Acyclic, Star, and Injective Colouring: Bounding the Diameter

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Abstract. We examine the effect of bounding the diameter for well-studied variants of the Colouring problem. A colouring is acyclic, star, or injective if any two colour classes induce a forest, star forest or disjoint union of vertices and edges, respectively. The corresponding decision problems are Acyclic Colouring, Star Colouring and Injective Colouring. The last problem is also known as $L(1,1)$-Labelling and we also consider the framework of $L(a,b)$-Labelling. We prove a number of (almost-)complete complexity classifications, in particular, for Acyclic 3-Colouring, Star 3-Colouring and $L(1,2)$-Labelling.

1 Introduction

A natural way of increasing our understanding of NP-complete graph problems is to restrict the input. The *diameter* of a graph G is the maximum distance between any two vertices of G. We look at graph classes of *bounded* diameter, that is, with diameter at most d for some constant d. Such a graph class is closed under vertex deletion (hereditary) only if $d = 1$. Many graph problems stay NP-complete even if $d = 2$. The reason usually is that from a general instance we can obtain an instance of diameter 2 by adding a dominating vertex. For example, in this way, Clique, Independent Set and Colouring all stay NP-complete for graphs of diameter 2. The latter problem is to decide if for a graph G and integer k, there is a mapping $c : V(G) \to \{1,\dots,k\}$ with $c(u) \neq c(v)$ for each $uv \in E(G)$. If k is *fixed*, i.e., not part of the input, we write k-Colouring.

Let $d \geq 2$ and $k \geq 3$. It is readily seen that k-Colouring for graphs of diameter at most d is NP-complete for every $(d,k) \notin \{(2,3),(3,3)\}$. Mertzios and Spirakis [18] gave a highly non-trivial NP-hardness proof for the case $(3,3)$. The case $(2,3)$ is a notorious open problem, see, for example, [2, 8, 16–19].
The ith *colour class* in a graph $G = (V,E)$ with a colouring c is the set $V_i = \{u \in V \mid c(u) = i\}$. For $i \neq j$, let $G_{i,j}$ be the (bipartite) subgraph of G induced by $V_i \cup V_j$. If every $G_{i,j}$ is a forest, then c is an *acyclic colouring*. If every $G_{i,j}$ is P_4-free, i.e., a disjoint union of stars, then c is a *star colouring*. If

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every $G_{i,j}$ is P_3-free, i.e., a disjoint union of vertices and edges, then c is an *injective colouring*. The three decision problems are Acyclic Colouring, Star Colouring and Injective Colouring, respectively; for the last problem it is sometimes allowed for adjacent vertices to be coloured alike (see, e.g., [12–14]) but we do *not* permit this: as can be observed from the aforementioned definitions, all colourings considered in this paper are proper. If k is fixed we write Acyclic k-Colouring, Star k-Colouring and Injective k-Colouring.

Injective colourings are also known as *distance*-2 *colourings* and as $L(1,1)$-labelings. Namely, a colouring of a graph G is injective if the neighbours of every vertex of G are coloured differently, i.e., also vertices of distance 2 from each other must be coloured differently. The distance constrained labelling problem $L(a_1,\ldots,a_p)$-Labelling is to decide if a graph G has an $L(a_1,\ldots,a_p)$-(k-)labelling, i.e., a mapping $c : V(G) \rightarrow \{1,\ldots,k\}$ for some $k \geq 1$, such that for every two vertices u and v and every integer $1 \leq i \leq p$: if G contains a path of length i between u and v, then $|c(u) - c(v)| \geq a_i$; see also [9] (if $a_1 \geq a_2 \geq \ldots \geq a_p$, the condition is equivalent to "if u and v are of distance i").

The above problems are NP-complete, even for very restricted graph classes, see the survey [9] and very recent papers, such as [4, 5, 15, 20]. We consider graph classes of bounded diameter. In contrast to many other problems, bounding the diameter does help for colouring variants. For instance, the problem Near Bipartiteness is to determine if a graph has a 3-colouring such that (only) two colour classes induce a forest. This problem, on graphs of diameter at most d, is polynomial-time solvable if $d \leq 2$ [21] and NP-complete if $d \geq 3$ [6]. Or consider the $L(a_1,\ldots,a_p)$-Labelling problem. The degree of every vertex of a graph G with an $L(a_1,\ldots,a_p)$-k-labelling is at most k. Hence, $|V(G)| \leq 1 + k + \ldots + k^d$, where d is the diameter of G, and we can make the following observation.

Proposition 1. *Let $a_1,\ldots,a_p,d \geq 1$. Then, for every $k \geq 1$, $L(a_1,\ldots,a_p)$-k-*Labelling *is constant-time solvable for graphs of diameter at most d.*

This led us to the question: *How much does bounding the diameter help for obtaining polynomial-time algorithms for well-known graph colouring variants?*

Our Results. By using a very recent NP-completeness result on Acyclic 3-Colouring for graphs of diameter at most 4 [7] we obtain the following two almost-complete dichotomies; note that the case where $k \leq 2$ is trivial.

