# A SEMI-STABLE CASE OF THE SHAFAREVICH CONJECTURE

VICTOR ABRASHKIN

Abstract. Suppose $K = W(k)[1/p]$, where $W(k)$ is the ring of Witt vectors with coefficients in algebraically closed field $k$ of characteristic $p \neq 2$. We discuss an explicit construction of $p$-adic semi-stable representations of the absolute Galois group of $K$ with Hodge-Tate weights from $[0, p)$. This theory is applied to projective algebraic varieties over $\mathbb{Q}$ with good reduction outside 3 and semi-stable reduction modulo 3.

## Introduction

In this expository paper we discuss the following result in the spirit of the Shafarevich conjecture about non-existence of non-trivial abelian schemes over $\mathbb{Z}$.

**Theorem 0.1.** *If $Y$ is a projective algebraic variety over $\mathbb{Q}$ with good reduction outside 3 and semi-stable reduction modulo 3 then $h^2(Y_{\mathbb{C}}) = h^{1,1}(Y_{\mathbb{C}})$.*

In particular, above Theorem implies that there are no such (non-trivial) abelian varieties $Y$ (first proved in [13, 27]). Our result also eliminates a great deal of other varieties, e.g. all K3-surfaces.

The proof of Theorem 0.1 is given in [11] and is based on a:
— study of torsion subquotients of the Galois module $H^2_{et}(Y_{\bar{Q}}, \mathbb{Q}_3)$;
— modification of Breuil's torsion theory of semi-stable $p$-adic represenations with HT (Hodge-Tate) weights from $[0, p-1]$ over $W(k)$, where $k$ is algebraically closed field of characteristic $p$;
— formalism of pre-abelian categories (short exact sequences, 6-terms Hom$-$Ext exact sequences, $p$-divisible group objects, devissage);
— study of the group of fundamental units in $\mathbb{Q}(\sqrt[3]{3}, e^{2\pi i/9})$ (via the computing package SAGE).

The strategy of the proof is very close to the strategy used in the following "crystalline case" of the Shafarevich conjecture [23, 7].

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**Theorem 0.2.** *Suppose $X$ is a projective algebraic variety over $\mathbb{Q}$ with everywhere good reduction. Then*

*a)* $h^1(X_{\mathbb{C}})=0$, $h^2(X_{\mathbb{C}})=h^{1,1}(X_{\mathbb{C}})$ *and* $h^3(X_{\mathbb{C}})=0$*;*

*b)* $h^4(X_{\mathbb{C}})=h^{2,2}(X_{\mathbb{C}})$ *under Generalized Riemann Hypothesis (GRH).*

Part a) of this Theorem was obtained in [7] by studying the finite subquotients of the Galois modules $H^i_{et}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_5)$ with $1\leqslant i\leqslant 3$. These Galois modules are unramified outside 5 and their local behaviour at 5 is described by the Fontaine-Laffaille theory [19] of $p$-adic torsion crystalline representations with HT weights from $[0,p-2]$. The approach in [7] is essentially similar to the approach from [23] but Fontaine considers etale cohomology with coefficients in $\mathbb{Q}_7$. (Of course, these results would be not possible without great achievements of Fontaine's theory of $p$-adic periods.)

Part b) was proved by the author in [7]. The proof requires the study of the Galois module $H^4_{et}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_5)$, where the tools of the Fontaine-Laffaille theory are not sufficient. For this reason, we developed in [6] a modification of the Fontaine-Laffaille theory for crystalline representations with HT weights from $[0,p-1]$. Note that our modification of Breuil's theory works also in the context of crystalline representations and can be applied to reprove part b) of Theorem 0.2 (and similar results for varieties over $\mathbb{Q}(i),\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$ from [7]). The appropriate comments will be given in due course below.

The constructions in [11] are very technical and we just sketch and discuss their basic steps. Most of them can be illustrated by earlier results related to the Shafarevich Conjecture, cf. Subsection 1.

In Subsections 2–4 we work with a local field $K=\operatorname{Frac}W(k)$, where $W(k)$ is the ring of Witt vectors with coefficients in algebracally closed field $k$ of characteristic $p$, $p>2$. Let $\bar{K}$ be an algebraic closure of $K$ and $\Gamma_K=\operatorname{Gal}(\bar{K}/K)$. In Subsection 2 we outline the construction of the functor $\mathcal{V}^*$ from an appropriate category of filtered modules to the category of $\mathbb{F}_p[\Gamma_K]$-modules. This construction is based on the introduction of a modulo $p$ "truncated" version of Fontaine's ring of $p$-adic semi-stable periods. We associate to $\mathcal{V}^*$ the functor $\mathcal{CV}^*$ with values in the category of co-filtered $\mathbb{F}_p[\Gamma_K]$-modules and prove that this functor is fully faithful. In Subsection 3 we obtain the ramification estimates for the Galois modules $H$ from the image of $\mathcal{V}^*$: if $v>2-1/p$ then the higher ramification subgroups $\Gamma_K^{(v)}$ act trivially on $H$. We also obtain the ramification estimate for the Galois modules which are associated with the modulo $p$ subquotients of crystalline representations with HT weights from $[0,p)$ and prove that both estimates are sharp. The methods we use here are close to the methods from [8, 9, 10]; one can use also our constructions to show that the estimates from [24] are sharp if $e=n=1$. In Subsection 4 we explain the construction of our modification of Breuil's functor $\mathcal{V}^{ft}$. In fact, it is very close to the construction of the modification of the Fontaine-Laffaille functor from [6] but it can be developed in a simpler way due to advantages of Breuil's theory. One of main features of this construction is that on the level of modulo $p$ subquotients, $\mathcal{V}^{ft}$ essentially coincides with the functor $\mathcal{V}^*$ from Subsection 2. This gives the ramification estimates for modulo $p$ subquotients of semi-stable and crystalline representations with HT weights from $[0, p)$. Finally, in Subsection 5 we outline the proofs of Theorems 0.1 and 0.2.

## 1. The Shafarevich conjecture

**Conjecture** (I. R. Shafarevich, 1962). *There are no projective algebraic curves over $\mathbb{Q}$ of genus $g \geqslant 1$ with everywhere good reduction,* [29].

The case $g = 1$ was considered by Shafarevich himself. He has just listed explicitly 22 elliptic curves over $\mathbb{Q}$ with good reduction outside 2 and verified that all these curves have bad reduction at 2. Later his PhD student (Volynsky) studied the case of curves of genus 2. This approach resulted in enormous calculations and was not published. In both cases the approach was based on the study of canonical equations for these curves. It became clear later that one should study the problem in a more general setting.

**Conjecture.** *There are no abelian varieties $A$ over $\mathbb{Q}$ of dimension $g \geq 1$ with everywhere good reduction.*

This statement is easier to approach. The existence of such abelian variety would have provided examples of non-trivial $p$-divisible groups over $\mathbb{Z}$ (for all prime numbers $p$). The question about the existence of such $p$-divisible groups was asked by J.Tate in [31]. On this way the conjecture was proved in [21, 3] in 1985. Main features of used methods will be described below.

### 1.1. **Small values of $g$.**

In [1, 2] it was proved that any 2-divisible group over $\mathbb{Z}$ of height $h \leqslant 6$ is isogeneous to the trivial 2-divisible group. This gave the cases $g = 2$ and $g = 3$ of the Shafarevich conjecture. The method can be explained as follows.

Suppose $G$ is a f.f.g.s. (finite flat group scheme) over $\mathbb{Z}$ such that $2\mathrm{id}_G = 0$. Then

a) if the order $|G| = 2$ then $G$ is either etale $(\mathbb{Z}/2)_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z} \oplus \mathbb{Z})$ or multiplicative $\mu_2 = \mathrm{Spec}\,\mathbb{Z}[x]/(x^2 - 1)$ f.f.g.s. over $\mathbb{Z}$, [31];

b) if $|G| = 4$ and $G = \mathrm{Spec}\,A(G)$ is not a product of f.f.g.s. of order 2 then there is a short exact sequence of f.f.g.s.

$$0 \longrightarrow \mu_2 \longrightarrow G \longrightarrow (\mathbb{Z}/2)_{\mathbb{Z}} \longrightarrow 0$$

and $A(G) = A(\mu_2) \oplus \mathbb{Z}[i]$, [1]. In particular, $A(G)_{\mathbb{Q}} \neq \mathbb{Q} \oplus K$, where $[K : \mathbb{Q}] = 3$. (Use that $A(G)_{\mathbb{Q}}$ is etale over $\mathbb{Q}$ and there are no cube field extensions $K/\mathbb{Q}$ unramified outside 2.)

c) there are similar short exact sequences for f.f.g.s. $G$ over $\mathbb{Z}$ of order $2^n$ with $n = 3, 4, 5, 6$,

$$0 \longrightarrow \mu_2^a \longrightarrow G \longrightarrow (\mathbb{Z}/2)_{\mathbb{Z}}^b \longrightarrow 0,$$

where $a + b = n$, [2]. This statement is highly non-trivial because the Galois group of the field-of-definition $\mathbb{Q}(G)$ of $\bar{\mathbb{Q}}$-points of f.f.g.s. of order $2^n$ is not generally soluble if $n \geqslant 4$. On the one hand, we used the Tate formula for the discriminant of $A(G)$ from [31], $v_2(D(A(G)) = d2^n$, where $d = \dim(G \otimes \mathbb{F}_2)$ (it implies that $v_2(D(A(G))) \leqslant 192$ because we can assume that $d \leqslant 3$ by switching, if necessary, from $G$ to its Cartier dual). On the other hand, we used the Odlyzko lower bounds for the minimal discriminants of algebraic number fields, cf. [30, 18, 25];

d) in the special pre-abelian category of f.f.g.s. $G$ over $\mathbb{Z}$ such that $2\mathrm{id}_G = 0$, one has

$$\mathrm{Ext}(\mu_2, (\mathbb{Z}/2)_{\mathbb{Z}}) = \mathrm{Ext}((\mathbb{Z}/2)_{\mathbb{Z}}, (\mathbb{Z}/2)_{\mathbb{Z}}) = \mathrm{Ext}(\mu_2, \mu_2) = 0.$$

Therefore, the above exact sequences for $G$ and devissage in the pre-abelian category of finite flat 2-group schemes over $\mathbb{Z}$ give the following exact sequence of 2-divisible groups over $\mathbb{Z}$

$$0 \longrightarrow \{\mu_{2^n}\}_{n \geq 1}^a \longrightarrow \mathcal{G} \longrightarrow (\mathbb{Q}_2/\mathbb{Z}_2)^b \longrightarrow 0, \tag{1.1}$$

where $\mathcal{G}$ is of height $a + b \leqslant 6$ (for more details about devissage in pre-abelian categories cf. Appendix, especially Theorem A.1);

e) such 2-divisible group $\mathcal{G}$ never comes from a non-trivial abelian scheme $A$ over $\mathbb{Z}$. Otherwise, looking at dimensions we obtain $b \neq 0$, but the exact sequence of 2-divisible groups from d) splits over $\mathbb{F}_2$ and, therefore, $A$ has infinitely many $\mathbb{F}_2$-points. The contradiction.

The above method does not work in higher dimensions.

Indeed, suppose $A$ is an abelian scheme over $\mathbb{Z}$ and $G = \mathrm{Ker}(2\mathrm{id}_A)$ is a group scheme of points of order $\leqslant 2$ on $A$. Then $|G| = 2^{2g}$, $\dim(G \otimes \mathbb{F}_2) = g$ and Tate's formula gives $v_2(D(A(G))^{1/2g}) = g$. Note that $A(G) \otimes \mathbb{Q}$ is the product of algebraic number fields (because $G \otimes \mathbb{Q}$ is etale) and these fields are unramified outside 2 (because $G \otimes \mathbb{Z}_l$ is etale if $l \neq 2$). Therefore, the normalized discriminant of $A(G)$ equals $2^g$ and tends to infinity if $g \to \infty$.

