $\theta = 2$, $P_{\underline{\lambda}}(\mathbf{z}; \theta = 2)$. This correspondence continues for any Young diagram $\underline{\lambda}$.

Note that this derivation of superblocks and super Jacks from group theory makes manifest the stability in m, n of both, since they arise from formulae derived from matrices Z of arbitrary dimensions. A combinatoric description of (super)Jack polynomials with $\theta = 2$ along these lines has also been discussed in [92].

A.3 $\theta = \frac{1}{2}$: $OSp(4n|2m)$

The other value of θ for which there is a supergroup interpretation for any m, n is $\theta = \frac{1}{2}$. Here the relevant (complexified) supergroup is $OSp(4n|2m)$, the same complexified group as previously but with the roles of m, n reversed. This has bosonic subgroup $Sp(2m) \times SO(4n)$ but now with $Sp(2m)$ non-compact (for $m = 2$ this is the complexified 3d conformal group $Sp(4) \sim SO(5) \sim SO(2, 3)$) and $SO(4n)$ compact the internal subgroup.

The following is then essentially identical to the $\theta = 2$ case but with the role of Grassmann odd and even exchanged. The supercoset space defined by the marked Dynkin diagram (3.12)c (or (3.13)c if $n = 0$) can also be realised in the block 2×2 matrix form of (3.4) by realising the group $osp(4n|2m)$ in the following form[46]:

$$osp(4n|2m) = \left\{ M \in \mathbb{C}^{(m|4n|n)\times(m|4n|m)} : MJM^T = J = \left(\begin{array}{c|cc|c} & & & 1_m \\ \hline & & 1_{2n} & \\ & 1_{2n} & & \\ \hline -1_m & & & \end{array}\right) \right\} \tag{A.8}$$

The Levi subgroup (under which the operators transform explicitly) is isomorphic to $GL(m|2n)$ and the coordinates are $(m|2n) \times (m|2n)$ *symmetric* supermatrices

$$X = X^T \ . \tag{A.9}$$

Four-point functions are written in terms of a function of four such Xs, X_1, X_2, X_3, X_4 invariant under $OSp(4n|2m)$. We can use the symmetry to set $X_3 \to 0$ and then $X_2^{-1} \to 0$. Then we can use the remaining symmetry to set

$$X_1 \to K = \left(\begin{array}{c|cc} 1_m & & \\ \hline & & 1_n \\ & -1_n & \end{array}\right) \ . \tag{A.10}$$

We then are left with a function of X_4 which invariant under $X_4 \to AX_4A^T$ where $AKA^T = K$ i.e. under $OSp(m|2n)$. Letting $Z = X_4K$ we seek functions of Z invariant under conjugation under $OSp(m|2n) \subset GL(m|2n)$. Thus ultimately, an invariant function of X_1, X_2, X_3, X_4 will be a function of the eigenvalues of this matrix Z. The eigenvalues of the internal $n \times n$ piece of the matrix Z (3.8) are repeated and the independent eigenvalues yield the $m|n$ arguments of the superblocks (2.7)

$$Z = X_{12}X_{23}^{-1}X_{34}X_{41}^{-1} \sim \mathrm{diag}(x_1, .., x_m|y_1, y_1, y_2, y_2, .., y_n, y_n) \ . \tag{A.11}$$

As for $\theta = 2$ this construction provides a completely manifest way of solving the superconformal Ward identities naturally giving $\theta = \frac{1}{2}$ super Jack polynomials! Take $|\underline{\lambda}|$ copies of $W = X_4$, and symmetrise all indices according to the Young symmetriser of $\hat{\underline{\lambda}}$ which is this time the Young diagram obtained from $\underline{\lambda}$ by duplicating all *columns* rather than rows. Then contract all indices with $|\underline{\lambda}|$ copies of K.

In the simplest examples:

$$\begin{array}{llll} \underline{\lambda} = \square & \Rightarrow \quad \hat{\underline{\lambda}} = \boxed{A}\boxed{B} & \to & W^{AB}K_{AB} = \mathrm{Tr}(Z) \\ \underline{\lambda} = \square\square & \Rightarrow \quad \hat{\underline{\lambda}} = \boxed{A}\boxed{B}\boxed{C}\boxed{D} & \to & W^{(AB}W^{CD)}K_{AB}K_{CD} = \frac{1}{3}\left(\mathrm{Tr}(Z)^2 + 2\,\mathrm{Tr}(Z^2)\right) \\ \underline{\lambda} = \begin{array}{c}\square\\ \square\end{array} & \Rightarrow \quad \hat{\underline{\lambda}} = \begin{array}{c}\boxed{A}\boxed{B}\\ \boxed{C}\boxed{D}\end{array} & \to & W^{(B[A}W^{C]D)}K_{AB}K_{CD} = \frac{1}{2}\left(\mathrm{Tr}(Z)^2 - \mathrm{Tr}(Z^2)\right) \end{array} \tag{A.12}$$

Inputting the eigenvalues of Z (A.11) into these we obtain (up to an overall normalisation) precisely the corresponding Jack polynomials with $\theta = \frac{1}{2}$, $P_{\underline{\lambda}}(\mathbf{z}; \theta = \frac{1}{2})$. This correspondence works for any Young diagram $\underline{\lambda}$.

[46] This is a change of basis of the matrix form in (A.3)

As for $\theta = 1, 2$ this derivation of superblocks and super Jacks from group theory makes manifest the stability in m, n of both, since they arise from formulae derived from matrices Z of arbitrary dimensions.

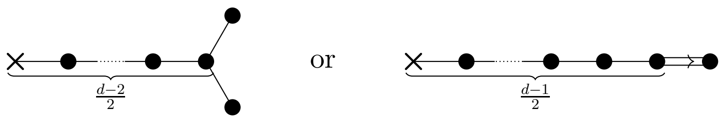
A.4 Non-supersymmetric conformal and internal blocks

We conclude our discussion by considering non-supersymmetric conformal and compact blocks. These were first analysed in [11] using an embedding space formalism for Minkowski space. Here we see how an unusual coset representation of Minkowski space relates these cases to the general matrix formalism we have presented.

Conformal blocks $(m, n) = (2, 0)$ with any $\theta \in \mathbb{Z}^+/2$

Complexified Minkowski space in d dimensions, M_d, can be viewed as a coset of the complexified conformal group $SO(d + 2; \mathbb{C})$ divided by the subgroup consisting of Lorentz transformations, dilatations and special conformal transformations. It is both an orthogonal Grassmannian and a flag manifold and can be conveniently denoted by taking the Dynkin diagram of $SO(d + 2; \mathbb{C})$ and putting a single cross through the first node (see [60])

M_d:



This crossed-through node represents the group of dilatations and the remaining Dynkin diagram (with the crossed node omitted) represents the Lorentz subgroup $SO(d; \mathbb{C})$ with $d = 2\theta + 2$.

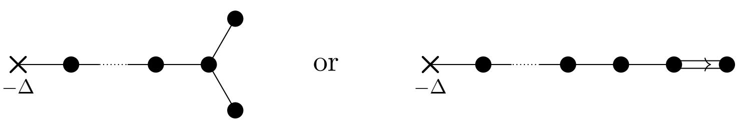
The coset construction for $\theta = \frac{1}{2}, 1, 2$ in the previous section suggests to start now from the spinorial representation of $SO(d + 2)$, with $d = 2\theta + 2$, rather than the fundamental representation. This representation is $2^{d/2}$ dimensional for d even (the Weyl representation) and $2^{(d+1)/2}$ dimensional for d odd. In this representation, the coset space can be written in the block form (3.6) and the coordinates X are $2^{\lceil\theta\rceil} \times 2^{\lceil\theta\rceil}$ matrices. The matrices will be highly constrained in general. The cross-ratios arise as eigenvalues of the matrix Z which are now repeated (due to the constraints, this matches (A.6) for the case $\theta = 2$ when $m = 2, n = 0$):

$$Z = X_{12} X_{23}^{-1} X_{34} X_{41}^{-1} \sim \text{diag}(x_1, .., x_1, x_2, .., x_2) \ . \tag{A.13}$$

thus there are always just two independent cross-ratios x_1, x_2 which become the variables of the conformal blocks in (2.7).

Representations of the conformal group are specified by placing Dynkin weights below the nodes, so for example a fundamental scalar is given by placing a -1 by the first node and zero's everywhere else. The scalars of dimension Δ, $\mathcal{O}_\Delta$ appearing as external operators in the four-point function (2.5), have a $-\Delta$ by the first node

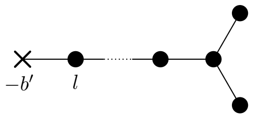
$\mathcal{O}_\Delta(X)$:



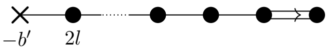
The fact that it has a negative Dynkin label is consistent with the fact that this corresponds to an infinite dimensional representation of the (complexified) conformal group (positive Dynkin labels give finite reps). One reads off from the diagram the representation as a field on Minkowski space: it has dilatation weight Δ and is a scalar under the Lorentz subgroup (since it has zero's under all nodes of the uncrossed part of the Dynkin diagram).

The only reps which occur in the OPE of two such scalars has dimension Δ and Lorentz spin l. The corresponding Dynkin diagram is

$O_{\gamma,\underline{\lambda}}(X)$:



or



where the Young tableau $\underline{\lambda}$ is at most two row (the only shape consistent with $(m,n)=(2,0)$) and

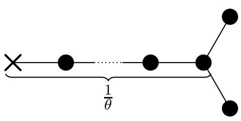
$$b' = \frac{\Delta+l}{\theta} = \gamma + \frac{2}{\theta}\lambda_1, \qquad ; \qquad \Delta = \theta\gamma + \lambda_1 + \lambda_2, \qquad ; \qquad l = \lambda_1 - \lambda_2 \ . \tag{A.14}$$

Notice again the redundancy in the description in terms of γ and $\underline{\lambda}$. The two operators $\mathcal{O}_{\gamma,[\lambda_1,\lambda_2]} = \mathcal{O}_{\gamma-2k,[\lambda_1+\theta k,\lambda_2+\theta k]}$ give the same representation for any k as long as it leaves a valid Young tableau. We could use this to set $\gamma=0$ in this case and just have a description in terms of the Young tableau $\underline{\lambda}$ only.

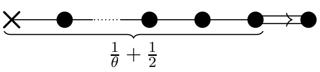
Note that there are different ways of realising M_d as a coset. The above way is a bit complicated for large d. Had we started instead from the fundamental representation of $SO(d+2)$, the coset construction would be equivalent to the embedding space formalism, in which M_d is viewed as the space of null $d+2$ vectors in projective space P_{d+1}. This is the approach used in [11]. This approach does not fit directly into the general (m,n,θ) matrix formalism we worked out in the previous cases with $\theta = \frac{1}{2}, 1, 2$ however. In particular the coset is not in the 2×2 block form of (3.6), and the independent cross-ratios have no natural origin arising from a diagonal matrix.

Internal blocks: $SO(2\theta+4)$ **blocks:** $(m,n)=(0,2)$ **with** $\theta \in 2/\mathbb{Z}^+$

A very similar case occurs for $m=0, n=2$, corresponding to internal blocks. Namely instead of four conformal scalars we have four finite dimensional reps of $SO(4+\frac{2}{\theta})$. Blocks for these were again discussed in Dolan and Osborn [11]. Here the complexified coset space is the same as in the previous case with $\theta \leftrightarrow \frac{1}{\theta}$ (although the real forms will be different, previously $SO(2,2+2\theta)$, now $SO(4+\frac{2}{\theta})$).

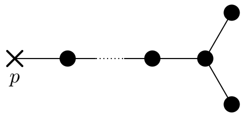


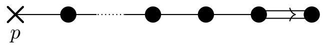
or



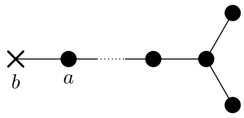
The discussion of these spaces as explicit matrices is also identical to the previous section.

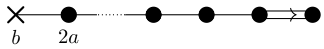
The external states are a specific representation of $SO(4+2/\theta)$ specified by placing p above the first node and zeros everywhere else. Note that this time the Dynkin label is positive as it is a finite dimensional representation.

$\mathcal{O}_p(X)$: 

or 

The only reps which occur in the 'OPE' (in this case equivalent to the tensor product) of two such reps have Dynkin labels

$O_{\gamma,\underline{\lambda}}(X)$: 

or 

.

The Young tableau $\underline{\lambda}$ is at most two column (the only shape consistent with $(m, n) = (0, 2)$ this is a rotation of a standard $SL(2)$ Young tableau) and we have

$$b = \gamma - 2\lambda'_1 \qquad ; \qquad a = \lambda'_1 - \lambda'_2 \; . \tag{A.15}$$

Again there is redundancy in the description in terms of γ and $\underline{\lambda}$ with $\mathcal{O}_{\gamma,[\lambda'_1,\lambda'_2]'} = \mathcal{O}_{\gamma+2\delta,[\lambda'_1+\delta,\lambda'_2+\delta]'}$ for any δ as long as it leaves a valid Young tableau. Here we could use this to force $\lambda'_2 = 0$, so only allow 1 column Young tableaux.

B From CMS Hamiltonians to the superblock Casimir

Jack polynomials (in suitable variables) are eigenfunctions of the Hamiltonian of the Calogero-Moser-Sutherland system for A_n root sytem whilst Jacobi polynomials (as well as dual Jacobi functions/ blocks) are eigenfunctions of the BC_n root system. Similarly *deformed* (i.e. supersymmetric) Calogero-Moser-Sutherland (CMS) Hamiltonians of the *generalised* root systems $A_{n|m}$ and $BC_{n|m}$ yield super Jack polynomials and dual super Jacob fucntions (superblocks) as eigenfunctions.

In this section we review the *deformed* Calogero-Moser-Sutherland (CMS) Hamiltonian associated to *generalised* root systems and show explicitly how the BC type relates to the super Casimir for superblocks in our theories.

B.1 Deformed CMS Hamiltonians.

The (non-deformed) Calogero-Moser-Sutherland (CMS) operator for *any* (not necessarily reduced) root system was first given in [93] via the Hamiltonian

$$\mathcal{H} = -\partial_I \partial^I + \sum_{\alpha \in R_+} \frac{k_\alpha(1 + k_\alpha + 2k_{2\alpha})\alpha^2}{\sin^2 \alpha_I u^I} \; , \tag{B.1}$$

where

- the u^I are coordinates in a vector space, $\partial_I := \partial/\partial u^I$, and the indices are raised and lowered using the Euclidean metric g_{IJ}.
- the roots $\{\alpha_I\}$ are a set of covectors (the roots of the root system) and R_+ is the set of positive roots. α^2 denotes the length squared of root α, $\alpha^2 = \alpha_I \alpha_J g^{IJ}$.

- Finally to each root α is associated a parameter k_α, which is constant under Weyl transformations, so $k_{w\alpha} = k_\alpha$ for w in the Weyl group. For any covector α which is not a root then we have $k_\alpha = 0$.

In [35] this story was generalised and *deformed* CMS operators were defined in terms of *generalized* (or supersymmetric) root systems. The irreducible generalized root systems were first classified by Serganova [94]. The generalized root systems are no longer required to have a Euclidean metric but can have non-trivial signature, and they have the analogous relation to Lie superalgebras as root systems do to Lie algberas. In the classification there are just two infinite series, $A_{m,n}$ and $BC_{m,n}$ (which we will concentrate on) together with a handful of exceptional cases.

The deformed Hamiltonian has the same form as (B.1) but with some additional subtleties

- The metric g_{IJ} need not be Euclidean.
- There are odd (also known as imaginary) roots which must have $k_\alpha = 1$.
- There will be relations between some of the parameters k_α even when they are not related by Weyl reflections.

We will give the two main cases, $A_{n|m}$ and $BC_{n|m}$ explicitly in the next subsections.

In all cases there is a special eigenfunction (the ground state) of $\mathcal{H}$ which takes the universal form,

$$\Psi_0(u_I, \{k_\alpha\}) = \prod_{\alpha \in R^+} \sin^{-k_\alpha} \alpha_I u^I \ . \tag{B.2}$$

This is automatic in the non-deformed case but in the deformed case it requires the additional relations between parameters k_α.