Theorem 1. *Let $d \geq 1$ and $k \geq 3$. Then* Acyclic k-Colouring *on graphs of diameter at most d is*

- *polynomial-time solvable if $d \leq 2$, $k = 3$ and* NP*-complete if $d \geq 4$, $k = 3$.*
- *polynomial-time solvable if $d = 1$, $k \geq 4$ and* NP*-complete if $d \geq 2$, $k \geq 4$.*

Theorem 2. *Let $d \geq 1$ and $k \geq 3$. Then* Star k-Colouring *on graphs of diameter at most d is*

- *polynomial-time solvable if $d \leq 3$, $k = 3$ and* NP*-complete if $d \geq 8$, $k = 3$.*
- *polynomial-time solvable if $d = 1$, $k \geq 4$ and* NP*-complete if $d \geq 2$, $k \geq 4$.*

Finally, we consider $L(a,b)$-Labelling for the most studied values of (a,b), namely when $1 \leq a \leq b \leq 2$. We now assume that k is part of the input, due to Proposition 1. Every two non-adjacent vertices in a graph G of diameter 2 have a common neighbour. Hence, an $(1,1)$-labelling of G colours each vertex uniquely, and $L(1,1)$-Labelling, on graph of diameter $d \leq 2$, is trivial. The problem is NP-complete if $d = 3$, as it is NP-complete for the subclass of split graphs [3]. Griggs and Yeh [11] proved that $L(2,1)$-Labelling is NP-complete for graphs of diameter 2 via a relation with Hamiltonian Path. We also connect the remaining case $(a,b) = (1,2)$ to Hamiltonian Path in order to prove NP-completeness in Section 4. To summarize, we obtained the following dichotomy:

Theorem 3. *Let $a, b \in \{1,2\}$ and $d \geq 1$. Then $L(a,b)$-*Labelling *on graphs of diameter at most d is*

- *polynomial-time solvable if $a = b$ and $d \leq 2$, or $d = 1$.*
- NP*-complete if either $a = b$ and $d \geq 3$, or $a \neq b$ and $d \geq 2$.*

Future Work. It would be interesting to close the gaps in Theorems 1 and 2, but this seems challenging. The NP-hardness construction of Mertzios and Spirakis [18] for 3-Colouring of graphs of diameter 3 does lead to NP-hardness for Near Bipartiteness for graphs of diameter 3 [6]. However, it cannot be used for Acyclic 3-Colouring and Star 3-Colouring.

2 The Proof of Theorem 1

We show the following result (proof omitted) and also recall a very recent result.

Lemma 1. Acyclic 3-Colouring *is polynomial-time solvable for graphs of diameter at most* 2*.*

Lemma 2 ([7]). Acyclic 3-Colouring *is* NP*-complete on triangle-free* 2*-degenerate graphs of diameter at most* 4*.*

The Proof of Theorem 1. The first statement follows from Lemmas 1 and 2. For the second statement, the case $d = 1$ is trivial, and for the case $d \geq 2$, $k \geq 4$ we reduce from Acyclic 3-Colouring: to an instance G of Acyclic k-Colouring, we add a clique of $k - 3$ vertices, which we make adjacent to every vertex of G.

3 The Proof of Theorem 2

A *list assignment* of a graph G is a function L that gives each vertex $u \in V(G)$ a *list of admissible colours* $L(u) \subseteq \{1, 2, \ldots\}$. A colouring c *respects* L if $c(u) \in L(u)$ for every $u \in V$. If $|L(u)| \leq 2$ for each $u \in V$, then L is a 2-*list assignment.* The 2-List Colouring problem is the corresponding decision problem.

Theorem 4 ([10]). *The* 2-List Colouring *problem is solvable in time $O(n+m)$ on graphs with n vertices and m edges.*

We will use Theorem 4 in the proof of Lemma 6, which is the main result of the section. In order to do this, we must first be able to modify an instance of Star 3-Colouring into an equivalent instance of 3-Colouring. We can do this as follows. Let $G = (V, E)$ be a graph. We construct a supergraph G_s of G as follows. For each edge $e = uv$ of G we add a vertex z_{uv} that we make adjacent to both u and v. We also add an edge between two vertices z_{uv} and $z_{u'v'}$ if and only if u, v, u', v' are four distinct vertices such that G has at least one edge with one end-vertex in $\{u, v\}$ and the other one in $\{u', v'\}$. We say that G_s is the *edge-extension* of G. Observe that we constructed G_s in $O(m^2)$ time. It is readily seen that G has a star 3-colouring if and only if G_s has a 3-colouring.