On the other hand, if $\mathbb{Q}(G)$ is the field-of-definition of $\bar{\mathbb{Q}}$-points of $G$, then $\mathrm{Gal}(\mathbb{Q}(G)/\mathbb{Q}) \subset \mathrm{SL}(2g, \mathbb{F}_2)$ is not generally soluble if $g \geqslant 2$, and the only global idea we can use in this situation is related to lower bounds of minimal discriminants of algebraic number fields. The best known bounds are the Odlyzko estimates and they are given by the tables of real numbers $\{d_N \mid N \in \mathbb{N}\}$ such that if $[K : \mathbb{Q}] = N$ then $|D(K/\mathbb{Q})|^{1/N} \geq d_N$. For large $N$, $d_N \approx d_\infty \approx 22.3$; under *GRH* there are better estimates $\{d_N^* \mid N \in \mathbb{N}\}$ in this case $d_\infty^* \approx 44.76$, [30, 18, 25].

Unfortunately, an analogue of Odlyzko estimates under additional assumption that $K/\mathbb{Q}$ is ramified only over 2, does not exist. Nonetheless, $A(G)$ is considerably smaller than its integral closure and Tate's formula can be replaced by much better upper estimate for the 2-adic valuation of the normalised discriminant of $\mathbb{Q}(G)$. The evidence for its existence is illustrated in the next Subsection.

1.2. **The Shafarevich Conjecture, the ordinary case.** Suppose our abelian variety $A$ has good ordinary reduction at 2. Then:

a) $G := \mathrm{Ker}(2\mathrm{id}_A)$ is a f.f.g.s. over $\mathbb{Z}$ of order $2^{2g}$;

b) there is a short exact sequence of f.f.g.s. over $\mathbb{Z}_2$

$$0 \longrightarrow H^{mult} \longrightarrow G \otimes_{\mathbb{Z}} \mathbb{Z}_2 \longrightarrow H^{et} \longrightarrow 0,$$

where $H^{mult}$ is multiplicative and $H^{et}$ is etale group schemes over $\mathbb{Z}_2$ of order $2^g$;

c) because $H^{et} \otimes W(\bar{\mathbb{F}}_2) = \prod_j (\mathbb{Z}/2)_{W(\bar{\mathbb{F}}_2)}$ and $H^{mult} \otimes W(\bar{\mathbb{F}}_2) = \prod_i \mu_{2,W(\bar{F}_2)}$, we have

$$G \otimes W(\bar{\mathbb{F}}_2) = \sum_{i,j} G_{ij} \in \oplus_{i,j} \mathrm{Ext}((\mathbb{Z}/2)_{W(\bar{\mathbb{F}}_2)}, \mu_{2,W(\bar{\mathbb{F}}_2)}),$$

where for all $i, j$, there are short exact sequences of f.f.g.s.

$$0 \longrightarrow \mu_{2,W(\bar{\mathbb{F}}_2)} \longrightarrow G_{ij} \longrightarrow (\mathbb{Z}/2)_{W(\bar{\mathbb{F}}_2)} \longrightarrow 0;$$

d) the field-of-definition of geometric points of $G_{ij}$ over the maximal unramified extension $\mathbb{Q}_{2,ur}$ of $\mathbb{Q}_2$, is $\mathbb{Q}_{2,ur}(\sqrt{v_{ij}})$, where all $v_{ij}$ are principal units in $\mathbb{Q}_{2,ur}$, cf. Appendix by J.Tate in [26]. Therefore, for all $v > 1$, the higher ramification subgroups $\Gamma_{\mathbb{Q}_2}^{(v)}$ of $\Gamma_{\mathbb{Q}_2} = \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2)$ act trivially on the field-of-definition $\mathbb{Q}_2(G)$ of all $\bar{\mathbb{Q}}_2$-points of $G$;

e) the triviality of $\Gamma_{\mathbb{Q}_2}^{(v)}$-action, where $v > 1$, implies the inequality $|D(\mathbb{Q}(G)/\mathbb{Q})|^{1/[\mathbb{Q}(G):\mathbb{Q}]} < 2^2$ (e.g. use Prop 9.4 of Ch. 1, [12]). But the Odlyzko estimate $d_4 < 4$ and we obtain $[\mathbb{Q}(G) : \mathbb{Q}] < 4$. Therefore, $\mathbb{Q}(G) \subset \mathbb{Q}(i)$, we can use devissage to obtain exact sequence (1.1) for $a = b = g$ and finish the proof similarly to the case of small $g$.

In the above discussion, the prime number 2 can be replaced by arbitrary prime number $p$. If $A \otimes \mathbb{F}_p$ is ordinary and $G = \mathrm{Ker}(p\,\mathrm{id}_A)$ then for $v > 1/(p-1)$, the ramification subgroups $\Gamma_{\mathbb{Q}_p}^{(v)}$ act trivially on $\mathbb{Q}_p(G)$ and using the Odlyzko estimates we can see that for $3 \leqslant p \leqslant 17$, $\mathbb{Q}(G) \subset \mathbb{Q}(\sqrt[p]{1})$. This implies that $G$ is the product of constant etale and multiplicative f.f.g.s. over $\mathbb{Z}$, the corresponding $p$-divisible group of $A$ will be just the product of several copies of trivial etale $(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbb{Z}}$

and multiplicative $\{\mu_{2^n,\mathbb{Z}}\}_{n\geqslant 1}$ $p$-divisible groups over $\mathbb{Z}$ and, therefore, such abelian variety does not exist.

The above case of the Shafarevich Conjecture was not published but gave a right direction towards the proof of the general case.

1.3. **The Shafarevich Conjecture, the general case.** In this case the same ramification estimates are proved in general situation [21, 5]: if $G$ is a finite flat group scheme over $W(k)$, where $k$ is a perfect field of characteristic $p$, $p\,\mathrm{id}_G = 0$ and $\mathrm{Frac}\, W(k) = K$ then the higher ramification subgroups $\Gamma_K^{(v)}$ act trivially on the field-of-definition of $\bar{K}$-points of $G$ for all $v > 1/(p-1)$.

Essentially, Fontaine found ramification estimates for any finite flat $p$-group schemes over the valuation ring $O_L$ of complete discrete valuation field $L \supset \mathbb{Q}_p$. His method uses the rigidity properties of $p$-divisible groups defined over valuation rings and a very elegant interpretation of Krasner's Lemma. The methods in [3, 5] are much more computational and use Fontaine's theory of f.f.g.s. over Witt vectors, [20]. In Subsection 3 we present an alternative proof of ramification estimates. It works also equally well for the subquotients of crystalline and semi-stable $p$-adic representations.

In our approach from [3, 5] we treated systematically also the case $p = 2$. Here the category of f.f.g.s. over $W(k)$ is not abelian contrary to the case $p \neq 2$, but one can still proceed with the devissage. This gave us not only the bigger list of algebraic number fields where the Shafarevich conjecture about the non-existence of abelian varieties with everywhere good reduction holds. Our main idea [4] of removing the restriction to unipotent objects in Fontaine's classification of 2-group schemes in [20] gave later a right approach to the constructions of modifications of the Fontaine-Laffaille [6] and Breuil [11] functors. These modifications allow us to obtain the ramification estimates for all modulo $p$ subquotients of representations with HT weights from $[0, p)$. They also provide us with the nulity of some groups of extensions in the category of Galois modules appeared as such subquotients. As a matter of fact, these two key ingredients resulted finally in proving Theorem 0.1 and part $b$) of Theorem 0.2.

## 2. The functor $\mathcal{CV}^*$

Let $\mathcal{W}_1 = k[[u]]$, where $u$ is an indeterminate. Denote by $\sigma$ the automorphism of $k$ induced by the $p$-th power map on $k$ and agree to use the same symbol for the continuous extension of $\sigma$ to $\mathcal{W}_1$ such that $\sigma(u) = u^p$. Denote by $N : \mathcal{W}_1 \longrightarrow \mathcal{W}_1$ the unique continuous $k$-differentiation such that $N(u) = -u$.

2.1. **Categories of filtered modules.** Introduce the following categories:

• the category $\widetilde{\underline{\mathcal{L}}}_0^*$ — its objects are $\mathcal{L} = (L, F(L), \varphi)$, where $L$ and $F(L)$ are $\mathcal{W}_1$-modules, $L \supset F(L)$ and $\varphi : F(L) \longrightarrow L$ is a $\sigma$-linear morphism of $\mathcal{W}_1$-modules; the morphisms are $\mathcal{W}_1$-linear maps of filtered modules which commute with the corresponding $\sigma$-linear maps $\varphi$;

• the category $\widetilde{\underline{\mathcal{L}}}^*$ — its objects are $\mathcal{L} = (L, F(L), \varphi, N)$, where $(L, F(L), \varphi) \in \widetilde{\underline{\mathcal{L}}}_0^*$ and $N : L \longrightarrow L/u^pL$ is such that for $w \in \mathcal{W}_1$ and $l \in L$, $N(wl) = N(w)l + wN(l)$ (we use the same notation $l$ for the image of $l$ in $L/u^pL$); the morphisms are the morphisms from $\widetilde{\underline{\mathcal{L}}}_0^*$ which commute with the corresponding differentiations $N$;

• the category $\underline{\mathcal{L}}_0^*$ is a full subcategory of $\widetilde{\underline{\mathcal{L}}}_0^*$ consisting of $\mathcal{L} = (L, F(L), \varphi)$ such that the module $L$ is free of finite rank, $u^{p-1}L \subset F(L)$ and the natural embedding $\varphi(F(L)) \subset L$ induces the identification $\varphi(F(L)) \otimes_{\sigma(\mathcal{W}_1)} \mathcal{W}_1 = L$;

• the category $\underline{\mathcal{L}}^*$ is a full subcategory of $\widetilde{\underline{\mathcal{L}}}^*$ consisting of $\mathcal{L} = (L, F(L), \varphi, N)$ such that $(L, F(L), \varphi) \in \underline{\mathcal{L}}_0^*$, for any $l \in F(L)$, one has $uN(l) \in F(L) \bmod u^pL$ and $N(\varphi(l)) = \varphi(uN(l))$ (we use the same notation $\varphi$ for the morphism $\varphi \bmod u^pL$).

The above categories are analogs of the categories of filtered modules from [14], Subsection 2.1.2, but we work with the category of $\mathcal{W}_1$-modules. (Breuil uses modules over the appropriate divided power envelope of $W(k)[[u]]$).) Note that in the context of $\mathcal{W}_1$-modules the monodromy operator $N$ can't be defined as a map with values in $L$. In [11], Subsection 1.1, we proved that $N$ can be defined as a map from $L$ to $L/u^{2p}L$ and it appears as a unique lift of its reduction $N_1 = N \bmod u^pL$. (We used the existence of such lift when proving in [11] that the category $\underline{\mathcal{L}}^*$ is pre-abelian; we also need this property when defining the functor $\mathcal{V}^*$ in Subsection 2.3 below.) In this paper we use the notation $N$ for this $(\bmod\, u^p)$-map $N_1$;

• the category $\underline{\mathcal{L}}_{cr}^*$ is a full subcategory in $\underline{\mathcal{L}}^*$ consisting of the objects $(L, F(L), \varphi, N)$ such that $N(\varphi(F(L))) = 0$.

For obvious reasons, $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}_{cr}^*$ is completely determined by $(L, F(L), \varphi) \in \underline{\mathcal{L}}_0^*$. Note that the category $\underline{\mathcal{L}}_{cr}^*$ is very closely related to the category of Fontaine-Laffaille modules, cf. [11], Subsection 1.3.

According to above definitions the objects of the categories $\underline{\mathcal{L}}_0^*$, $\underline{\mathcal{L}}^*$ and $\underline{\mathcal{L}}_{cr}^*$ are filtered free $\mathcal{W}_1$-modules with additional structures. The category of filtered free $\mathcal{W}_1$-modules is a typical example of a special pre-abelian category, i.e. it is additive category with kernels and cokernels and nicely behaving bifunctor Ext, cf. Appendix. In Subsection 1.1 of [11] we verified that $\underline{\mathcal{L}}_0^*$, $\underline{\mathcal{L}}^*$ and $\underline{\mathcal{L}}_{cr}^*$ inherit the property to be special pre-abelian.