It is useful to then write other eigenfunctions of $\mathcal{H}$ as the product $\Psi_0 f$ for some function $f(u_I, \{k_\alpha\})$. The function f is then an eigenfunction of the conjugate operator $\Psi_0^{-1}\mathcal{H}\Psi_0$. This is the one we shall relate to the super Casimir. This conjugate operator also has a universal (and indeed simpler) form for all such (generalised) root systems

$$\mathcal{L} := \Psi_0^{-1}\mathcal{H}\Psi_0 + c = -\partial_I \partial^I + \sum_{\alpha \in R_+} 2k_\alpha \cot(\alpha_J u^J)\alpha^I \partial_I \tag{B.3}$$

for some constant c.

Our presentation in this appendix collects a number of known results and follows [35]. In addition, we will rewrite the differential operators $\mathcal{L}$, by using the (orthogonal) measures associated to the root systems.

B.2 A-type Hamiltonians and Jack polynomials

The $A_{n-1|m-1}$ generalised root system can be placed inside an $n+m$ dimensional vector space. The positive roots will be parametrised as

$$A_{n-1|m-1}: \qquad R^+ = \left\{ \begin{array}{l} e_i - e_j \\ e_i - e_{j'} \\ e_{i'} - e_{j'} \end{array} \quad ; \quad 1 \le i < j \le n, \;\; n+1 \le i' < j' \le m+n \right\} \tag{B.4}$$

where $e_{I=1,\dots m+n}$ give the basis of unit vectors. The (inverse) metric is

$$g^{IJ} = \left\{ \begin{array}{ll} \delta^{ij} & i,j = 1,..,n \\ -\theta\delta^{i'j'} & i',j' = n+1,..,m+n \end{array} \right. \tag{B.5}$$

R^+ splits into three separate Weyl orbits (therefore there will be three distinct k_α) and the three families of roots are

$$\begin{array}{llll} \alpha = e_i - e_j & : & \alpha^2 = 2, & k_\alpha = -\theta \\ \alpha = e_i - e_{j'} & : & \alpha^2 = 1-\theta, & k_\alpha = 1 \\ \alpha = e_{i'} - e_{j'} & : & \alpha^2 = -2\theta, & k_\alpha = -\frac{1}{\theta} \end{array} \tag{B.6}$$

Plugging in these, the CMS operator (B.1) thus becomes

$$\begin{aligned} \mathcal{H}^A &= -\sum_i \partial_i^2 + \theta \sum_{i'} \partial_{i'}^2 - \sum_{i<j} \frac{2\theta(1-\theta)}{\sin^2 u_{ij}} + \sum_i \sum_{j'} \frac{2(1-\theta)}{\sin^2(u_i - u_{j'})} + \sum_{i'<j'} \frac{2(1-1/\theta)}{\sin^2 u_{i'j'}} \\ &= -\sum_{I=1}^{n+m} (-\theta)^{\pi_I} \partial_i^2 + 2(1-\theta) \sum_{1 \le I < J \le n+m} \frac{(-\theta)^{1-\pi_I-\pi_J}}{\sin^2(u_I - u_J)} \end{aligned} \tag{B.7}$$

where the parity assignment is $\pi_{i=1,\dots n} = 0$ and $\pi_{i'=n+1,\dots n+m} = 1$.

The ground state is given by

$$\Psi_0 = \frac{\prod_{i<j} \sin^\theta(u_i - u_j) \prod_{i'<j'} \sin^{\frac{1}{\theta}}(u_{i'} - u_{j'})}{\prod_i \prod_{j'} \sin(u_i - u_{j'})} \tag{B.8}$$

and the conjugated operator (B.3) is

$$\begin{aligned} \mathcal{L}^A = \Psi_0^{-1} \left(\mathcal{H}^A - |\rho_\theta|^2 \right) \Psi_0 = &-\sum_{I=1}^{n+m} (-\theta)^{\pi_I} \partial_I^2 + 2 \sum_{i,j'} \cot(u_i - u_{j'})(\partial_i + \theta \partial_{j'}) \\ &- 2\theta \sum_{i<j} \cot(u_i - u_j)(\partial_i - \partial_j) + 2 \sum_{i'<j'} \cot(u_i - u_j)(\partial_{i'} - \partial_{j'}) \end{aligned}$$

where $\rho_\theta = \sum_\alpha k_\alpha \alpha = \sum_{I<J} (-\theta)^{1-\pi_I-\pi_J}(e_I - e_J)$, and $|\rho_\theta|^2$ is the norm of ρ_θ under g^{IJ}.

Notice that $\mathcal{L}^A$ contains in principle all terms coming from $\sum_I (-\theta)^{\pi_I} \partial_I^2 \Psi_0$ and in particular the terms $\sum_\beta \sum_{\alpha \neq \beta} k_\alpha k_\beta \, (\alpha_I g^{IJ} \beta_J) \, \cot(\alpha_I u^I) \cot(\beta_I u^I)$, which have mixed nature and do not cancel immediately with the ones from $\mathcal{H}^A$. However these can be replaced in favour of a θ dependent constant, $-\sum_\beta \sum_{\alpha\neq\beta} k_\alpha k_\beta\, (\alpha_I g^{IJ}\beta_J)$, because of the following relation

$$\sum_\beta \sum_{\alpha\neq\beta} k_\alpha k_\beta\, (\alpha_I g^{IJ}\beta_J)\, \big(\cot(\alpha_I u^I)\cot(\beta_I u^I)+1\big)=0 \tag{B.9}$$

This condition is automatically satisfied by the k_α assignment of the roots, and can be understood as a consistency condition on Ψ_0 being the ground state.

Finally, by changing variables to $z_I = e^{2iu_I}$, we find $\cot(u_I-u_J)=i(z_I+z_J)/(z_I-z_J)$ and $\partial_{u_I}=2iz_I\partial_{z_I}$. Therefore,

$$\tfrac{1}{4}\mathcal{L}^A=\sum_{I=1}^{n+m}(-\theta)^{\pi_I}(z_I\partial_I)^2+\theta\sum_{I<J}^{n+m}\frac{z_I+z_J}{z_I-z_J}((-\theta)^{-\pi_J}z_I\partial_I-(-\theta)^{-\pi_I}z_J\partial_J)\ . \tag{B.10}$$

and one can check that

$$\tfrac{1}{4}\mathcal{L}^A=\mathbf{H}+(\theta(m-1)-(n-1))\sum_I z_I\partial_I \tag{B.11}$$

where $\mathbf{H}$ is defined in (5.9) and is the defining operator of super Jacks (5.12) .

At this point it is also interesting to note that superJack operators are orthogonal in an $A_{(m,n)}$ measure [95]

$$\mathcal{S}_{(m,n)}(\mathbf{z};\theta)=\prod_I\prod_{J\neq I}\left(1-\frac{z_I}{z_J}\right)^{-(-\theta)^{1-\pi_I-\pi_J}} \tag{B.12}$$

(where the parity assignment is $\pi_{i=1,\dots n}=0$ and $\pi_{i'=n+1,\dots n+m}=1$) and the A-type CMS operator (B.10) can be rewritten in a very simple way in terms of this measure as

$$\tfrac{1}{4}\mathcal{L}^A=\mathcal{S}^{-1}\sum_I(-\theta)^{\pi_I}z_I\partial_I[\mathcal{S}z_I\partial_I]\ . \tag{B.13}$$

Finally, we point out that Jack polynomials are also eigenfunctions of the one-parameter family of Sekiguchi differential operators [67].

B.3 BC-type Hamiltonians and superblocks

We now move on to the CMS operator for the generalised BC root system and relate it to the Casimir which give superblocks.

The positive roots of the $BC_{n|m}$ root system live in an $m+n$ dimensional vector space and are as follows

$$R^{BC+}=\left\{\begin{matrix} e_i \\ e_{i'}\end{matrix}\ ;\ \begin{matrix}2e_i\\ 2e_{i'}\end{matrix}\ ;\ \begin{matrix}e_i\pm e_j\\ e_i\pm e_{j'}\\ e_{i'}\pm e_{j'}\end{matrix}\quad ;\quad 1\le i<j\le n,\ n+1\le i'<j'\le m+n\right\} \tag{B.14}$$

where again $e_{I=1,\dots m+n}$ are the basis of unit vectors and the inverse metric is the same as in the $A_{n|m}$ case (B.5). The parameters are assigned as follows,

$$\begin{aligned} \alpha = e_i \pm e_j \quad &: \quad \alpha^2 = 2, \qquad k_\alpha = -\theta \\ \alpha = e_i \pm e_{j'} \quad &: \quad \alpha^2 = 1-\theta, \qquad k_\alpha = 1 \\ \alpha = e_{i'} \pm e_{j'} \quad &: \quad \alpha^2 = -2\theta \qquad k_\alpha = -\tfrac{1}{\theta} \end{aligned} \tag{B.15}$$

and

$$\begin{array}{llll} \alpha = e_i : & \alpha^2 = 1, & k_\alpha = p & \\ & & & ; \\ \alpha = e_{i'} : & \alpha^2 = -\theta, & k_\alpha = r & \end{array} \quad \begin{array}{lll} \alpha = 2e_i : & \alpha^2 = 4, & k_\alpha = q \\ \alpha = 2e_{i'} : & \alpha^2 = -4\theta, & k_\alpha = s \end{array} \tag{B.16}$$

Plugging these values into the general formula (B.1) we obtain the Hamiltonian

$$\begin{aligned} \mathcal{H}^{BC} = &-\sum_{I=1}^{n+m} (-\theta)^{\pi_I} \partial_I^2 + 2(1-\theta) \sum_{1\le I<J\le n+m} \frac{(-\theta)^{1-\pi_I-\pi_J}}{\sin^2(u_I \pm u_J)} \\ &+ \sum_{i=1}^{n} \left(\frac{p(p+2q+1)}{\sin^2 u_i} + \frac{4q(q+1)}{\sin^2 2u_i} \right) - \theta \left(\sum_{i'=n+1}^{n+m} \frac{r(r+2s+1)}{\sin^2 u_{i'}} + \frac{4s(s+1)}{\sin^2 2u_{i'}} \right) \end{aligned} \tag{B.17}$$

where the first line is a trivial modification of (B.7).

The ground state (B.2) becomes

$$\Psi_0 = \frac{\prod_{i<j} \sin^\theta(u_i \pm u_j) \prod_{i'<j'} \sin^{\frac{1}{\theta}}(u_{i'} \pm u_{j'})}{\prod_{i=1}^{n} \prod_{j'=n+1}^{m+n} \sin(u_i \pm u_{j'}) \prod_{i=1}^{n} (\sin^p(u_i) \sin^q(2u_i)) \prod_{i'=n+1}^{n+m} (\sin^r(u_{i'}) \sin^s(2u_{i'}))} \tag{B.18}$$

and the conjugated operator is

$$\begin{aligned} \mathcal{L}^{BC} = \Psi_0^{-1} \left(\mathcal{H}^{BC} - |\rho_\theta|^2 \right) \Psi_0 = &-\sum_{I=1}^{n+m} (-\theta)^{\pi_I} \mathcal{D}_I \partial_I + 2\sum_{i,j'} \cot(u_i \pm u_{j'})(\partial_i \mp \theta \partial_{j'}) \\ &- 2\theta \sum_{i<j} \cot(u_i \pm u_j)(\partial_i \pm \partial_j) + 2\sum_{i'<j'} \cot(u_{i'} \pm u_{j'})(\partial_{i'} \pm \partial_{j'}) \end{aligned}$$

where $\rho_\theta = \sum_\alpha k_\alpha \alpha$ and $\mathcal{D}_I = (\partial_I + 2\cot(2u_I)) - 2(-\theta)^{-\pi_I}(p \cot u_I + (2q+1)\cot(2u_I))$.

Going through the computation notice that for the ground state to be so, the following relation had to be satisfyied

$$\sum_\beta \sum_{\beta \nsim \alpha} k_\alpha k_\beta \left(\alpha_I g^{IJ} \beta_J \right) \left(\cot(\alpha_I u^I) \cot(\beta_I u^I) + 1 \right) = 0 \qquad ; \qquad \begin{aligned} p &= -\theta r \\ 2q+1 &= -\theta(2s+1) \end{aligned} \tag{B.19}$$

This was not automatic for the k_α assignment but puts a constraint which we used to solve for r and s. The summation over $\beta \nsim \alpha$ excludes roots which are parallel in the vector space. The origin of this constraint again can be understood as a rewriting of the potential terms arising from $\sum(-\theta)^{\pi_I}\partial_I^2\Psi_0$ coming from mixed products which do not cancel immediately against the potential terms in $\mathcal{H}^{BC}$.

We now change variables to exponential coordinates $\hat{z}_I = e^{2iu_I}$, and we get

$$\begin{aligned}\tfrac{1}{4}\mathcal{L}^{BC} = &\sum_{I=1}^{n+m}\left((-\theta)^{\pi_I}\left(\hat{z}_I\hat{\partial}_I + \frac{\hat{z}_I^2+1}{\hat{z}_I^2-1}\right) - \left(p\frac{\hat{z}_I+1}{\hat{z}_I-1} + (2q+1)\frac{\hat{z}_I^2+1}{\hat{z}_I^2-1}\right)\right)(\hat{z}_I\hat{\partial}_I)\\ &+\theta\sum_{I<J}^{n+m}\frac{\hat{z}_I+\hat{z}_J^{\pm}}{\hat{z}_I-\hat{z}_J^{\pm}}((-\theta)^{-\pi_J}\hat{z}_I\hat{\partial}_I \mp (-\theta)^{-\pi_I}\hat{z}_J\hat{\partial}_J)\end{aligned} \tag{B.20}$$

We can rearrange the sum over many-body interactions as a sum over $I \neq J$, then for a single derivative we can add up the two terms $\frac{\hat{z}_I+\hat{z}_J^{\pm}}{\hat{z}_I-\hat{z}_J^{\pm}}$ to find $\frac{2\hat{z}_J(1-\hat{z}_I^2)}{(\hat{z}_I-\hat{z}_J)(1-\hat{z}_i\hat{z}_J)}$.

Finally we need to change variables again to[47]

$$z_I = \frac{1}{2} - \frac{1}{4}\left(\hat{z}_I + \frac{1}{\hat{z}_I}\right) \tag{B.21}$$

and we obtain

$$\begin{aligned}\tfrac{1}{4}\mathcal{L}^{BC}(\mathbf{z},\theta) = &\sum_{I=1}^{n+m}\left((-\theta)^{\pi_I}\partial_I z_I(z_I-1)\partial_I - \left(p(z_I-1) + (2q+1)\left(z_I-\tfrac{1}{2}\right)\right)\partial_I\right)\\ &-2\theta\sum_{I\neq J}\frac{(-\theta)^{-\pi_J}}{z_I-z_J}z_I(1-z_I)\partial_I\end{aligned} \tag{B.22}$$

At this point we can understand the relation between the super Casimir $\mathbf{C}$, defining the superblocks, and $\mathcal{L}^{BC}$. One can check that they are closely related after conjugation and a shift

$$\begin{aligned}&\left(\frac{\prod_{i=1}^m x_i^\theta}{\prod_{j=1}^n y_j}\right)^{-\beta}\tfrac{1}{4}\mathcal{L}^{BC}\left(\tfrac{1}{x_1}\ldots\tfrac{1}{x_m};\tfrac{1}{y_1}\ldots\tfrac{1}{y_n};\theta\right)\left(\frac{\prod_{i=1}^m x_i^\theta}{\prod_{j=1}^n y_j}\right)^{+\beta} =\\ &\qquad\mathbf{C}^{(\theta,\frac{1}{2}(\gamma-p_{12}),\frac{1}{2}(\gamma-p_{43}),\gamma)} - \left[\beta(n-m\theta)((n-1)-(m-1)\theta+(\gamma-\beta)\theta)\right]\end{aligned} \tag{B.23}$$

with $\beta = \min\left(\frac{1}{2}(\gamma-p_{12}),\frac{1}{2}(\gamma-p_{43})\right)$. We have thus shown that the differential operator corresponding to BC root system is equivalent to our Casimir operator.

Before concluding this section however, we point that there is an analogue of the measure based relation (B.13) in the super BC case which we have not seen in the literature previously.