Now suppose G has a 2-list assignment L. We extend L to a list assignment L_s of G_s. We first set $L_s(u) = L(u)$ for every $u \in V(G)$. Initially, we set $L_s(z_e) = \{1, 2, 3\}$ for each edge $e \in E(G)$. We now adjust a list $L_s(z_e)$ as follows. Let $e = uv$. If $L(u) = L(v)$ or $L(u)$ has size 1, then we set $L_s(z_{uv}) = \{1, 2, 3\} \setminus L(u)$. If $L(v)$ has size 1, then we set $L_s(z_{uv}) = \{1, 2, 3\} \setminus L(v)$. If $z_{u'v'}$ is adjacent to a vertex z_{uv} with $|L'(z_{uv})| = 1$, then we set $L_s(z_{u'v'}) = \{1, 2, 3\} \setminus L'(z_{uv})$. We apply the rules exhaustively. We call the resulting list assignment L_s of G_s the *edge-extension* of L. We say that an edge uv of G is *unsuitable* if $|L(u)| = |L(v)| = 2$ but $L(u) \neq L(v)$, whereas uv is *list-reducing* if $|L(u)| = |L(v)| = 1$ and $L(u) \neq L(v)$. Note that in G_s, we may have $|L_s(z_e)| = 3$ if e is unsuitable, whereas $|L_s(z_e)| = 1$ if e is list-reducing. We say that an end-vertex u of an unsuitable edge e is a *fixer* for e if u is adjacent to an end-vertex of a list-reducing edge $u'v'$ (note that $\{u, v\} \cap \{u', v'\} = \emptyset$). We make the following observation.

Lemma 3. *Let G be a graph on m edges with a 2-list assignment L. Then we can construct in $O(m^2)$ time the edge-extension G_s of G and the edge-extension L_s of L. Moreover, G has a star 3-colouring that respects L if and only if G_s has a 3-colouring that respects L_s. Furthermore, L_s is a 2-list assignment of G_s if every unsuitable edge uv of G has a fixer.*

Let $d_G(u)$ be the degree of a vertex u in G. We omit the proofs of two lemmas.

Lemma 4. *Let G be a graph of diameter at most 3. If G has a star 3-colouring, then*

1. *for every 4-cycle $v_0v_1v_2v_3v_0$ of G, $d_G(v_0) = d_G(v_2) = 2$ or $d_G(v_1) = d_G(v_3) = 2$, and*
2. *there is no 5-cycle in G.*

Lemma 5. *Let G be a graph of diameter at most 3 that has two vertices u and v with at least three common neighbours. Let $w \in N(u) \cap N(v)$. Then G has a star 3-colouring if and only if $G - w$ has a star 3-colouring. Moreover, $G - w$ has diameter at most 3 as well.*

Two non-adjacent vertices in a graph G that have the same neighbourhood are *false twins* of G. We are now ready to give our algorithm.

Lemma 6. STAR 3-COLOURING *is polynomial-time solvable for graphs of diameter at most* 3.

Proof. Let G be a graph of diameter 3. We may assume without loss of generality that G is connected. We first determine in $O(nm^2)$ time all 4-cycles and all 5-cycles in G. If G has a 4-cycle with two adjacent vertices of degree at least 3 in G or if G has a 5-cycle, then G is not star 3-colourable by Lemma 4. We continue by assuming that G satisfies the two properties of Lemma 4. We reduce G by applying Lemma 5 exhaustively. Let G' be the resulting graph, which has diameter at most 3 (by Lemma 5). We can determine in $O(n)$ time all vertices of degree 2 in G. For each vertex of degree 2 we can compute in $O(n)$ time all its false twins. Hence, we found G' in $O(n^2)$ time. As we only removed vertices, G' also satisfies the two properties of Lemma 4.

If G' has maximum degree at most 4, then $|V(G')| \leq 53$, as G' has diameter at most 3. We check in constant time if $|V(G')| \leq 53$ and if so, whether G' has a star 3-colouring. Otherwise, we found a vertex v of degree at least 5 in G'.

Let N_i be the set of vertices of distance i from v. Then, $N_1 = N(v)$ and as G' has diameter at most 3, $V(G') = \{v\} \cup N_1 \cup N_2 \cup N_3$. We assume without loss of generality that if G' has a star 3-colouring c, then $c(v) = 1$. We will examine the following situations: c gives each vertex in N_1 colour 3; or c gives at least one vertex of N_1 colour 2 and at least three vertices of N_1 colour 3. As v has degree at least 5, at least one of colours 2, 3 must occur three times on $N(v)$, and we may assume without loss of generality that this colour is 3. Hence, G' has a star 3-colouring if and only if one of these two cases holds.

