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Short communication

Proportional resource allocation in dynamic n-player Blotto games

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ABSTRACT

In this note, we introduce a general model of dynamic n-player multi-battle Blotto contests in which asymmetric resources and non-homogeneous battlefield prizes are possible. Each player's probability of winning the prize in a battlefield is governed by a ratio-form contest success function and players' resource allocation on that battlefield. We show that there exists a pure subgame perfect equilibrium in which players allocate their resources in proportion to the battlefield prizes for every history. We also give a sufficient condition that if there are two players and the contest success function is of Tullock type, then the subgame perfect equilibrium is unique.

1. Introduction

Many social, economic, and political interactions can be modeled as contests. Examples include rent-seeking, political campaigns, sports competitions, litigation, lobbying, and wars. A Colonel Blotto game is a two-person static game in which each player allocates a limited resource over a number of identical "battlefields". The first contest model of the "Colonel Blotto" game was introduced by Borel (1921). Fast forward to the present day; the literature on contests is now enormous, and a Blotto contest denotes any contest in which two or more players allocate a limited resource over a number of battlefields.

In this note, we extend Sela and Erez's (2013) (henceforth S&E) model to a more general prize structure. S&E study two-player sequential multi-battle Blotto contests with heterogeneous budgets, Tullock contest success functions (CSFs), and specific prize structures. Suppose that the battlefield prizes are equal across the stages and that for each resource unit a player allocates on a battlefield, the player's budget decreases in proportion to the budget allocation in that battlefield. S&E show the existence of a subgame perfect equilibrium such that the players' resource allocations are weakly decreasing over the stages.

We extend S&E's model in three notable ways: (i) the prizes vary arbitrarily across battlefields, (ii) the number of players is $n \geq 2$, and (iii) the winner of the prize in a battlefield is determined by a generalized Tullock CSF satisfying the axioms (A1–A6) of Skaperdas (1996). In our sequential multi-battle n-player Blotto game model, both asymmetric resources and distinct battlefield prizes are possible. Each player's probability of winning the prize in a battlefield is governed by a CSF and players' resource allocation on that battlefield. Each player starts the dynamic contest with a limited budget and distributes this budget over a finite number of battlefields. Since the battles take place in sequential order, players can condition their strategies on the outcomes of previous battles. At time t, players simultaneously choose their allocation on battlefield t to win $v^t > 0$, which is the battlefield prize. The winner and the resulting resource allocations are revealed to every player before the next battle. As in S&E, players maximize the total expected prizes in this dynamic game.

Studying a static, simultaneous-move model of resource allocation in U.S. presidential campaigns in a prominent paper, Brams and Davis (1974) highlighted the concepts of 'population of states' and 'population proportionality' in campaign resource allocation.[1] Brams and Davis (1974, p.113) showed that populous states receive disproportionately more investments with regards to their population. More specifically, the winner-take-all feature of the Electoral College—i.e., that the popular-vote winner in each battle wins all the electoral votes of that battle—induces candidates to allocate campaign resources roughly in proportion to the 3/2's power of the electoral votes of each state. The question of why some small states "punch above their weight"—i.e., attract

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[1] As noted by Brams and Davis (1974), the population of a state need not exactly reflect the proportion of the voting-age population who are registered and actually vote in a presidential election.

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attention and resources more than proportional to their weight—in political campaigns has been puzzling researchers (for an analysis in a non-Blotto setting, paying attention to "momentum", see, e.g., (Klumpp and Polborn, 2006)).

The proportional allocation of resources is generally considered a benchmark in resource distribution games, and especially in Blotto contests it is not only one of the prominent strategies but arguably the most salient heuristic. In a symmetric experimental Blotto game, Arad and Rubinstein (2012) consider the equal distribution of resources as level-0 behavior; it also seems to be the first strategy that comes to mind because of the low response time associated with it. (For level-k reasoning, see Stahl (1993) and Nagel (1995).) As we discuss next, we find that this prominent heuristic is an equilibrium outcome in our setting.[2]

Our solution concept is subgame perfect equilibrium. We find that the strategy profile in which players allocate their resources proportional to the battlefield prizes at every history is a subgame perfect equilibrium (see Theorems 1 and 2). This overall result does not depend on the number of players, asymmetry in the resources, the number or the battlefield prizes, or the type of contest success functions satisfying Skaperdas's axioms. We also show the uniqueness of the subgame perfect equilibrium in two-player dynamic Blotto games with Tullock CSFs (Proposition 1).

1.1. Relevant literature

As mentioned above, our paper primarily extends S&E's to more general prize structure, and it also contributes to the more general literature on dynamic contests and campaign resource allocation in sequential elections. This brief sub-section summarizes related work apart from the ones mentioned earlier.

Friedman (1958) first shows that proportional allocation is a Nash equilibrium in static 2-player Blotto contests with Tullock CSF (see Eq. (5) in Section 2.2). Osorio (2013) extends Friedman's result to the case in which battlefield prizes are asymmetric. Duffy and Matros (2015) extend Friedman's result to static n-player Blotto contests with Tullock CSF. Contributions to static generalized Blotto games with asymmetric and heterogeneous battlefield prizes include Kim et al. (2018) who show the existence of Nash equilibrium, Kovenock and Arjona (2019) who characterize best-response functions, and more recently Li and Zheng (2022a), who show the existence of pure Nash equilibrium in these Blotto games and give a characterization of such an equilibrium. Li and Zheng (2022a, p. 5) also study the conditions for the proportionality of pure Nash equilibrium.