[47]Note the following useful relations

$$\begin{cases} z_I &= -\frac{1}{4}\left(\frac{1}{\sqrt{\hat{z}_I}} - \sqrt{\hat{z}_I}\right)^2\\ z_I - 1 &= -\frac{1}{4}\left(\frac{1}{\sqrt{\hat{z}_I}} + \sqrt{\hat{z}_I}\right)^2\end{cases} \quad ; \quad \begin{cases}\hat{z}_I\frac{\partial z_I}{\partial \hat{z}_I} = & \frac{1}{4}\frac{(1-\hat{z}_I^2)}{\hat{z}_I} &= \sqrt{z_I(z_I-1)}\\ & \frac{1}{4}\frac{(1+\hat{z}_I^2)}{\hat{z}_I} &= -(z_I-\frac{1}{2})\end{cases}$$

Macdonald used a measure to define BC_n Jacobi polynomials as orthogonal (see for example [28] page 52). This measure has a natural generalisation to the $BC_{n|m}$ case as follows

$$\mathcal{S}^{(p^-,p^+)}(\mathbf{z};\theta) = \prod_{I=1}^{n+m} (z_I)^{p^-(-\theta)^{1-\pi_I}} (1-z_I)^{p^+(-\theta)^{1-\pi_I}} \prod_{I<J} (z_I - z_J)^{-2(-\theta)^{1-\pi_I-\pi_J}} . \tag{B.24}$$

We then find that the operator $\frac{1}{4}\mathcal{L}^{BC}$ has the simple form

$$\tfrac{1}{4}\mathcal{L}^{BC}(\mathbf{z};\theta,p,q) = -\mathcal{S}^{-1}\sum_{I=1}^{n+m} (-\theta)^{\pi_I}\partial_{z_I}\Big[z_I(1-z_I)\ \mathcal{S}\ \partial_{z_I}\Big] \tag{B.25}$$

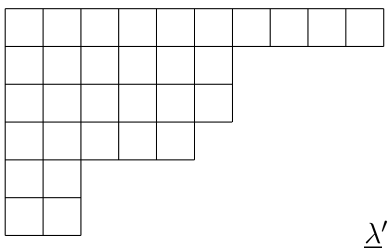
where $p=\theta(p^- - p^+)$ and $q=-\frac{1}{2}+\theta p^+$. Recall that $p^\pm = \frac{|p_{43}\pm p_{12}|}{2}$ in terms of the external charges.

C Symmetric and supersymmetric polynomials

In this section we review relevant background regarding Young diagrams and symmetric polynomials (Jacks and interpolation Jacks), both in the bosonic and supersymmetric case. We shall view multivariate Jack polynomials and interpolation polynomials as fundamental, in the sense that they are homogeneous polynomials in certain variables. In particular they can be constructed as a sum over semistandard Young tableaux, or filling, as we will see. On the other hand, Jacobi polynomials (and blocks) we view as 'composite': they naturally can be thought of as a sum of Jacks as we have done throughout the paper. In this appendix we focus on the fundamental objects themselves.

C.1 Symmetric polynomials

A Young diagram is a collection of boxes drawn consecutively on rows and columns, with the number of boxes on each row decreasing as we go down, for example



$$\underline{\lambda} = [10,6,6,5,2,2] \qquad \underline{\lambda}' = [6,6,4,4,4,3,1,1,1,1] \tag{C.1}$$

By counting the number of boxes on the rows we define a representation of the Young diagram of the form $\underline{\lambda}=[\lambda_1,\ldots]$. Equivalently, by counting the number of boxes on the columns we define the transposed representation $\underline{\lambda}'=[\lambda'_1,\ldots]$. Both $\underline{\lambda}$ and $\underline{\lambda}'$ are partitions of the total number of boxes. A box $\square$ in the diagram has coordinates (i,j) where for each row index i, $1\le j\le\lambda_i$, and for each column index j, $1\le j\le\lambda'_i$.

Physics-wise, Jack and Jacobi polynomials are best known as eigenfunctions of known differential operators, but more abstractly, the theory of symmetric polynomials associates polynomials to Young diagrams in such a way that polynomials are characterised by properties and uniqueness theorems [27, 28, 32, 33], which are equivalent to solving the corresponding differential equations.[48]

In this appendix we will highlight a combinatorial definition for Jack and interpolation polynomials, which is very efficient in actual computations. We focus on bosonic polynomials first, for which there is *no* distinction among $\{x_1, \dots x_m\}$ variables, differently from the supersymmetric case discussed afterwards.

For a polynomial P of m variables, the combinatorial formula takes the form

$$P_{\underline{\lambda}}(x_1, \dots x_m; \vec{s}) = \sum_{\{\mathcal{T}\}} \Psi_{\mathcal{T}}(\vec{s}) \prod_{(i,j)\in\underline{\lambda}} f\Big(x_{\mathcal{T}(i,j)}; \vec{s}\Big) \ . \tag{C.2}$$

The functions Ψ and f depend on the polynomials under consideration (i.e. Jack or Interpolation polynomials). Note that in some cases the function f may depend explicitly on the integers (i, j) and the integer $\mathcal{T}(i, j)$, in addition to the variable $x_{\mathcal{T}(i,j)}$, but we will usually suppress this dependence in order to avoid cluttering the notation. Both Ψ and f may also depend on various external parameters, which we denoted collectively by $\vec{s}$. As a concrete example to have in mind: Jack polynomials have simply $f(x, i, j; \theta) = x$ and for the special case with $\theta = 1$ (corresponding to Schur polynomials) the coefficient $\Psi_{\mathcal{T}}(\theta = 1) = 1$.

The sum in (C.2) is over all *fillings* (also known as semistandard Young tableaux) of the Young diagram $\underline{\lambda}$, denoted here and after by $\{\mathcal{T}\}$. A filling $\mathcal{T}$ assigns to a box with coordinates $(i, j) \in \underline{\lambda}$, a number in $\{1, \dots m\}$ in such a way that $\mathcal{T}(i, j)$ is weakly decreasing in j, which means from left-to-right, and strongly decreasing in i, from top-to-bottom, which means $\mathcal{T}(i, j) > \mathcal{T}(i - 1, j)$, $\mathcal{T}(i, j) \geq \mathcal{T}(i, j - 1)$.

Note that the fillings precisely correspond to the independent states of the $U(m)$ representation $\underline{\lambda}$ which one typically views as a tensor with $|\underline{\lambda}|$ indices symmetrised via a Young symmetrizer.

For example, if $\underline{\lambda} = [3, 1]$ and $m = 2$, the fillings $\{\mathcal{T}\}$ are

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \tag{C.3}$$

which correspond to the three states in the corresponding rep of $U(2)$, the independent states in a tensor of the form $S_{abcd} = T_{(abc)d} - T_{(dbc)a}$ where the indices $a, b, c, d = 1, 2$. Then for example the Schur polynomial can be directly read off from (C.2) with $f(x) = x$ and $\Psi = 1$ as

$$P_{[3,1]}(x_1, x_2; \theta = 1) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 \ . \tag{C.4}$$

We immediately see from this definition that if the number of rows of $\underline{\lambda}$ is greater than the number of variables m then $P_{\underline{\lambda}} = 0$ since it is not possible to construct a valid semistandard tableaux and the sum is empty (just as for $U(m)$ reps).

[48] See [29] for a comprehensive introduction.

In order to define Ψ it is first useful to note that there is a simple way to generate all the fillings $\{\mathcal{T}\}$ given a Young diagram $\underline{\lambda}$, via recursion in the number of variables m. (See for example the discussion in [67].) This recursion in fact gives a way of generating the polynomial itself also, equivalent to the combinatorial formula (C.2). It reads

$$\begin{aligned} P_{\underline{\lambda}}(x_1,\dots x_m,x_{m+1};\vec{s}) &= \sum_{\underline{\kappa}\prec\underline{\lambda}} \psi_{\underline{\lambda},\underline{\kappa}}(\vec{s}) \left(\prod_{(i,j)\in\underline{\lambda}/\underline{\kappa}} f(x_{m+1};\vec{s})\right) P_{\underline{\kappa}}(x_1,\dots x_m;\vec{s}) \\ P_{[\varnothing]} &= 1 \end{aligned} \tag{C.5}$$

where $\underline{\lambda}/\underline{\kappa}$ is the skew Young diagram obtained by taking the Young diagram of $\underline{\lambda}$ and deleting the boxes of the sub Young diagram $\underline{\kappa}$ (see section C.2). Here ψ is closely related to Ψ mentioned above (we will give the precise relation shortly), and the symbol $\underline{\kappa}\prec\underline{\lambda}$ means that $\underline{\kappa}$ belongs to the following set,[49]

$$\{\ [\kappa_1,\dots,\kappa_m]:\quad \lambda_{m+1}\le\kappa_m\le\lambda_m\ ,\ \dots\ ,\ \lambda_2\le\kappa_1\le\lambda_1\ \}\ . \tag{C.6}$$

In this formula, if $\underline{\lambda}$ is a partition with less than $m+1$ rows, it is extended with trailing zeros. The recursion generates sequences of Young diagrams of the form

$$[\varnothing]\equiv\underline{\kappa}^{(0)}\prec\underline{\kappa}^{(1)}\prec\dots\prec\underline{\kappa}^{(m)}\prec\underline{\kappa}^{(m+1)}\equiv\underline{\lambda}, \tag{C.7}$$

with a strict inclusion, i.e. $\underline{\kappa}^{(i-1)}\subset\underline{\kappa}^{(i)}$. This is the same as considering the set of fillings $\{\mathcal{T}\}$. For example, for the above case with $\underline{\lambda}=[3,1]$ in (C.3) the sum in (C.5) would be over $\underline{\kappa}\in\{[3],[2],[1]\}$. These sub Young diagrams are then filled with x_1's and the remaining bits $\underline{\lambda}/\underline{\kappa}$ filled with x_2 reproducing (C.3).

It follows that for a filling $\mathcal{T}$ corresponding to a sequence (C.7),

$$\Psi_{\mathcal{T}}(\vec{s}) = \prod_{i=1}^{m+1}\psi_{\underline{\kappa}^{(i)},\underline{\kappa}^{(i-1)}}\ , \qquad \prod_{(i,j)\in\underline{\lambda}} f(x_{\mathcal{T}(i,j)};\vec{s}) = \prod_{l=1}^{m+1}\prod_{(i,j)\in\kappa^{(l)}/\kappa^{(l-1)}} f(x_l;\vec{s}) \tag{C.8}$$

and the recursion and the combinatorial formula (C.2) are the same. In particular l is the level of nesting in the recursive formula. In the example above with $m+1$ variables, the relation between l and $\mathcal{T}(i,j)$ is

$$l=(m+1)-\mathcal{T}(i,j)+1. \tag{C.9}$$

Given the above facts it is enough for us to define $\psi_{\underline{\lambda},\underline{\kappa}}$ and the function $f(x;\vec{s})$ in order to fully define the symmetric function.

We will now examine the various specific cases, beginning with the Jack polynomials.

[49]The skew diagram $\underline{\lambda}/\underline{\kappa}$ is sometimes known as a *horizontal strip.*

C.2 Jack polynomials

Jack polynomials depend on one parameter, θ. The defining function is (for example [67])

$$\psi_{\underline{\lambda},\underline{\kappa}}(\theta) = \prod_{1\leq i\leq j<m+1} \frac{(\kappa_i - \lambda_{j+1} + 1 + \theta(j-i))_{\lambda_i-\kappa_i}}{(\kappa_i - \lambda_{j+1} + \theta(j-i+1))_{\lambda_i-\kappa_i}} \frac{(\kappa_i - \kappa_j + \theta(j-i+1))_{\lambda_i-\kappa_i}}{(\kappa_i - \kappa_j + 1 + \theta(j-i))_{\lambda_i-\kappa_i}} \ . \quad \text{(C.10)}$$

with,

$$f(x;\theta) = x \quad \text{(C.11)}$$

Considering the top-left Pochhammer symbol, notice that $\psi_{\underline{\lambda},\underline{\kappa}}$ vanishes when,

$$(\kappa_i - \lambda_{i+1} + 1)_{\lambda_i-\kappa_i} = (\kappa_i + 1 - \lambda_{i+1}) \dots (\lambda_i - \lambda_{i+1}) = 0 \quad \text{(C.12)}$$

i.e. when one of the terms vanishes. This happens precisely when $\underline{\kappa} \not\prec \underline{\lambda}$.

As a simple example then consider $P_{[2,1]}(x_1, x_2; \theta)$. The case with $\theta = 1$ (Schur polynomial) is given in (C.4). For arbitrary θ we need the coefficient $\Psi_{\mathcal{T}}(\theta)$ for the three fillings $\mathcal{T}$ in (C.3). The first and third filling have $\Psi = 1$ whereas the second has $\Psi = \frac{2\theta}{1+\theta}$ and we thus obtain

$$P_{[3,1]}(x_1, x_2; \theta) = x_1^3 x_2 + \tfrac{2\theta}{1+\theta} x_1^2 x_2^2 + x_1 x_2^3 \ . \quad \text{(C.13)}$$

Notice that Jack polynomials are stable, i.e. $P_{\underline{\lambda}}(x_1, \dots x_m, 0; \theta) = P_{\underline{\lambda}}(x_1, \dots x_m; \theta)$ as can be shown directly from the combinatorial formula (as well as from their definition as the unique polynomial eigenfunctions of the differential equation (5.12) with eigenvalues and differential operator independent of m, and with the same normalisation).

Also note that Jack polynomials have the following property

$$P_{\underline{\lambda}+\tau^m} = (x_1 \dots x_m)^\tau P_{\underline{\lambda}} \ . \quad \text{(C.14)}$$

It is instructive to prove (C.14) by showing again that both RHS and LHS are eigenfunctions of the A_n CMS operator $\mathbf{H}$ in (5.12). Applying $\mathbf{H}$ on the LHS we simply find the corresponding eigenvalue $h_{\underline{\lambda}+\tau^m}$. On the RHS we need to consider what happens upon conjugation,

$$(x_1 \dots x_m)^{-\tau} \cdot \mathbf{H} \cdot (x_1 \dots x_m)^\tau = \mathbf{H} + 2\tau \sum_i x_i \partial_i + h^{(\theta)}_{\tau^m} \quad \text{(C.15)}$$

Thus applying $\mathbf{H}\cdot$ to (C.14) we find

$$h^{(\theta)}_{\underline{\lambda}+\tau^m} = h^{(\theta)}_{\underline{\lambda}} + 2\tau|\underline{\lambda}| + h^{(\theta)}_{\tau^m} \quad \text{(C.16)}$$

which is an identity, and proves (C.14), since both RHSand l.h.s have the same small variable expansion.

Remark. When there is an ambiguity in the notation, in relation to the supersymmetric case, we will specify $P^{(m,0)}(\mathbf{z};\theta)$ to mean the bosonic Jack polynomial.

Dual Jack polynomials

Jack polynomials are orthogonal but not orthonormal (except when $\theta = 1$ where they reduce to Schur polynomials) under the Hall inner product. The *dual* Jack polynomials (where dual here denotes the usual vector space dual under the Hall inner product) are thus simply a normalisation of the Jacks, defined so that they have unit inner product with the corresponding Jack. Given a Jack polynomial, the *dual* Jack polynomial has the form [29]

$$Q_{\underline{\kappa}}(\mathbf{x};\theta) = \frac{C^-_{\underline{\kappa}}(\theta;\theta)}{C^-_{\underline{\kappa}}(1;\theta)} P_{\underline{\kappa}}(\mathbf{x};\theta) \quad ; \quad C^-_{\underline{\kappa}}(t;\theta) = \prod_{(ij)\in\underline{\kappa}} \big(\kappa_i - j + \theta(\kappa'_j - i) + t\big) \tag{C.17}$$

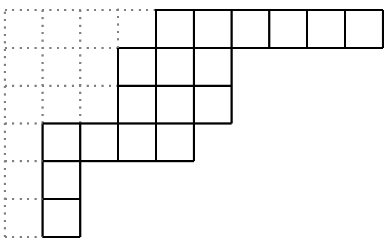
where, as throughout

$$\Pi_{\underline{\kappa}}(\theta) = \frac{C^-_{\underline{\kappa}}(\theta;\theta)}{C^-_{\underline{\kappa}}(1;\theta)} \qquad ; \qquad \Pi_{\underline{\kappa}}(\tfrac{1}{\theta}) = \big(\Pi_{\underline{\kappa}'}(\theta)\big)^{-1} \ . \tag{C.18}$$

Skew Jack polynomials

For one way of defining super Jack polynomials shortly we will also need the concept of skew Jack polynomials.