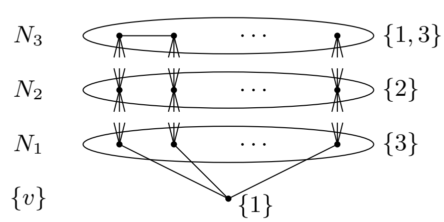

Fig. 1. The pair (G', L') in Case 1.

Case 1. Check if G' has a star 3-colouring that gives every vertex of N_1 colour 3. As $|N_1| \geq 5$, such a star 3-colouring c must assign each vertex of N_2 colour 2. This means that every vertex of N_3 gets colour 1 or 3. Hence, we obtained, in $O(n)$ time, a 2-list assignment L' of G'. We construct the pair (G'_s, L'_s). By Lemma 3 this take $O(m^2)$ time. As every list either has size 1 or is equal to $\{1, 3\}$, we find that the edge-extension L'_s of L' is a 2-list assignment of G'_s. By Lemma 3, it remains to solve 2-LIST-COLOURING on (G'_s, L'_s). We can do this in $O(m^2)$ time using Theorem 4 as the size of G'_s is $O(m^2)$. Hence, the total running time for dealing with Case 1 is $O(m^2)$. See also Figure 1.

Case 2. Check if G' has a star 3-colouring that gives at least one vertex of N_1 colour 2 and at least three vertices of N_1 colour 3.

We set $L'(v) = \{1\}$. This gives us the property: **P0.** $N_0 = \{v\}$ and $L'(v) = \{1\}$.

We now select four arbitrary vertices of $N(v)$. We consider all possible colourings of these four vertices with colours 2 and 3, where we assume without loss of generality that colour 3 is used on these four vertices at least as many times as colour 2. For the case where colour 2 is not used we consider each of the $O(n)$ options of colouring another vertex from $N(v)$ with colour 2. For the cases where colour 3 is used exactly twice, we consider each of the $O(n)$ options of colouring another vertex from $N(v)$ with colour 3. Hence, the total number of options is $O(n)$, and in each option we have a neighbour x of v with colour 2 and a set $W = \{w_1, w_2, w_3\}$ of three distinct neighbours of v with colour 3. That is, we set $L'(x) = \{2\}$ and $L'(w_i) = \{3\}$ for $1 \le i \le 3$.

For each set $\{x\} \cup W$ we do as follows. We first check if W is independent; otherwise we discard the option. If W is independent, then initially we set $L'(u) = \{1, 2, 3\}$ for each $u \notin \{x, v\} \cup W$. We now show that we can reduce the list of every such vertex u by at least 1. As an *implicit step*, we will discard the instance (G', L') if one of the lists has become empty. In doing this we will use the following *Propagation Rule*:

Whenever a vertex has only one colour in its list, we remove that colour from the list of each of its neighbours.

By the Propagation Rule, we obtain the following property, in which we updated the set W:

P1. N_1 can be partitioned into sets W, X, Y with $|W| \ge 3$, $|X| \ge 1$ and $|Y| \ge 0$, such that no vertex of Y is adjacent to any vertex of $X \cup W$, and moreover, X is an independent set with $x \in X$ and W is an independent set with $\{w_1, w_2, w_3\} \subseteq W$, such that
- every vertex $w \in W$ has list $L'(w) = \{3\}$,
- every vertex $x \in X$ has list $L'(x) = \{2\}$, and
- every vertex $y \in Y$ has list $L'(y) = \{2, 3\}$.

Note that by the Propagation Rule, we removed colour 3 from the list of every neighbour of a vertex of W in N_2. We now also remove colour 1 from the list of every neighbour of a vertex of W in N_2; the reason for this is that if a neighbour y of, say, w_1 is coloured 1, then the vertices y, w_1, v, w_2 form a bichromatic P_4. Hence, any neighbour of every vertex in W in N_2 has list $\{2\}$.

Now consider a vertex $z \in N_2$ that still has a list of size 3. Then z is not adjacent to any vertex in N_1 with a singleton list (as otherwise we applied the Propagation Rule), but by definition z still has a neighbour z' in N_1. This means that $z' \in Y$ and thus z' has list $\{2, 3\}$. Hence, z cannot be coloured 1: if z' gets colour 2, the vertices x, v, z', z will form a bichromatic P_4, and if z' gets colour 3, the vertices w_1, v, z', z will form a bichromatic P_4. Hence, we may remove colour 1 from $L'(z)$, so $L'(z)$ will have size at most 2.

We make some more observations. First, we recall that every neighbour of a vertex in W in N_2 has list $\{2\}$, and every vertex in X has list $\{2\}$ as well. Hence, no vertex in N_2 has both a neighbour in W and a neighbour in X; otherwise this vertex would have an empty list by the Propagation Rule and we would have discarded this option.