Duffy and Matros (2015) study static contests (stochastic asymmetric Blotto games) in up to four battlefields with two players having asymmetric yet similar budgets and generalizing Lake (1979)'s paper, which we discuss below.[3] In a similar setting, Deck et al. (2017) study symmetric static contests with two players who do not have budget constraints. They identified the Nash equilibrium of the symmetric game (Electoral College).

In another static presidential campaign model, Lake (1979) argues that one would need to assume that the candidates maximize only their probability of winning the election, i.e., one would simply try to receive a majority of electoral votes, instead of complying with Brams and Davis (1974) and Brams and Davis (1973) assumption that they maximize their expected electoral vote. Nevertheless, Lake's (1979) main result echoes Brams and Davis (1974) impossibility of population proportionality result in that in Lake's model too it turns out that presidential candidates find it optimal to spend a disproportionately large amount of their funds in the larger states.[4]

In a recent closely related paper, Klumpp et al. (2019) consider dynamic Blotto games where two players fight in odd number of battlefields, which are identical.[5] The player who wins the majority of battles wins the game. Accordingly, they show that under general contest success functions players allocate their resources evenly (i.e., proportionally by default) across battlefields in all subgame perfect equilibria, one of which is in pure strategies.[6] A more recent follow-up paper by Li and Zheng (2021) study Klumpp et al.'s even-split strategy in a more general setting. More recently, Xie and Zheng (2022) study resource allocation in two-player Blotto-type tug-of-war games with the win-by-n rule, where $n \geq 2$.

Acharya, Grillo, Sugaya, and Turkel's (2022) recent paper builds on Klumpp et al.'s (2019) by studying dynamic electoral campaigns as dynamic contests with two players whose 'relative popularity' changes over time. Acharya et al.'s contests are also of Blotto type in the sense that the two players have fixed resources to allocate. However, their model differs from Blotto contests in that players' investments affect the evolution of popularity via a Brownian motion. In their setting, Acharya et al. also confirm the even-split result of Klumpp et al. (2019).

Harris and Vickers (1985) construed a patent race as a multi-battle contest, in which two players alternate in expending resources in a sequence of single battles. These battles or subcontests serve as the components of the overall R&D contest. Just like in a singles tennis match, the player who is first to win a given number of battles wins the contest, by obtaining the patent.[7]

Additional work on dynamic resource allocation contests is as follows. Dziubiński et al. (2021) have recently studied multi-battle dynamic contests on networks in which neighboring 'kingdoms' battle in a sequential order. Li and Zheng (2022b) study resource allocation in Blotto games under general network structure. Hinnosaar (2023) characterizes the equilibria of sequential contests in which efforts are exerted sequentially to win a (single-battle) contest. Ewerhart and Teichgräber (2019) study multi-battle dynamic non-Blotto contests and show the existence of a unique symmetric Markov perfect equilibrium. In a two-player and two-stage campaign resource allocation game, Kovenock and Roberson (2009) characterize the unique subgame perfect equilibrium.

In a two-player best-of-three multi-battle dynamic contest, Konrad (2018) analyzes resource carryover effects between the battles. Brams and Davis (1982) examined a model of resource allocation in the U.S. presidential primaries to study the effects of momentum transfer from one primary to another. As alluded to before, Klumpp and Polborn (2006) also focused on momentum

[2] In both the original Blotto game and the one considered by Arad and Rubinstein (2012), all battlefields have the same prize and the CSF is an all-pay auction. In the heterogeneous battlefields we consider, equal distribution of resources corresponds to a distribution of resources proportional to different battlefield prizes.

[3] For experimental results on Blotto games see, e.g., Deck and Sheremeta (2012), Montero et al. (2016) and Duffy and Matros (2017) and the references therein.

[4] Resource allocation frameworks are often used in modeling electoral competition; see, e.g., Laslier and Picard (2002), Duggan (2007), Barelli et al. (2014), Thomas (2017), and the references therein.

[5] Among others, recent contributions to Blotto games include Roberson (2006), Kvasov (2007), and Rinott et al. (2012). There is also a huge literature on non-Blotto contests initiated by Tullock (1967) and Tullock (1974), and see also, e.g., Krueger (1974) and more recently Konrad and Kovenock (2009). The early literature on non-Blotto contests is motivated by rent-seeking.

[6] For a discussion of dynamics in contests, see Konrad (2009).

[7] In the PGA Tour, which brings professional male golfers together to play in a number of tournaments each year (LPGA does so for female golfers), each tournament consists of multiple battles in that golfers attempt to minimize the total number of shots they take across 72 holes.

issues; they considered a two-player model in which an early primary victory increases the likelihood of victory for one player and creates an asymmetry in campaign spending, which in turn magnifies the player's advantage. This asymmetry of campaign spending generates a momentum which can propel an early winner to the overall victory. Strumpf (2002), on the other hand, discussed a countervailing force to momentum, which favors later winners.

2. The model and results

2.1. Model

We consider dynamic Blotto contests where there are m heterogeneous battlefields with a predetermined *sequential* order, indexed by $t = 1, 2, \dots, m$, and n players, indexed by $i = 1, 2, \dots, n$. Players have possibly asymmetric (sunk) budgets: Each player i has a budget $X_i \geq 0$ that he or she can allocate over the battlefields. The *prize* of each battlefield t is denoted by $v^t > 0$. Each time period t, the battle at t takes place, and each player i *simultaneously* chooses a pure action (allocation) denoted by x_i^t which is smaller than or equal to the budget, X_i, minus the already spent allocation by player i until battle t. Given the chosen actions in battle t, $x^t := (x_1^t, \dots, x_n^t)$, the probability of player i winning battle t is defined by a CSF, which has the following form:

$$p_i^t(x^t) = \begin{cases} \frac{f(x_i^t)}{\sum_j f(x_j^t)} & \text{if } \sum_j x_j^t > 0 \\ \frac{1}{n} & \text{if } \sum_j x_j^t = 0, \end{cases} \tag{1}$$

where $f(\cdot)$ satisfies Skaperdas's (1996) axioms (A1–A6), which characterize a wide range of contest success functions used in the literature. More specifically, it is of the following form: $f(x_j^t) = \beta(x_j^t)^\alpha$ for some $0 < \alpha < \infty$ and $0 < \beta < \infty$.[8]

To avoid trivial cases, we assume that for any t, $v^t < \sum_{t' \neq t} v^{t'}$, that is, there is no "dictatorial" battlefield. Let v_i^t be the prize player i wins at battle t, which is v^t with probability $p_i^t(x^t)$ or 0 with probability $1 - p_i^t(x^t)$.

