CYCLIC QUADRILATERALS AND SMOOTH JORDAN CURVES

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Abstract. For every smooth Jordan curve γ and cyclic quadrilateral Q in the Euclidean plane, we show that there exists an orientation-preserving similarity taking the vertices of Q to γ. The proof relies on the theorem of Polterovich and Viterbo that an embedded Lagrangian torus in $\mathbb{C}^2$ has minimum Maslov number 2.

A quadrilateral Q *inscribes* in a smooth Jordan curve γ in the Euclidean plane if there exists an orientation-preserving similarity of the plane taking the vertices of Q to γ; it is *cyclic* if it inscribes in a circle. The result of this paper is the solution of the cyclic quadrilateral peg problem [5, Conjecture 9]:

Theorem. *Every cyclic quadrilateral inscribes in every smooth Jordan curve in the Euclidean plane.*

The result is best possible, by considering the case in which the smooth Jordan curve is itself a circle. Moreover, some regularity hypothesis on the Jordan curve is necessary in order for the Theorem to hold, as the only cyclic quadrilaterals that inscribe in all triangles are the isosceles trapezoids [8, § 3.6]. See [2, 3, 4, 5] for earlier progress towards this result.

Proof. For a fixed cyclic quadrilateral Q and smooth Jordan curve γ, we construct a pair of Lagrangian tori T_1 and T_2 in standard symplectic $\mathbb{C}^2$. They intersect cleanly along $\gamma \times \{0\}$ and in a disjoint set of points P which parametrize the inscriptions of Q in γ. By smoothing the intersection along $\gamma \times \{0\}$, we obtain an immersed Lagrangian torus T whose set of self-intersections is P. As we show, T has minimum Maslov number 4. On the other hand, a theorem independently due to Polterovich and Viterbo asserts that an embedded Lagrangian torus in $\mathbb{C}^2$ has minimum Maslov number 2 [9, 12]. Therefore P is non-empty, so Q inscribes in γ. $\square$

The strategy of proof of the Theorem resembles that of our earlier result, which treated the case in which Q is a rectangle [2]. In that case, we additionally arranged that T is invariant under a symplectic involution τ of $\mathbb{C}^2$. Passing to the quotient by τ, we obtained an immersed Lagrangian Klein bottle $K = T/\tau$ in $\mathbb{C}^2$ whose self-intersections P/τ parametrize inscriptions of Q in γ up to rotation by π. A theorem independently due to Shevchishin and Nemirovski asserts that there is no embedded Lagrangian Klein bottle in $\mathbb{C}^2$ [7, 11], thereby ensuring that P is non-empty, so Q inscribes in γ. In the more general case of a cyclic quadrilateral, T does not admit any apparent symmetry, which impedes reusing the same approach. Our revised approach produces a stronger result and somewhat more directly.

Cyclic quadrilaterals. We begin by characterizing the set of cyclic quadrilaterals. Let Q denote a convex quadrilateral in the plane whose vertices are labeled $ABCD$ in counterclockwise order. Its diagonals AC and BD intersect in a point X. Euclid's chord theorem asserts that Q is cyclic if and only if $|AX| \cdot |CX| = |BX| \cdot |DX|$ [1, Theorem III.35].[1]

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[1] Euclid proves the forward direction, which can be used to prove the reverse.

By a cyclic permutation of the vertex labels, we may assume that $|AX| \le |CX|$ and $|BX| \le |DX|$. We thereby obtain real values $s = |AX|/|AC|$ and $t = |BX|/|BD|$ in $(0, 1/2]$ and an angle $\phi = \angle AXB$ in $(0, \pi)$. The triple of values (s, t, ϕ) uniquely determines the oriented similarity class of Q, unless one of s and t equals $1/2$, in which case (s, t, ϕ) and $(t, s, \pi - \phi)$ determine the same oriented similarity class.

We reformulate the preceding description for our present purposes. Identify the Euclidean plane with the complex numbers $\mathbb{C}$. Define $\mathbb{C}$-linear automorphisms of $\mathbb{C}^2$ by the matrices

$$F_r = \begin{pmatrix} 1-r & r \\ \sqrt{r(1-r)} & -\sqrt{r(1-r)} \end{pmatrix} \quad \text{and} \quad R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

for values $r \in (0, 1/2]$ and $\phi \in (0, \pi)$.