Due to the above, we can partition N_2 into sets W^*, X^*, and Y^* such that the vertices of W^* are the neighbours of W and the vertices of X^* are the neighbours of X, whereas $Y^* = N_2 \setminus (X^* \cup W^*)$. Consequently, the neighbours in N_1 of every vertex of Y^* belong to Y.

Recall that G' has no 5-cycles. Hence, there is no edge between vertices from two different sets of $\{W^*, X^*, Y^*\}$. Furthermore, every vertex $w^* \in W^*$ has list $L'(w^*) = \{2\}$, every vertex $x^* \in X^*$ has list $L'(x^*) = \{1, 3\}$, and every vertex $y^* \in Y^*$ has list $L'(y^*) = \{2, 3\}$. If a vertex $y \in Y$ has a neighbour $w^* \in W^*$, then vww^*yv is a 4-cycle where $w \in W$ is a neighbour of w^*. Recall that G' satisfies the properties of Lemma 4. As v has degree at least 5 in G', this means that y has degree 2 in G'. Hence, v and w^* are the only neighbours of y. In particular, we find that every vertex in Y with a neighbour in W^* has no neighbour in $X^* \cup Y^*$.

We now apply the Propagation Rule again. As a consequence, we update the lists of the vertices in $Y \cup N_3$, the sets Y and W in **P1**. The latter is because some vertices might have moved from Y to W; in particular it now holds that no vertex in W^* is adjacent to any vertex in Y.

We summarize the above in the following property:

P2. N_2 can be partitioned into sets W^*, X^* and Y^*, such that

- every vertex $w^* \in W^*$ has list $L'(w^*) = \{2\}$ and all its neighbours in N_1 belong to W,
- every vertex $x^* \in X^*$ has list $L'(x^*) \subseteq \{1, 3\}$ and at least one of its neighbours in N_1 belong to X and none of them belong to W,
- every vertex $y^* \in Y^*$ has list $L'(y^*) \subseteq \{2, 3\}$ and all its neighbours in N_1 belong to Y, and
- there is no edge between vertices from two different sets of $\{W^*, X^*, Y^*\}$.

We now consider N_3. We let T_1 be the set consisting of all vertices in N_3 that have at least two neighbours in W^*. We let T_2 be the set consisting of all vertices in N_3 that have exactly one neighbour in W^*. Moreover, we let S_1 be the set of vertices of $N_3 \setminus (T_1 \cup T_2)$ that have at least one neighbour in T_1. We let S_2 be the set of vertices of $N_3 \setminus (T_1 \cup T_2)$ that have no neighbours in T_1 but at least two neighbours in T_2. If for a vertex $s \in N_3$, there is a vertex $w \in W$ and a 4-path from s to w whose internal vertices are in X and X^*, then we let $s \in R$.

We note that the sets S_1, S_2, T_1 and T_2 are pairwise disjoint by definition, whereas the set R may intersect with $S_1 \cup S_2 \cup T_1 \cup T_2$. We now show that $N_3 = R \cup S_1 \cup S_2 \cup T_1 \cup T_2$. For contradiction, assume that s is a vertex of N_3 that does not belong to any of the five sets R, S_1, S_2, T_1, T_2. As $s \notin T_1 \cup T_2$, we find that the distance from s to every vertex of W is at least 3. Then, as G' has diameter 3, there exists a 4-path P_i from s to each $w_i \in W$ (by **P1** we can write

$W^* = \{w_1, \ldots, w_a\}$ for some $a \geq 3$). Every P_i must be of one of the following forms: $s - N_2 - N_1 - w_i$ or $s - N_2 - N_2 - w_i$ or $s - N_3 - N_2 - w_i$.

First assume there is some P_i that is of the form $s - N_2 - N_1 - w_i$, that is, $P_i = szz'w_i$ for some $z \in N_2$ and $z' \in N_1$. As z' is a neighbour of both w_i and v, we find that $z' \in X$ and $z' \in X^\star$, and consequently, $s \in R$, a contradiction.

Now assume that there exists some P_i that is of the form $s - N_2 - N_2 - w_i$, that is, $P_i = szz'w_i$ for some z and z' in N_2. By definition, z must have a neighbour in N_1. As G' has no 5-cycle, this is only possible if z is adjacent to w_i. However, now s is no longer of distance 3 from w_i in G', a contradiction.