A skew Young diagram $\underline{\lambda}/\underline{\mu}$, where $\underline{\mu} \subseteq \underline{\lambda}$ is obtained by erasing $\underline{\mu}$ from $\underline{\lambda}$. The Figure below gives a simple example,



$\underline{\lambda} = [10, 6, 6, 5, 2, 2]$

$\underline{\mu} = [4, 3, 3, 1, 1, 1]$

Skew Jack polynomials are then defined by a similar combinatorial formula but where one sums only over semi standard skew Young tableaux. Or equivalently by the recursion formula

$$P_{\underline{\lambda}/\underline{\mu}}(x_1, \ldots x_m, x_{m+1}; \theta) = \sum_{\underline{\mu} \preceq \underline{\kappa} \prec \underline{\lambda}} \psi_{\underline{\lambda},\underline{\kappa}}(\theta)\ x_{m+1}^{|\underline{\lambda}|-|\underline{\kappa}|} P_{\underline{\kappa}/\underline{\mu}}(x_1, \ldots x_m; \theta) \ . \tag{C.19}$$

Notice that $m + 1 \geq \lambda'_1 - \mu'_1$. The number of variables here is the number of variables that can fill in the Young diagram according to recursion in (C.6). The recursion goes on as long as $\underline{\mu} \prec \underline{\kappa}^{(1)} \prec \ldots \prec \underline{\kappa}^{(m)} \prec \underline{\kappa}^{(m+1)} \equiv \underline{\lambda}$ and the condition $\underline{\mu} \prec \underline{\kappa}^{(1)}$ is non trivial, since it implies that a skew Jack polynomial has the vanishing property

$$P_{\underline{\lambda}/\underline{\mu}}(x_1, \ldots x_m) = 0 \qquad \text{if} \quad \lambda_{m+i} > \mu_i \ . \tag{C.20}$$

For example, imagine a $\underline{\lambda}$ with very long rows, and take a very small $\underline{\mu}$, then no horizontal strip of $\underline{\lambda}$ will contain $\underline{\mu}$. Indeed the minimal diagram $\underline{\kappa}$ generated by the table (C.6) at each step of the recursion is given by components of $\underline{\lambda}$ properly shifted upwards.

An example of a skew Jack polynomial is

$$P_{[3,1,1]/[1]}(x_1, x_2; \theta) = x_1^3 x_2 + x_1 x_2^3 + \frac{2\theta}{1+\theta} x_1^2 x_2^2 \tag{C.21}$$

Skew polynomials have the property that

$$P_{\underline{\lambda}}(\mathbf{x}; \theta) = \sum_{\underline{\mu} \subset \underline{\lambda}} P_{\underline{\mu}}(x_1, \dots x_{m-n}; \theta) P_{\underline{\lambda}/\underline{\mu}}(x_{m-n+1}, \dots x_m; \theta) \tag{C.22}$$

Structure constants and decomposition formulae for Jacks

The Jack structure constants $\mathcal{C}^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(\theta)$ are defined as follows

$$P_{\underline{\lambda}} P_{\underline{\mu}} = \sum_{\underline{\nu}} \mathcal{C}^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(\theta) P_{\underline{\nu}} \ . \tag{C.23}$$

For $\theta = 1$ they are just the Littlewood-Richardson coefficients.

Then there are related coefficients $\mathcal{S}^{\underline{\lambda}\,\underline{\mu}}_{\underline{\nu}}(\theta)$ obtained from decomposing skew Jack polynomials into Jack polynomials

$$P_{\underline{\nu}/\underline{\lambda}} = \sum_{\underline{\mu}} \mathcal{S}^{\underline{\lambda}\,\underline{\mu}}_{\underline{\nu}}(\theta) P_{\underline{\mu}} \ . \tag{C.24}$$

The property (C.22) then yields the decomposition formula for decomposing higher dimensional Jacks into sums of products of lower dimensional Jacks

$$P_{\underline{\lambda}}(x_1, .., x_{m+n}) = \sum_{\underline{\mu},\underline{\nu}} P_{\underline{\mu}}(x_1, .., x_m) \mathcal{S}^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta) P_{\underline{\nu}}(x_{m+1}, .., x_{m+n}) \ . \tag{C.25}$$

For $\theta = 1$ these decomposition coefficients are also the Littlewood-Richardson coefficients, $\mathcal{C}^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(1) = \mathcal{S}^{\underline{\lambda}\,\underline{\mu}}_{\underline{\nu}}(1)$, but for general θ they are related via normalisation:

$$\mathcal{S}^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta) = \frac{\Pi_{\underline{\mu}}(\theta)\Pi_{\underline{\nu}}(\theta)}{\Pi_{\lambda}(\theta)} \mathcal{C}^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}}(\theta) \ . \tag{C.26}$$

Note that the structure constants $\mathcal{C}^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(\theta)$ and $\mathcal{S}^{\underline{\lambda}\,\underline{\mu}}_{\underline{\nu}}(\theta)$ do not depend on the dimensions of the Jack polynomials in any of the above formulae.

Also note that if the Young diagram $\underline{\lambda}$ is built from two Young diagrams $\underline{\mu}, \underline{\nu}$ on top of each other then the corresponding structure constant is 1:

$$\mathcal{S}^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta) = 1 \qquad \text{if} \qquad \underline{\lambda} = [\underline{\mu}, \underline{\nu}] \ . \tag{C.27}$$

This can be seen by (C.24) noting that the Jack polynomial of dimension equal to the height of $\underline{\mu}$ is equal to the skew Jack

$$P_{\underline{\mu}}(x_1, .., x_{\mu'_1}) = P_{\underline{\lambda}/\underline{\nu}}(x_1, .., x_{\mu'_1}) \qquad \text{if} \qquad \underline{\lambda} = [\underline{\mu}, \underline{\nu}] \ . \tag{C.28}$$

This can be verified from the respective combinatoric formulae.

C.3 Interpolation polynomials

The interpolation polynomials of relevance here (i.e. the BC-type which appear in section 7) depend on two parameters, θ, and a new one denoted here by u. They can be defined in a very similar way to the Jack polynomials and we will denote them $P^{ip}(\mathbf{x}; \theta, u)$. They are symmetric polynomials in the variables $\mathbf{x} = (x_1, .., x_m)$. Generically P^{ip} is a complicated, non factorisable polynomial and it is uniquely defined by the following vanishing property

$$P^{ip}_{\underline{\kappa}}(\underline{\nu} + \theta\underline{\delta} + u; \theta, u) = 0 \qquad \text{if} \qquad \underline{\lambda} \subset \underline{\mu} \tag{C.29}$$

where $\underline{\delta} = (m-1, .., 1, 0)$. This idea generalises what happens for the Pochhammer symbol $(-z)_\lambda = (-z)(-z+1)\dots(-z+\lambda-1)$ which vanishes if $0 \leq z < \lambda$.[50]

The interpolation polynomial is defined via the combinatorial formula (C.2) (or the equivalent recursion (C.5)) with ψ exactly the same as for the Jacks (C.10), but with a more complicated x dependence arising from a modified f

$$\psi_{\underline{\lambda},\underline{\kappa}}(\theta, u) = \psi_{\underline{\lambda},\underline{\kappa}}(\theta) \tag{C.30}$$

$$f(x_l, i, j, l; \theta, u) = x_l^2 - ((j-1)-\theta(i-1) + \theta(l-1) + u)^2 \tag{C.31}$$

where $\psi_{\underline{\lambda},\underline{\kappa}}(\theta)$ is the defining function for Jack polynomials (C.10). Instead, note that f here depends explicitly on (i, j) and l. As in (C.7) the index l labels the nesting and is related to $\mathcal{T}(i, j)$ via (C.9).

One can see an example of the vanishing property from the above definition: take the last variable x_m and the last row $i = m$ in (C.31), this contributes with terms which vanish for all $x_m = j - 1 + u$, where $j = 1, \dots \kappa_m$.

Because of the various shifts in the vanishing property (C.29), it is useful to define the non-symmetric version of the interpolation polynomial $P^*(\mathbf{x}; \theta, u)$ by

$$P^*(\mathbf{x}; \theta, u) \equiv P^{ip}(\mathbf{x} + \theta\underline{\delta} + u; \theta, u) \qquad \underline{\delta} = (m-1, .., 1, 0) \tag{C.32}$$

So the vanishing property (C.29) takes the form

$$P^*_{\underline{\mu}}(\underline{\lambda}; \theta, u) = 0 \qquad \text{if} \quad \underline{\lambda} \subset \underline{\mu} \ . \tag{C.33}$$

(Recalling however that $P^*_{\underline{\mu}}$ here is no longer symmetric in its variables $\mathbf{z}$.)

This interpolation polynomial is $\mathbb{Z}_2$ invariant under $x_i \leftrightarrow -x_i$. It was introduced by Okounkov [33], and re-obtained by Rains [32], using a different approach.

[50] Indeed the Pochhammer symbol $(-z)_k$ is precisely a case of interpolating polynomial, giving the standard one-variable binomial expansion $(1+x)^n = \sum_k \binom{n}{k} x^k = \sum_k \frac{(-n)_k}{(-k)_k} x^k$.

C.4 Supersymmetric polynomials

The general combinatorial formulation of symmetric polynomials outlined in section C.1 has a natural supersymmetric generalisation [34]. This allows in particular to define super Jack polynomials and super interpolation Jack polynomials. The only real modification in the general story is that the definition of a filling (semistandard tableaux) is modified to become a supersymmetric filling (or bitableau) since it has to take into account two alphabets, $x_1, \ldots x_m$ and $y_1, \ldots y_n$. Written in terms of the letters $\mathbf{z}$, the labelling is $z_i = x_i$ for $i = 1, \ldots m$, and $z_{m+j} = y_j$ for $j = 1, \ldots n$ and the combinatorial formula then looks the same as in (C.2)

$$P_{\underline{\lambda}}(z_1, \ldots z_{m+n}; \vec{s}) = \sum_{\{\mathcal{T}\}} \Psi_{\mathcal{T}}(\vec{s}) \prod_{(i,j)\in\underline{\lambda}} f(z_{\mathcal{T}(i,j)}; \vec{s}) \tag{C.34}$$

but with the sum now over all *supersymmetric* fillings.

The supersymmetric filling assigns to a box of coordinate (i, j) a number $\{1, \ldots n + m\}$ such that $\mathcal{T}(i, j)$ is weakly decreasing[51] in j, i.e. from left-to-right, and is weakly decreasing in i, i.e. from top-to-bottom. But, if $\mathcal{T}(i, j) \in \{1, \ldots m\}$, then $\mathcal{T}(i, j)$ is strictly decreasing in i, i.e. from top-to-bottom, and if $\mathcal{T}(i, j) \in \{n+1, \ldots n+m\}$, then $\mathcal{T}(i, j)$ is strictly decreasing in j, i.e. from left-to-right.

Note that the above notion of a supersymmetric filling has a direct relation with states of the supergroup $U(m|n)$ just as the ordinary fillings relate to states of $U(m)$. This can again be clearly seen by representing the $U(m|n)$ irrep $\underline{\lambda}$ as a tensor $T_{A_1A_2..A_{|\underline{\lambda}|}}$ with superindices $A \in \{1, .., m|m + 1, .., m + n\}$ and symmetrising the indices in the standard fashion via a Young symmetriser. The only caveat is that when the index $A \in \{m + 1, .., m + n\}$ the index is viewed as 'fermionic' and so symmetrising two of them becomes anti-symmetrising and vice versa. The resulting independent states obtained in this way will have a precise correspondence with the supersymmetric fillings.

A first example, $\underline{\lambda} = [3, 1]$ **with** $(z_1, z_2|z_3)$**.**

This has two rows, so we can fill it in as in (C.3), namely

$$\begin{array}{|c|c|c|}\hline 2&2&2\\\hline 1\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 2&2&1\\\hline 1\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 1\\\cline{1-1}\end{array} \tag{C.35}$$

Then we introduce the variable $z_3 = y_1$, once

$$\begin{array}{|c|c|c|}\hline 3&2&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&2\\\hline 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 1\\\cline{1-1}\end{array}$$

and twice

$$\begin{array}{|c|c|c|}\hline 3&2&2\\\hline 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 3\\\cline{1-1}\end{array} \tag{C.36}$$

[51] Conventionally the supersymmetric filling is also reversed, hence decreasing rather than increasing.

These correspond to the states of the $U(2|1)$ rep $[3, 1]$ as can be seen via Young symmetrising as described above.

A second example, $\underline{\lambda} = [3, 1]$ with $(z_1, z_2|z_3, z_4)$.

Again we can fill $\underline{\lambda}$ as in the previous example, with both 3 and $3 \to 4$. Then we have new fillings in which both z_3 and z_4 appear. These are

$$\begin{array}{|c|c|c|}\hline 4&3&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&2\\\hline 4\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&1\\\hline 4\\\cline{1-1}\end{array} \tag{C.37}$$

and

$$\begin{array}{|c|c|c|}\hline 4&1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&2&1\\\hline 3\\\cline{1-1}\end{array} \tag{C.38}$$

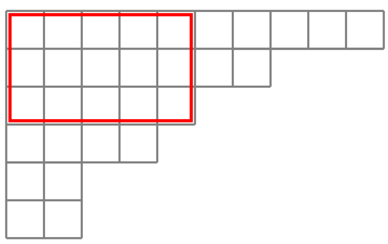
$$\begin{array}{|c|c|c|}\hline 4&3&1\\\hline 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&2\\\hline 1\\\cline{1-1}\end{array} \tag{C.39}$$

for a total of $3 + 2 \times 9 + 4 + 7$ fillings. These correspond to the states of the $U(2|2)$ rep $[3, 1]$ as can be seen via Young symmetrising.

The variables $\mathbf{z} = (x_1, \dots x_m|y_{m+1}, \dots y_{m+n})$ are defined to have a parity $\pi_i = 0$ if x_i and $\pi_i = 1$ if y_i.

A Young diagram $\underline{\lambda}$ endowed with an (m, n) structure is a Young diagram $\underline{\lambda}$ that has to satisfy the condition $\lambda_{m+1} \leq n$. One can easily check that only then will the diagram allow any supersymmetric filling, and correspondingly only then can it yield a non vanishing $U(m|n)$ rep. This condition implies that the Young diagram has at most a *hook* shape, with only m arbitrarily long rows and only n arbitrarily long columns. The reps split into two cases,

- *typical* Young diagram, i.e. $\lambda_{m+1} \leq n$ such that it contains the rectangle n^m. These correspond to long representations of $U(m|n)$.

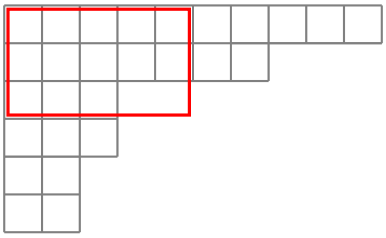


$(m, n) = (3, 5)$

$\underline{\lambda} = [10, 7, 5, 4, 2, 2]$

(C.40)

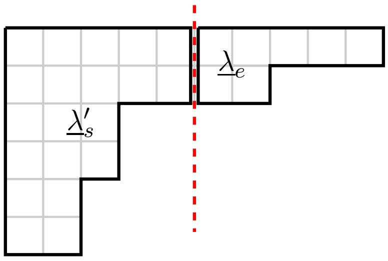
- *atypical* Young diagram, i.e. $\lambda_{m+1} \leq n$ such that it *does not* contain n^m. These correspond to short representations of $U(m|n)$.



$$(m,n) = (3,5) \qquad \underline{\lambda} = [10,7,3,3,2,2] \tag{C.41}$$

Typical representations are such that the box with coordinates (m,n) lies in the diagram. The simplest example of an atypical representation is the empty diagram.

In both cases it can be useful to define two sub Young diagrams, by essentially cutting open the diagram vertically after the nth column. We then define the Young diagram obtained from the first n columns and transposing as $\underline{\lambda}_s$ and the remaining diagram as $\underline{\lambda}_e$. So in the above atypical example:



$$(m,n) = (3,5) \qquad \underline{\lambda} = [10,7,3,3,2,2] \qquad \underline{\lambda}_s = [6,6,4,2,2] \qquad \underline{\lambda}_e = [5,2] \tag{C.42}$$

These sub Young tableaux have a direct interpretation in terms of the corresponding $U(m|n)$ rep. They are simply the corresponding representations of the $U(n)$ and $U(m)$ subgroups of the highest weight state. Or put another way, the highest term in the supersymmetric filling will have only y entries in $\underline{\lambda}_s$ and only x entries in $\underline{\lambda}_e$.