The set of histories of length t is denoted by H^t. A history of length $t \geq 1$ is a sequence

$$h^t := (((x_1^1, v_1^1), \dots, (x_n^1, v_n^1)), \dots, ((x_1^t, v_1^t), \dots, (x_n^t, v_n^t))) \tag{2}$$

satisfying the following conditions

(i) For each $1 \leq i \leq n$ and for each $1 \leq t' \leq t$, $x_i^{t'} \in [0, X_i - \sum_{j<t'} x_i^j]$.
(ii) For each battle $t' \leq t$, there exists a unique player i such that $v_i^{t'} = v^{t'}$ and for all $j \neq i$, $v_j^{t'} = 0$.

The first property states that each action at any given battle t is bounded by the budget set which diminishes after each action taken in previous battles. The second property states that each battle has a winner-take-all structure.

The history H^0 consists of only the empty sequence ø. Let $H = H^0 \cup H^1 \cup \dots \cup H^m$. Note that, the history h^{t-1} is presented to all players at time t. There is a subset $\overline{H^t} \subset H$ consisting of histories of length t where the game comes to an end at battle t. We call $\overline{H^t}$ the set of terminal histories of length t. If the game has not ended before battle m then the game ends at battle m. We will specify terminal histories in detail later on.

The remaining budget of player i after history $h^t \in H$ is defined as $B_i(h^t) = X_i - \sum_{j \leq t} x_i^j$ where for every $j \leq t$, x_i^j is a realized spending of history h^t. The realized winning schedule of a given history $h^t \in H$, denoted by $V(h^t)$, is the sequence of players that won the battles at battlefields $1, \dots, t$. Thus $V(h^t) \in \{1, \dots, n\}^t$. For example, if $h^3 = (((x_1^1, v_1^1), (x_2^1, 0)), ((x_1^2, 0), (x_2^2, v_2^2)), ((x_1^3, 0), (x_2^3, v_2^3)))$ in a two-player dynamic contest with $m > 3$ battlefields, then $V(h^3) = (1, 2, 2)$.

For player i, a pure strategy σ_i is a sequence of σ_i^t's such that for each t, σ_i^t assigns, to every $h^{t-1} \in H^{t-1}$, allocation $\sigma_i^t(h^{t-1}) \in [0, B_i(h^{t-1})]$. A pure strategy profile is denoted by $\sigma = (\sigma_i)_{i \leq n}$. The set of pure strategies of player $i \leq n$ is denoted by Σ_i and the set of pure strategy profiles by $\Sigma = \times_{i \leq n} \Sigma_i$. For any $\sigma \in \Sigma$, let $(\sigma|h) = ((\sigma_1|h), \dots, (\sigma_n|h))$ denote the strategy profile induced by σ in the subgame starting from history h.

Players maximize the expected payoff which is defined as the sum of expected battlefield prizes. So, the terminal histories are exactly the histories with length m. The set of terminal histories is denoted by $\overline{H^m}$, which is equal to H^m.

For any $\bar{h}^m \in \overline{H^m}$, player i receives a payoff equal to

$$u_i(\bar{h}^m) = \sum_{t \leq m} v_i^t, \tag{3}$$

where $\bar{h}^m := (((x_1^1, v_1^1), \dots, (x_n^1, v_n^1)), \dots, ((x_1^m, v_1^m), \dots, (x_n^m, v_n^m)))$.

The set of terminal histories induced by a strategy profile σ conditional on reaching history h is denoted by $\rho(\sigma|h)$, which is a subset of $\overline{H^m}$. The expected payoff for player $i \leq n$ induced by a pure strategy profile $\sigma \in \Sigma$ at any $h^t \in H^t$ is

$$\pi_i(\sigma|h^t) = \sum_{\bar{h}^m \in \rho(\sigma|h^t)} q(\sigma, \bar{h}^m|h^t) u_i(\bar{h}^m). \tag{4}$$

Our solution concept is subgame perfect equilibrium in pure strategies.

Subgame perfect equilibrium: A pure strategy profile $\sigma \in \Sigma$ is a *subgame perfect equilibrium* if for every battle $t \leq m$, for every history $h \in H^t$, for every player $i \leq n$, and for every strategy $\sigma_i' \in \Sigma_i$

$$\pi_i(\sigma|h) \geq \pi_i(\sigma_{-i}, \sigma_i'|h).$$

A strategy profile $\sigma \in \Sigma$ is a subgame perfect equilibrium if and only if for every $h \in H$, σ induces an equilibrium in the subgame starting with history h.

2.2. Tullock contest success function

In this sub-section, we first define the well-known Tullock CSF (i.e. $\alpha = 1$).

$$p_i^t(x^t) = \begin{cases} \frac{x_i^t}{\sum_j x_j^t} & \text{if } \sum_j x_j^t > 0 \\ \frac{1}{n} & \text{if } \sum_j x_j^t = 0. \end{cases} \tag{5}$$

A dynamic Tullock contest is a dynamic contest in which the contest success functions are of the Tullock variety.

