



3 Teaching for understanding/ understanding for teaching

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Introduction: the context for understanding

The issue of understanding in mathematics has been a particular focus for educational policy in England and Wales in recent years. A report published by Ofsted (2008: 5) highlighted the lack of development of mathematical understanding in the classroom:

The fundamental issue for teachers is how better to develop pupils' mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently.

The *Independent Review of the Primary Curriculum: Final Report* (the Rose Review) (DCSF 2009) recommended that one of the proposed strands of learning should be 'mathematical understanding'. It highlighted the need to develop children's thinking and discussion in mathematics, and opportunities to use and apply mathematics – areas that perhaps are being neglected in the classroom through a conception of 'numeracy' that is too narrow. The *Independent Review of Mathematics Teaching in the Early Years and Primary Schools* (the Williams Review) (DCSF 2008: 7) specifically recommended the provision of mathematics specialist teachers 'with deep mathematical subject and pedagogical knowledge', with a focus on impacting on standards and attainment in mathematics. A recent Nuffield review of mathematical learning (Nuñez et al. 2009: 3) aimed 'to identify the issues that are fundamental to understanding children's mathematics learning' and focused throughout on 'key understandings in mathematics'.

However, although the importance of understanding is agreed upon, a





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vital issue is what we mean by understanding in mathematics and how we teach in order to develop it in children. We have already seen that it draws on ideas and terms such as ‘discussion’ (see Monaghan, Chapter 4), ‘using and applying’ (see Ollerton, Chapter 6) and ‘deep subject and pedagogical knowledge’. But what is this deep knowledge? Why are discussion and using and applying important? In this chapter, we set out to clarify exactly what we mean by understanding in mathematics. In doing so, we look at the implications of our definition on teaching for understanding and the understanding that teachers need to bring to the classroom. We hope that this discussion of understanding will help teachers and prospective teachers of mathematics to be clear about why they are doing what they are doing in the classroom, and also help them to develop their practice in the future.

Defining understanding

An important characteristic of understanding is that it involves connections between different ideas or concepts. More specifically, Hiebert and Carpenter (1992: 67) defined mathematical understanding as involving the building up of a conceptual ‘network’:

The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of its connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections.

The mental representations that make up this network are defined by Davis (1984: 203) as follows: ‘Any mathematical concept, or technique, or strategy – or anything else mathematical that involves either information or some means of processing information – if it is to be present in the mind at all, must be represented in some way.’

Goldin (1998) highlighted the fact that we have a variety of internal representations, including verbal, imagistic, symbolic, planning (for example, problem solving approaches) and affective (that is, attitudes about maths) representations. Therefore, we have this picture of understanding as being this variety of internal or mental representations associated with mathematical concepts, being connected together to form a complex network. This view of mathematical understanding is closely related to the ‘connectionist’ view of mathematics teaching (see Askew, Chapter 2). The question then is how are these mental representations connected together?

Sierpinska (1994) identified the ‘processes of understanding’ as involving connections being made between mental concepts through reasoning





processes. For example, in our minds, the concepts of ‘addition’ and ‘multiplication’ may be connected because we can show that multiplication can be the same as ‘repeated addition’ – perhaps because calculations give the same answer, or perhaps through pictures such as the number line (we will say a little more about the ‘quality’ of this reasoning later). Therefore, the overall picture or model of understanding that we have adopted is a ‘representational–reasoning’ model of understanding, as shown in Figure 3.1. The model shows the different representations (the circles) connected together by different levels of reasoning (the lines).

It has to be emphasized here that this is not meant to be a picture of what is actually happening inside our minds. Rather, we emphasize that this is a ‘model’ of understanding: a picture that helps us to make sense of this concept. However, because we have this picture to work with, we can start to look at what such a model means for what teachers do in the mathematics classroom. In the next section, we examine the implications of the model for teaching for understanding.