C.5 Super Jack polynomials

Having outlined the general structure of supersymmetric polynomials let us now specify to the main interest, the super Jack polynomials. We just need to define Ψ and f in (C.34). To define Ψ we split the superfilling $\mathcal{T}$ of $\underline{\lambda}$ into $\mathcal{T}_1$, the part containing $m+1,..,m+n$, and the rest $\mathcal{T}_0 = \mathcal{T}/\mathcal{T}_1$. Say that $\underline{\mu}$ is the shape of $\mathcal{T}_1$. Then $\Psi(\mathcal{T};\theta)$ is defined in terms of the Jack Ψ as [34]

$$\Psi_{\mathcal{T}}(\theta) = (-)^{|\underline{\mu}|}\Pi_{\underline{\mu}'}(\tfrac{1}{\theta})\Psi_{\mathcal{T}_1'}(\tfrac{1}{\theta})\Psi_{\mathcal{T}_0}(\theta) \tag{C.43}$$

$$f(z;\theta) = z \; . \tag{C.44}$$

According to this definition the superJack is given as a sum over all decompositions of the form

$$P_{\underline{\lambda}}(\mathbf{z};\theta) = \sum_{\underline{\mu}\subseteq\underline{\lambda}} (-)^{|\underline{\mu}|} Q_{\underline{\mu}'}(y_1,\dots y_n,\tfrac{1}{\theta})\, P_{\underline{\lambda}/\underline{\mu}}(x_1,\dots x_m;\theta) \; . \tag{C.45}$$

where the sum can also be restricted to $\underline{\mu}$ such that $\max(\lambda'_j - m, 0) \leq \mu'_j \leq \lambda'_j$ with $j = 1, \ldots n$.

Note also that we could construct the superJacks directly from Jack polynomials via their decomposition into $U(m)$ and $U(n)$ reps following [34] (see also [95] for a nice review). Quite nicely [34] showed that this decomposition can be brought to (C.45). From this point of view it might be useful to compare this with the bosonic decomposition (C.22).

Remark. A super Jack polynomial is denoted by $P^{(m,n)}_{\underline{\lambda}}(\mathbf{z};\theta)$. However, the (m,n) dependence can be read off the variables $\mathbf{z}$, when there are no ambiguities in the notation. When this is the case, we will simply use $P_{\underline{\lambda}}(\mathbf{z};\theta)$.

Some simple observations which can be seen directly from this formula:

1) If $\underline{\lambda}$ fails to have an (m,n) structure, so $(i,j) = (m{+}1, n{+}1)$ is in the Young diagram $\underline{\lambda}$, then the superJack vanishes. For any $\underline{\mu}$ in the summation, either the box $(m+1, n+1)$ is in $\underline{\mu}$, in which case $Q_{\underline{\mu}'} = 0$ as $\underline{\mu}'$ will have $n+1$ rows, or $(m+1, n+1)$ is in $\underline{\lambda}/\underline{\mu}$ in which case $P_{\underline{\lambda}/\underline{\mu}} = 0$ as $\underline{\lambda}/\underline{\mu}$ will have a full column with $m+1$ elements.

2) When $m = 0$ the sum localises on $\underline{\mu} = \underline{\lambda}$, and since $P_{\underline{\lambda}/\underline{\lambda}} = P_{\varnothing} = 1$, the superJack polynomial reduces to a (normalised) dual Jack polynomial $(-)^{|\underline{\lambda}'|} Q_{\underline{\lambda}'}(\mathbf{y}; \frac{1}{\theta})$. When $n = 0$ the sum localises on $\underline{\mu} = \varnothing$ and the superJack polynomials reduces to $P_{\underline{\lambda}}(\mathbf{x};\theta)$.

C.6 Properties of Super Jack polynomials

Stability

The combinatorial formula makes stability of superJacks manifest. Alternatively, from the fact that the eigenvalue of the A-type CMS differential depends only on the Young diagram, and the uniqueness of the polynomial solution for given Young diagram, we also infer that the super Jack polynomials are stable.

The m and n switch

From the bosonisation of the eigenvalue and a property of the differential operator $\mathbf{H}$, namely

$$h^{(\theta)}_{\underline{\lambda}} = -\theta \sum_{j=1} \lambda'_j(\lambda'_j - 1 - \tfrac{2}{\theta}(j-1)) = -\theta h^{(\frac{1}{\theta})}_{\underline{\lambda}'} \qquad ; \qquad \mathbf{H}^{(\frac{1}{\theta})}(\mathbf{y}|\mathbf{x}) = -\tfrac{1}{\theta}\mathbf{H}^{(\theta)}(\mathbf{x}|\mathbf{y}) \tag{C.46}$$

we deduce that $P_{\underline{\lambda}'}(\mathbf{y}|\mathbf{x}; \frac{1}{\theta})$ has to be proportional to $P_{\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta)$. Looking at the precise normalisations we arrive at

$$P^{(m,n)}_{\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta) = (-1)^{|\underline{\lambda}'|} \Pi_{\underline{\lambda}'}(\tfrac{1}{\theta}) P^{(n,m)}_{\underline{\lambda}'}(\mathbf{y}|\mathbf{x}; \tfrac{1}{\theta}) \tag{C.47}$$

We can also show this more directly from either the combinatorial formula or the decomposition formula (where we further decompose the skew Jacks into Jacks via structure constants, and use some known properties of the structure constants).

Shift transformation of superJacks

We have emphasised that blocks should be invariant under the shift symmetry (2.20). In appendix D we will show that the coefficients T_γ in the expansion of blocks over Jacks possesses this symmetry, but we thus also require the symmetry for the Jacks themselves in order that the blocks have this symmetry. We consider this here.

Let $\underline{\lambda}$ be a typical (m,n) representation, i.e. $\lambda_m \geq n, \lambda_{m+1} \leq n$. Then we consider: 1) the Young diagram obtained by a horizontal shift of $\theta\tau'$ to the first m rows of $\underline{\lambda}$ (on the east), and 2) the Young diagram obtained by a vertical shift of τ' to the first n columns of $\underline{\lambda}'$ (on the south). We will show that

$$P^{(m,n)}_{\underline{\lambda}+(\theta\tau')^m}(\mathbf{x}|\mathbf{y};\theta) \qquad ; \qquad \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\tau'} P^{(m,n)}_{(\underline{\lambda}'+\tau'^n)'}(\mathbf{x}|\mathbf{y};\tfrac{1}{\theta}) \tag{C.48}$$

are proportional to each other, then we will find the proportionality factor.

For the bosonic $n=0$ Jack polynomials, the above claim follows from (C.16) with $\tau = \theta\tau'$. We will thus use the same argument, and show that both the LHS and r.h.s in (C.48) are eigenfunctions of $\mathbf{H}$ with both m and n turned on.

So let us see what happens when we apply $\mathbf{H}$ on (C.48). On the LHS we simply find the eigenvalue $h^{(\theta)}_{\underline{\lambda}+(\tau)^m}$. On the RHS we use

$$\left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{-\tau'} \cdot \mathbf{H} \cdot \left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\tau'} = \mathbf{H} + 2\theta\tau' \sum_I z_I \partial_I + \tau'(m\theta - n)(n-1-\theta(m-1)+\tau'\theta) \tag{C.49}$$

Note also that the constant term can be written in terms of the same eigenvalue of $\mathbf{H}$,

$$\tau'(m\theta-n)(n-1-\theta(m-1)+\tau'\theta) = +2\theta\tau'|m^n| + h^{(\theta)}_{(\theta\tau')^m} - \theta h^{(\frac{1}{\theta})}_{(-\tau')^n} \tag{C.50}$$

(In the bosonic case, $n=0$, this was $h^{(\theta)}_{\tau^m}$). Then, the action of $\mathbf{H}$ on (C.48) matches because of the identity

$$h^{(\theta)}_{\underline{\lambda}+(\theta\tau')^m} = +h^{(\theta)}_{(\theta\tau')^m} - \theta h^{(\frac{1}{\theta})}_{\underline{\lambda}'+\tau'^n} - \theta h^{(\frac{1}{\theta})}_{(-\tau')^n} + 2\theta\tau'\Big(|m^n| + |\lambda'| + |(\tau')^n|\Big) \,. \tag{C.51}$$

Note that this is true *only* if $\lambda_m \geq n, \lambda_{m+1} \leq n$ ie only for typical, long representations. For example it is clearly false for $\underline{\lambda} = [\varnothing]$, which would read "$0 = 2\theta\tau'|m^n|$".[52]

Having established that the polynomials in (C.48) are proportional to each other, we only need to fix the proportionality. By considering the first term in the decomposition formula (C.45) the only change to take into account is the Π function inside Q for the lowest

[52] We can also rewrite the above equality as

$$\frac{h^{(\theta)}_{\underline{\lambda}+(\theta\tau')^m} - h^{(\theta)}_{(\theta\tau')^m}}{2\theta\tau'} + \frac{h^{(\frac{1}{\theta})}_{\underline{\lambda}'+\tau'^n} + h^{(\frac{1}{\theta})}_{(-\tau')^n}}{2\tau'} = |m^n| + |\lambda'| + |(\tau')^n| \tag{C.52}$$

In the limit $\tau' \to 0$ the RHS is simply $\sum_{i=1}^m \lambda_i + \sum_{j=1}^n \lambda'_j$.

representation. Following this through we arrive at the shift formula (D.8)

$$\Pi_{(\underline{\lambda}_s-m^n)'}(\theta)P^{(m,n)}_{\underline{\lambda}+(\theta\tau')^m}(\mathbf{x}|\mathbf{y};\theta)=\Pi_{(\underline{\lambda}_s-m^n+\tau'^n)'}(\theta)\left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\tau'}P^{(m,n)}_{(\underline{\lambda}'+\tau'^n)'}(\mathbf{x}|\mathbf{y};\theta) \tag{C.53}$$

where we are using $\Pi^{-1}_{\underline{\nu}}(\frac{1}{\theta})=\Pi_{\nu'}(\theta)$, and we introduced

$$\underline{\lambda}_s=[\lambda'_1,\ldots\lambda'_n] \tag{C.54}$$

as in (C.42).

Super Jacks and the superconformal Ward identity

The uplift of Jack to superJack polynomials follows from a characterisation theorem, cf. Theorem 2 of [34]. In particular, a superJack polynomial $P_{\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta)$ is generated by the map $\varphi_{(m,n)}$ which acts on the power sum decomposition of $P_{\underline{\lambda}}(\mathbf{z};\theta)$ as

$$\varphi_{(m,n)}\left(p_r\equiv\sum_j z^r\right)=\sum_{j=1}^m x_j^r-\tfrac{1}{\theta}\sum_{j=1}^n y_j^r \quad\rightarrow\quad P_{\underline{\lambda},(m,n)}(;\theta)=\varphi_{(m,n)}(P_{\underline{\lambda}}(\mathbf{z};\theta)) \tag{C.55}$$

Note that this map is very easy to understand in the cases $\theta=1,2,\frac{1}{2}$ where there is a group theoretic interpretation. For example, when $\theta=1$ we consider the symmetric polynomials as functions of the $n\times n$ matrix Z, invariant under conjugation $f(Z)=f(G^{-1}ZG)$, with z_i the eigenvalues of Z. Then the supersymmetric case just corresponds to the case where Z is a $(m|n)\times(m|n)$ supermatrix and the map φ is just taking the supertrace (see (A.1) and the following discussion). So for example the power sums

$$p_r=\mathrm{tr}(Z^r)=\sum_j z^r \qquad\rightarrow\qquad \mathrm{str}(Z^r)=\sum_{i=1}^m x_i^r-\sum_{j=1}^n y_j^r\ . \tag{C.56}$$

A similar discussion also follows in the cases $\theta=2$ and $\theta=\frac{1}{2}$ (which is dual under the m,n swap) with the only real difference as far as power sums is concerned being that there are repeated eigenvalues of Z in these cases, which accounts for the factors of θ appearing. (See (A.6) and the following for the $\theta=2$ case and (A.11) for the $\theta=\frac{1}{2}$ case.)

The characterisation (C.55) implies that superJack polynomials satisfy the condition

$$\left[\left(\frac{\partial}{\partial x_i}+\theta\frac{\partial}{\partial y_i}\right)P(\mathbf{z};\theta)\right]_{x_i=y_i}=0 \tag{C.57}$$

From the point of view of the four point functions invariant under the superconformal group, this condition is a consequence of the super-conformal Ward identity. We understand in this way that our construction of the superconformal blocks, as given by the series over super Jack polynomials, automatically satisfy the super-conformal Ward identity. Our approach here is alternative to various other approaches which instead use the superconformal Ward identity to fix the superblock, given an ansatz of $m\otimes n$ bosonic blocks. We will consider this approach in appendix 9.

Structure constants and decomposition formulae for super Jacks

The super Jacks (from stability) have exactly the same structure constants as the Jacks. So (C.23) is true for superJacks

$$P_{\underline{\lambda}} P_{\underline{\mu}} = \mathcal{C}^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(\theta) P_{\underline{\nu}} \ . \tag{C.58}$$

and furthermore so is the decomposition formula for decomposing higher dimensional super Jacks into sums of products of lower dimensional super Jacks

$$P^{(m+m'|n+n')}_{\underline{\lambda}} = \sum_{\underline{\mu},\underline{\nu}} P^{(m|n)}_{\underline{\mu}} \mathcal{S}^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta) P^{(m'|n')}_{\underline{\nu}} \ . \tag{C.59}$$

Here the arguments of the superJack on the LHS are split between the two superJacks on the RHS e.g. $P^{(m|n)}_{\underline{\mu}}(x_1,..,x_m|y_1,..,y_n)$ and $P^{(m'|n')}_{\underline{\nu}}(x_{m+1},..,x_{m+m'}|y_{n+1},..,y_{n+n'})$. The coefficients $\mathcal{S}^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta)$ are completely independent of m, m', n, n' and are related to the structure constants just as before (C.26)

$$\mathcal{S}^{\underline{\mu}\underline{\nu}}_{\underline{\lambda}}(\theta) = \frac{\Pi_{\underline{\mu}}(\theta)\Pi_{\underline{\nu}}(\theta)}{\Pi_{\lambda}(\theta)} \mathcal{C}^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}}(\theta) \ . \tag{C.60}$$

Indeed the definition of superJacks itself in terms of Jacks (C.45) is just an example of this decomposition with $n = m' = 0$ (after using (C.47) with $n = 0$).

Note that in the $\theta = 1$ case this corresponds to the decomposition of $U(m+m'|n+n') \to U(m|n) \otimes U(m'|n')$ for which the coefficients are the Littlewood Richardson coefficients.

C.7 Super interpolation polynomials

The last polynomials we review are the super interpolation polynomials which we first discuss below (7.29). These were first introduced in [38], and we will repeat below the definition given in [38] following our conventions.