Finally, assume that no path from s to any w_i is of one of the two forms above. Hence, every P_i is of the form $s - N_3 - N_2 - w_i$. We write $P_i = st_iw_i^*w_i$ where $t_i \in T_1 \cup T_2$ and $w_i^* \in W^*$. We consider the paths P_1, P_2, P_3, which exist as $|W| \geq 3$. As $s \notin S_1$, we find that $t_i \notin T_1$. Moreover, as $s \notin S_2$, we find that $t_1 = t_2 = t_3$, and so $w_1^* = w_2^* = w_3^*$. In particular, the latter implies that w_1^* is adjacent to w_1, w_2 and w_3 and thus has degree at least 3. Recall that G' satisfies Property 1 of Lemma 4. As w_1^* and v each have degree at least 3 in G', this means that each w_i must only be adjacent to v and w_1^*. However, then w_1, w_2 and w_3 are three false twins of degree 2 in G', and by construction of G' we would have removed one of them, a contradiction. We conclude that $N_3 = R \cup S_1 \cup S_2 \cup T_1 \cup T_2$.

We now reduce the lists of the vertices in N_3. Let $s \in N_3$. If $s \in T_1 \cup T_2$ (that is, s is adjacent to a vertex $w^* \in W^*$) then, as $L'(w^*) = \{2\}$, we find that $L'(s) \subseteq \{1, 3\}$. If $s \in T_1$, then we can reduce the list of s as follows. By the definition of T_1, s is adjacent to a second vertex $w' \neq w^*$ in W^*. By **P2**, we find that w' has a neighbour $w \in W$. We find that $L'(w^*) = L(w') = \{2\}$ and $L(w) = \{3\}$. Then s cannot be assigned colour 3, as otherwise w^*, s, w', w would form a bichromatic P_4. Hence, we can reduce the list of s from $\{1, 3\}$ to $\{1\}$.

Now suppose that $s \in S_1$. Then, by the definitions of the sets S_1 and T_1 and **P2**, there exists a path $P = stw^*w$ where $t \in T_1$, $w^* \in W^*$ and $w \in W$. We deduced above that t has list $L'(t) = \{1\}$. Consequently, we can delete colour 1 from the list of s by the Propagation Rule, so $L'(s) \subseteq \{2, 3\}$. Now suppose that $s \in S_2$. Then, by the definition of S_2 and **P2**, there exist two paths $P_1 = st_1w_1^*w_1$ and $P_2 = st_2w_2^*w_2$ where $t_1, t_2 \in T_2$, $w_1^*, w_2^* \in W^*$, $w_1, w_2 \in W$, and $t_1 \neq t_2$. We claim that s cannot be assigned colour 2. For contradiction, suppose that s has colour 2. Then t_1, which has list $\{1, 3\}$, must receive colour 1, as otherwise t_1 will have colour 3 and s, t_1, w_1^*, w_1 is a bichromatic P_4 (recall that w_1^* and w_1 can only be coloured with colours 2 and 3, respectively). For the same reason, t_2 must get colour 1 as well. However, now w_1^*, t_1, s, t_2 is a bichromatic P_4, a contradiction. Hence, we can remove colour 2 from $L'(s)$. Afterwards, $L'(s) \subseteq \{1, 3\}$.

Finally, suppose that $s \in R$. By the definition of R, there is some path $P_i = sx^*x'w$ where $x^* \in X^*$, $x' \in X$, and $w \in W$. By **P1** and **P2**, respectively, it holds that $L'(x') = \{2\}$ and $L'(x^*) \subseteq \{1, 3\}$. Hence, s cannot be coloured 2: if x^* gets colour 1, the vertices v, x', x^*, s will form a bichromatic P_4, and if x^* gets colour 3, the vertices w_1, x', x^*, s will form a bichromatic P_4. In other words, we may remove colour 2 from $L'(s)$, so $L'(s) \subseteq \{1, 3\}$.

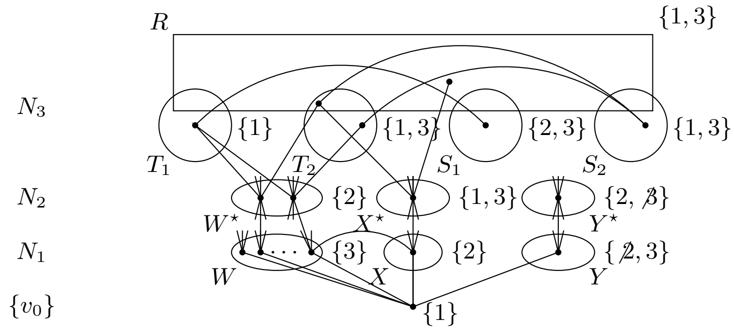

Fig. 2. An example of a pair (G', L') in Case 2a. The colours crossed out show the difference between the general situation in Case 2 and what we show holds in Case 2a.