Implications of the model: teaching for understanding

The first issue that we can note from the model of understanding is that there is no limit to the connections we can make – there is no ‘boundary’ to our understanding:

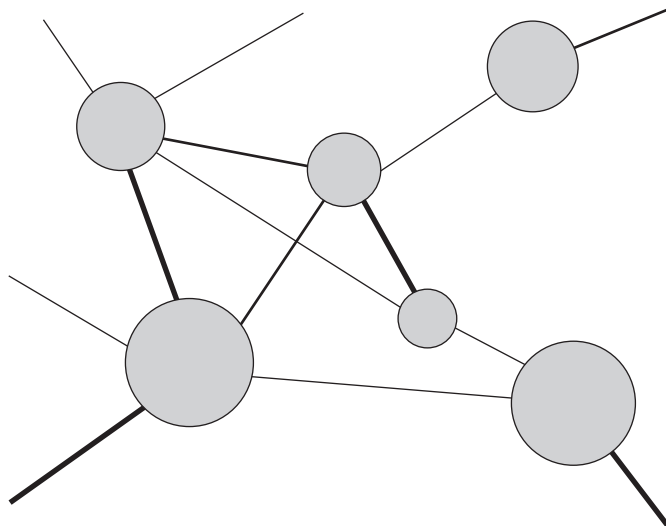


Figure 3.1 Representational–reasoning model of understanding.





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Understanding is not a dichotomous state, but a continuum... Everyone understands to some degree anything that they know about. It also follows that understanding is never complete; for we can always add more knowledge, another episode, say, or refine an image, or see new links between things we know already.

(White and Gunston 1992: 6)

Understanding of a mathematical topic is not something that we suddenly attain at the end of some programme of learning. Rather, it is a continuously evolving process – a process rather different from the often conveyed perception of mathematics as being about ‘right’ and ‘wrong’ answers. The latter issue of ‘wrong’ answers is also challenged by the idea that ‘everyone understands to some degree anything that they know about’. Certainly, we all have ‘misconceptions’, but to simply label them as ‘wrong’ fails to recognize that there are often good reasons for us to possess these misconceptions (see Ryan and Williams, Chapter 11). For example, a child that states that we cannot do $2 \div 5$ because ‘5 into 2 doesn’t go’, is basing their reasoning on a conception of division as repeated subtraction. Within this limited understanding, their misconception is entirely reasonable. In order to tackle this misconception, we need to build upon their conception of division (that is, increase the variety of mental representations) and to develop their subsequent reasoning.

In examining the issue of misconceptions, we have touched upon the issue of how we develop understanding in the mathematics classroom. And because our model of understanding has two components – the representations and the reasoning – this provides us with two areas to explore with regards to how we teach for understanding. Let us examine first the issue of representations. In order to develop the range of mental representations available to a person, we can provide them with a variety of external representations (for example, concrete manipulatives, pictures, symbolic representations, procedures). Let us give an example of this from some work that we have been carrying out with a group of experienced primary mathematics teachers. We were working with the array representation for multiplication (Figure 3.2).

One of the strengths of the array is that we can easily show the distributive properties of multiplication. For example, from the diagram, we can see that 8×6 is the same as $(5 + 3) \times (5 + 1)$, which in turn is $(5 \times 5) + (5 \times 1) + (3 \times 5) + (3 \times 1)$. Now, let us look at another representation for multiplication, that of the grid method (Figure 3.3).

Figure 3.3 shows the grid method to calculate 18×16 . By comparing the two representations of the grid method and the array, we can see why we can split the numbers in the grid method as we do. But one of the teachers we were working with went further, with the realization that the grid method did not have to be based on units, tens, and so on. For example, the grid could be split into multiples of 5 (Figure 3.4):