Introduce the bosonic polynomial,

$$\hat{I}^{(m)}_{\underline{\lambda}}(x_1, \ldots x_m; \theta, h) = P^{ip}_{\underline{\lambda}}(\ldots, x_i - \theta i + h, \ldots; \theta, h - \theta m) \tag{C.61}$$

$$= P^{*}_{\underline{\lambda}}(\mathbf{x}; \theta, h - \theta m) \tag{C.62}$$

where the corresponding function f follows from (C.31), i.e.

$$f(x_l, i, j, l; \theta, h) = (x_l - \theta l + h)^2 - ((j-1) - \theta(i-1) + h - \theta \mathcal{T}(i,j))^2 \tag{C.63}$$

where $\mathcal{T}(i,j) = 1 + m - l$. In the main text, see (7.29), we wrote

$$\tilde{P}^{*(m)}_{\underline{\lambda}}(\mathbf{x}; \theta, h) \equiv \hat{I}^{(m)}_{\underline{\lambda}}(\mathbf{x}; \theta, h) = P^{*(m)}_{\underline{\lambda}}(\mathbf{x}; \theta, h - \theta m) \, . \tag{C.64}$$

The supersymmetrised version of $\hat{I}$ has the same structure as (C.45), simply reflecting the underlying sum over superfillings, and it is simply

$$\hat{I}^{(m,n)}_{\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta,h)=\sum_{\underline{\mu}\subseteq\underline{\lambda}}(-)^{|\underline{\mu}|}\left[(\theta^2)^{|\underline{\mu}'|}\Pi_{\mu'}(\tfrac{1}{\theta})\hat{I}_{\underline{\mu}'}(y_1,\dots y_n;\tfrac{1}{\theta},\tfrac{1}{2}+\tfrac{1}{2\theta}-\tfrac{h}{\theta}+m)\right]\hat{I}_{\underline{\lambda}/\underline{\mu}}(x_1,\dots x_m;\theta,h) \tag{C.65}$$

In particular, we find $\hat{I}^{(0,n)}_{\underline{\lambda}}(|\mathbf{y};\theta,h)=(-\theta^2)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}'}(\frac{1}{\theta})\tilde{P}^{*(n)}_{\underline{\lambda}'}(\dots y_n;\frac{1}{\theta},\frac{1}{2}+\frac{1}{2\theta}-\frac{h}{\theta})$, and for $n=0$ we obvious recover $\hat{I}^{(m)}_{\underline{\lambda}}$. Note that if we take the scaling limit $z\to\epsilon z$ and we look at the leading contribution as $\epsilon\to\infty$, that polynomial is a superJack, i.e.

$$\textit{lead. contr.}\left[\hat{I}^{(m,n)}_{\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta,h)\right]=P^{(m,n)}_{\underline{\lambda}}(\dots,x_m^2|\dots,\theta^2y_n^2;\theta) \tag{C.66}$$

Following Veselov and Sergeev, we now define

$$I^{(m,n)}_{\underline{\lambda}}(\mathbf{x}|\mathbf{y};\theta,h)=(-\theta^2)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}'}(\tfrac{1}{\theta})\hat{I}^{(n,m)}_{\underline{\lambda}'}(\mathbf{y}|\mathbf{x};\tfrac{1}{\theta},\tfrac{1}{2}+\tfrac{1}{2\theta}-\tfrac{h}{\theta}) \tag{C.67}$$

This object is such that

$$I^{(m,0)}_{\underline{\lambda}}=\tilde{P}^{*(m)}_{\underline{\lambda}}(\dots x_m;\theta,h)\qquad;\qquad I^{(m,n)}_{\underline{\lambda}}(\underline{\mu}_e|\underline{\mu}_s;\theta,h)=0\qquad\text{if}\qquad\underline{\mu}\subset\underline{\lambda} \tag{C.68}$$

where $\underline{\mu}_s=[\mu'_1,..,\mu'_N]$ and $\underline{\mu}_e\equiv\underline{\mu}/\underline{\mu}'_s=[(\mu_1-N)_+,..,(\mu_M-N)_+]$ where $(x)_+\equiv\max(x,0)$. Note also that the leading contribution of I now gives a superJack polynomial through the relation in (C.47). In the main text, see (7.30), we wrote

$$\tilde{P}^{(m,n)}_{\underline{\lambda}}(\mathbf{z};\theta,h)\equiv I^{(m,n)}_{\underline{\lambda}}(\mathbf{z};\theta h) \tag{C.69}$$

with I being the supersymmetrisation we were looking for. Finally

$$I^{(m,n)}_{\underline{\lambda}}(\underline{\lambda}_e|\underline{\lambda}_s;\theta,h)=\prod_{(i,j)\in\underline{\lambda}}(1+\lambda_i-j+\theta(\lambda'_j-i))(2h-1+\lambda_i+j-\theta(\lambda'_j+i)) \tag{C.70}$$

The r.h.s. of course does not depend on (m,n) and it becomes (7.7) upon matching the parameters.

D More properties of analytically continued superconformal blocks

D.1 Shift symmetry of the supersymmetric form of the recursion

We would like the supersymmetric recursion (6.8) to be invariant under the supersymmetric shift:

$$\begin{array}{ccccc}\lambda_i\to\lambda_i-\theta\tau' & & \lambda'_j\to\lambda'_j+\tau' & & \\ \mu_i\to\mu_i-\theta\tau' & ; & \mu'_j\to\mu'_j+\tau' & ; & \gamma\to\gamma+2\tau'\ . \\ i=1,\dots m & & j=1,\dots n & & \end{array} \tag{D.1}$$

This is *almost* the case for (6.8). The issue is to do with normalisation and we will fix it below. Let us emphasise first that this shift symmetry arises directly from the group theory in all cases with a group theory interpretation, $\theta=\frac{1}{2},1,2$ (see (3.18)) but we expect it for any

θ. Note that in writing (D.1), we are implicitly assuming that the Young diagram, prior to analytic continuation, contains the box with coordinates (m,n). Thus in a supersymmetric theory we are looking at typical or long representations.

We claim that the following normalisation of T_γ gives a shift invariant result:

$$(T_\gamma^{\texttt{long}})^{[\underline{\mu}_s;\underline{\mu}_e]}_{[\underline{\lambda}_s;\underline{\lambda}_e]} = \frac{\Pi_{\underline{\mu}_s - m^n}(\frac{1}{\theta})}{\Pi_{\underline{\lambda}_s - m^n}(\frac{1}{\theta})}(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} \quad ; \quad \begin{array}{l} \underline{\lambda}_s = [\lambda'_1, \dots \lambda'_n] \quad ; \quad \underline{\lambda}_e = [\lambda_1, \dots \lambda_m] \\ \underline{\mu}_s = [\mu'_1, \dots \mu'_n] \quad ; \quad \underline{\mu}_e = [\mu_1, \dots \mu_m] \end{array} \tag{D.2}$$

where we split the diagrams into east and south components by taking the corresponding first m rows and first n columns. We might consider $\underline{\lambda}_e \to \underline{\lambda}_e - n^m$, however this is not important here.

So the claim is that $T_\gamma^{\texttt{long}}$ is invariant under the shift (D.1). In particular, it solves the following recursion

$$\Big(h_{\underline{\mu}} - h_{\underline{\lambda}} + \theta\gamma\,(|\underline{\mu}| - |\underline{\lambda}|)\Big)(T_\gamma^{\texttt{long}})^{[\underline{\mu}_s;\underline{\mu}_e]}_{[\underline{\lambda}_s;\underline{\lambda}_e]} =$$

$$\sum_{i=1}^{m} (\mu_i - 1 - \theta(i-1-\alpha))\,(\mu_i - 1 - \theta(i-1-\beta))\,\mathbf{f}^{(i)}_{\underline{\mu}_e - \square_i}(\theta)(T_\gamma^{\texttt{long}})^{[\underline{\mu}_s;\underline{\mu}_e - \square_i]}_{[\underline{\lambda}_s;\underline{\lambda}_e]} +$$

$$+\theta\sum_{j=1}^{n}\Big(\mu'_j - 1 - \alpha - \tfrac{j-1}{\theta}\Big)\Big(\mu'_j - 1 - \beta - \tfrac{j-1}{\theta}\Big)\,\mathbf{f}^{(j)}_{\underline{\mu}_s - \square_j}(\tfrac{1}{\theta})\mathbf{c}^{(j)}_{\underline{\mu} - \square_j}(\tfrac{1}{\theta})(T_\gamma^{\texttt{long}})^{[\underline{\mu}_s - \square_j;\underline{\mu}_e]}_{[\underline{\lambda}_s;\underline{\lambda}_e]} \tag{D.3}$$

where

$$\mathbf{c}^{(j)}_{\underline{\mu}}(\tfrac{1}{\theta}) = \prod_{i=1}^{m} \frac{\Big(\mu_i - j + 1 + \theta(\mu'_j - i)\Big)\Big(\mu_i - j + \theta(\mu'_j - i + 2)\Big)}{\Big(\mu_i - j + 1 + \theta(\mu'_j - i + 1)\Big)\Big(\mu_i - j + \theta(\mu'_j - i + 1)\Big)} \tag{D.4}$$

Note that $\mathbf{c}^{(j)}_{\underline{\mu}}(\theta)$ is built out of arm and leg length symbols, $c_{\underline{\nu}}(i,j) = \nu_i - j + (\nu'_j - i + 1)/\theta$ and $c'_{\underline{\nu}}(i,j) = \nu_i - j + 1 + (\nu'_j - i)/\theta$ for $\underline{\nu} = \underline{\mu}, (\underline{\mu}' + \square_j)'$. This $\mathbf{c}^{(j)}_{\underline{\mu}}(\theta)$ is now invariant under (D.1) since it mixes rows and columns coherently, i.e.

$$\mu_i + \theta\mu'_j \qquad \text{is invariant under (D.1) if} \qquad 1 \le i \le m \quad ; \quad 1 \le j \le n. \tag{D.5}$$

The origin of the normalisation in (D.2) can be understood starting from the Casimir differential equations and the super Jack polynomials. Indeed, a quick way to see the invariance of the superblocks under (D.1) is to consider that the eigenvalue of the original Casimir operator on $B_{\gamma,\underline{\lambda}}$ (2.11)

$$E^{(m,n;\theta)}_{\gamma,\underline{\lambda}} = h^{(\theta)}_{\underline{\lambda}} + \theta\gamma|\underline{\lambda}| + \Big[\gamma\theta|m^n| + h^{(\theta)}_{[e^m]} - \theta h^{(\frac{1}{\theta})}_{[s^n]}\Big] \tag{D.6}$$

with $e = +\frac{\theta\gamma}{2}$ and $s = -\frac{\gamma}{2}$, is indeed invariant under (D.1). To see this, the hybrid form of the RHS is useful, therefore $|\gamma| = \sum_{i=1}^{m}\lambda_i + \sum_{j=1}^{n}\lambda'_j - nm$, and for $h_{\underline{\lambda}}(\theta)$ the expression given in (5.26).

The precise statement about shift invariance of the long superconformal blocks is

$$\Pi_{(\underline{\lambda}_s-m^n)'}(\theta)\,B_{\gamma,\underline{\lambda}+(\theta\tau')^m}=\Pi_{(\underline{\lambda}_s-m^n+(\tau')^n)'}(\theta)\,B_{\gamma+2\tau',(\underline{\lambda}'+(\tau')^n)'} \tag{D.7}$$

The Π factors follow from the fact that we normalise our blocks so that the coefficient of the leading superJack $P_{\underline{\lambda}}$ is unity, i.e. $B_{\gamma,\underline{\lambda}}=(\prod x_i^\theta/\prod_j y_j)^{\frac{\gamma}{2}}(P_{\underline{\lambda}}+\dots)$, and from the analogous relation proved in appendix C.6 for super Jack polynomials, namely

$$\Pi_{(\underline{\lambda}_s-m^n)'}(\theta)\,P^{(m,n)}_{\underline{\lambda}+(\theta\tau')^m}=\Pi_{(\underline{\lambda}_s+(\tau')^n-m^n)'}(\theta)\left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\tau'}P^{(m,n)}_{(\underline{\lambda}'+(\tau')^n)'}\,. \tag{D.8}$$

From (D.7) and (D.8) we find the corresponding transformation on the coefficients

$$(T_\gamma)^{\underline{\mu}+(\theta\tau')^m}_{\underline{\lambda}+(\theta\tau')^m}=(T_{\gamma+2\tau'})^{(\underline{\mu}'+(\tau')^n)'}_{(\underline{\lambda}'+(\tau')^n)'}\times\frac{\Pi_{\lambda_s-m^n}(\frac{1}{\theta})}{\Pi_{\lambda_s-m^n+(\tau')^n}(\frac{1}{\theta})}\frac{\Pi_{\underline{\mu}_s-m^n+(\tau')^n}(\frac{1}{\theta})}{\Pi_{\underline{\mu}_s-m^n}(\frac{1}{\theta})}\,. \tag{D.9}$$

This gives the normalisation of $T^{\texttt{long}}_\gamma$ in (D.2), so that it is invariant under the shift

$$(T^{\texttt{long}}_\gamma)^{[\underline{\mu}_s;\underline{\mu}_e+(\theta\tau')^m]}_{[\underline{\lambda}_s;\underline{\lambda}_e+(\theta\tau')^m]}=(T^{\texttt{long}}_\gamma)^{[\underline{\mu}_s+(\tau')^n;\underline{\mu}_e]}_{[\underline{\lambda}_s+(\tau')^n;\underline{\lambda}_e]}\,. \tag{D.10}$$

We could then define normalised long super Jack polynomial $P^{\texttt{long}}_{\underline{\lambda}}$ so that $(T^{\texttt{long}}_\gamma)$ gives the expansion coefficients and all formulae are manifestly shift symmetric.

Finally, let us point out that the ratio of Π functions in (D.9) should simplify because it does not depend on γ. Indeed, by using a property of C^- under shifts[53] and rearranging the expression we obtain

$$(T_\gamma)^{\underline{\mu}+(\theta\tau')^m}_{\underline{\lambda}+(\theta\tau')^m}=(T_{\gamma+2\tau'})^{(\underline{\mu}'+(\tau')^n)'}_{(\underline{\lambda}'+(\tau')^n)'}\times\frac{C^0_{\underline{\mu}_s/\underline{\lambda}_s}(\tau'+\frac{n}{\theta}-m)C^0_{\underline{\mu}_s/\underline{\lambda}_s}(1+\frac{n-1}{\theta}-m)}{C^0_{\underline{\mu}_s/\underline{\lambda}_s}(\frac{n}{\theta}-m)C^0_{\underline{\mu}_s/\underline{\lambda}_s}(\tau'+1+\frac{n-1}{\theta}-m)} \tag{D.11}$$

As expected the ratio of Π functions above is more explicitly just a function of the skew diagrams.

D.2 Truncations of the superconformal block

An (m,n) superconformal block $B_{\gamma,\underline{\lambda}}$ has been so far defined as the multivariate series

$$B_{\gamma,\underline{\lambda}}=\left(\frac{\prod_i x_i^\theta}{\prod_j y_j}\right)^{\frac{\gamma}{2}}\sum_{\underline{\mu}\supseteq\underline{\lambda}}(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}P_{\underline{\mu}}(\mathbf{z};\theta)\,. \tag{D.12}$$

In this section we discuss how the infinite sum over $\underline{\mu}$ depends effectively on the external parameters α,β and $\underline{\lambda}$. Let us recall indeed that the α and β dependence of T_γ is solved by

[53] This is

$$C^-_{\underline{\kappa}+(\tau')^n}(w;\theta)/C^-_{\underline{\kappa}}(w;\theta)=C^-_{(\tau')^n}(w;\theta)\times C^0_{\underline{\kappa}}(w+\tau'+\theta(n-1);\theta)/C^0_{\underline{\kappa}}(w+\theta(n-1);\theta)$$

the C^0 factors, as we pointed out already in section 5.4. Thus

$$(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha;\theta) C^0_{\underline{\mu}/\underline{\lambda}}(\theta\beta;\theta) \times (T^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{\underline{\lambda}} \tag{D.13}$$

where $T^{\texttt{rescaled}}_\gamma$ is α, β *independent.* The observation we will elaborate on is that the C^0 factors have certain vanishing properties which in practise truncate the sum.

Given the various forms of the recursions, spelled out in previous sections, it is useful to put the C^0 factors accordingly. To do so consider the equivalent rewritings[54]

$$C^0_{\underline{\mu}/\underline{\lambda}}(w;\theta) = (-\theta)^{|\underline{\mu}|-|\underline{\lambda}|} C^0_{\underline{\mu}'/\underline{\lambda}'}(-\tfrac{w}{\theta};\tfrac{1}{\theta}) \tag{D.14}$$

$$= (-\theta)^{|\underline{\mu}_s|-|\underline{\lambda}_s|} C^0_{\underline{\mu}_e/\underline{\lambda}_e}(w;\theta) C^0_{\underline{\mu}'_s/\underline{\lambda}'_s}(-\tfrac{w}{\theta};\tfrac{1}{\theta}) \tag{D.15}$$

which follow from the definition in (5.38). Then, let us note that C^0 has the following analytic continuation,

$$C^0_{[\kappa_1,\ldots,\kappa_\ell]}(w;\theta) = \prod_{i=1}^{\ell} (w - \theta(i-1))_{\kappa_i} \tag{D.16}$$

therefore, if we consider the $(m,0)$ analytic continuation of section 6, we are keeping fixed the number of rows of $\underline{\lambda}, \underline{\mu}$, as for the Young diagrams, and we find that $C^0_{\underline{\mu}/\underline{\lambda}}(w;\theta)$ has the correct $(m,0)$ analytic continuation. If instead we consider the $(0,n)$ analytic continuation, we are keeping fixed the number of columns of $\underline{\lambda}, \underline{\mu}$, and thus it is $C^0_{\underline{\mu}'/\underline{\lambda}'}(-\frac{w}{\theta};\frac{1}{\theta})$ on the RHS of (D.14) the one with the correct $(0,n)$ analytic continuation. Finally, the rewriting on the RHS of (D.15) is the one with the correct (m,n) analytic continuation.