There are the concepts of etale, unipotent, connected and multiplicative objects in our categories defined in the following way, for more details cf. Subsection 1.2 of [11].

Suppose $\mathcal{L}=(L,F(L),\varphi,N)\in\underline{\mathcal{L}}^*$.

Introduce a $\sigma$-linear map $\phi:L\longrightarrow L$ via $\phi:l\mapsto\varphi(u^{p-1}l)$. The module $\mathcal{L}$ is etale (resp., connected) if $\phi\bmod u$ is invertible (resp., nilpotent) on $L/uL$. Denote by $\underline{\mathcal{L}}^{*et}$ (resp, $\underline{\mathcal{L}}^{*c}$) the full subcategory of $\underline{\mathcal{L}}^*$ consisting of etale (resp. connected) objects. Then any $\mathcal{L}\in\underline{\mathcal{L}}^*$ contains a unique maximal etale subobject $(\mathcal{L}^{et},i^{et})$ and a unique maximal connected quotient object $(\mathcal{L}^c,j^c)$ and the sequence

$$0\longrightarrow\mathcal{L}^{et}\xrightarrow{i^{et}}\mathcal{L}\xrightarrow{j^c}\mathcal{L}^c\longrightarrow 0$$

is short exact.

Note that $\varphi(F(L))$ is a $\sigma(\mathcal{W}_1)$-module and $L=\varphi(F(L))\otimes_{\sigma(\mathcal{W}_1)}\mathcal{W}_1$. If $l\in L$ and for $0\leqslant i<p$, the elements $l^{(i)}\in F(L)$ are such that $l=\sum_{0\leqslant i<p}\varphi(l^{(i)})\otimes u^i$, set $V(l)=l^{(0)}$. Then $V\bmod u$ is a $\sigma^{-1}$-linear endomorphism of the $k$-vector space $L/uL$.

The module $\mathcal{L}$ is multiplicative (resp., unipotent) if $V\bmod u$ is invertible (resp., nilpotent) on $L/uL$. Denote by $\underline{\mathcal{L}}^{*m}$ (resp, $\underline{\mathcal{L}}^{*u}$) the full subcategory of $\underline{\mathcal{L}}^*$ consisting of multiplicative (resp. unipotent) objects. Then any $\mathcal{L}\in\underline{\mathcal{L}}^*$ contains a unique maximal multiplicative quotient object $(\mathcal{L}^m,j^m)$ and a unique maximal unipotent subobject $(\mathcal{L}^u,i^u)$ and the sequence

$$0\longrightarrow\mathcal{L}^{u}\xrightarrow{i^{u}}\mathcal{L}\xrightarrow{j^m}\mathcal{L}^m\longrightarrow 0$$

is short exact.

Note that $\underline{\mathcal{L}}^{*c}$ and $\underline{\mathcal{L}}^{*u}$ are abelian categories: it follows easily from the description of simple objects of $\underline{\mathcal{L}}^*$ in Subsection 1.4 of [11].

2.2. **The object $\mathcal{R}^0_{st}\in\widetilde{\underline{\mathcal{L}}}^*$.** Let $R=\varprojlim_n(\bar O/p)_n$ be Fontaine's ring; it has a natural structure of $k$-algebra via the map $k\longrightarrow R$ given by $\alpha\mapsto\varprojlim_n([\sigma^{-n}\alpha]\bmod p)$, where $[\gamma]\in W(k)\subset\bar O$ denotes the Teichmüller representative of $\gamma\in k$. Let $\mathrm{m}_R$ be the maximal ideal of $R$.

Choose $x_0=(x_0^{(n)}\bmod p)_{n\geqslant 0}\in R$ and $\varepsilon=(\varepsilon^{(n)}\bmod p)_{n\geqslant 0}$ such that for all $n\geqslant 0$, $x_0^{(n+1)p}=x_0^{(n)}$ and $\varepsilon^{(n+1)p}=\varepsilon^{(n)}$ with $x_0^{(0)}=-p$, $\varepsilon^{(0)}=1$ but $\varepsilon^{(1)}\ne 1$. Denote by $v_R$ the valution on $R$ such that $v_R(x_0)=1$.

Let $Y$ be an indeterminate.

Consider the divided power envelope $R\langle Y\rangle$ of $R[Y]$ with respect to the ideal $(Y)$. If for $j\geqslant 0$, $\gamma_j(Y)$ is the $j$-th divided power of $Y$ then $R\langle Y\rangle=\oplus_{j\geqslant 0}R\gamma_j(Y)$. Denote by $R_{st}$ the completion $\prod_{j\geqslant 0}R\gamma_j(Y)$ of $R\langle Y\rangle$ and set, $\mathrm{Fil}^pR_{st}=\prod_{j\geqslant p}R\gamma_j(Y)$. Define the $\sigma$-linear morphism of the $R$-algebra $R_{st}$ by the correspondence $Y\mapsto x_0^pY$; it will be denoted below by the same symbol $\sigma$.

Introduce a $\mathcal{W}_1$-module structure on $R_{st}$ by the $k$-algebra morphism $\mathcal{W}_1 \longrightarrow R_{st}$ such that $u \mapsto \iota(u) := x_0 \exp(-Y) = x_0 \sum_{j\geqslant 0}(-1)^j\gamma_j(Y)$.

Set $F(R_{st}) = \sum_{0\leqslant i<p} x_0^{p-1-i} R\gamma_i(Y) + \mathrm{Fil}^p R_{st}$.

Define the continuous $\sigma$-linear morphism of $R$-modules $\varphi : F(R_{st}) \longrightarrow R_{st}$ by setting for $0 \leqslant i < p$, $\varphi(x_0^{p-1-i}\gamma_i(Y)) = \gamma_i(Y)(1-(i/2)x_0^pY)$, and for $i \geqslant p$, $\varphi(\gamma_i(Y)) = 0$.

Let $N$ be a unique $R$-differentiation of $R_{st}$ such that $N(Y) = 1$.

Note that $(R_{st}, F(R_{st}), \varphi, N)$ is not an object of $\widetilde{\underline{\mathcal{L}}}^*$, e.g. $\varphi$ is not a $\sigma$-linear morphism of $\mathcal{W}_1$-modules. Nevertheless, all appropriate compatibilities between above introduced additional structures on $R_{st}$ hold modulo $x_0^{2p}R_{st}$, cf. Proposition 2.1 in [11], and we can introduce

$$\mathcal{R}_{st}^0 = (R_{st}^0, F(R_{st}^0), \varphi, N) \in \widetilde{\underline{\mathcal{L}}}^*,$$

where $R_{st}^0 = R_{st} \operatorname{mod} x_0^p \mathrm{m}_R$ and $F(R_{st}^0) = F(R_{st}) \operatorname{mod} x_0^p\mathrm{m}_R$ with the appropriate induced maps $\varphi$ and $N$.

In our theory $\mathcal{R}_{st}^0$ plays a role of the ring $\hat{A}_{st}$ from the theory of $p$-adic semi-stable representations [14], Subsection 3.1.1. In particular, $R_{st}^0$ can be provided with continuous Galois action as follows. For any $\tau \in \Gamma_K$, let $k(\tau) \in \mathbb{Z}$ be such that $\tau(x_0) = \varepsilon^{k(\tau)}x_0$ and let $\widetilde{\log}(1+X) = X - X^2/2 + \cdots - X^{p-1}/(p-1)$ be the truncated logarithm. Define a map $\tau : R_{st} \longrightarrow R_{st}$ by extending the natural action of $\tau$ on $R$ and setting for all $j \geqslant 0$,

$$\tau(\gamma_j(Y)) := \sum_{0\leqslant i\leqslant \min\{j,p-1\}} \gamma_{j-i}(Y)\gamma_i(\widetilde{\log}\varepsilon).$$

Then the correspondences $\gamma_j(Y) \mapsto \tau(\gamma_j(Y))$ induce a $\Gamma_K$-action on the $\mathcal{W}_1$-algebra $R_{st}^0$ which extends the natural $\Gamma_K$-action on $R$ and respects the structure of $\mathcal{R}_{st}^0$ as an object of the category $\widetilde{\underline{\mathcal{L}}}^*$, cf. Proposition 2.2 in [11].

2.3. **The functor $\mathcal{V}^*$.** For any $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$, consider the $\Gamma_K$-module $\mathcal{V}^*(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}, \mathcal{R}_{st}^0)$. Note that in this definition we need $N$ to be defined slightly better than just modulo $u^pL$ (we work modulo $x_0^p\mathrm{m}_R$ rather than modulo $x_0^pR$) but such lift exists and unique, cf. Subsection 2.1. The Galois module $\mathcal{V}^*(\mathcal{L})$ can be studied via the following method from [15], Subsection 2.3.

Let $\mathcal{R}^0 = (R^0, F(R^0), \varphi) \in \widetilde{\underline{\mathcal{L}}}_0^*$, where $R^0 = R/x_0^p\mathrm{m}_R$, $F(R^0) = x_0^{p-1}R^0$, the $\mathcal{W}_1$-module structure on $R^0$ is given via $u \mapsto x_0$ and $\phi$ is induced by the map $r \mapsto r^p/x_0^{p(p-1)}$, $r \in x_0^{p-1}R$.

If $f \in \mathcal{V}^*(\mathcal{L})$ and $i \geqslant 0$, introduce $k$-linear morphisms $f_i : L \longrightarrow R^0$ such that for any $l \in L$, $f(l) = \sum_{i\geqslant 0} f_i(l)\gamma_i(Y)$. The correspondence $f \mapsto f_0$ gives the homomorphism of abelian groups $\mathrm{pr}_0 : \mathcal{V}^*(\mathcal{L}) \longrightarrow \mathcal{V}_0^*(\mathcal{L}) := \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{R}^0)$. Then, cf. Subsection 2.2 of [11],

— $\mathrm{pr}_0$ is isomorphism of abelian groups;

— if $\mathrm{rk}_{\mathcal{W}_1} L = s$ then $|\mathcal{V}_0^*(\mathcal{L})| = p^s$.

Therefore, $\mathcal{V}^*$ is exact functor from $\underline{\mathcal{L}}^*$ to the category of finite $\mathbb{F}_p[\Gamma_K]$-modules.