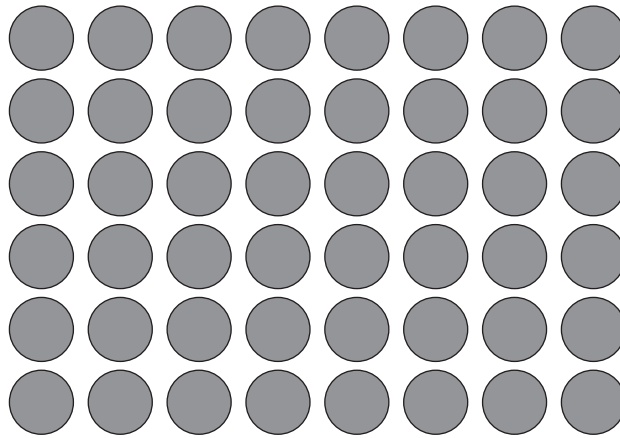


Figure 3.2 Array representation for multiplication.

The teacher felt that this modified grid method might be better for lower-attaining pupils as they could change more difficult times tables such as the 8 times tables into the 5 and 3 times tables. By increasing the range of representations available to the teacher, this led to the development of their understanding in multiplication.

One aspect of teaching for understanding can be seen to be this development of the range of mental representations of mathematical concepts. However, this in itself is not sufficient. This has been highlighted by Sowell (1989) who concluded that the long-term use of concrete materials led to

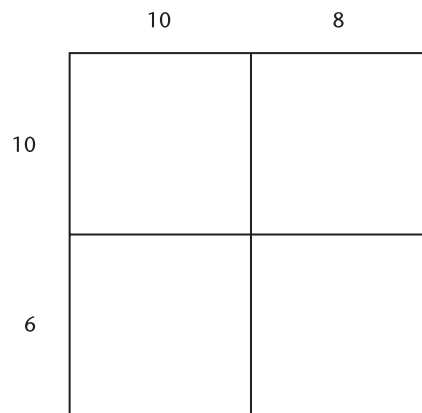


Figure 3.3 Grid method for multiplication.





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	10	5	3
10			
5			
1			

Figure 3.4 Modified grid method.

benefits for children. The emphasis was on *long-term* use, and significant benefits were not observed for pictorial representations. In fact, we have emphasized in our model of understanding that we need to develop our reasoning between representations as well.

The process of trying to make connections between our existing understanding of a concept and alternative representations for that concept is what brings about the processes of assimilation of the new representation (if our existing understanding is compatible) or accommodation of the new representation (through restructuring of our understanding) that was highlighted by Piaget (1968). Therefore, an additional implication for teaching for understanding in the classroom is to provide opportunities for children to develop their ‘reasonings’ – for example, explaining why they do a calculation in a certain way (Nuñez et al. 2009: 10–11). Now this reasoning may not necessarily be formal mathematical reasoning – for example, a child might split up a multiplication calculation according to the distributive law simply because their teacher had told them that they could do this. Nevertheless, this is the reasoning that is used by the child. We can now bring in here the issue of discussion that we highlighted at the beginning of the chapter. Hoyles (1985) suggested three aspects to pupil–pupil discussion:

- articulating ideas brings about reflection on those ideas;
- discussion involves framing ideas in a way that will be accepted by others;
- listening to others modifies your own thoughts.

All three of these aspects of discussion bring about an examination of one’s own reasonings. Discussion and explanation of methods by children can be





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used in our teaching for understanding so that children can reflect upon the quality of their reasoning, and thereby strengthen or change their connections (see Monaghan, Chapter 4).

The representational–reasoning model of understanding provides us with a basis for how we can approach teaching for understanding in the mathematics classroom. Interestingly, both ‘representing’ and ‘reasoning’ appears in the five themes within ‘using and applying’ identified by guidance notes provided by the Department for Education and Skills (DfES 2006: 4):

- Solving problems;
- Representing – analyse, record, do, check, confirm;
- Enquiring – plan, decide, organize, interpret, reason, justify;
- Reasoning – create, deduce, apply, explore, predict, hypothesize, test;
- Communicating – explain methods and solutions, choices, decisions, reasoning.