To appreciate the vanishing properties of the C^0 factors in (D.13), consider for example $C^0_{\underline{\mu}/\underline{\lambda}}(\theta\alpha;\theta)$. It will vanish when

$$(\lambda_i + \theta\alpha - \theta(i-1)) \ldots (\mu_i - 1 + \theta\alpha - \theta(i-1)) = 0 \tag{D.17}$$

and similarly for $C^0_{\underline{\mu}/\underline{\lambda}}(\theta\beta;\theta)$. In particular, it is enough that only one factor becomes zero. This of course depends on the values of α, β, θ, as well as $\underline{\lambda}$. The general situation is summarised by the following tables. On the east

	`truncation`
if $\exists\, i \leq m$ such that $(i-1) - w \geq \frac{\lambda_i}{\theta} \in \mathbb{Z}$	$\mu_i = \theta(i-1-w)$

(D.18)

with w here being either β or α. This means that when the condition is satisfied on a row index i, by progressively increasing $\mu_i = \lambda_i + n_i$ by integers n_i we will hit a vanishing point. Similarly on the south

	`truncation`
if $\exists\, j \leq n$ such that $(j-1) + \theta w \geq \theta\lambda'_j \in \mathbb{Z}$	$\mu'_j = \frac{(j-1)}{\theta} + w$

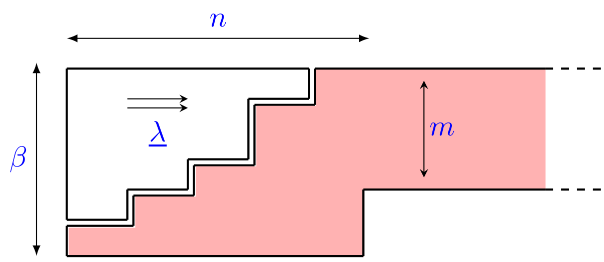
(D.19)

where again w is either β or α.

[54]For the last rewriting consider that $((\mu_s - m^n)')_i = \mu_{m+i}\ \forall i \geq 1$.

Consider now a situation with a group theory interpretation, thus relevant for superconformal blocks. Assume first that $\underline{\lambda}$ is a Young diagram, then $\underline{\lambda}$ has at most $\beta \in \mathbb{N}$ rows (since in our conventions $\beta \leq \alpha$). In particular, $\lambda'_j - \beta \leq 0$. From (D.19) with $j = 1$ we thus find $\mu'_1 \leq \beta$, and we conclude that there is a vertical cut off on the Young diagrams $\underline{\mu}$ over which we sum. On the contrary, note that by construction $\lambda_i + \theta\beta \geq \theta(i-1)$, $\forall\ i \leq m \leq \beta$, i.e. the opposite of the condition in (D.18), precisely because β sets the value for the maximal number of rows. Therefore there is no truncation on the horizontal east directions. The picture to have in mind for the superconformal blocks is thus

$$F_{\gamma,\underline{\lambda}}(\mathbf{x}|\mathbf{y}) = \sum_{\underline{\mu}:\underline{\lambda}\subseteq\underline{\mu}} (T_\gamma)^{\underline{\mu}}_{\underline{\lambda}} P_{\underline{\mu}}(\mathbf{x}|\mathbf{y})$$



(D.20)

A posteriori, the cut-off on the south is expected, because the internal subgroup of the superconformal algebra is compact. Note also that for Young diagrams, since T only depends on diagrams, we can see that a solution $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ where $\underline{\mu}$ has more than β rows does not exist, by considering an equivalent reasoning on the east. In fact, when $i = \beta + 1$ and $\lambda_{i=\beta+1} = 0$, we find from (D.18) that $\mu_i = 0$.

Consider now a long block with analytically continued $\underline{\lambda}$, according to an (m, n) structure, in such a way to describe an anomalous dimension. In this case we still require finiteness of the sum (D.12) on the south, as expected for a compact group. From (D.19) we see then that even when β and λ'_j are generic, if however $0 \leq \beta - \lambda'_{j=1,\ldots n} \in \mathbb{Z}^+$, the condition on the truncation can be satisfied. Note instead that our Young diagram argument on the east (D.18) now will not work, precisely because β is not integer for cases of physical interest where $\lambda_i > 0$. We notice however that taking the $\lambda_i < 0$ it woud be possible to have truncation on the east and the south simultaneously and perhaps this reproduces the Sergeev-Veselov super Jacobi polynomials.

The generic 'shape' of a physical superconformal block is thus the one illustrated in (D.20).

E Revisiting known blocks with the binomial coefficient

Our formula for the coefficients $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ in (7.12), written in terms of interpolation polynomials, is valid for any Young diagram. We find instructive to test (7.12) against known analytic solutions of the recursion, such as the rank-one and rank-two bosonic blocks. Then, we will repeat a similar check for the determinantal solution found in [9] for $\theta = 1$.

We will use the `rescaled` version of it, where we omit the dependence on α and β. Therefore,

$$(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} = (\mathcal{N}^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} \times \frac{P^*_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\lambda};\theta,u)}{P^*_{N^M\backslash\underline{\lambda}}(N^M\backslash\underline{\lambda};\theta,u)}\Bigg|_{u=\frac{1}{2}-\theta\frac{\gamma}{2}-N} \tag{E.1}$$

where

$$(\mathcal{N}^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} = \frac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \, \frac{C^0_{\underline{\mu}/\underline{\lambda}}(1-\theta+M\theta;\theta)}{C^0_{\underline{\mu}/\underline{\lambda}}(M\theta;\theta)} \tag{E.2}$$

As pointed out already, the interpolation polynomials encode the non-factorisable γ dependence of $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$, which we saw emerging experimentally from the recursion. However, the way the interpolation polynomials are evaluated is quite different compared with the recursion. In fact, in order to match $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ we will need to compute P^* of $N^M \backslash \underline{\mu}$, rather than looking recursing over $\underline{\mu}/\underline{\lambda}$, as we do in the recursion. Our exercise here will show in simple cases how these two combinatorics actually produce the same result.

Through this section, we shall take the minimal choice for N and M in the above formulae, i.e.

$$N = \mu_1 \qquad ; \qquad M = \beta \tag{E.3}$$

where β as usual fixes the maximal height of both $\underline{\lambda}$ and $\underline{\mu}$.

E.1 The half-BPS solution

The simplest case to start with is the derivation from (E.1) of the half-BPS solution in (5.37),

$$(T^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{[\varnothing]} = \frac{1}{C^0_{\underline{\mu}}(\theta\gamma;\theta)} \frac{1}{C^-_{\underline{\mu}}(1;\theta)} \ . \tag{E.4}$$

To obtain the γ dependence we need consider the ratio of interpolation polynomials in (E.1) and since $\underline{\lambda} = [\varnothing]$, the numerator P^* is evaluated on μ_1^β, which is a "constant" Young diagram. This evaluation is known. From [32] we find

$$P^{ip}_{\underline{\kappa}}(c + \theta\delta_\ell + u; \theta, u) = (-)^{|\underline{\lambda}|} C^0_{\underline{\kappa}}(c + 2u + \theta(\ell - 1), -c; \theta) \frac{C^0_{\underline{\kappa}}(\ell\theta;\theta)}{C^-_{\underline{\kappa}}(\theta;\theta)} \tag{E.5}$$

which we need here for $c = \mu_1$ and $\ell = \beta$. For what concerns the denominator instead, this is always factorised for any $\underline{\lambda}$ and it is given by (7.7). Putting together we find,

$$\frac{P^*_{\mu_1^\beta \backslash \underline{\mu}}(\mu_1{+}\theta\delta_\beta{+}u;\theta,u)}{C^+_{\mu_1^\beta}(-2\mu_1{-}\theta\gamma{+}2\theta\beta)C^-_{\mu_1^\beta}(1;\theta)} = \frac{C^0_{\mu_1^\beta \backslash \underline{\mu}}(-\theta\gamma{-}(\mu_1{-}1){+}\theta(\beta{-}1);\theta)}{C^0_{\mu_1^\beta}(-\theta\gamma{-}(\mu_1{-}1){+}\theta(\beta{-}1);\theta)} \times \frac{C^0_{\mu_1^\beta \backslash \underline{\mu}}(-\mu_1;\theta)\, C^0_{\mu_1^\beta \backslash \underline{\mu}}(\beta\theta;\theta)}{(-)^{|\underline{\mu}|-\beta\mu_1} C^-_{\mu_1^\beta}(1;\theta) C^-_{\mu_1^\beta \backslash \underline{\mu}}(\theta;\theta)} \tag{E.6}$$

where we rewrote C^+ in terms of C^0 using the explicit definition (7.9), and the fact that the Young diagram involved is a rectangle μ_1^β. At this point, the $1/C^0(\theta\gamma;\theta)$ comes out thanks to the identity,

$$\frac{C^0_{\underline{\mu}}(w - (\mu_1 - 1) + \theta(\beta - 1);\theta) C^0_{\mu_1^\beta - \underline{\mu}}(-w;\theta)}{C^0_{\mu_1^\beta}(-w;\theta)} = (-1)^{|\underline{\mu}|} \quad ; \quad \forall\, w \tag{E.7}$$

taken with $w = -\theta\gamma$, and the $(-1)^{|\underline{\mu}|}$ cancels against the one in (E.2). Finally, the contribution $1/C^-_{\underline{\mu}}(1;\theta)$ comes from $\Pi_{\underline{\mu}}(\theta)$ in (E.2). The remaining terms in the binomial coefficient necessarily have to simplify to unity. When we collect them all,[55] upon using another identity

$$\frac{C^-_{\underline{\mu}}(w;\theta)}{C^0_{\mu_1^\beta}(\theta(\beta-1)+w;\theta)C^-_{\mu_1^\beta\backslash\underline{\mu}}(w;\theta)} = \frac{C^0_{\underline{\mu}}(-w-(\mu_1-1);\theta)}{(-)^{|\underline{\mu}|}C^-_{\mu_1^\beta}(w;\theta)} \quad ; \quad \forall\, w \tag{E.8}$$

we finally obtain our desired result.

Already in this simple example we needed several identities between $C^{\pm,0}$ coefficients. This shows that the rewriting of T in terms of interpolation polynomials is quite non trivial.

E.2 Rank-one and rank-two

After the half-BPS solution, we will consider the rank-one cases, since these are fully factorised. Next, the rank-two case, which involves a ${}_4F_3$ and has non trivial dependence on γ. Thus it is the most interesting case for our purpose. Our task will be to revisit the solution found by Dolan and Osborn in [11].

Rank-one $(1,0)$

For the single row case $\beta=1$, with $\underline{\mu}=[\mu]$ and $\underline{\lambda}=[\lambda]$, we find

$$\begin{aligned}(\mathcal{N}^{\texttt{reduced}})_{\underline{\lambda}}^{\underline{\mu}} &= \tfrac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \quad\times\quad \tfrac{C^0_{\underline{\mu}/\underline{\lambda}}(1;\theta)}{C^0_{\underline{\mu}/\underline{\lambda}}(\theta;\theta)} \\ &= \tfrac{(-)^{\mu}(\theta)_{\mu}(1)_{\lambda}}{(-)^{\lambda}(1)_{\mu}(\theta)_{\lambda}} \quad\times\quad \tfrac{(\lambda+1)_{\mu-\lambda}}{(\lambda+\theta)_{\mu-\lambda}}\end{aligned} \tag{E.9}$$

which simplifies to a sign. Then, for the interpolation polynomials we find

$$\frac{P^*_{[\varnothing]}(\mu-\lambda;\theta,u)}{P^*_{[\mu-\lambda]}(\mu-\lambda;\theta,u)} = \frac{1}{(-)^{\mu-\lambda}(2\lambda+\theta\gamma)_{\mu-\lambda}(1)_{\mu-\lambda}} \tag{E.10}$$

in particular, the numerator is trivial, and the denominator follows straightforwardly from (7.7) and (7.9). Putting together the two results above we obtain the formula

$$(T_\gamma^{\texttt{rescaled}})^{[\mu]}_{[\lambda]} = \frac{1}{(\mu-\lambda)!(2\lambda+\theta\gamma)_{\mu-\lambda}} \tag{E.11}$$

which coincides with the one derived in section 4.1.

This case was quite immediate and the reason is that the formula for the P^* is oriented on the east, as the $(1,0)$ theory. We will see now what happens in the $(0,1)$ theory.

[55] These terms are

$$\frac{(-)^{|\underline{\mu}|+\beta\mu_1}C^-_{\underline{\mu}}(\theta;\theta)}{C^0_{\underline{\mu}}(\beta\theta;\theta)C^-_{\mu_1^\beta\backslash\underline{\mu}}(\theta;\theta)} \times \frac{C^0_{\mu_1^\beta\backslash\underline{\mu}}(\beta\theta;\theta)}{C^-_{\mu_1^\beta}(1;\theta)} \times C^0_{\mu_1^\beta\backslash\underline{\mu}}(-\mu_1;\theta)C^0_{\underline{\mu}}(1+\theta(\beta-1);\theta) = 1$$

using $w=\theta$ in (E.8) and $w=\mu_1$ in (E.7), then again $w=-\beta\theta$ in (E.7). Finally $(-)^{\beta\mu_1}C^0_{\mu_1^\beta}(-\mu_1;\theta)/C^-_{\mu_1^\beta}(1;\theta)=1$ and $C^0_{\mu_1^\beta}(\beta\theta;\theta)/C^-_{\mu_1^\beta}(\theta;\theta)=1$, directly from their definitions.

Rank-one $(0,1)$

For the single column case, the diagrams are $\underline{\lambda} = [1^{\lambda'}]$ and $\underline{\mu} = [1^{\mu'}]$. This is a case in which $\lambda', \mu' \leq \beta$. Let us proceed with a direct computation starting with (E.1). The normalisation (E.2) becomes

$$\begin{aligned}(\mathcal{N}^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} &= \frac{(-)^{|\underline{\mu}|}\Pi_{\underline{\mu}}(\theta)}{(-)^{|\underline{\lambda}|}\Pi_{\underline{\lambda}}(\theta)} \times \frac{C^0_{\underline{\mu}/\underline{\lambda}}(1+(\beta-1)\theta;\theta)}{C^0_{\underline{\mu}/\underline{\lambda}}(\beta\theta;\theta)} \\ &= \frac{(-)^{\mu'}(1)_{\mu'}(\frac{1}{\theta})_{\lambda'}}{(-)^{\lambda'}(\frac{1}{\theta})_{\mu'}(1)_{\lambda'}} \times \frac{(\frac{1}{\theta}+\beta-\mu')_{\mu'-\lambda'}}{(1+\beta-\mu')_{\mu'-\lambda'}}\end{aligned} \tag{E.12}$$

For the ratio of interpolation polynomials the diagrams are $\mu_1^\beta \backslash \underline{\mu} = [1^{\beta-\mu'}]$ and $\mu_1^\beta \backslash \underline{\lambda} = [1^{\beta-\lambda'}]$ and the polynomials have β variables. This time the numerator involves a non trivial Young diagram, which however has less than β rows. To simplify the result we use the non-trivial reducibility property for the interpolation polynomials, which reads

$$P^{ip}_{[\kappa_1,\ldots\kappa_s]}(\underbrace{\ldots, w_s, u, u+\theta, \ldots u+(r-1)\theta}_{\text{variables}}; \theta, u) = P^{ip}_{[\kappa_1,\ldots\kappa_s]}(\ldots, w_s; \theta, u+r\theta) \tag{E.13}$$

where r is counting the excess between the number of variables β and the non zero parts of $\underline{\kappa}$.[56] In our case, the above result gives back an evaluation formula to

$$P^{ip}_{[1^{\beta-\mu'}]}(1^{\beta-\lambda'} + \delta_{\beta-\lambda'} + u'; \theta, u')\Bigg|_{u'=u+\lambda'\theta} = \frac{(\mu'-\lambda'+1)_{\beta-\mu'}}{(1)_{\beta-\mu'}}(\theta)^{2(\beta-\mu')}(\tfrac{1}{\theta})_{\beta-\mu'}(\lambda'+\mu'-\gamma)_{\beta-\mu'} \tag{E.14}$$

and in fact the RHS follows from (E.5). Thus, the ratio of interpolation polynomials contributes as,

$$\frac{P^*_{[1^{\beta-\mu'}]}(1^{\beta-\lambda'};\theta,u)}{P^*_{[1^{\beta-\lambda'}]}(1^{\beta-\lambda'};\theta,u)} = \frac{(\theta)^{2(\beta-\mu')}(\lambda'+\mu'-\gamma)_{\beta-\mu'}(\frac{1}{\theta})_{\beta-\mu'}}{(\theta)^{2(\beta-\lambda')}(2\lambda'-\gamma)_{\beta-\lambda'}(\frac{1}{\theta})_{\beta-\lambda'}} \,. \tag{E.15}$$

Putting it all together, the final result for $T^{\texttt{rescaled}}_\gamma$ (defined in (5.45)) is

$$(T^{\texttt{rescaled}}_\gamma)^{[1^{\mu'}]}_{[1^{\lambda'}]} = \frac{(\theta)^{2(\lambda'-\mu')}}{(\mu'-\lambda')!(2\lambda'-\gamma)_{\mu'-\lambda'}} \frac{\mu'!(\frac{1}{\theta})_{\lambda'}}{\lambda'!(\frac{1}{\theta})_{\mu'}}(-1)^{\mu'-\lambda'}. \tag{E.16}$$

This coincides precisely with the one derived in section 4.1 for $(T_\gamma)^{[1^{\mu'}]}_{[1^{\lambda'}]}$ when we re-insert

$$C^0_{\underline{\mu}/\underline{\lambda}}\left(\tfrac{\theta(\gamma-p_{12})}{2}, \tfrac{\theta(\gamma-p_{43})}{2}; \theta\right) = (-\theta)^{2(\mu'-\lambda')}\big(\lambda' - \tfrac{(\gamma-p_{12})}{2}, \lambda' - \tfrac{(\gamma-p_{43})}{2}\big)_{\mu'-\lambda'} \tag{E.17}$$

properly oriented towards the south.