The following theorem provides our first main result in which we show that a subgame perfect equilibrium in "proportional strategies" exists in every dynamic Tullock contest. Note that in the following sub-section we generalize our model and extend this result to more general contest success functions.

Theorem 1 (*Existence and Characterization: Tullock Contests*)**.** *For any n-player dynamic Tullock contest, there exists a subgame perfect equilibrium, σ, which is given as follows. For any t, for any nonterminal history $h^{t-1} \in H - \overline{H}$, and for any player i, let*

$$\sigma_i^t(h^{t-1}) = B_i(h^{t-1}) \frac{v^t}{v^t + \dots + v^m}. \tag{6}$$

[8] Note that we do not restrict α to, say, below 2 in part because we assume Blotto-type fixed and "use-it or lose-it" budgets. For a discussion of the restrictions on α in a non-Blotto model, see, e.g., Baye et al. (1994).

Proof. We show that the proportional strategy profile $\sigma = (\sigma_i)_{i\leq n}$ given above is robust to one-shot deviations, which implies that σ is a subgame perfect equilibrium. That is, any player i at any nonterminal history h^t cannot improve his payoff by changing σ_i^t, given that all other players, $j \neq i$, follow the proportional strategy. If player i switches to a strategy $\bar{\sigma}_i = (\bar{\sigma}_i^{t+1}, \sigma_i^{t+2}, \ldots, \sigma_i^m)$ after history h^t such that $\bar{\sigma}_i^{t+1}(h^t) \neq \sigma_i^{t+1}(h^t)$, then the expected prize that player i wins after battle t given the history h^t is denoted as $\pi_{i,t+1}(\bar{\sigma}_i, \sigma_{-i}|h^t)$, which satisfies

$$\pi_{i,t+1}(\bar{\sigma}_i, \sigma_{-i}|h^t) = \frac{v^{t+1}\bar{\sigma}_i^{t+1}(h^t)}{\bar{\sigma}_i^{t+1}(h^t) + \sum_{j\neq i}\sigma_j^{t+1}(h^t)} + \pi_{i,t+2}(\sigma|h^{t+1}_{dev}), \tag{7}$$

where h^{t+1}_{dev} is a successor of h^t with the property that at battle $t+1$ player i spent $\bar{\sigma}_i^{t+1}(h^t)$, and each player $j \neq i$ spent proportionally. And the expected payoff of player i after history h^t if she follows σ,

$$\pi_{i,t+1}(\sigma|h^t) = \frac{v^{t+1}\sigma_i^{t+1}(h^t)}{\sum_{1\leq j\leq n}\sigma_j^{t+1}(h^t)} + \pi_{i,t+2}(\sigma|h^{t+1}), \tag{8}$$

where h^{t+1} is a successor of h^t with the property that at battle $t+1$, each player spent proportionally. For simplicity, we take

$$\frac{v^{t+1} + \cdots + v^m}{v^{t+1}} = k,$$

$$B_i(h^t) = a,$$

$$\sum_{j\neq i} B_j(h^t) = b,$$

$$\bar{\sigma}_i^{t+1}(h^t) = \sigma_i^{t+1}(h^t) + \Delta = \frac{a}{k} + \Delta.$$

where Δ is a real number. We can rewrite player i's probability of winning battle $t+1$ if he plays $\bar{\sigma}_i^{t+1}(h^t)$ as

$$\frac{\bar{\sigma}_i^{t+1}(h^t)}{\bar{\sigma}_i^{t+1}(h^t) + \sum_{j\neq i}\sigma_j^{t+1}(h^t)} = \frac{\frac{a}{k} + \Delta}{\frac{a}{k} + \Delta + \frac{b}{k}} = \frac{a + \Delta k}{a + \Delta k + b},$$

and player i's probability of winning battle $t+1$ if he plays $\sigma_i^{t+1}(h^t)$ as

$$\frac{\sigma_i^{t+1}(h^t)}{\sum_{1\leq j\leq n}\sigma_j^{t+1}(h^t)} = \frac{\frac{a}{k}}{\frac{a}{k} + \frac{b}{k}} = \frac{a}{a+b}.$$

Since σ is a proportional strategy profile, for any t, for any h^t, and for successor of histories where h^{t+1} is a successor of h^t, h^{t+2} is a successor of h^{t+1}, and so on up to and including h^m is a successor of h^{m-1}, given that players follow proportional strategy profile, we have

$$\frac{\sigma_i^{t+1}(h^t)}{\sum_{1\leq j\leq n}\sigma_j^{t+1}(h^t)} = \frac{\sigma_i^{t+2}(h^{t+1})}{\sum_{1\leq j\leq n}\sigma_j^{t+2}(h^{t+1})} = \cdots = \frac{\sigma_i^{m}(h^{m-1})}{\sum_{1\leq j\leq n}\sigma_j^{m}(h^{m-1})} = \frac{a}{a+b},$$

which means that player i wins each battle after h^t with equal probability if he/she follows σ_i. That is, if players follow the proportional strategy profile, the proportions of the remaining budgets stay constant throughout the battles. The same property satisfies for the strategy profile $(\bar{\sigma}_i, \sigma_{-i})$ after history h^{t+1}_{dev}. Hence for successor of histories where h^{t+2}_{dev} is a successor of h^{t+1}_{dev}, h^{t+3}_{dev} is a successor of h^{t+2}_{dev}, and so on up to and including h^m_{dev} is a successor of h^{m-1}_{dev}, given that players follow proportional strategy profile after history h^{t+1}_{dev}, we have

$$\frac{\sigma_i^{t+2}(h^{t+1}_{dev})}{\sum_{1\leq j\leq n}\sigma_j^{t+2}(h^{t+1}_{dev})} = \frac{\sigma_i^{t+3}(h^{t+2}_{dev})}{\sum_{1\leq j\leq n}\sigma_j^{t+3}(h^{t+2}_{dev})} = \cdots = \frac{\sigma_i^{m}(h^{m-1}_{dev})}{\sum_{1\leq j\leq n}\sigma_j^{m}(h^{m-1}_{dev})}.$$