Introduce the ideal $\widetilde{J} = \sum_{0\leqslant i<p} x_0^{p-i}\mathrm{m}_R\gamma_i(Y) + \mathrm{Fil}^p R^0_{st}$ in $R^0_{st}$. Then $F(R^0_{st}) \supset \widetilde{J}$ and $\varphi|_{\widetilde{J}}$ is nilpotent. Therefore, we can introduce $\widetilde{\mathcal{R}}^0_{st} = (R^0_{st}/\widetilde{J}, F(R^0_{st})/\widetilde{J}, \varphi \bmod \widetilde{J}) \in \widetilde{\underline{\mathcal{L}}}_0^*$, there is a natural projection $\mathcal{R}^0_{st} \longrightarrow \widetilde{\mathcal{R}}^0_{st}$ in $\widetilde{\underline{\mathcal{L}}}_0^*$ and for any $\mathcal{L} \in \underline{\mathcal{L}}_0^*$, $\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{R}^0_{st}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \widetilde{\mathcal{R}}^0_{st})$. This implies the following description of the $\Gamma_K$-modules $\mathcal{V}^*(\mathcal{L})$, $\mathcal{L} \in \underline{\mathcal{L}}^*$,

$$\mathcal{V}^*(\mathcal{L}) = \left\{ \sum_{0\leqslant i<p} N^{*i}(f_0)\gamma_i(Y) \bmod \widetilde{J} \mid f_0 \in \mathcal{V}_0^*(\mathcal{L}) \right\} \tag{2.1}$$

Note that for $i \geqslant 1$, it is sufficient to know the maps $N^{*i}(f_0)$ modulo $x_0^p$ and this requires just the $(\bmod\, u^p)$-version of $N$, cf. discussion in Subsection 2.1. For future applications also notice the following two special cases of above general description (2.1).

a) Let $\Gamma_{K,1} = \mathrm{Gal}(\bar{K}/K(\sqrt[p]{p}) \subset \Gamma_K$. Then this group acts trivially on $Y \bmod \widetilde{J}$ and $x_0 \bmod x_0^p\mathrm{m}_R$. Therefore, for any $\mathcal{L} \in \underline{\mathcal{L}}^*$, the map $\mathrm{pr}_0 : \mathcal{V}^*(\mathcal{L}) \longrightarrow \mathcal{V}_0^*(\mathcal{L})$ is isomorphism of $\Gamma_{K,1}$-modules.

b) Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}_{cr}^*$. Then there is a $\mathcal{W}_1$-basis $l_1, \dots, l_s \in \varphi(F(L))$ and integers $0 \leqslant a_1, \dots, a_s \leqslant p-1$ such that $l_1' = u^{a_1}l_1, \dots, l_s' = u^{a_s}l_s$ is a $\mathcal{W}_1$-basis of $F(L)$, cf. Subsection 1.4 of [11]. Then there is a matrix $A \in \mathrm{GL}_s(k)$ such that

$$(\varphi(l_1'), \dots, \varphi(l_s')) = (l_1, \dots, l_s)A \bmod u^pL,$$

and for $f \in \mathcal{V}^*(\mathcal{L})$, $f_0 = \mathrm{pr}_0(f)$ and all $i$, we have

— $f(l_i) \equiv f_0(l_i) \bmod \widetilde{J}$;

— $x_0^{a_i} f_0(l_i) \equiv f(u^{a_i}l) \bmod \widetilde{J}$.

Let $b_i = p-1-a_i$, where $1 \leqslant i \leqslant s$. Then the Galois module $\mathcal{V}^*(\mathcal{L})$ is isomorphic to the Galois module of all $(r_1, \dots, r_s) \bmod x_0^p\mathrm{m}_R \in R^s \bmod x_0^p\mathrm{m}_R$ such that

$$(r_1^p/x_0^{pb_1}, \dots, r_s^p/x_0^{pb_s}) \equiv (r_1, \dots, r_s)(\sigma A) \bmod x_0^p\mathrm{m}_R.$$

2.4. **The category $\underline{\mathrm{CM}\Gamma_K}$ and the functor $\mathcal{CV}^*$.** Let $\underline{\mathrm{M}\Gamma_K}$ be the category of continuous $\mathbb{Z}_p[\Gamma_K]$-modules. The objects of the category $\underline{\mathrm{CM}\Gamma_K}$ are the triples $\mathcal{H} = (H, H^0, j)$, where $H, H^0 \in \underline{\mathrm{M}\Gamma_K}$ are finite, $\Gamma_K$ acts trivially on $H^0$ and $j : H \longrightarrow H^0$ is an epimorphic map in $\underline{\mathrm{M}\Gamma_K}$. If $\mathcal{H}_1 = (H_1, H_1^0, j_1) \in \underline{\mathrm{CM}\Gamma_K}$ then $\mathrm{Hom}_{\underline{\mathrm{CM}\Gamma_K}}(\mathcal{H}_1, \mathcal{H})$ consists of the couples $(f, f^0)$, where $f : H_1 \longrightarrow H$ and $f^0 : H_1^0 \longrightarrow H^0$ are morphisms in $\underline{\mathrm{M}\Gamma_K}$ such that $jf = f^0j_1$.

The category $\underline{\mathrm{CM}\Gamma}_K$ is special pre-abelian and its objects have a natural group structure.

**Definition.** Suppose $\mathcal{L}\in\underline{\mathcal{L}}^*$ and $i^{et}:\mathcal{L}^{et}\longrightarrow\mathcal{L}$ is the maximal etale subobject. Then $\mathcal{CV}^*:\underline{\mathcal{L}}^*\longrightarrow\underline{\mathrm{CM}\Gamma}_K$ is the functor such that $\mathcal{CV}^*(\mathcal{L})=(\mathcal{V}^*(\mathcal{L}),\mathcal{V}^*(\mathcal{L}^{et}),\mathcal{V}^*(i^{et}))$.

The simple objects in $\underline{\mathrm{CM}\Gamma}_K$ are of the form either $(H,0,0)$, where $H$ is a simple $\mathbb{Z}_p[\Gamma_K]$-module, or $(\mathbb{F}_p,\mathbb{F}_p,\mathrm{id})$, where $\mathbb{F}_p$ is provided with the trivial $\Gamma_K$-action. The functor $\mathcal{CV}^*$ establishes a bijection of the families of simple objects in $\underline{\mathcal{L}}^*$ and $\underline{\mathrm{CM}\Gamma}_K$, cf. Proposition 2.8 of [11].

In particular, let $\mathcal{L}_0=(\mathcal{W}_1,u^{p-1}\mathcal{W}_1,\varphi)\in\underline{\mathcal{L}}^*_{cr}$ be such that $\varphi(u^{p-1})=1$, and $\mathcal{L}_{p-1}=(\mathcal{W}_1,\mathcal{W}_1,\varphi)\in\underline{\mathcal{L}}^*_{cr}$ be such that $\varphi(1)=1$. Then $\mathcal{CV}^*(\underline{\mathcal{L}}_0)=\mathcal{F}_0:=(\mathbb{F}_p,\mathbb{F}_p,\mathrm{id})$ and $\mathcal{CV}^*(\mathcal{L}_{p-1})=\mathcal{F}_{p-1}:=(\mathbb{F}_p,0,0)$.

The functor $\mathcal{CV}^*$ is fully faithful, cf. Proposition 2.13 in [11].

By devissage the proof of this result is reduced to the fact that for any two simple objects $\mathcal{L}',\mathcal{L}''\in\underline{\mathcal{L}}^*$, $\mathcal{CV}^*$ induces an injective map from $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}',\mathcal{L}'')$ to $\mathrm{Ext}_{\underline{\mathrm{CM}\Gamma}_K}(\mathcal{CV}^*(\mathcal{L}''),\mathcal{CV}^*(\mathcal{L}'))$. The first group was explicitly described in Subsection 1.5 of [11] and the corresponding objects of $\underline{\mathrm{CM}\Gamma}_K$ were studied in Subsections 2.5-2.8 of [11] by the use of (2.1).

**Example.** One can verify that (remind that $p>2$)

$$\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_{p-1},\mathcal{L}_0)=\mathrm{Ext}_{\underline{\mathcal{L}}^*_{cr}}(\mathcal{L}_{p-1},\mathcal{L}_0)\simeq k$$

Explicitly this isomorphism is described via $\mathcal{L}[\gamma]\mapsto\gamma$, where for $\gamma\in k$, $\mathcal{L}[\gamma]=(L,F(L),\varphi,N)\in\underline{\mathcal{L}}^*_{cr}$ is such that $L=\mathcal{W}_1l_0\oplus\mathcal{W}_1l_1$, $F(L)=\mathcal{W}_1(u^{p-1}l_0)+\mathcal{W}_1(l_1+\gamma l_0)$, $\varphi(u^{p-1}l_0)=l_0$ and $\varphi(l_1+\gamma l_0)=l_1$.

Then $\mathcal{CV}^*:\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_{p-1},\mathcal{L}_0)\longrightarrow\mathrm{Ext}_{\underline{\mathrm{CM}\Gamma}_F}(\mathcal{F}_0,\mathcal{F}_{p-1})$ is injective.

Indeed, for any $\gamma\in k$, $\mathcal{CV}^*(\mathcal{L}[\gamma])=(V[\gamma],\mathbb{F}_p,j)\in\underline{\mathrm{CM}\Gamma}_K$, where the Galois module $V[\gamma]$ is identified with the module of all vectors $\bar r=(r_0,r_1)\mathrm{mod}x_0^p\mathrm{m}_R\in R^2\,\mathrm{mod}\,x_0^p\mathrm{m}_R$ such that $r_0^p\equiv r_0\,\mathrm{mod}x_0^p\mathrm{m}_R$ and $(r_1/x_0^p)-(r_1/x_0^p)^p\equiv\gamma^pr_0^p/x_0^{p^2}\,\mathrm{mod}\,\mathrm{m}_R$.

Then $V[\gamma]$ can be included into the short exact sequence of $\Gamma_K$-modules $0\longrightarrow\mathbb{F}_ph^1\longrightarrow V[\gamma]\xrightarrow{j}\mathbb{F}_pj(h^0)\longrightarrow 0$, where $h^0,h^1\in V[\gamma]$ are such that $h^0=(1,\alpha)\,\mathrm{mod}\,x_0^p\mathrm{m}_R$, $h^1=(0,x_0^p)\,\mathrm{mod}\,x_0^p\mathrm{m}_R$. Here $\alpha\in R$ is such that $\alpha-\alpha^p=\gamma^p/x_0^{p^2}$. So, $V[\gamma]$ can be described as an element of $\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}_K}(\mathbb{F}_pj(h^0),\mathbb{F}_ph^1)$ via the cocycle $\Theta_\gamma\in\mathrm{Hom}(\Gamma_K,\mathbb{F}_p)$ such that $\Theta_\gamma(\tau)=(\tau\alpha-\alpha)\,\mathrm{mod}\,\mathrm{m}_R$. Clearly, $\Theta_\gamma=0$ iff $\gamma=0$.

## 3. Ramification estimates

3.1. **Ramification estimates.** For any rational number $v\geqslant 0$, denote by $\Gamma_K^{(v)}$ the higher ramification subgroup of $\Gamma_K$ in upper numbering, [28]. In this Section we prove the following Theorem.

**Theorem 3.1.** a) *If $\mathcal{L}\in\underline{\mathcal{L}}^*$ and $v>2-\frac{1}{p}$ then $\Gamma_K^{(v)}$ acts trivially on $\mathcal{V}^*(\mathcal{L})$.*

b) *If* $\mathcal{L} \in \underline{\mathcal{L}}^*_{cr}$ *and* $v > 1$ *then* $\Gamma_K^{(v)}$ *acts trivially on* $\mathcal{V}^*(\mathcal{L})$.

c) *The above ramification estimates are sharp.*

The proof of part a) was only outlined in Subsection 2.9 of [11]. In Subsections 3.3–3.5 we shall give a proof based on our characteristic $p$ approach from [8, 9, 10]. One can also apply the methods from [24].

3.2. **Review of ramification theory.** The following brief sketch of ramification theory of complete discrete valuation fields with perfect residue field is based on the papers [17, 32, 33].

Let $E$ be a complete discrete valuation field with perfect residue field $k_E$ and the maximal ideal $m_E$. Let $E_{sep}$ be a separable closure of $E$. Denote by $v_E$ a unique extension of the normalized valuation on $E$ to $E_{sep}$.

Let $\mathcal{I}_E$ be the group of all continuous automorphisms of $E_{sep}$ which are compatible with $v_E$ and induce the identity map on the residue field of $E_{sep}$. If $F$ is a finite extension of $E$ in $E_{sep}$ then we always assume that $F_{sep} = E_{sep}$ and, therefore, we have a natural identification $\mathcal{I}_E = \mathcal{I}_F$.

Note that $\Gamma_E = \mathrm{Gal}(E_{sep}/E) \supset \{\iota \in \mathcal{I}_E \mid \iota|_E = \mathrm{id}\}$ and if $E$ is unramified over $\mathbb{Q}_p$ then $\mathcal{I}_E$ is identified with the inertia subgroup of $\Gamma_E$. If characteristic of $E$ is $p$ then $\mathcal{I}_E$ is considerably bigger: it contains the subgroup $\mathrm{Aut}^0_E E_{sep} = \{\ \iota \in \mathcal{I}_E \mid \iota(E) = E\}$ which is mapped onto the group of "analytic" automorphisms $\mathrm{Aut}^0 E$ of $E$ via $\iota \mapsto \iota|_E$.