Lemma 1. *Points $A, B, C, D \in \mathbb{C}$ correspond as above to vertices of a cyclic quadrilateral with parameters (s, t, ϕ) if and only if*

$$R_\phi \circ F_s(A, C) = F_t(B, D) \quad \text{and} \quad A \neq C \text{ (equivalently } B \neq D). \tag{1}$$

Proof. Equality in the first coordinate of (1) is equivalent to the assertion that segments AC and BD intersect at a point X so that $|AX| = s \cdot |AC|$ and $|BX| = t \cdot |BD|$. Equality in the second coordinate given the first then ensures that $\angle AXB = \phi$ and that $|AX| \cdot |CX| = s(1-s) \cdot |AC|^2 = t(1-t) \cdot |BD|^2 = |BX| \cdot |DX|$. Insisting that $A \neq C$ or $B \neq D$ ensures that Q does not degenerate to a point. □

Two embedded Lagrangian tori. Suppose that Q is a cyclic quadrilateral with parameters (s, t, ϕ) as above and that γ is a smooth Jordan curve in $\mathbb{C}$. Note that $\gamma \times \gamma$ is a smoothly embedded torus in $\mathbb{C}^2$. Define tori

$$T_1 = R_\phi \circ F_s(\gamma \times \gamma) \quad \textit{and} \quad T_2 = F_t(\gamma \times \gamma).$$

Note that both $R_\phi \circ F_s$ and F_t map the point (z, z) to $(z, 0)$ for all $z \in \mathbb{C}$. From Lemma 1 we see that the set of inscriptions of Q in γ is parametrized by the set of points

$$P = T_1 \cap T_2 - \gamma \times \{0\}.$$

In addition, the hypothesis that γ is a smooth Jordan curve ensures that $\gamma \times \{0\}$ is a smooth submanifold of both T_1 and T_2. Let $\omega = dz \wedge d\overline{z} + dw \wedge d\overline{w}$ denote the standard symplectic form on $\mathbb{C}^2$, up to scale.

Lemma 2. *The tori T_1 and T_2 are Lagrangian with respect to ω and intersect cleanly along $\gamma \times \{0\}$:*

$$T_{(p,0)}T_1 \cap T_{(p,0)}T_2 = T_{(p,0)}(\gamma \times \{0\}), \quad \textit{for all } p \in \gamma.$$

Proof. A direct calculation shows that

$$\omega_r := F_r^* \omega = (1 - r) \cdot dz \wedge d\overline{z} + r \cdot dw \wedge d\overline{w}$$

for $r \in (0, 1/2]$. Note that $\gamma \times \gamma$ is Lagrangian with respect to ω_r and $R_\phi^* \omega = \omega$. It follows that T_1 and T_2 are Lagrangian with respect to ω.

If $p \in \gamma$ is a point on the Jordan curve, then $T_p\gamma \subset \mathbb{C}$ is a 1-dimensional real subspace. A direct calculation shows that

$$T_{(p,0)}T_1 = T_p\gamma \times \{0\} \oplus \{0\} \times e^{i\phi} T_p\gamma \text{ and } T_{(p,0)}T_2 = T_p\gamma \times \{0\} \oplus \{0\} \times T_p\gamma,$$

so

$$T_{(p,0)}T_1 \cap T_{(p,0)}T_2 = T_p\gamma \times \{0\} = T_{(p,0)}(\gamma \times \{0\}),$$

and the intersection along $\gamma \times \{0\}$ is clean, as required. □

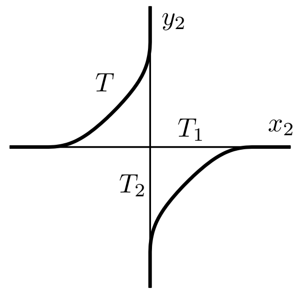

FIGURE 1. Cross-section of smoothing in the $x_1 =$ constant, $y_1 = 0$ plane.

A surgered immersed Lagrangian torus. Because T_1 and T_2 intersect cleanly along $\gamma \times \{0\}$, a version of the Weinstein neighborhood theorem due to Poźniak [10, Proposition 3.4.1] implies that we can select coordinates (x_1, y_1, x_2, y_2) in a neighborhood $N \approx (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^3$ of $\gamma \times \{0\}$ such that

- $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$,
- $T_1 \cap N = \{y_1 = y_2 = 0\}$, and
- $T_2 \cap N = \{y_1 = x_2 = 0\}$.