Now we can simply calculate player i's probability of winning any battle after history h^{t+1}_{dev}, if player i follows the strategy $\bar{\sigma}_i$

$$\frac{\sigma_i^{t+2}(h^{t+1}_{dev})}{\sum_{1\leq j\leq n}\sigma_j^{t+2}(h^{t+1}_{dev})} = \frac{a - \frac{a}{k} - \Delta}{a - \frac{a}{k} - \Delta + b - \frac{b}{k}}.$$

Therefore we can rewrite Eq. (7) as

$$\pi_{i,t+1}(\bar{\sigma}_i, \sigma_{-i}|h^t) = v^{t+1}\frac{a + \Delta k}{a + \Delta k + b} + \frac{a - \frac{a}{k} - \Delta}{a - \frac{a}{k} - \Delta + b - \frac{b}{k}}(v^{t+2} + \cdots + v^m),$$

And we can rewrite Eq. (8) as

$$\pi_{i,t+1}(\sigma|h^t) = \frac{a}{a+b}(v^{t+1} + \cdots + v^m).$$

We show that $\pi_i(\sigma|h^t) - \pi_i(\bar{\sigma}_i, \sigma_{-i}|h^t) \geq 0$; in other words, we show that

$$v^{t+1}\left(\frac{a}{a+b} - \frac{a+\Delta k}{a+\Delta k+b}\right) + (v^{t+2} + \cdots + v^m)\left(\frac{a}{a+b} - \frac{a - \frac{a}{k} - \Delta}{a - \frac{a}{k} - \Delta + b - \frac{b}{k}}\right) \geq 0. \tag{9}$$

Since $k - 1 = (v^{t+2} + \cdots + v^m)/(v^{t+1})$, we can rewrite inequality (9) as

$$\left(\frac{a}{a+b} - \frac{a+\Delta k}{a+\Delta k+b}\right) + (k-1)\left(\frac{a}{a+b} - \frac{a - \frac{a}{k} - \Delta}{a - \frac{a}{k} - \Delta + b - \frac{b}{k}}\right) \geq 0. \tag{10}$$

We can simplify inequality (10) as

$$\frac{b\Delta^2 k^3}{(a+b)(a(k-1) + b(k-1) - \Delta k)(a+b+\Delta k)} \geq 0. \tag{11}$$

Inequality (11) satisfies because we have the following conditions

$$a \geq \frac{a}{k} + \Delta,$$

$$b(k-1) \geq 0,$$

$$\frac{a}{k} + \Delta \geq 0.$$

Thus, for any Δ we have $\pi_i(\sigma|h^t) - \pi_i(\bar{\sigma}_i, \sigma_{-i}|h^t) \geq 0$. □

2.3. More general contest success functions

In this sub-section, we provide an extension of our main result to a more general setting with any CSF satisfying Skaperdas's (1996) axioms A1–A6.

We now state our second main result, which extends Theorem 1 to the case with more general contest success functions.

Theorem 2 (*General Existence and Characterization*). *For any n-player dynamic contest with CSF satisfying Skaperdas's (1996) axioms A1–A6, the following proportional strategy profile, σ, is a* *subgame perfect equilibrium. For any t, for any nonterminal history $h^{t-1} \in H - \overline{H}$, and for any player i,*

$$\sigma_i^t(h^{t-1}) = B_i(h^{t-1})\frac{v^t}{v^t + \cdots + v^m}. \tag{12}$$

Proof. First, we show that a proportional strategy profile $\sigma^* \in \Sigma$ in the dynamic contest is a Nash equilibrium by proving that for any player i a strategy $\sigma_i \in \Sigma_i$ is a best response to σ^*_{-i} whenever $\sigma_i = \sigma_i^*$. Let x be the spending sequence associated with $(\sigma_i, \sigma^*_{-i})$ and $x_{-i} = (x^t_{-i})_{t\leq m}$ denote the spending sequence excluding player i, where for all t, $x^t_{-i} = (x^t_1, \ldots, x^t_{i-1}, x^t_{i+1}, \ldots, x^t_n)$.[9] We show that

$$\sigma_i \in \arg\max_{\sigma_i'} \pi_i(\sigma^*_{-i}, \sigma_i'|\emptyset), \tag{13}$$

that is, player i's best response to σ^*_{-i} associated with x_{-i} is σ_i associated with x_i. Given that all players but i follow the spending sequence x_{-i}, player i's expected prize from a 1-prize battle t_1 for any $x_i^{t_1}$, which we treat as a variable, is given by

$$\frac{\beta(x_i^{t_1})^\alpha}{(\beta(x_i^{t_1})^\alpha + \beta\sum_{j\neq i}(x_j^{t_1})^\alpha)}. \tag{14}$$