Denote by $\mathcal{I}_{F/E}$ the set of all continuous embeddings of $F$ into $E_{sep}$ which induce the identity map on $E$ and $k_F$. For $v \geqslant 0$, let

$$\mathcal{I}^{(v)}_{F/E} = \{\iota \in \mathcal{I}_{F/E} \mid v_F(\iota(a) - a) \geqslant 1 + v \quad \forall a \in \mathrm{m}_F\}.$$

If $\iota_1, \iota_2 \in \mathcal{I}_{F/E}$ then they are *$v$-equivalent* iff for any $a \in \mathrm{m}_F$, it holds $v_F(\iota_1(a) - \iota_2(a)) \geqslant 1 + v$. The number of $v$-equivalent classes in $\mathcal{I}_{F/E}$ we shall denote by $(\mathcal{I}_{F/E} : \mathcal{I}^{(v)}_{F/E})$. Then the Herbrand function can be defined as $\varphi_{F/E}(x) = \int_0^x (\mathcal{I}_{F/E} : \mathcal{I}^{(v)}_{F/E})^{-1} dv$, $x \geqslant 0$. It has the following properties:

- $\varphi_{F/E}$ is a piece-wise linear function with finitely many edges;
- if $L \supset F \supset E$ is a tower of finite field extensions then for any $x \geqslant 0$, $\varphi_{L/E}(x) = \varphi_{F/E}(\varphi_{L/F}(x))$.

We define the ramification filtration $\{\mathcal{I}^{(v)}_E\}_{v \geqslant 0}$ on $\mathcal{I}_E$ as follows:

**Definition.** The subset $\mathcal{I}^{(v)}_E$ of $\mathcal{I}_E$ consists of $\iota \in \mathcal{I}_E$ such that for any finite extension $F$ of $E$ in $E_{sep}$ and $a \in \mathrm{m}_F$, $v_F(\iota(a) - a) \geqslant 1 + \varphi^{-1}_{F/E}(v)$.

**Remark.** a) If $\varphi_{F/E}(v_1) = v$ then $\mathcal{I}^{(v)}_E = \mathcal{I}^{(v_1)}_F$ (with respect to the natural identification $\mathcal{I}_E = \mathcal{I}_F$); b) $\Gamma^{(v)}_E = \Gamma_E \cap \mathcal{I}^{(v)}_E$ is just the usual higher ramification subgroup of $\Gamma_E$ with upper number $v$.

The ramification theory is perfectly compatible with the field-of-norms functor of Fontaine-Wintenberger, [32]. Suppose $\widetilde{E}/E$ is an infinite strictly APF-extension in $E_{sep}$. Then one can define the Herbrand function $\widetilde{\varphi} = \varphi_{\widetilde{E}/E}$ as the limit of Herbrand functions of all finite extensions of $E$ in $\widetilde{E}$. In this situation the field-of-norms functor $\mathcal{X}$ gives a complete discrete valuation field $\mathcal{E} = \mathcal{X}(E)$ of characteristic $p$, its separable closure $\mathcal{E}_{sep} = \mathcal{X}(E_{sep})$ and the embedding $\mathcal{X} : \mathcal{I}_E \longrightarrow \mathcal{I}_{\mathcal{E}}$.

With the above notation, the compatibility of the field-of-norms functor $\mathcal{X}$ with the ramification filtration means that for any $v \geqslant 0$,

$$\mathcal{X}(\mathcal{I}_E^{(\widetilde{\varphi}(v))}) = \mathcal{X}(\mathcal{I}_E) \cap \mathcal{I}_{\mathcal{E}}^{(v)}.$$

We apply this general theory in the following situation.

Fix an algebraic closure $\bar{K}$ of $K$ and set for $n \geqslant 0$, $K_n = K(\sqrt[p^n]{p}) \subset \bar{K}$. Then $\widetilde{K} = \bigcup_n K_n$ is a strictly APF-extension and by [32],

— $\mathcal{K} = \mathcal{X}(\widetilde{K}) = k((x_0)) \subset \operatorname{Frac} R$;

— $\mathcal{X}(\bar{K}) = \mathcal{K}_{sep}$ is a separable closure of $\mathcal{K}$ in $\operatorname{Frac} R$;

— $\mathcal{X}$ transforms the action of $\Gamma_K$ on $\bar{K}$ to the natural action of $\Gamma_K$ on Frac$R$ and $\Gamma_K \simeq \mathcal{X}(\Gamma_K) \subset \mathcal{I}_{\mathcal{K}}$ (remind that the residue field $k$ of $K$ is assumed to be algebraically closed).

Note that for the derivative of the Herbrand function $\varphi_{\widetilde{K}/K}$ it holds

$$\varphi'_{\widetilde{K}/K}(x) = \begin{cases} 1, & \text{if } 0 < x < p/(p-1) \\ 1/p, & \text{if } p(p-1) < x < p^2/(p-1). \end{cases}$$

Therefore,

$$\mathcal{X}(\Gamma_K^{(2-1/p)}) = \mathcal{X}(\Gamma_K) \cap \mathcal{I}_{\mathcal{K}}^{(p-1)}, \quad \mathcal{X}(\Gamma_K^{(1)}) = \mathcal{X}(\Gamma_K) \cap \mathcal{I}_{\mathcal{K}}^{(1)}. \tag{3.1}$$

3.3. **Proof of part a) of Theorem 3.1.** Consider a filtered module $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$. Then its structure can be specified as follows.

Choose a $\mathcal{W}_1$-basis $f_1, \dots, f_s$ of $F(L)$, let $l_i = \varphi(f_i)$ for $1 \leqslant i \leqslant s$, and set $\bar{f} = (f_1, \dots, f_s)$ and $\bar{l} = (l_1, \dots, l_s)$. Let $C \in M_s(\mathcal{W}_1)$ be such that $\bar{f} = \bar{l}C$. Note that $\mathcal{W}_1$ is identified with a subring of $R$ by $u \mapsto x_0$. Therefore, $C$ can be considered as $(s \times s)$-matrix with coefficients in $k[[x_0]] \subset R$. Note, $C$ divides the scalar matrix $x_0^{p-1} I_s$.

Let $H = \mathcal{V}^*(\mathcal{L})$. Because $\Gamma_{K,1} = \operatorname{Gal}(\bar{K}/K(\sqrt[p]{p})) \supset \Gamma_K^{(2-1/p)}$, we can assume that $H = \mathcal{V}_0^*(\mathcal{L})$.

**Lemma 3.2.** *There is a natural identification of $\mathbb{F}_p[\Gamma_{K,1}]$-modules*

$$H = \{\bar{r} \in R^s \bmod x_0 \mathrm{m}_R \mid \sigma(\bar{r}) C \equiv x_0^{p-1} \bar{r} \bmod x_0^p \mathrm{m}_R\}.$$

*Proof of Lemma.* Indeed, if $h \in H$, then $h(\bar{l}) = \bar{r}^* \bmod x_0^p \mathrm{m}_R$, where $\bar{r}^* \in R^s$ is such that

$$\frac{\sigma(\bar{r}^*)\sigma(C)}{x_0^{p(p-1)}} \equiv \bar{r}^* \bmod x_0^p \mathrm{m}_R.$$

Then $\sigma^{-1}(\bar{r}^*) = \bar{r}$ satisfies the congruence $\sigma(\bar{r})C \equiv x_0^{p-1}\bar{r} \bmod x_0^p \mathrm{m}_R$. It remains to verify that $h \mapsto \bar{r}$ gives the required identification. □

On the other hand, for any $\tau \in \mathcal{I}_{\mathcal{K}}^{(v)}$ with $v > p-1$, it holds $\tau(C) \equiv C \bmod x_0^p \mathrm{m}_R$ and this implies (use Lemma 3.2) for any $h \in H$, that $\tau(h) \in H$. Therefore, our proposition will be proved if we show that for any such $\tau$ and any $h \in H$, $\tau(h) = h$.

From the left-continuity of the ramification filtration $\{\mathcal{I}_{\mathcal{K}}^{(v)} \mid v \geqslant 0\}$ it follows the existence of a minimal $v^* = v^*(H)$ such that for any $v > v^*$ and $\tau \in \mathcal{I}_{\mathcal{K}}^{(v)}$, $\tau|_H = \mathrm{id}$.

If $v^* \leqslant p-1$ there is nothing to prove.

Otherwise, choose $r^* \in (p-1, v^*)$ such that $v_p(r^*) = 0$. Such $r^*$ can be always written in the form $r^* = m/(q-1)$, where $m \in \mathbb{N}$ is prime to $p$ and $q$ is an integral power of $p$. For the following Lemma cf. [8], Subsection 1.5.

**Lemma 3.3.** *With above chosen $r^*$ and $q$ there is a field extension $\mathcal{K}' = k((x_0'))$ of $\mathcal{K} = k((x_0))$ such that*

a $[\mathcal{K}' : \mathcal{K}] = q$;

b) $\varphi'_{\mathcal{K}'/\mathcal{K}}(x) = \begin{cases} 1, & \textit{if} \quad 0 < x < r^* \\ 1/p, & \textit{if} \quad x > r^* \end{cases}$

c) $x_0 \equiv x_0'^{\,q}(1 - x_0'^{\,r^*(q-1)}) \bmod x_0'^{\,q+2r^*(q-1)}$.

Note that for above chosen $r^*$, the appropriate $m$ and $q$ are not defined uniquely, e.g. for any $a \in \mathbb{N}$, it holds aslo that $r^* = m_a/(q^a - 1)$, where $m_a = m(1+q+\cdots+q^{a-1})$. Therefore, we can assume additionally that $q$ is large enough to provide us with the following inequality

$$r^*(1 - 1/q) > p - 1. \tag{3.2}$$

Choose a field isomorphism $\kappa : \mathcal{K} \longrightarrow \mathcal{K}'$ such that $\kappa(x_0) = x_0'$ and $\kappa|_k = \sigma^{-q}$. Note that by Lemma 3.3 c) and assumption (3.2), for any $\gamma \in k[[x_0]]$, $\kappa(\gamma)^q \equiv \gamma \bmod x_0^p \mathrm{m}_R$. The isomorphism $\kappa$ can be extended to an isomorphism of separable closures of these fields in $R$. Therefore, we have the bijection $\kappa^* : \mathcal{I}_{\mathcal{K}} \longrightarrow \mathcal{I}_{\mathcal{K}'}$ such that for any $v \geqslant 0$, $\kappa^*(\mathcal{I}_{\mathcal{K}}^{(v)}) = \mathcal{I}_{\mathcal{K}'}^{(v)}$. In particular, if

$$h' \in H' = \{\bar{r}' \in R^s \bmod x_0' \mathrm{m}_R \mid \sigma(\bar{r}')\kappa(C) \equiv x_0'^{\,p-1}\bar{r}' \bmod x_0'^{\,p} \mathrm{m}_R\}$$

then for any $v > v^*$ and $\tau \in \mathcal{I}_{\mathcal{K}'}^{(v)}$, $\tau(h') = h'$.

On the other hand, from Lemma 3.3 c) it follows that

$$H=\{\bar r\in R^s \bmod x_0\mathrm{m}_R \mid \sigma(\bar r)\sigma^q(\kappa(C))\equiv x_0^{p-1}\bar r \bmod x_0^p\mathrm{m}_R\}.$$

Therefore, the map $\bar r'\mapsto\sigma^q(\bar r')$ establishes a Galois equivariant bijection of $H'$ and $H$. Because, $\mathcal{I}_{\mathcal{K}'}^{(v^*)}=\mathcal{I}_{\mathcal{K}}^{(\varphi_{\mathcal{K}'/\mathcal{K}}(v^*))}$, this implies that for any $\tau\in\mathcal{I}_{\mathcal{K}}^{(v)}$ with $v>\varphi_{\mathcal{K}'/\mathcal{K}}(v^*)$, $\tau|_H=\mathrm{id}$. But $\varphi_{\mathcal{K}'/\mathcal{K}}(v^*)<v^*$. The contradiction.