As $N_3 = R \cup S_1 \cup S_2 \cup T_1 \cup T_2$, we obtained the following property:

P3. N_3 only consists of vertices whose lists are a subset of $\{1,3\}$ or $\{2,3\}$, and N_3 can be split into sets R, S_1, S_2, T_1, T_2, such that S_1, S_2, T_1 and T_2 are pairwise disjoint, and
- every vertex $r \in R$ has list $L'(r) \subseteq \{1,3\}$ and there is a 4-path from r to a vertex in W that has its two internal vertices in X^* and X, respectively,
- every vertex $t \in T_1$ has list $L'(t) = \{1\}$ and has at least two neighbours in W^*,
- every vertex $t \in T_2$ has list $L'(t) \subseteq \{1,3\}$ and has exactly one neighbour in W^*,
- every vertex $s \in S_1$ has list $L'(s) \subseteq \{2,3\}$, has no neighbours in W^* but is adjacent to at least one vertex in T_1, and
- every vertex $s \in S_2$ has list $L'(s) \subseteq \{1,3\}$ and has no neighbours in $T_1 \cup W^*$ but at least two neighbours in T_2.

Hence, we constructed a set $\mathcal{L}'$ of 2-list assignments of G', such that $\mathcal{L}'$ is of size $O(n)$ and G' has a star 3-colouring if and only if G' has a star 3-colouring that respects L' for some $L' \in \mathcal{L}'$. Moreover, we can find each $L' \in \mathcal{L}$ in $O(m+n)$ time by a bread-first search for detecting the 4-paths. For each $L' \in \mathcal{L}$, we do as follows. We still need to construct the edge-extension G'_s of G'. However, the edge-extension L'_s of L' might not be a 2-list assignment. The reason is that G' may have an edge ss' for some vertex $s \in N_2$ with $L'(s) = \{2,3\}$ and some vertex $s' \in N_3$ with $L'(s') = \{1,3\}$ such that $L'_s(z_{ss'}) = \{1,2,3\}$. We distinguish between two cases; see also Figure 2 and Figure 3.

Case 2a. Check if G' has a star 3-colouring that gives x colour 2 and every other vertex of N_1 colour 3.

We only consider this case if $|X| = 1$. We give every vertex in Y list $\{3\}$. Then, by the Propagation Rule, we can delete colour 3 from every list of a vertex in Y^*. We construct G'_s and L'_s in $O(m^2)$ time by Lemma 3. Then L'_s is a 2-list

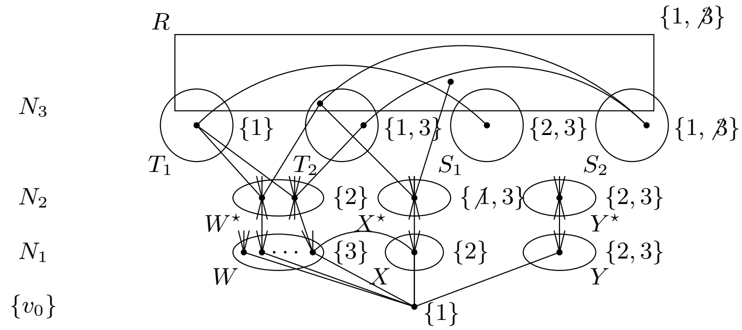

Fig. 3. An example of a pair (G', L') in Case 2b. The colours crossed out show the difference between the general situation in Case 2 and what we show holds in Case 2b.

assignment of G'_s. This can be seen as follows. Let $e = ss'$ be an unsuitable edge of G'. As G' has no vertices with list $\{1, 2\}$, we find that $L'(s) = \{2, 3\}$ and $L'(s') = \{1, 3\}$. Then s must be in S_1. By definition, it follows that there exist vertices $t \in T_1$ and $w^* \in W^*$ such that st and tw^* are edges of G'. As $L'(t) = \{1\}$ and $L'(w^*) = \{2\}$, the edge tw^* is list-reducing. Hence, s is a fixer for the edge ss'. The claim now follows from Lemma 3, and by the same lemma, it remains to check if G'_s has a 3-colouring that respects L'_s. We can do the latter in $O(m^2)$ time by Theorem 4.

Case 2b. Check if G' has a star 3-colouring that gives at least one other vertex of N_1, besides x, colour 2.

If $|X| \geq 2$, then we found a vertex of $N_1 \setminus \{x\}$ that gets colour 2. If $X = \{x\}$, we will not try to find this vertex; for our algorithm its existence will suffice. By **P2**, every $x^* \in X^*$ has list $L(x^*) \subseteq \{1, 3\}$ and a neighbour $x' \in X$ with $L'(x') = \{2\}$. By the Case 2b assumption, there is at least one other vertex x'' in N_1 that gets colour 2. Then x^* cannot be coloured 1, as otherwise x'', v, x', x^* would form a bichromatic P_4. Hence, we remove colour 1 from the list of every vertex of X^* so that afterwards $L(x^*) = \{3\}$ for every $x^* \in X^*$. We remove colour 3 from the list of every neighbour of a vertex of X^*. As L' is a 2-list assignment that does not assign any vertex of G' the list $\{1, 2\}$, afterwards every neighbour of every vertex of X^* in N_3 has list $\{1\}$ or $\{2\}$. Moreover, X^* is an independent set (as otherwise we discard (G', L')). No vertex of $W^* \cup Y^*$ is adjacent to any vertex in X^* (by **P2**). Hence, every vertex in X^* has no neighbours in N_2.