Differentiating (14) with respect to $x_i^{t_1}$ gives

$$\frac{\alpha(x_i^{t_1})^{\alpha-1}\sum_{j\neq i}(x_j^{t_1})^\alpha}{((x_i^{t_1})^\alpha + \sum_{j\neq i}(x_j^{t_1})^\alpha)^2}, \tag{15}$$

which is player i's marginal gain from the battle t_1. We next consider a k-prize battle t_k for some $k \in \{1, \ldots, m\}$. By our supposition each player except player i spends in proportion to the prize of the battle, i.e., for each $j \neq i$, $x_j^{t_k} = kx_j^{t_1}$. We next show that player i's best response to proportional allocation is also to spend in proportion to the prize at battle t_k, i.e., $x_i^{t_k} = kx_i^{t_1}$. In this case, player i's expected prize for any $x_i^{t_k}$ from k-prize battle t_k is

$$\frac{k\beta(x_i^{t_k})^\alpha}{(\beta(x_i^{t_k})^\alpha + \beta\sum_{j\neq i}x_j^{t_k})} = \frac{k(x_i^{t_k})^\alpha}{((x_i^{t_k})^\alpha + k^\alpha\sum_{j\neq i}(x_j^{t_1})^\alpha)}. \tag{16}$$

Differentiating (16) with respect to $x_i^{t_k}$ gives

$$\frac{\alpha k^{\alpha+1}(x_i^{t_k})^{\alpha-1}\sum_{j\neq i}(x_j^{t_1})^\alpha}{((x_i^{t_k})^\alpha + k^\alpha\sum_{j\neq i}(x_j^{t_1})^\alpha)^2}, \tag{17}$$

which is player i's marginal gain from the k-prize battle t_k. Next, we show that for $x_i^{t_k} = kx_i^{t_1}$, Expression (17) equals Expression (15). First, Expression (17) equals

$$\frac{\alpha k^{\alpha+1}(x_i^{t_k})^{\alpha-1}\sum_{j\neq i}(x_j^{t_1})^\alpha}{((x_i^{t_k})^\alpha + k^\alpha\sum_{j\neq i}(x_j^{t_1})^\alpha)^2} = \frac{\alpha k^{\alpha+1}(kx_i^{t_1})^{\alpha-1}\sum_{j\neq i}(x_j^{t_1})^\alpha}{((kx_i^{t_1})^\alpha + k^\alpha\sum_{j\neq i}(x_j^{t_1})^\alpha)^2}. \tag{18}$$

Cancelling out k's leads to

$$\frac{\alpha(x_i^{t_1})^{\alpha-1}\sum_{j\neq i}(x_j^{t_1})^\alpha}{((x_i^{t_1})^\alpha + \sum_{j\neq i}(x_j^{t_1})^\alpha)^2}, \tag{19}$$

which is Expression (15). We showed that if player i allocates proportionally to k-prize battle, then his marginal gain from that battle is equal to his marginal gain from 1-prize battle provided that others allocate proportionally. Thus, there is no incentive for player i to deviate from proportional allocation when others allocate proportionally. Hence, the proportional strategy profile σ is a Nash equilibrium of the dynamic contest.

Now we show that σ^* is a subgame perfect equilibrium. In other words, for every $h \in H$, σ^* induces an equilibrium in the subgame starting with history h. By definition, the subgame starting with history h is a game (i.e., dynamic contest) and $(\sigma^*|h)$ is a proportional strategy profile. Thus, by an analogous argument used in the first part of the proof, $(\sigma^*|h)$ is a Nash equilibrium in the subgame starting with history h.[10] That is, we obtain for every i and every h

$$(\sigma_i^*|h) \in \arg\max_{\sigma_i'} \pi_i(\sigma^*_{-i}, \sigma_i'|h), \tag{20}$$

Therefore, σ^* is a subgame perfect equilibrium, so the dynamic contest satisfies proportionality. □

2.4. Uniqueness of subgame perfect equilibrium

In addition to existence and characterization, the third natural question regarding the properties of subgame perfect equilibrium is whether and under what conditions it is unique. The next proposition gives a sufficient condition in two-player dynamic contests.

Proposition 1 (*Uniqueness of equilibrium*)**.** *In every two-player dynamic Blotto contest with Tullock CSF, the subgame perfect equilibrium shown in Theorem 1 is unique.*

Proof. Let σ^* be a "proportional" subgame perfect equilibrium as shown in Theorem 1. The proof strategy is to show that for every player $i \neq j$ in a two-player dynamic Tullock contest the best response of player i against σ^*_{-i} is unique. This will conclude the proof of Proposition 1 because in two-player zero-sum games every equilibrium strategy must be a best response against every equilibrium strategy of the opponent by von Neumann (1928) minimax theorem. Thus, if for every player i, σ_i^* is the unique best response to σ^*_{-i}, then the subgame perfect equilibrium σ^* must be unique.

In a two-player contest with Tullock CSF (i.e. $\alpha = 1$), (14) in the proof of Theorem 2 reduces to

$$\frac{x_i^{t_1}}{x_i^{t_1} + x_j^{t_1}},$$

and (15) to

$$\frac{x_j^{t_1}}{(x_i^{t_1} + x_j^{t_1})^2}, \tag{21}$$

which is player i's marginal gain from the 1-prize battle t_1. Similarly, (16) simplifies to

$$\frac{kx_i^{t_k}}{x_i^{t_k} + x_j^{t_k}},$$

and (17) to

$$\frac{k^2x_j^{t_1}}{(x_i^{t_k} + kx_j^{t_1})^2}, \tag{22}$$

which is player i's marginal gain from the k-prize battle t_k. Then, (21) equals (22) if and only if

$$\frac{1}{(x_i^{t_1} + x_j^{t_1})^2} = \frac{k^2}{(x_i^{t_k} + kx_j^{t_1})^2}. \tag{23}$$

[9] To be sure, one may condition his strategy on the winners of the previous battles and also on the previous battle spendings. However, without loss of generality, we can confine attention to the spending sequence, x, that is associated with the given strategy profile, because the payoff received from the previous battles does not affect the payoff that can be received from the remaining ones as the payoff function is additive.

