Robust Uncertainty Quantification for Measurement Problems with Limited Information.*

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Metrology has an important role in modern science and relies on the accuracy and repeatability of a measurement. However, these measurements are the outcomes of different expensive experiments and noisy due to the epistemic uncertainty associated with these experiments. We express our model by $y = f(\mu_1, \mu_2, \cdots, \mu_m)$, where $\mu := (\mu_1, \ldots, \mu_m)$ are m different inputs. Our main goal is to obtain a confidence interval for $f(\mu)$, based on some estimates for μ.

We use the delta method [3] for uncertainty quantification, which is based on the multivariate normal approximation. Let $\hat{X} := (\hat{X}_1, \ldots, \hat{X}_m)$ be an estimator of μ such that approximately $\hat{X} \sim N(\mu, \Sigma)$ where $\Sigma := \mathrm{Cov}(\hat{X})$. If f is differentiable, then by first order Taylor expansion, we have

$$f(\hat{X}) \approx f(\mu) + \nabla f(\mu)^T (\hat{X} - \mu). \tag{1}$$

Now, if f is approximately linear around μ for the distributional range of $\hat{X}$, then we approximately have that $f(\hat{X}) \sim N(f(\mu), \nabla f(\mu)^T \Sigma \nabla f(\mu))$, by Eq. (1) and by the usual linear transformation rule for the covariance matrix. We can use this approximate distribution of $f(\hat{X})$ to construct a 95% confidence interval for $f(\mu)$. However, $f(\hat{X})$ may not be necessarily Gaussian especially if f is highly non-linear. Additionally, for the variance term $\nabla f(\mu)^T \Sigma \nabla f(\mu)$, we may need to use the sample standard deviation as we do not know Σ, and we may need to use $\nabla f(\hat{X})$ as we do not know $\nabla f(\mu)$.

To avoid these issues, we propose using imprecise probability for uncertainty quantification in metrology, which is a new contribution to the field. Specifically, we propose using p-boxes [1].This helps us to relax distributional assumptions and thereby leads to more robust estimates. Additionally, uncertainty expressed as a p-box can be easily propagated through a range of standard non-linear operators.

We illustrate our results by analysing the uncertainty associated with end gauge calibration [2]. Here, we try to determine the length (ℓ_M) of an end gauge (M) by comparing it with length (ℓ_S) of a known standard (S) using the relation, $\ell_M = \frac{\ell_S(1+\alpha_S\theta_S)+d}{1+\alpha_M\theta_M}$. Here, α_M and θ_M (α_S and θ_S) are thermal expansion coefficient and temperature deviation of M (S) and d is the difference between ℓ_M and ℓ_S. In practice, α_M and θ_M (α_S and θ_S) often have weak correlation between them. Therefore, we use p-boxes to characterise these variables. We inspect their dependence structure for uncertainty propagation and obtain a robust estimate. Finally, we compare our results with the delta method.

References

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* This work is funded by the European Commission's H2020 programme, through the UTOPIAE Marie Curie Innovative Training Network, H2020-MSCA-ITN-2016, Grant Agreement number 722734.