However, based on the explanation of the themes provided by the DfES, where aspects of representation and reasoning seem to appear in different themes, we could simplify the picture of using and applying mathematics in terms of the two aspects of understanding that we have already identified. Let us draw on some example problems from the guidance paper to illustrate this. First, we have a word problem such as ‘How much will seven oranges cost if four oranges cost £1?’ Based on our existing understanding, we could represent this problem in a variety of ways (Figure 3.5).

We can then examine each alternative representation to see which is most useful for providing an answer. Perhaps multiplying by $1\frac{3}{4}$, and therefore multiplying £1 to £1.75, provides the most direct answer. The important issue here is that the act of representing the problem in alternative ways provides us with a solution. Also, the act of solving the problem results in different representations of the concept being linked (for example, multiplication by fractions with ratios). The sharing of problem-solving approaches in the classroom through discussion will hopefully result in new connections being made

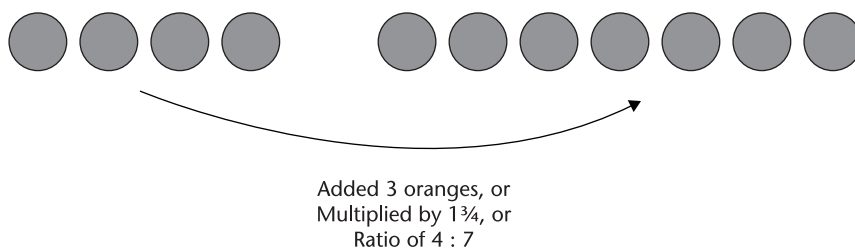


Figure 3.5 Representing a problem in a variety of ways.





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within children's understanding. Taking a broader view of mental representations as well, this will also result in new 'planning' representations (for example, what to do when faced with a ratio problem) within this understanding.

Let us look at one more problem – this time a sequence problem. What is the 51st number in the sequence 2, 7, 12 . . .? We can start by representing the problem in a different way again (Figure 3.6).

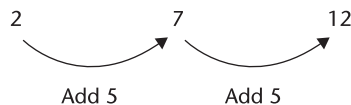


Figure 3.6 Representing a sequence.

We could keep adding 5 until we reach the 51st number. However, this is rather laborious – let us use another representation. The first number is 2 with no fives added; the second number is 2 with 1 five added; the third number is 2 with 2 fives added, and so on. From this pattern, we can reason that the 51st number would be 2 with 50 fives added, or 252. However, this is not the end of the problem. We can also examine the reasoning we have used. Are we correct in reasoning that 'adding 5' is the only way in which the sequence 2, 7, 12 can be obtained? Are there other representations that we could have used? Likewise, in the previous problem, are there other ways of approaching this problem? Calling into question our reasoning in a problem, as well as using a variety of representations, can lead to a development in our understanding either through strengthening connections or developing new ones. We can therefore start to see the role of using and applying in developing our understanding of mathematical concepts.

Let us examine one last implication of the earlier model on teaching for understanding – this time a particular difficulty that the model implies. The complex network that makes up our model means that if we are to assess a child's understanding of a mathematical concept, then we need to try and assess the variety of representations and reasonings associated with that concept in a child's mind. This is no simple task: 'Understanding usually cannot be inferred from a single response on a single task; any individual task can be performed correctly without understanding. A variety of tasks, then, are needed to generate a profile of behavioural evidence' (Hiebert and Carpenter 1992: 89).

A broader approach to assessment then, rather than one simply based on an examination of procedural calculations, is required. Perhaps we can use problem-solving tasks where we have greater access to the representations that the child can draw upon or greater access to their reasoning. Davis (1984) advocates the use of task-based interviews where children are questioned while they tackle mathematical tasks about the approach they are using,





the reasons for their approach, the possibility of other approaches, and so on. Alternatively, we could use other approaches such as mind maps to access the variety of representations that a child associates with a concept.