[10] Note that in the first part, we showed that a proportional strategy profile is a Nash equilibrium in any dynamic contest.

Because all values in Eq. (23) must be strictly positive in any equilibrium (allocating zero resources is never a best response), (23) holds if and only if

$$k^2(x_i^{t_1}+x_j^{t_1})^2-(x_i^{t_k}+kx_j^{t_1})^2=0,$$

if and only if

$$(kx_i^{t_1}+kx_j^{t_1}-x_i^{t_k}-kx_j^{t_1})(kx_i^{t_1}+kx_j^{t_1}+x_i^{t_k}+kx_j^{t_1})=0,$$

if and only if $x_i^{t_k} = kx_i^{t_1}$. Thus, player i's "proportional" subgame perfect equilibrium strategy σ_i^* is the unique best response to player j's subgame perfect equilibrium strategy. Therefore, the subgame perfect equilibrium must be unique, because the dynamic Tullock contest is a two-person zero-sum game. □

Note that the proof strategy used in Proposition 1 does not immediately extend to two-player dynamic contests with non-Tullock contest success functions because if $\alpha \neq 1$, then "$x_i^{t_k} = kx_i^{t_1}$" need not be the unique solution when we set (15) and (17) equal.[11] This implies that a player may have multiple best responses to opponent's equilibrium strategy. Proposition 1 does not extend to n-player dynamic contests either because we cannot use the minimax theorem to prove the uniqueness of the equilibrium in n-player games. While we did not assume symmetric budgets in Proposition 1, making this assumption would not resolve the aforementioned difficulties in proving the uniqueness of equilibrium. For these reasons, we leave the uniqueness of subgame perfect equilibrium in n-player dynamic contests as an open problem.

3. Discussion and concluding remarks

In this paper, we provide an extension of S&E's two-player multi-battle sequential Blotto model to n-player multi-battle sequential Blotto games with arbitrary prize structures and more general CSFs. Players' budgets and battlefield prizes may be asymmetric, and there are no restrictions on the number of players or the number of battlefields (e.g., odd or even). In this context, we study the proportional allocation of resources and the equilibrium behavior. We show that the strategy profile in which players proportionally allocate their resources at every history is a subgame perfect equilibrium. Moreover, the results do not depend on the specific CSF used in the competition as long as it satisfies Skaperdas's (1996) axioms. An open question for future research is whether or not the proportional strategy profile is the unique subgame perfect equilibrium in n-player dynamic contests.

Blotto games can be applied to a variety of economic and political situations, as Borel (1921) himself envisioned. As an example, consider sequential elections as an n-player dynamic multi-battle contest where political candidates choose how to distribute their limited resources over multiple "battlefields" or states, as in the U.S. presidential primaries. In this context, our results imply that proportionality is immediately rectified once one has candidates who maximize their electoral vote instead of simply maximizing their probability of winning, despite the presence of the winner-take-all feature.

To achieve proportionality, at least in the U.S. presidential primaries, which operate on a winner-take-all system, a viable policy suggestion could be providing additional incentives to induce players to win as many delegates as possible in all of the presidential primaries overall. For instance, the electoral system could provide players with additional funding in the ensuing presidential race, where these incentives are positively linked to the number of delegates won by the presidential player in the primaries. Such incentives can be very effective at the margin. However, even in the absence of any such additional pecuniary incentives, players themselves seem to already exhibit the behavioral trait of maximizing their expected number of delegates and do not appear to want to stop pumping campaign funding into the remaining primaries, even when they have already guaranteed winning the majority of the delegates.

The main reason players may try to win additional delegates beyond those they need to guarantee their presidential candidacy (i.e., the main reason they might continue investing in the remaining primaries even though they know that it will not affect their chances of winning further delegates) could be that they care about entering the U.S. presidential race with an impressive momentum gained in the presidential primaries. This is reminiscent of the strategy Hillary Clinton tried to employ against Bernie Sanders' late surge in the 2016 U.S. Democratic primaries, even though she had already accumulated more than enough delegates to win her party's presidential candidacy up to that point. Nevertheless, to ensure proportionality, the parties or the electoral system might consider boosting players' tendency to maximize their expected delegates via some additional pecuniary incentives, which may help at the margin, at least for the players who may simply try to maximize their probability of winning in their U.S. presidential primaries.

Data availability

No data was used for the research described in the article.

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References

Acharya, Avidit, Grillo, Edoardo, Sugaya, Takuo, Turkel, Eray, 2022. Electoral campaigns as dynamic contests. Preprint available at https://economia.unipd.it/sites/economia.unipd.it/files/20220293.pdf.
Arad, A., Rubinstein, A., 2012. Multi-dimensional iterative reasoning in action: The case of the Colonel Blotto game. J. Econ. Behav. Organ. 84 (2), 571–585.
Barelli, P., Govindan, S., Wilson, R., 2014. Competition for a majority. Econometrica 82 (1), 271–314.
Baye, M.R., Kovenock, D., De Vries, C.G., 1994. The solution to the Tullock rent-seeking game when R > 2: Mixed-strategy equilibria and mean dissipation rates. Public Choice 81 (3–4), 363–380.
Borel, E., 1921. La théorie du jeu et les équations intégralesa noyau symétrique. C. R. Acad. Sci. 173 (1304–1308), 58.
Brams, S.J., Davis, M.D., 1973. Resource-allocation models in presidential campaigning: Implications for democratic representation. Ann. New York Acad. Sci. 219 (1), 105–123.
Brams, S.J., Davis, M.D., 1974. The 3/2's rule in presidential campaigning. Am. Political Sci. Rev. 68 (1), 113–134.
Brams, S.J., Davis, M.D., 1982. Optimal resource allocation in presidential primaries. Math. Social Sci. 3 (4), 373–388.
Deck, C., Sarangi, S., Wiser, M., 2017. An experimental investigation of simultaneous multi-battle contests with strategic complementarities. J. Econ. Psychol. 63, 117–134.