We now prove that no vertex in S_2 can receive colour 3. For contradiction, assume that c is a star 3-colouring of G that respects L' and that assigns a vertex $s \in S_2$ colour $c(s) = 3$. As G' has diameter 3, there is a path P from s to $x \in X$ of length at most 3. Then P is of the form $s - N_2 - x$ or $s - N_3 - N_2 - x$ or $s - N_2 - N_2 - x$ or $s - N_2 - N_1 - x$. If P is of the form $s - N_2 - x$, then s has a neighbour in X^*, which has list $\{3\}$. Hence, as s received colour 3, this is not possible. We show that the other three cases are not possible either.

First suppose that P is of the form $s - N_3 - N_2 - x$, say $P = szx^*x$ for some $z \in N_3$ and $x^* \in N_2$. As no vertex of $W^* \cup Y^*$ is adjacent to any vertex in X, we find that $x^* \in X^*$. This means that z must receive colour 1, as otherwise the vertices x, x^*, z, s would form a bichromatic P_4. As $s \in S_2$, we find that s has two neighbours t_1 and t_2 in T_2. Both t_1 and t_2 have list $\{1, 3\}$, so they must receive colour 1. At least one of them, say t_1, is not equal to z. However, now x^*, z, s, t_1 form a bichromatic P_4, a contradiction. Hence, this case cannot happen.

Now suppose that P is of the form $s - N_2 - N_2 - x$, say $P = szx^*x$ for some $z, x^* \in N_2$. As no vertex of $W^* \cup Y^*$ is adjacent to any vertex in X, $x^* \in X^*$. However, no vertex in X^* has a neighbour in N_2. Hence, this case cannot happen.

Finally, suppose that P is of the form $s - N_2 - N_1 - x$, say $P = sw^*wx$ for some $w^* \in N_2$ and $w \in N_1$. As X is independent and no vertex of Y is adjacent to a vertex of X, we find that $w \in W$ and thus $w^* \in W^*$. However, this is not possible, as $s \in S_2$ is not adjacent to any vertex in W^* by definition. Hence, this case cannot happen either, so we have proven the claim. So, we can remove colour 3 from the list of every vertex $s \in S_2$. Hence, $L'(s) = \{1\}$ for every $s \in S_2$.

We construct G'_s and L'_s in $O(m^2)$ time by Lemma 3. We claim that L'_s is a 2-list assignment of G'_s. This can be seen as follows. Let $e = ab$ be an unsuitable edge of G'. As G' has no vertices with list $\{1, 2\}$, we may assume that $L'(a) = \{1, 3\}$ and $L'(b) = \{2, 3\}$. As every vertex in R is adjacent to a vertex in X^* with list $\{3\}$, no vertex in R has list $\{1, 3\}$. We just deduced that no vertex in S_2 has list $\{1, 3\}$ either. Hence, the only vertices with list $\{1, 3\}$ belong to T_2, so $a \in T_2$. Then, by definition, we find that a has a neighbour $w \in W^*$, which has a neighbour $w \in W$. As w^* has list $\{2\}$ and w has list $\{3\}$, the edge w^*w is list-reducing. Hence, a is a fixer for the edge ab. The claim now follows from Lemma 3, and by the same lemma, it remains to check if G'_s has a 3-colouring that respects L'_s. We can do the latter in $O(m^2)$ time by Theorem 4.

This concludes the description of our algorithm. The correctness of our algorithm follows from the correctness of the branching steps. Its running time is $O(nm^2)$, as there are $O(n)$ branches, and we deal with each branch in $O(m^2)$ time. □

We also need an observation on a known construction [1] (proof omitted).

Lemma 7. Star 3-Colouring *is* NP*-complete on graphs of diameter at most* 8.

The Proof of Theorem 2. The first statement follows from Lemmas 6 and 7. For the second statement, the case $d = 1$ is trivial, and for the case $d \geq 2$, $k \geq 4$ we reduce from Star 3-Colouring: to an instance G of Star k-Colouring, we add a clique of $k - 3$ vertices, which we make adjacent to every vertex of G.

4 L(1, 2)-Labelling for Graphs of Diameter 2

We show that an n-graph G of diameter 2 has an $L(1, 2)$-n-labelling if and only if G has a Hamiltonian path, no edge of which is contained in a triangle, and that the latter problem is NP-complete (proofs omitted). This yields:

Theorem 5. $L(1, 2)$-Labelling *is* NP*-complete for graphs of diameter at most* 2.

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