We smooth the intersection of T_1 and T_2 in N as suggested by Figure 1 and let T denote the result. The tangent plane to T at a point in N is spanned by $\partial/\partial x_1$ and a vector of the form $a \cdot \partial/\partial x_2 + b \cdot \partial/\partial y_2$, which are ω-orthogonal. Thus, T is an immersed Lagrangian torus in $(\mathbb{C}^2, \omega)$, and its set of self-intersections equals P, which parametrizes the set of inscriptions of Q in γ.

The minimum Maslov number. Equip $\mathbb{C}^n$ with a product symplectic form $\omega_0 = \sum_{i=1}^n c_i \cdot dz_i \wedge d\overline{z}_i$. An immersed Lagrangian submanifold $i : L \to (\mathbb{C}^n, \omega_0)$ has a Maslov class $\mu \in H^1(L; \mathbb{Z})$, given as follows (cf. [6, pp.117-118]). The tangent planes to $i(L)$ along the image of an embedded loop $\alpha \subset L$ determine a loop $\alpha^\sharp$ in $\mathcal{L}(\omega_0)$, the Grassmannian of Lagrangian n-planes in $(\mathbb{C}^n, \omega_0)$. The Maslov index of α is the value $\mu([\alpha]) := [\alpha^\sharp] \in H_1(\mathcal{L}(\omega_0); \mathbb{Z}) \approx \mathbb{Z}$, and the minimum Maslov number of L is the non-negative integer $m(L)$ such that $\mu(H_1(L; \mathbb{Z})) = m(L) \cdot \mathbb{Z}$.

Proposition. *The minimum Maslov number of T is 4.*

Proof. Orienting $\gamma \subset \mathbb{C}$ counterclockwise, its Maslov index equals 2 with respect to $c \cdot dz \wedge d\overline{z}$. Hence $\gamma \times \{\text{pt.}\}$ and $\{\text{pt.}\} \times \gamma$ both have Maslov index 2 in $\gamma \times \gamma$ with respect to the product form ω_r. Since their homology classes generate $H_1(\gamma \times \gamma; \mathbb{Z})$, we obtain $m(\gamma \times \gamma) = 2$. The diagonal loop $\{(z, z) : z \in \gamma\}$ is homologous to their sum, so it has Maslov index 4 in $\gamma \times \gamma$ with respect to ω_r. Applying $R_\phi \circ F_s$ and F_t, we deduce that $\gamma \times \{0\}$ has Maslov index 4 in both T_1 and T_2 with respect to ω and that $m(T_1) = m(T_2) = 2$. Let δ denote a push-off of $\gamma \times \{0\}$ in T_1 away from the site of surgery. A neighborhood of δ survives the surgery, so the Maslov index of $[\delta]$ in T is 4 with respect to ω.

Next, select oriented loops $\lambda_1 \subset T_1$, $\lambda_2 \subset T_2$, and $\lambda \subset T$ such that $\lambda_1 \cup \lambda_2$ and λ coincide outside the neighborhood N above and meet it in a single slice $x_1 =$ constant, $y_1 = 0$, as displayed in Figure 1. The tangent planes to $T \cup T_1 \cup T_2$ along the difference 1-cycle $\lambda - \lambda_1 - \lambda_2$ describe a nullhomotopic loop in $\mathcal{L}(\omega)$. Consequently, $[\lambda^\sharp] = [\lambda_1^\sharp] + [\lambda_2^\sharp] \in H_1(\mathcal{L}(\omega); \mathbb{Z}) \approx \mathbb{Z}$. The class $[\lambda_j]$ completes to a basis of $H_1(T_j; \mathbb{Z})$ with $[\gamma \times \{0\}]$ for $j = 1, 2$. Since $m(T_j) = 2$ and $[\gamma \times \{0\}]$ has Maslov index 4 in T_j, $j = 1, 2$, it follows that $[\lambda_j]$ has Maslov index 2 (mod 4), $j = 1, 2$. Therefore, the Maslov index of $[\lambda]$ in T is a multiple of 4.

Since $[\delta]$ and $[\lambda]$ form a basis for $H_1(T; \mathbb{Z})$, it follows that $m(T) = 4$. □

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