This need for a broader approach to assessment is seen in intervention situations where it is important to access a child's understanding so that gaps in understanding and misconceptions can be identified. In the Every Child Counts intervention (see Dunn, Matthews and Dowrick, Chapter 17), teachers spend an initial period of around two weeks carrying out a broad assessment of a child in order to identify particular difficulties in numeracy. This assessment involves traditional tests, classroom observations and task-based diagnostic work with the child. In the broader context of teaching for understanding, we need to be aware of the limitations of our traditional forms of assessment, and look for opportunities within our teaching where a broader assessment of children's understanding can be gained. This approach is very much part of the recent introduction of APP or 'Assessing Pupils' Progress' by the National Strategies in England and Wales (see Hodgen and Askew, Chapter 10).

Implications of the model: understanding for teaching

Through the model of understanding that we have adopted, we have been able to highlight some implications for how we can approach 'teaching for understanding' in the mathematics classroom. Another area that we would like to explore is the understanding that is required by teachers themselves – what the Williams Review (DCSF 2008: 3) referred to as 'deep mathematical subject and pedagogical knowledge'. We can use the picture of understanding that we have, alongside existing research in this area, to gain further insight into this 'deep knowledge'.

Previous research by Shulman (1986) has been very influential in providing a theoretical view for the categories of knowledge possessed by teachers. With regard to subject-specific knowledge (for example, mathematics), these categories are:

- subject-matter content knowledge;
- pedagogical content knowledge;
- curricular knowledge.

Subject-matter content knowledge includes not only the organized factual content of the subject, but also how the subject functions as a discipline in terms of establishing the validity of ideas. In mathematics, this is the need for proof and deductive reasoning in order to establish as strongly as possible the connections between ideas. This view of subject-matter content knowledge is mirrored by what constitutes our understanding and also how we improve the quality of our reasonings that we touched upon before. Pedagogical knowledge





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is the content knowledge required to teach the subject. Shulman proposed the following components to this area of content knowledge: how we represent ideas, including the most powerful representations (models, illustrations, analogies, examples, and so on) for teaching, and also an understanding of how pupils generally learn the subject (including what makes topics easy or hard, typical conceptions and misconceptions, and how to tackle typical misconceptions). Curricular knowledge, then, is knowledge about the teaching programmes and teaching materials used in the subject.

Although Shulman's work is best remembered for these categories of teacher knowledge, he also strongly emphasized the importance of 'understanding', that is teachers' understanding, in the teaching of a subject: 'With Aristotle we declare that the ultimate test of understanding rests with the ability to transform one's knowledge into teaching. Those who can, do. Those who understand, teach' (Shulman 1986: 14).

Returning to pedagogical content knowledge, this emphasis on understanding is also highlighted by the need for a variety of representations required to teach a subject: 'Since there are no single most powerful forms of representations, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas other originate through practice' (Shulman 1986: 9).

This last quote also emphasizes the 'forms' or sources of knowledge for teachers. Shulman emphasized *propositional* knowledge in the form of principles that are taught to teachers, what is introduced to them through research on teaching (this chapter is mainly propositional!). We also have *case* knowledge which is more detailed reporting of specific events or sequences of events which are presented to the teacher (for example through research, through courses, through colleagues, and so on) Then we have *strategic* knowledge where the teachers experience particular events themselves, drawing on propositional and case knowledge but developing these in light of practice.

The understanding of mathematics that a teacher has is based then on their subject-matter content knowledge and pedagogical content knowledge, where concepts and ideas are connected, not just through mathematical reasoning, but also reasoning based on principles, examples of specific cases and personal experience in the classroom. In fact, research has shown that for more experienced teachers, their subject-matter content knowledge and their pedagogical content knowledge are indeed more connected (see, for example, Krauss et al. 2008). To illustrate these connections, let us provide an example from our own experience in the classroom.