**3.4.** Prove that the ramification estimate from Theorem 3.1 a) is sharp.

Introduce $\mathcal{L}=(L,F(L),\varphi,N)\in\widetilde{\underline{\mathcal{L}}}^*$ such that:

— $L=\sum_{p>i\geqslant 0}\mathcal{W}_1 l_i$;

— $F(L)=\sum_{p>i\geqslant 0}\mathcal{W}_1 f_i$, where for $p>i\geqslant 1$, $f_i=u^i(l_{p-1}+\dots+l_i)$ and $f_0=u\sum_{p-1\geqslant i\geqslant 2}(i-1)l_i+l_1+l_0$;

— for all $i$, $\varphi(f_i)=l_i$;

— $N(l_{p-1})=0$, if $p-1>i\geqslant 1$ then $N(l_i)=l_{i+1}\bmod u^pL$, and $N(l_0)=-l_2\bmod u^pL$.

A direct verification shows that $\mathcal{L}\in\underline{\mathcal{L}}^*$. In particular:

— for $p>i\geqslant 2$, $\varphi(uN(f_i))=\varphi(f_{i+1})=l_{i+1}=N(l_i)=N(\varphi(f_i))$;

— $N(f_0)=-u(l_{p-1}+\dots+l_2)$ and, therefore, $\varphi(uN(f_0))=\varphi(-f_2)=-l_2=N(l_0)=N(\varphi(f_0))$.

By Lemma 3.2, the $\Gamma_{K,1}$-module $\mathcal{V}_0^*(\mathcal{L})=H$ is identified with the $\Gamma_{K,1}$-module of all $\bar r=(r_{p-1},\dots,r_1,r_0)\in R^p\bmod x_0\mathrm{m}_R$ such that

$$r_{p-1}^p\equiv r_{p-1}\bmod x_0\mathrm{m}_R$$

$$r_{p-1}^p+r_{p-2}^p\equiv x_0r_{p-2}\bmod x_0^2\mathrm{m}_R$$

$$\dots$$

$$r_{p-1}^p+\dots+r_1^p\equiv x_0^{p-2}r_1\bmod x_0^{p-1}\mathrm{m}_R$$

$$r_{p-1}^p(p-2)x_0+\dots+r_2^px_0+r_1^p+r_0^p\equiv x_0^{p-1}r_0\bmod x_0^p\mathrm{m}_R$$

Let $v^*\geqslant 0$ be the minimal such that for all $v>v^*$, $\mathcal{I}_{\mathcal{K}}^{(v)}$ acts trivially on $H$. We must prove that $v^*=p-1$.

Suppose $v^*<p-1$.

Choose $r^*\in(v^*,p-1)$ such that $v_p(r^*)=0$. As earlier, we can assume that $r^*=m/(q-1)$, where $m\in\mathbb{N}$, $q$ is a power of $p$ and $r^*(1-1/q)>p-2$.

Apply Lemma 3.3 and consider the appropriate fields isomorphism $\kappa:\mathcal{K}\longrightarrow\mathcal{K}'$. If $\kappa(x_0)=x_0'$ then for $i\geqslant 2$, $x_0'^{iq}\equiv x_0^i\bmod x_0^p\mathrm{m}_R$, $x_0'^q\equiv x_0+x_0'^{q+r^*(q-1)}\bmod x_0^p\mathrm{m}_R$ and $x_0'^{q+r^*(q-1)}\in x_0^{p-1}\mathrm{m}_R$.

This implies that for $p-1\geqslant i\geqslant 1$, $r_i\equiv\sigma^q(r_i')\bmod x_0\mathrm{m}_R$ and $r_i^p\equiv\sigma^q(r_i')^p\bmod x_0^p\mathrm{m}_R$. Therefore,

$$\bar r\equiv\sigma^q(\bar r')+(0,\dots,0,Y_0)\bmod x_0\mathrm{m}_R,$$

where $x_0^{p-1}Y_0 - Y_0^p \equiv (r_{p-1}^p(p-2) + \cdots + r_2^p){x'_0}^{q+r^*(q-1)}$

$$\equiv r_{p-1}{x'_0}^{q+r^*(q-1)} \bmod x_0^p\mathrm{m}_R.$$

Note that this relation can be rewritten in the following form

$$(Y_0/x_0) - (Y_0/x_0)^p \equiv r_{p-1}{x'_0}^{-a} \bmod \mathrm{m}_R,$$

where $\varphi_{\mathcal{K}'/\mathcal{K}}(a) = p-1$. Therefore, $\mathcal{I}_{\mathcal{K}'}^{(a)} = \mathcal{I}_{\mathcal{K}}^{(p-1)}$ acts non-trivially on $Y_0 \bmod x_0\mathrm{m}_R$.

On the other hand, we assumed that $v^* < p-1$ and, therefore, for $\tau \in \mathcal{I}_{\mathcal{K}'}^{(p-1)}$, we have $\tau(\bar{r}') = \bar{r}'$. But $a > p-1$ and this implies $\mathcal{I}_{\mathcal{K}}^{(p-1)} = \mathcal{I}_{\mathcal{K}'}^{(a)} \subset \mathcal{I}_{\mathcal{K}'}^{(p-1)}$. Therefore, any $\tau \in \mathcal{I}_{\mathcal{K}}^{(p-1)}$ acts trivially on $(\bar{r} - \sigma^q(\bar{r}')) \bmod x_0\mathrm{m}_R = (0, \dots, Y_0) \bmod x_0\mathrm{m}_R$. The contradiction.

**3.5. Proof of estimate b) of Theorem 3.1.** Suppose $\mathcal{L} \in \underline{\mathcal{L}}_{cr}^*$ is given in notation of Subsection 2.3 b). By (3.1) we must prove that $\mathcal{I}_{\mathcal{K}}^{(v)}$ acts trivially on $H = \mathcal{V}_0^*(\mathcal{L})$ if $v > 1$.

Let $v^*$ be the maximal such that $\mathcal{I}_{\mathcal{K}}^{(v^*)}$ acts non-trivially on $H$.

Assume that $v^* > 1$. Choose $r^* \in (1, v^*)$ such that $r^* = m/(q-1)$, where $m \in \mathbb{N}$, $q$ is a power of $p$ and $r^*(1 - 1/q) > 1$. Consider the appropriate field isomorphism $\kappa : \mathcal{K} \longrightarrow \mathcal{K}'$ and its extension to $\mathcal{K}_{sep} = \mathcal{K}'_{sep}$. Let $x'_0 = \kappa(x_0)$ and $\bar{r}' = (r'_1, \dots, r'_s) = \kappa(\bar{r})$. If

$$H' = \left\{ \bar{r}' \bmod {x'_0}^p\mathrm{m}_R \mid \left( \frac{{r'_1}^p}{{x'_0}^{pb_1}}, \dots, \frac{{r'_s}^p}{{x'_0}^{pb_s}} \right) = \bar{r}'(\sigma^{1-q}A) \bmod {x'_0}^p\mathrm{m}_R \right\}$$

then for any $v > v^*$, $\mathcal{I}_{\mathcal{K}'}^{(v)}$ acts trivially on $H'$.

Note that the assumption $r^*(1-1/q) > 1$ implies that ${x'_0}^{r^*(q-1)} \in x_0\mathrm{m}_R$ and for all $1 \leqslant i \leqslant s$,

$$\frac{r_i^p}{x_0^{pb_i}} \equiv \frac{(\sigma^q r'_i)^p}{{x'_0}^{qpb_i}} \bmod x_0^p\mathrm{m}_R.$$

Therefore, $\bar{r}' \mapsto \sigma^q\bar{r}'$ induces $\mathrm{Aut}^0_{\mathcal{K}}\mathcal{K}_{sep}$-equivariant isomorphism of $H'$ and $H$.

If $v' \geqslant 0$ is such that $\varphi_{\mathcal{K}'/\mathcal{K}}(v') = v^*$ then $v' > v^*$ and $\mathcal{I}_{\mathcal{K}'}^{(v')} = \mathcal{I}_{\mathcal{K}}^{(v^*)}$ acts trivially on $H'$ but not on $H$. The contradiction.

**3.6.** The example in Subsection 2.4 shows that for all $\gamma \neq 0$, $\mathcal{I}_{\mathcal{K}}^{(1)}$ acts non-trivially on $h^0 \in V[\gamma] = \mathcal{V}^*(\mathcal{L}[\gamma])$. Therefore, the estimate from b) is sharp.

## 4. A construction of modification of Breuil's functor.

Generalize slightly the initial data from Section 2 as follows.

Let $\mathcal{W} = W(k)[[u]]$, where $u$ is an indeterminate. Denote by $\sigma$ the automorphism of $W(k)$ induced by the $p$-th power map on $k$ and agree to use the same symbol for the continuous extension of $\sigma$ to $\mathcal{W}$ such that $\sigma(u) = u^p$. Denote by $N : \mathcal{W} \longrightarrow \mathcal{W}$ the unique continuous $W(k)$-differentiation such that $N(u) = -u$. We denote by $S$ the divided power envelope of $\mathcal{W}$ with respect to the ideal $(u + p)$.

**4.1. Breuil's functor.** We work with Breuil's theory of semi-stable $p$-adic representations of $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$, [14]-[16]. This theory allows to construct $\Gamma_K$-invariant lattices in semi-stable $\mathbb{Q}_p[\Gamma_K]$-modules with Hodge-Tate weights from $[0, p)$. The construction is done via Breuil's functor $\mathcal{S}_{p-1} \longrightarrow \underline{\mathrm{M}\Gamma}_K$, where $\mathcal{S}_{p-1}$ is a suitable category of free $S$-modules $M$ with filtration by a submodule $F(M)$ and additional structures involving $\sigma$-linear morphisms $\varphi : F(M) \longrightarrow M$ and differentiations $N : M \longrightarrow M$, [16], Subsection 2.2. The objects of $\mathcal{S}_{p-1}$ satisfy the properties similar to those from the definition of the category $\underline{\mathcal{L}}^*$ from Subsection 2.1. Breuil's functor appears in the form $\mathcal{M} \mapsto \mathrm{Hom}_{F,\varphi,N}(M, \hat{A}_{st})$, where $\hat{A}_{st}$ is the ring of semi-stable $p$-adic periods [14], Subsection 3.1. Note that $\hat{A}_{st}$ is provided with the appropriate $S$-module structure, filtration, morphisms $\varphi$ and $N$, and $\Gamma_K$-module structure. The notation $\mathrm{Hom}_{F,\varphi,N}$ means the set of all $S$-linear homomorphisms compatible with filtrations and the morphisms $\varphi$ and $N$. Breuil's theory allows also to construct crystalline representations of $\Gamma_K$ with HT weights from $[0, p)$ by the use of the appropriate subcategory $\mathcal{S}^{cr}_{p-1}$ of $\mathcal{S}_{p-1}$. (The objects of $\mathcal{S}^{cr}_{p-1}$ come from the Fontaine-Laffaille modules with filtration of length $p$.)

Similarly to the Fontaine-Laffaille theory the Breuil theory perfectly describes all $\Gamma_K$-invariant lattices of semi-stable representations with HT weights from $[0, p-2]$ but does not give generally all such lattices for representations with weights from $[0, p)$.

**4.2. Modification of Breuil's functor.** In Subsection 4 of [11] we constructed a modification of Breuil's functor which allows us to construct all Galois invariant lattices and study all subquotients modulo $p$ of semi-stable representations with weights from $[0, p)$. We shall give below a brief explanation of our construction from Subsection 4 of [11] together with a modelled example.