[11] Klumpp et al. (2019) use a similar proof strategy of using the minimax theorem to prove the uniqueness of their equilibrium in two-player zero-sum games.

Deck, C., Sheremeta, R.M., 2012. Fight or flight? defending against sequential attacks in the game of siege. J. Confl. Resolut. 56 (6), 1069–1088.
Duffy, J., Matros, A., 2015. Stochastic asymmetric Blotto games: Some new results. Econom. Lett. 134, 4–8.
Duffy, J., Matros, A., 2017. Stochastic asymmetric Blotto games: An experimental study. J. Econ. Behav. Organ. 139, 88–105.
Duggan, J., 2007. Equilibrium existence for zero-sum games and spatial models of elections. Games Econom. Behav. 60 (1), 52–74.
Dziubiński, M., Goyal, S., Minarsch, D.E., 2021. The strategy of conquest. J. Econom. Theory 191, 105161.
Ewerhart, C., Teichgräber, J., 2019. Multi-battle contests, finite automata, and the tug-of-war. Preprint available at https://www.zora.uzh.ch/id/eprint/169032/1/econwp318.pdf.
Friedman, L., 1958. Game-theory models in the allocation of advertising expenditures. Oper. Res. 6 (5), 699–709.
Harris, C., Vickers, J., 1985. Patent races and the persistence of monopoly. J. Ind. Econ. 46, 1–481.
Hinnosaar, T., 2023. Optimal sequential contests. Theor. Econ. (forthcoming).
Kim, G.J., Kim, J., Kim, B., 2018. A lottery Blotto game with heterogeneous items of asymmetric valuations. Econom. Lett. 173, 1–5.
Klumpp, T., Konrad, K.A., Solomon, A., 2019. The dynamics of majoritarian Blotto games. Games Econom. Behav. 117, 402–419.
Klumpp, T., Polborn, M.K., 2006. Primaries and the new hampshire effect. J. Public Econ. 90 (6–7), 1073–1114.
Konrad, K.A., 2009. Strategy and Dynamics in Contests. Oxford University Press.
Konrad, K.A., 2018. Budget and effort choice in sequential colonel blotto campaigns. CESifo Econ. Stud. 64 (4), 555–576.
Konrad, K.A., Kovenock, D., 2009. Multi-battle contests. Games Econom. Behav. 66 (1), 256–274.
Kovenock, D., Arjona, D.R., 2019. A full characterization of best-response functions in the lottery Colonel Blotto game. Econom. Lett. 182, 33–36.
Kovenock, D., Roberson, B., 2009. Is the 50-state strategy optimal? J. Theor. Politics 21 (2), 213–236.
Krueger, A.O., 1974. The political economy of the rent-seeking society. Am. Econ. Rev. 64 (3), 291–303.
Kvasov, D., 2007. Contests with limited resources. J. Econom. Theory 136 (1), 738–748.
Lake, M., 1979. A new campaign resource allocation model. In: Applied Game Theory. Springer, pp. 118–132.
Laslier, J.-F., Picard, N., 2002. Distributive politics and electoral competition. J. Econom. Theory 103 (1), 106–130.
Li, X., Zheng, J., 2021. Even-split strategy in sequential Colonel Blotto games. Preprint available at SSRN: https://ssrn.com/abstract=3947995.
Li, X., Zheng, J., 2022a. Pure strategy Nash equilibrium in 2-contestant generalized lottery Colonel Blotto games. J. Math. Econom. 103, 102771.
Li, X., Zheng, J., 2022b. Resource allocation under general conflict structure. Preprint available at SSRN: https://ssrn.com/abstract=4001406.
Montero, M., Possajennikov, A., Turocy, T.L., 2016. Majoritarian Blotto contests with asymmetric battlefields: an experiment on apex games. Econom. Theory 61 (1), 55–89.
Nagel, R., 1995. Unraveling in guessing games: An experimental study. Am. Econ. Rev. 85 (5), 1313–1326.
Osorio, A., 2013. The lottery Blotto game. Econom. Lett. 120 (2), 164–166.
Rinott, Y., Scarsini, M., Yu, Y., 2012. A Colonel Blotto gladiator game. Math. Oper. Res. 37 (4), 574–590.
Roberson, B., 2006. The Colonel Blotto game. Econom. Theory 29 (1), 1–24.
Sela, A., Erez, E., 2013. Dynamic contests with resource constraints. Soc. Choice Welf. 41 (4), 863–882.
Skaperdas, S., 1996. Contest success functions. Econ. Theory 7 (2), 283–290.
Stahl, D.O., 1993. Evolution of smart$_n$ players. Games Econom. Behav. 5 (4), 604–617.
Strumpf, K.S., 2002. Strategic competition in sequential election contests. Public Choice 111 (3–4), 377–397.
Thomas, C., 2017. N-dimensional Blotto game with asymmetric battlefield values. Econom. Theory 65, 509–544.
Tullock, G., 1967. The welfare costs of tariffs, monopolies, and theft. Econ. Inq. 5 (3), 224–232.
Tullock, G., 1974. The Social Dilemma: The Economics of War and Revolution. University publications.
von Neumann, J., 1928. Zur Theorie der Gesellschaftsspiele. Math. Ann. 100, 295–320.
Xie, H., Zheng, J., 2022. Dynamic resource allocation in tug-of-war. Preprint available at SSRN: https://ssrn.com/abstract=4090793.