[56]Of course the same is true for a symmetric permutation of all variables on the LHS

Rank-two $(2,0)$

In the rank-one cases there is no room for any non trivial polynomial in γ. The first non trivial case in this sense is then rank-two, corresponding to T_γ for two row Young diagrams. The recursion (5.35) for this case was solved explicitly by Dolan and Osborn [11]. They did this first on a case by case basis in dimensions $d = 2, 4, 6$, then for general θ by using properties of the ${}_4F_3$ function, and manipulations inspired by those in [99].[57] We will show here how this relates to the interpolation polynomial of (7.12).

In our conventions the solution of [11] reads

$$(T_\gamma^{\texttt{rescaled}})_{\underline{\lambda}}^{\underline{\mu}} = \frac{(\theta)_\theta}{(\mu_- + \theta)_\theta} \times (D^{\texttt{rescaled}})_{\underline{\lambda}+\frac{\theta}{2}\gamma}^{\underline{\mu}+\frac{\theta}{2}\gamma} \tag{E.18}$$

with

$$(D^{\texttt{rescaled}})_{\underline{\lambda}}^{\underline{\mu}} = \frac{(2\theta)_{\lambda_-}(2\theta)_{\mu_-}}{(\theta)_{\lambda_-}(\theta)_{\mu_-}} \frac{\frac{(\mu_-+1)_\theta}{(\mu_1-\lambda_2+1)_\theta}}{(\mu_1-\lambda_1)!(\mu_2-\lambda_2)!} \times \tag{E.19}$$
$$\frac{\frac{(\lambda_+-2\theta)_\theta}{(\lambda_2+\mu_2-2\theta)_\theta}}{(2\lambda_1)_{\mu_1-\lambda_1}(2\lambda_2-2\theta)_{\mu_2-\lambda_2}} \, {}_4F_3\left[\begin{array}{c} -\mu_- \,,\, \theta \,,\, -\lambda_- \,,\, \lambda_+ - 1 \\ -(\mu_1-\lambda_2) \,,\, \lambda_2+\mu_2-\theta \,,\, 2\theta \end{array} ; 1\right]$$

where $\underline{\lambda} = [\lambda_1, \lambda_2]$, $\underline{\mu} = [\mu_1, \mu_2]$, and $\kappa_\pm = \kappa_1 \pm \kappa_2$, for $\underline{\kappa} = \underline{\lambda}, \underline{\mu}$.

The contribution denoted by D is the translation to our conventions of the result of [11], where we implemented the shift symmetry and slightly rewrote some Pochhammers. Note that the ${}_4F_3$ has a series expansion which truncates at $\min(\lambda_-, \mu_-)$ and can thus be written as the explicit finite sum

$$(D^{\texttt{rescaled}})_{\underline{\lambda}}^{\underline{\mu}} = \sum_{n=0}^{\min(\lambda_-,\mu_-)} \frac{(-\lambda_-)_n}{n!} \frac{(2\theta)_{\lambda_-}(\theta)_n(2\theta)_{\mu_-}}{(\theta)_{\lambda_-}(2\theta)_n(\theta)_{\mu_-}} \frac{(\mu_- - n + 1)_{n+\theta}}{(\mu_1 - \lambda_2 - n + 1)_{n+\theta}} \tag{E.20}$$
$$\frac{(\lambda_+ - 1)_n (\lambda_+ - 2\theta)_\theta}{(\mu_1-\lambda_1)!(\mu_2-\lambda_2)!(2\lambda_1)_{\mu_1-\lambda_1}(2\lambda_2-2\theta)_{\mu_2-\lambda_2+n+\theta}} \,.$$

Now compare this expression with our expression for T_γ written in terms of interpolation polynomials (7.12) (with $N = \mu_1$, $M = 2$):

$$(T_\gamma)_{\underline{\lambda}}^{\underline{\mu}} = (\mathcal{N})_{\underline{\lambda}}^{\underline{\mu}} \times \left. \frac{P^{ip}_{[\mu_1-\mu_2]}(w_1, w_2; \theta, \frac{1}{2} - \theta\frac{\gamma}{2} - \mu_1)}{P^{ip}_{[\mu_1-\lambda_2, \mu_1-\lambda_1]}(w_1, w_2; \theta, \frac{1}{2} - \theta\frac{\gamma}{2} - \mu_1)} \right|_{w_i = \frac{1}{2} - \theta(\frac{\gamma}{2} - i + 1) - \lambda_i} \tag{E.21}$$

(Note that the shift symmetry shows up nicely in this formula since we can always put together combinations of the form $\kappa_i + \frac{\theta}{2}\gamma$, and $\mathcal{N}$ is automatically invariant.) All contributions in (E.21) coming from the denominator and the normalisation, can be straightforwardly computed, and written as product of Pochhammers or Gamma functions. The BC_2 interpolation

[57] In [11], see formulae (3.11), (3.18) and (3.19).

polynomial in the numerator is also known explicitly, and described by a ${}_4F_3$ [41],

$$P^{ip}_{\underline{\kappa}}(w_1,w_2;\theta,u) =(-)^{|\kappa|}(u\pm w_1,u\pm w_2)_{\kappa_2}(\kappa_2+\theta+u\pm w_1)_{\kappa_-}\times \\ \times\ {}_4F_3\left[\begin{matrix} -\kappa_-\,,\,\theta\,,\,\kappa_2+u\pm w_2 \\ 1-\theta-\kappa_-\,,\,\kappa_2+\theta+u\pm w_1 \end{matrix};1\right] \tag{E.22}$$

Although similar, this ${}_4F_3$ is not identical to that of (E.19). Also note that it is not manifestly $w_1\leftrightarrow w_2$ invariant even though the polynomial is symmetric under this interchange. We thus have two possibilities leading to the same result. Using $P^{ip}_{[\mu_1-\mu_2]}(w_2,w_1)$ we find

$$\text{numerator of }(T_{\gamma=0})^{\underline{\mu}}_{\underline{\lambda}}\ \to\ {}_4F_3\left[\begin{matrix} -\mu_-\,,\,\theta\,,\,1-\lambda_1-\mu_1\,,\,\lambda_1-\mu_1 \\ \lambda_2-\mu_1\,,\,1-\theta-\mu_1+\mu_2,1+2\theta-\lambda_2-\mu_1 \end{matrix};1\right] \tag{E.23}$$

which becomes the same hypergeometric as in (E.19) upon using the Whipple identity

$${}_4F_3\left[\begin{matrix} -n\,,\,a\,,\,b\,,\,c \\ d\,,\,e\,,\,f \end{matrix};1\right]=\frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\,{}_4F_3\left[\begin{matrix} -n\,,\,a\,,\,d-b\,,\,d-c \\ d\,,\,a-e+1-n\,,\,a-f+1-n \end{matrix};1\right] \tag{E.24}$$

Similarly had we chosen $P^{*}_{[\mu_1-\mu_2]}(w_1,w_2)$ a different ${}_4F_3$ identity will give the same final result. Thus we see how the ${}_4F_3$ of [11] arises directly from the interpolation polynomial.

It is quite spectacular to implement on a computer the above representation of T_γ and the one arising directly from the recursion and check that they agree over many examples.

E.3 Revisiting the $\theta=1$ case and determinantals

The superconformal blocks for the case $\theta=1$ and any (m,n) were obtained in [9]. One of the main outcomes of that derivation is an explicit expression for the coefficients in an expansion over super Schur polynomials, that we quote here below,

$$(R_\gamma)^{\underline{\mu}}_{\underline{\lambda}}=\sum_\sigma(-)^{|\sigma|}\prod_{i=1}^m\frac{\left(\lambda_i-i+1+\frac{\gamma-p_{12}}{2}\right)_{\mu_{\sigma(i)}+i-\sigma(i)-\lambda_i}\left(\lambda_i-i+1+\frac{\gamma-p_{43}}{2}\right)_{\mu_{\sigma(i)}+i-\sigma(i)-\lambda_i}}{(\mu_{\sigma(i)}+i-\sigma(i)-\lambda_i)!(2\lambda_i-2i+2+\gamma)_{\mu_{\sigma(i)}+i-\sigma(i)-\lambda_i}}$$

We can also conveniently rewrite the above formula for the coefficients as a determinant

$$(R_\gamma)^{\underline{\mu}}_{\underline{\lambda}}=\det\left(\frac{(\lambda_i-i+\frac{\gamma-p_{12}}{2}+1)_{\mu_j-j-\lambda_i+i}(\lambda_i-i+\frac{\gamma-p_{43}}{2}+1)_{\mu_j-j-\lambda_i+i}}{(\mu_j-j-\lambda_i+i)!(2\lambda_i-2i+2+\gamma)_{\mu_j-j-\lambda_i+i}}\right)_{1\le i,j\le\beta} \tag{E.25}$$

In this section we want to directly show how $(R_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ also arises from the interpolation polynomials via (7.12), namely we want to show

$$(R_\gamma)^{\underline{\mu}}_{\underline{\lambda}}=(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}\Big|_{\theta=1}=(\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}}\times\left.\left.\frac{P^*_{N^M\backslash\underline{\mu}}(N^M\backslash\underline{\lambda};\theta,u)}{P^*_{N^M\backslash\underline{\lambda}}(N^M\backslash\underline{\lambda};\theta,u)}\right|_{u=\frac{1}{2}-\theta\frac{\gamma}{2}-N}\right|_{\theta=1} \tag{E.26}$$

Notice that for $\theta = 1$ the normalisation simplifies massively,

$$(\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}}\Big|_{\theta=1} = C^0_{\underline{\mu}/\underline{\lambda}}\left(\tfrac{(\gamma-p_{12})}{2}, \tfrac{(\gamma-p_{43})}{2}; 1\right) \tag{E.27}$$

and only the ratio of interpolation polynomials is important.

Let us begin by improving the expression of the determinant in such a way as to extract the same normalisation. Using $(a)_x = \Gamma[a+x]/\Gamma[a]$, it is simple to realise that various contributions depend solely on either row or column index, therefore can be factored out, and rearranged. We arrive at the expression

$$(R_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = (\mathcal{N})^{\underline{\mu}}_{\underline{\lambda}} \times (R^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{\underline{\lambda}} \tag{E.28}$$

$$(R^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \left(\prod_{i=1}^{\beta} \Gamma[2-2i+\gamma+2\lambda_i]\right) \det\left(\frac{\Gamma[1+i-j-\lambda_i+\mu_j]^{-1}}{\Gamma[2-i-j+\gamma+\lambda_i+\mu_j]}\right)_{1\le i,j\le\beta} \tag{E.29}$$

Our formula for T_γ in terms of interpolation polynomials can be manipulated quite explicitly when $\theta = 1$, since the BC interpolation polynomials for $\theta = 1$ themselves also have a determinantal representation [40],

$$P^{ip}_{[\kappa_1,\ldots\kappa_\ell]}(\mathbf{w};\theta=1,u) = \frac{\det\left(P^{ip}_{[\kappa_j+\ell-j]}(w_i;u)\right)_{1\le i,j\le\ell}}{\prod_{i<j}(w_i^2-w_j^2)} \tag{E.30}$$

The entries of the matrix are single-variable interpolation polynomials $P^{ip}_\kappa(w_i,u)$. These do not depend on θ and are given by a pair of Pochhammers,

$$\begin{aligned} P^{ip}_\kappa(w;u) = (-)^k(u+w)_k(u-w)_k &= (-w-u-k+1)_k(u-w)_k \\ &= (-)^k(-w-u-k+1)_k(w-u-k+1)_k \end{aligned} \tag{E.31}$$

The denominator of (E.30) is the $\mathbb{Z}_2$ invariant Vandermonde determinant.

We can now proceed and compute $(T_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$ for $\theta = 1$. Since we already identified the normalisation, we will focus on $(R^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{\underline{\lambda}}$. This has to agree with

$$(T^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{\underline{\lambda}} \equiv \frac{(-)^{|\mu|}}{(-)^{|\lambda|}} \frac{P^{ip}_{\underline{\mu}_1^\beta-\underline{\mu}}(\frac{1}{2}-\frac{\gamma}{2}+i-1-\lambda_i;1,\frac{1}{2}-\frac{\gamma}{2}-\mu_1)}{P^{ip}_{\mu_1^\beta-\underline{\lambda}}(\frac{1}{2}-\frac{\gamma}{2}+i-1-\lambda_i;1,\frac{1}{2}-\frac{\gamma}{2}-\mu_1)} \tag{E.32}$$

The idea is simple: we need to recognise the matrix in (E.29). To do so, we first use the expression of the BC_1 interpolation polynomials given in (E.31), and pass from Pochhammers to Γ functions. The result is

$$(T^{\texttt{rescaled}}_\gamma)^{\underline{\mu}}_{\underline{\lambda}} = \frac{(-)^{|\mu|}}{(-)^{|\lambda|}} \frac{(-)^{\beta\mu_1-|\underline{\mu}|+\frac{1}{2}\beta(\beta-1)}\det\left(\frac{\Gamma[i-\lambda_i+\mu_1]\Gamma[1-i+\gamma+\lambda_i-\mu_1]}{\Gamma[1+i-J-\lambda_i+\mu_J]\Gamma[2-i-J+\gamma+\lambda_i+\mu_J]}\right)_{1\le i,j\le\beta}}{(-)^{\beta\mu_1-|\underline{\lambda}|+\frac{1}{2}\beta(\beta-1)}\det\left(\frac{\Gamma[i-\lambda_i+\mu_1]\Gamma[1-i+\gamma+\lambda_i-\mu_1]}{\Gamma[1+i-J-\lambda_i+\lambda_J]\Gamma[2-i-J+\gamma+\lambda_i+\lambda_J]}\right)_{1\le i,j\le\beta}} \tag{E.33}$$

where $J = \beta + 1 - j$ and the signs have been factored out since only depended on the column index. For the same reason we can factor out $\Gamma[i - \lambda_i + \mu_1]\Gamma[1 - i + \gamma + \lambda_i - \mu_1]$ and cancel it between numerator and denominator. Moreover we can reverse the columns and switch $j \leftrightarrow J$. We arrive at the simple formula

$$(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} = \frac{\det\left(\frac{\Gamma[1+i-j-\lambda_i+\mu_j]^{-1}}{\Gamma[2-i-j+\gamma+\lambda_i+\mu_j]}\right)_{1\leq i,j\leq\beta}}{\det\left(\frac{\Gamma[1+i-j-\lambda_i+\lambda_j]^{-1}}{\Gamma[2-i-j+\gamma+\lambda_i+\lambda_j]}\right)_{1\leq i,j\leq\beta}} \tag{E.34}$$

Comparing this result with (E.29) we are left with the statement

$$\prod_{i=1}^{\beta}\Gamma[2 - 2i + \gamma + 2\lambda_i] = \frac{1}{\det\left(\frac{\Gamma[1+i-j-\lambda_i+\lambda_j]^{-1}}{\Gamma[2-i-j+\gamma+\lambda_i+\lambda_j]}\right)_{1\leq i,j\leq\beta}} \tag{E.35}$$

But notice that $\Gamma[1 + i - j - \lambda_i + \lambda_j]^{-1}$, because $\lambda_i \geq \lambda_{i+1}$, makes the matrix on the r.h.s a triangular matrix, and the identity follows. Thus $(T_\gamma^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}} = (R^{\texttt{rescaled}})^{\underline{\mu}}_{\underline{\lambda}}$ and our proof is concluded.

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