**4.2.1.** As a first step, we prove that Breuil's category of filtered $S$-modules $\underline{\mathcal{S}}_{p-1}$ can be replaced by a similar category $\underline{\mathcal{L}}^f$ of free filtered $\mathcal{W}$-modules $(M, F(M))$ with $\sigma$-linear maps $\varphi : F(M) \longrightarrow M$ and differentiations $N : M \longrightarrow M \otimes_{\mathcal{W}} S$. Then we define a torsion analogue $\underline{\mathcal{L}}^t$ of the category $\underline{\mathcal{L}}^f$. As a result, we can use Breuil's functor in the form $\mathcal{V}^t : \mathcal{M} \mapsto \mathrm{Hom}_{F,\varphi,N}(\mathcal{M}, A_{st,\infty})$, where $\mathcal{M} \in \underline{\mathcal{L}}^t$ and $A_{st,\infty}$ is a torsion analogue of Fontaine's ring of semi-stable periods, [14], Subsection 3.1. Note that $\underline{\mathcal{L}}^t$ contains the full subcategory $\underline{\mathcal{L}}^{ft}$ whose objects are subquotients of objects of $\underline{\mathcal{L}}^f$ and this subcategory is strictly smaller than $\underline{\mathcal{L}}^t$. This is very special feature of "semi-stable" theory: if we start with the subcategory $\mathcal{S}_{p-1}^{cr}$ then the appropriate categories $\mathcal{L}_{cr}^t$ and $\mathcal{L}_{cr}^{ft}$ coincide. Denote the restriction of $\mathcal{V}^t$ to $\underline{\mathcal{L}}^{ft}$ by $\mathcal{V}^{ft}$.

Following general formalism we prove that $\underline{\mathcal{L}}^t$ is special pre-abelian and there is a concept of $p$-divisible object in $\underline{\mathcal{L}}^t$ (just mimic Tate's definition of $p$-divisible groups in the pre-abelian category of group schemes). Such $p$-divisible objects will be called $p$-divisible groups if there is no risk of confusion. Then the objects of $\underline{\mathcal{L}}^f$ can be recovered as "Tate's modules" associated with $p$-divisible groups in $\underline{\mathcal{L}}^t$. In particular, a $p$-divisible group in $\underline{\mathcal{L}}^t$ is inductive limit of objects from $\underline{\mathcal{L}}^{ft}$. As we have just noted, there is no similar problem for the appropriate subcategory $\underline{\mathcal{L}}_{cr}^t$ of "crystalline" filtered modules in $\underline{\mathcal{L}}^t$: any such module comes as a subquotient of a "crystalline" module from $\underline{\mathcal{L}}_{cr}^f \subset \underline{\mathcal{L}}^f$.

**4.2.2.** If $\mathcal{M} = (M, F(M)) \in \underline{\mathcal{L}}^t$ then it is called multiplicative if $M = F(M)$ and etale if $F(M) = u^{p-1}M$. As usually, any $\mathcal{M} \in \underline{\mathcal{L}}^t$ has a unique maximal etale subobject $\iota^{et} : \mathcal{M}^{et} \longrightarrow \mathcal{M}$ and a unique maximal multiplicative quotient object $\iota^m : \mathcal{M} \longrightarrow \mathcal{M}^m$. We call $\mathcal{M} \in \underline{\mathcal{L}}^t$ unipotent if $\mathcal{M}^m = 0$.

By compairing $\mathcal{V}^t$ and the functor $\mathcal{V}^*$ from Subsection 2.3 we deduce that $\mathcal{V}^t$ is fully faithful on the subcategory $\underline{\mathcal{L}}^{t,u}$ of unipotent objects of $\underline{\mathcal{L}}^t$. Quite oppositely, if $\mathcal{M} \in \underline{\mathcal{L}}^t$, $p\mathcal{M} = 0$ and

$$0 \longrightarrow \mathcal{M}^u \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}^m \longrightarrow 0$$

is the standard short exact sequence with unipotent $\mathcal{M}^u$ and multiplicative $\mathcal{M}^m$, then the corresponding exact sequence

$$0 \longrightarrow \mathcal{V}^t(\mathcal{M}^m) \longrightarrow \mathcal{V}^t(\mathcal{M}) \longrightarrow \mathcal{V}^t(\mathcal{M}^u) \longrightarrow 0$$

has a functorial splitting in $\underline{\mathrm{M}\Gamma}_K$. Denote the appropriate splitting maps by $\Theta : \mathcal{V}^t(\mathcal{M}^u) \longrightarrow \mathcal{V}^t(\mathcal{M})$ and $\widetilde{\Theta} : \mathcal{V}^t(\mathcal{M}) \longrightarrow \mathcal{V}^t(\mathcal{M}^m)$.

**Example.** Let $A_{cr}$ be Fontaine's crystalline ring. It is the $p$-adic closure of the $DP$-envelope of $W(R)$ with respect to the ideal $([x_0] + p)$. Let $F(A_{cr})$ be the $(p-1)$-st divided power of $([x_0]+p)$ and $\psi : A_{cr} \longrightarrow A_{cr}$ be the map induced by $\sigma : R \longrightarrow R$. Set $\varphi = \psi/p^{p-1}$. Notice that $A_{cr}$ is provided with the natural continuous $\Gamma_K$-action. Then $A_{cr,1} = A_{cr}/pA_{cr}$ is provided with induced filtration $F(A_{cr,1})$, morphism $\varphi$ and $\Gamma_K$-action, and we obtain $\mathcal{A}_{cr,1} = (A_{cr,1}, F(A_{cr,1}), \varphi) \in \widetilde{\underline{\mathcal{L}}}_0^*$ by defining the $\mathcal{W}_1$-module structure on $A_{cr,1}$ via $u \mapsto [x_0]$.

If $A_{st}$ is Fontaine's ring of semi-stable periods then $A_{st}$ is obtained from $A_{cr}$ in the same way as $R_{st}^0$ was obtained from $R^0$ in Subsections 2.2 and 2.3. Therefore, if $\mathcal{L} \in \underline{\mathcal{L}}^t$ and $p\mathcal{L} = 0$ then we can illustrate the splitting phenomenon by studying the abstract module $\mathcal{V}_0(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{A}_{cr,1})$. Even more, we can treat $\underline{\mathcal{L}}_{cr}^*$ as a full subcategory of $\underline{\mathcal{L}}^t$ and then for $\mathcal{L} \in \underline{\mathcal{L}}_{cr}^*$, $\mathcal{V}_0(\mathcal{L}) = \mathcal{V}^t(\mathcal{L})$ even as Galois modules.

From the definition of $A_{cr}$ it follows that $A_{cr,1} = (R/x_0^p)[T_1, T_2, \dots]$, where for all $i \geqslant 1$, $T_i$ comes from the divided powers $\gamma_{p^i}([x_0] + p)$ and $T_i^p = 0$. Let $J$ be the ideal in $A_{cr,1}$ generated by $T_1^2$ and $T_i$ with $i \geqslant 2$. Then $\widetilde{A}_{cr,1} = A_{cr,1}/J = (R/x_0^p)T_1 \oplus (R/x_0^p)$ is provided with the induced filtration $F(\widetilde{A}_{cr,1}) = (R/x_0^p)T_1 \oplus (x_0^{p-1}R/x_0^p)$ and $\sigma$-linear $\varphi : F(\widetilde{A}_{cr,1}) \longrightarrow \widetilde{A}_{cr,1}$ such that $\varphi(x_0^{p-1}) = 1 - T_1$ and $\varphi(T_1) = 1$. This gives the object $\widetilde{\mathcal{A}}_{cr,1}$ in $\widetilde{\underline{\mathcal{L}}}_0^*$ together with the natural projection $j_{cr,1} : \mathcal{A}_{cr,1} \longrightarrow \widetilde{\mathcal{A}}_{cr,1}$. Using that $\varphi(J) = 0$ we obtain that $j_{cr,1*}$ induces the identification $\mathcal{V}_0(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \widetilde{\mathcal{A}}_{cr,1})$.

Consider $\mathcal{L}[\gamma] \in \underline{\mathcal{L}}_{cr}^*$ from Subsection 2.4.

Then we have the following standard exact sequence

$$0 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{L}[\gamma] \to \mathcal{L}_{p-1} \to 0$$

where $\mathcal{L}_0 = (\mathcal{W}_1 l_0, u^{p-1}\mathcal{W}_1 l_0, \varphi)$ is a simple etale subobject in $\mathcal{L}[\gamma]$ and $\mathcal{L}_{p-1} = (\mathcal{W}_1 \bar{l}_1, \mathcal{W}_1 \bar{l}_1, \varphi)$ is its simple multiplicative quotient object.

Consider the corresponding short exact sequence of $\mathbb{F}_p[\Gamma_K]$-modules

$$0 \longrightarrow H_{p-1} \longrightarrow H \longrightarrow H_0 \longrightarrow 0$$

where $H_{p-1} = \mathcal{V}_0(\mathcal{L}_{p-1})$, $H = \mathcal{V}_0(\mathcal{L}[\gamma])$, $H_0 = \mathcal{V}_0(\mathcal{L}_0)$. Note that $H_0 = H_{p-1} \simeq \mathbb{F}_p$ are trivial $\Gamma_K$-modules.

**Lemma 4.1.** *For any $\alpha \in R$, there is a unique $r^*(\alpha) \in R \bmod x_0^p$ such that $r^*(\alpha)^p/x_0^{p(p-1)} - r^*(\alpha) \equiv \alpha \bmod x_0^p R$.*

*Proof.* It follows from the congruence $A^p - A = \alpha x_0^{-p} \bmod R$, where $A = r^*(\alpha) x_0^{-p} \in \operatorname{Frac} R$. $\square$

Now direct calculations show that:

- $H_0 = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}_0, \widetilde{\mathcal{A}}_{cr,1})$ consists of $h : l_0 \mapsto r_{-1}T_1 + r_0$ such that $r_0 = -r_{-1} = f$, where $f_0$ runs over all elements of $\mathbb{F}_p \subset R/x_0^p$;

- $H_{p-1} = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}_{p-1}, \widetilde{\mathcal{A}}_{cr,1})$ consists of $h : l_1 \mapsto r_{-1}T_1 + r_0$ such that $r_0 = r^*(f_1)$ and $r_{-1} = -(r^*(f_1) + f_1)$, where $f_1$ runs over $\mathbb{F}_p$;

- $H = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}[\gamma], \widetilde{\mathcal{A}}_{cr,1})$ consists of

$$h : (l_0, l_1) \mapsto (-f_0 T_1 + f_0, r_{-1}T_1 + r_0),$$

where $r_0 = -(r_{-1} + f_1) = r^*(f_1) - r^*(\gamma^p f_0/x_0^{p(p-1)})$ and $f_0, f_1 \in \mathbb{F}_p$;

- the splitting $\Theta$ is defined via the submodule $\Theta(H_0)$ of $H$ consisting of $h \in H$ such that $f_1 = 0$.