We argued earlier that the array representation is a useful representation for multiplication, as it clearly shows the distributive properties of multiplication. However, when we have introduced this representation to children in the classroom, we have found that younger children (for example, Year 2) are very unlikely to recognize the array as a representation of multiplication. This is





despite the fact that according to the framework for teaching used in schools in England and Wales, it is suggested that the array is introduced to them in Year 2 for multiplication. Even in Year 4 children, only about half of the children will have this recognition. In Year 6, however, almost all the children recognize the array as a representation of multiplication. Therefore, despite our mathematical understanding that the array is a powerful representation for multiplication, and despite our knowledge of the curriculum, our pedagogical knowledge based on classroom experience suggests that we have to be careful about how we use the array in the classroom. Perhaps we need to introduce it alongside less abstract representations of multiplication (Figure 3.7). We can, in turn, examine this reasoning in the future through further work in the classroom to see whether this does help children to recognize the array as a representation of multiplication.

What we are emphasizing here is that although the understanding of mathematics subject matter that a teacher brings to the classroom is important, the understanding required for teaching is broader than that. In this understanding for teaching, the constituent representations and reasoning go beyond that of mathematical concepts alone, but include pedagogical principles, examples and experiences, informing our reasoning in building up our understanding.

Conclusion

In conclusion, what we have tried to do in this chapter is to look in detail at this concept of ‘understanding’ that seems to be of central importance for how we teach children in the mathematics classroom. What we hope is helpful for teachers and student teachers is that, by trying to clarify exactly what we mean by this idea, there are clear implications for how we should be approaching the teaching of the subject in the classroom. Of course, it is not as if the recommendations for using, for example, discussion or using and applying in the classroom are not already there. However, what we feel to be important

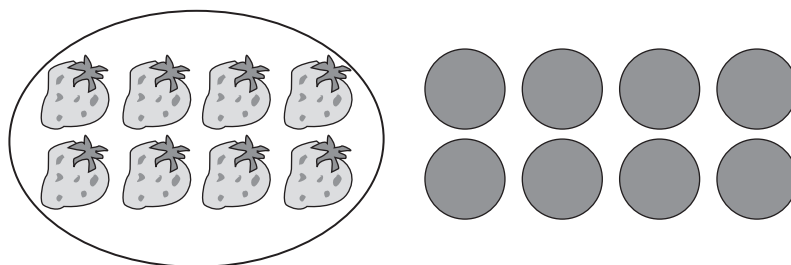


Figure 3.7 Array alongside less abstract representation (plate of strawberries).





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about this examination of understanding is that it makes clear why there are these recommendations. We also hope that it provides a model with which we can examine our own understanding of the subject.

We have further emphasized that the understanding required for teaching is more than that. Of course, we all accept that there is pedagogical knowledge that we need in order to teach the subject. However, our understanding is more integrated than separate bodies of subject and pedagogical knowledge. Our subject knowledge informs our pedagogy but, in turn, our pedagogical knowledge causes us to reflect on our subject knowledge. This then has implications for teachers' professional development and the training of teachers too. In developing their skills in teaching mathematics, we need to develop teachers' understanding; we need them to know how they can develop pupils' understanding; and we need to provide opportunities to reflect upon both of these so that they can further develop as teachers. We can illustrate this with two comments from student teachers of primary mathematics that we work with:

Although I know I have the subject knowledge to be able to teach to the children, I feel as if sometimes, although I have learnt so much on the course, I have not learnt things like, how to teach it, how to do the addition and the subtraction and how they are doing it in schools.

When it came to teaching it, I found it quite difficult to explain what I knew. I accept rules and . . . I apply it and it works. Trying to explain that to children, I found it at first a bit like, 'how am I going to break down what I just accept?' . . . But by the end of my placement this year, I felt much more confident in doing that. I would start to go back over what I knew and figure out how I had learnt it and how I had come to the point to be just doing it, which helped when it came to teaching it.

Both comments emphasize the importance of developing understanding in teaching mathematics. And this development will be an ongoing process, where, as we have seen from the model of understanding, it is a process without any end point, where we are continually developing as teachers of mathematics.

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