Let $A^0_{cr,1} = \{r_{-1}T_1 + r_0 \mid r_{-1} = -r_0\} \subset \widetilde{A}_{cr,1}$ with the induced filtration and the morphism $\varphi$, and denote by $\mathcal{A}^0_{cr,1}$ the appropriate object of $\widetilde{\underline{\mathcal{L}}}^*_0$. Then

$$\Theta(H_0) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*_0}(\mathcal{L}[\gamma], \mathcal{A}^0_{cr,1}) \tag{4.1}$$

**Remark.** Relation (4.1) determines the splitting $\Theta$ for any $\mathcal{L} \in \underline{\mathcal{L}}^*_{cr}$.

**4.2.3.** For $\mathcal{M} \in \underline{\mathcal{L}}^{ft}$, let $\mathcal{M}' \in \underline{\mathcal{L}}^{ft}$ be such that $p\mathcal{M}' = \mathcal{M}$ and let

$$C_p = \mathrm{Coker}\, p|_{\mathcal{M}'} : \mathcal{M}' \longrightarrow {}_p\mathcal{M}', \quad K_p = \mathrm{Ker}\, p|_{\mathcal{M}'} : \mathcal{M}'_p \longrightarrow \mathcal{M}'.$$

Consider the following sequence of objects and maps in $\underline{\mathrm{M}\Gamma}_K$

$$\mathcal{V}^{ft}({}_p\mathcal{M}'^u) \xrightarrow{\Theta} \mathcal{V}^{ft}({}_p\mathcal{M}') \xrightarrow{\mathcal{V}^{ft}(C_p)} \mathcal{V}^{ft}(\mathcal{M}') \xrightarrow{\mathcal{V}^{ft}(K_p)} \mathcal{V}^{ft}(\mathcal{M}'_p) \xrightarrow{\widetilde{\Theta}} \mathcal{V}^{ft}(\mathcal{M}'^m_p)$$

Then:

— $\widetilde{\Theta} \circ V^{ft}(K_p) \circ V^{ft}(C_p) \circ \Theta = 0$;

— $\Gamma_K$-module $\widetilde{\mathcal{V}}^{ft}(\mathcal{M}) := \mathrm{Ker}(\widetilde{\Theta} \circ \mathcal{V}^{ft}(K_p))/\mathrm{Im}\,(\mathcal{V}^{ft}(C_p) \circ \Theta)$ does not depend on a choice of $\mathcal{M}'$;

— $\widetilde{\mathcal{V}}^{ft}(\mathcal{M}^u) = \mathcal{V}^{ft}(\mathcal{M}^u)$, $\widetilde{\mathcal{V}}^{ft}(\mathcal{M}^m) = \mathcal{V}^{ft}(\mathcal{M}^m)$;

— there is a canonical epimorphism $\widetilde{\mathcal{V}}^{ft}(i^{et}) : \widetilde{\mathcal{V}}^{ft}(\mathcal{M}) \longrightarrow \widetilde{\mathcal{V}}^{ft}(\mathcal{M}^{et})$.

**Definition.** The modification of Breuil's functor $\widetilde{\mathcal{CV}}^{ft} : \underline{\mathcal{L}}^{ft} \longrightarrow \underline{\mathrm{CM}\Gamma}_K$ is induced by the correspondence $\mathcal{M} \mapsto (\widetilde{\mathcal{V}}^{ft}(\mathcal{M}), \widetilde{\mathcal{V}}^{ft}(\mathcal{M}^{et}), \widetilde{\mathcal{V}}^{ft}(i^{et}))$.

**Example.** Having in mind that $A_{cr}$ is related to the $DP$-envelope of $W(R)$ we can describe explicitly $A_{cr,2} = A_{cr}/p^2A_{cr}$ and similarly to the case of $A_{cr,1}$ introduce an appropriate simpler object $\widetilde{A}_{cr,2}$ as follows:

— the elements of $\widetilde{A}_{cr,2}$ are written in the form

$$[r_{-1}]T_1 + [r_0] + p[r_1],$$

where $r_{-1}, r_1 \in R/x_0^p$ and $r_0 \in R/x_0^{2p}$; the operations are induced by those on the Teichmuller representatives of $r_{-1}, r_0, r_1$ via the relations $pT = [x_0]^p$ and $p^2 = 0$;

— the $\mathcal{W}$-module structure on $\widetilde{A}_{cr,2}$ is induced by the $W(k)$-algebra morphism $\mathcal{W} \longrightarrow W(R)$ such that $u \mapsto [x_0] + p$;

— $F(\widetilde{A}_{cr,2})$ is generated over $W(R)$ by $f_1 = T_1$ and $f_2 = [x_0]^{p-1} - p[x_0]^{p-2}$;

— $\varphi : F(\widetilde{A}_{cr,2}) \longrightarrow \widetilde{A}_{cr,2}$ is uniquely determined by $\varphi(f_1) = 1 + [x_0]^p$ and $\varphi(f_2) = -T_1 + 1$.

Note that $p\widetilde{\mathcal{A}}_{cr,2} = \widetilde{\mathcal{A}}_{cr,1}$.

Consider again $\mathcal{L}[\gamma] \in \underline{\mathcal{L}}^*_{cr}$ from Subsection 2.4. Choose $\mathcal{L}' \in \underline{\mathcal{L}}^{ft}$ such that $p\mathcal{L}' = \mathcal{L}[\gamma]$. For simplicity assume that the corresponding $\mathcal{W}$-module is $(\mathcal{W}/p^2)l_0 \oplus (\mathcal{W}/p^2)l_1$. In this case $C_p : \mathcal{L}' \longrightarrow {}_p\mathcal{L}'$ is just the natural projection $\mathcal{L}' \longrightarrow \mathcal{L}'/p\mathcal{L}' = \mathcal{L}$ and $K_p : \mathcal{L}'_p \longrightarrow \mathcal{L}'$ is just the natural embedding $\mathcal{L} = p\mathcal{L}' \longrightarrow \mathcal{L}'$. Therefore,

- $\mathrm{Ker}(\widetilde{\Theta} \circ \mathcal{V}_0(K_p))$ appears as the kernel of the map

$$\mathcal{V}_0(\mathcal{L}') \longrightarrow \mathcal{V}_0(\mathcal{L}')/p\mathcal{V}_0(\mathcal{L}') = \mathcal{V}_0(\mathcal{L}) = H \xrightarrow{\widetilde{\Theta}} H_1$$

and equals $\mathrm{Hom}_{F,\varphi}(\mathcal{L}', \mathcal{A}^0_{cr,2}) \subset \mathrm{Hom}_{F,\varphi}(\mathcal{L}', \widetilde{\mathcal{A}}_{cr,2}) = \mathcal{V}_0(\mathcal{L}')$.

- $\mathrm{Im}(\mathcal{V}_0(C_p) \circ \Theta)$ appears as the image of the map

$$H_0 \xrightarrow{\Theta} H = p\mathcal{V}_0(\mathcal{L}') \subset \mathcal{V}_0(\mathcal{L}')$$

and equals $\mathrm{Hom}_{F,\varphi}(\mathcal{L}', p\mathcal{A}^0_{cr,2}) \subset \mathrm{Hom}_{F,\varphi}(\mathcal{L}', \widetilde{A}_{cr,2})$.

- $\widetilde{\mathcal{V}}^{ft}(\mathcal{L}) = \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{A}^0_{cr,2}/p\mathcal{A}^0_{cr,2})$.

Note that the correspondence

$$[r_0 \bmod x_0^p]T_1 + [r_0] + p[r_1] \mapsto (r_0 + x_0^p r_1) \bmod x_0^p \mathrm{m}_R$$

determines epimorphic map $\mathcal{A}^0_{cr,2}/p\mathcal{A}^0_{cr,2} \longrightarrow \mathcal{R}^0$ in the category $\underline{\widetilde{\mathcal{L}}}_0^*$ and this map induces isomorphism of $\Gamma_K$-modules $\widetilde{\mathcal{V}}^{ft}(\mathcal{L}[\gamma])$ and $\mathcal{V}^*(\mathcal{L}[\gamma])$.

**4.3. Properties of modified functor.** The following property was our main target.

**Theorem 4.2.** $\widetilde{\mathcal{CV}}^{ft}$ *is fully faithful.*

*Proof.* By devissage it will be sufficient to verify this statement on the level of the subcategories of killed by $p$ objects. The corresponding restriction of $\widetilde{\mathcal{CV}}^{ft}$ is equivalent then to the functor $\mathcal{CV}^*$ from Subsection 2 (cf. also the example in Subsection 4.2.3) but the functor $\widetilde{\mathcal{CV}}^*$ is fully faithful, cf. Subsection 2.4. □

Suppose $V$ is a finite dimensional vector space over $\mathbb{Q}_p$ with continuous $\Gamma_K$-action and $H$ a $\Gamma_K$-invariant lattice in $V$.

**Corollary 4.3.** *If $V$ is semi-stable (resp., crystalline) with HT weights from $[0,p)$ then the higher ramification subgroups $\Gamma_K^{(v)}$ act trivially on $H/pH$ for all $v > 2 - 1/p$ (resp., $v > 1$).*

*Proof.* As it was noted in Subsections 4.1–4.2, Breuil's functor allows us to obtain a Galois invariant lattice $H_0$ in $V$ in the form $H_0 = \varprojlim_n \mathcal{V}^t(\mathcal{M}_n)$, where $\{\mathcal{M}_n\}_{n\geqslant 0}$ is a $p$-divisible group in the category $\underline{\mathcal{L}}^{ft}$. Then $H_1 = \varprojlim_n \widetilde{\mathcal{V}}^{ft}(\mathcal{M}_n)$ is again a Galois invariant lattice in $V$. (Use that the $p$-divisible group $\{\widetilde{\mathcal{V}}^{ft}\}_{n\geqslant 0}$ is isogeneous to $\{\mathcal{V}^t(\mathcal{M}_n)\}_{n\geqslant 0}$, cf. Subsection 4.2.3.) We can assume that $H_1 \supset H \supset pH \supset p^m H_1$ with some $m \in \mathbb{N}$. Then by Theorem 4.2 there is a suibquotient modulo $p$,

$\mathcal{M} \in \underline{\mathcal{L}}^{ft}$ of $\mathcal{M}_m$, such that $\widetilde{\mathcal{CV}}^{ft}(\mathcal{M}) = (H/pH, \widetilde{\mathcal{V}}^{ft}(\mathcal{M}^{et}), \widetilde{\mathcal{V}}^{ft}(i^{et}))$. Therefore, the $\Gamma_K$-module $H/pH$ belongs to the image of the functor $\mathcal{V}^*$ and we can apply Theorem 3.1a). If $V$ is crystalline then $\mathcal{M} \in \underline{\mathcal{L}}_{cr}^{ft}$ and our assertion follows from Theorem 3.1b). $\square$

**Remark.** If in the above Corollary $V$ has HT weights from $[0, A]$, where $2 \leqslant A \leqslant p-2$. Then $\Gamma_K^{(v)}$ acts trivially on $H/pH$ for all $v > 1 + A/(p-1) - 1/p$ in the semi-stable case and for $v > A/(p-1)$ in the crystalline case. These ramification estimates have been proved in [24, 23, 7] and can be also obtained via methods from Subsection 3.

Let $\underline{\mathrm{CM}\Gamma}_{1,K}^{st}$, resp., $\underline{\mathrm{CM}\Gamma}_{1,K}^{cr}$, be the full subcategory in $\underline{\mathrm{CM}\Gamma}_K$ consisting of $\widetilde{\mathcal{CV}}^*(\mathcal{L})$ where $\mathcal{L}$ runs over the family of all objects of the category $\underline{\mathcal{L}}^*$, resp., $\underline{\mathcal{L}}_{cr}^*$.

Consider the simple objects $\mathcal{F}_j \in \underline{\mathrm{CM}\Gamma}_{1,K}^{cr} \subset \underline{\mathrm{CM}\Gamma}_{1,K}^{st}$ such that for $j=0$, $\mathcal{F}_0 = (\mathbb{F}_p, \mathbb{F}_p, \mathrm{id})$ and for $1 \leqslant j \leqslant p-1$, $\mathcal{F}_j = (\mathbb{F}_p(j), 0, 0)$, where $\mathbb{F}_p(j)$ is the $j$-th Tate twist. (The objects $\mathcal{F}_0$ and $\mathcal{F}_{p-1}$ have already appeared in Subsection 2.4.)

**Corollary 4.4.** a) *If $j_1 \geqslant j_2$ then* $\mathrm{Ext}_{\underline{\mathrm{CM}\Gamma}_{1,K}^{cr}}(\mathcal{F}_{j_1}, \mathcal{F}_{j_2}) = 0$.

b) *If $j_1 = 0$ or $j_2 = p-1$ then* $\mathrm{Ext}_{\underline{\mathrm{CM}\Gamma}_{1,K}^{st}}(\mathcal{F}_{j_1}, \mathcal{F}_{j_2}) = 0$.

*Proof.* This follows from the appropriate statements in $\underline{\mathcal{L}}_{cr}^*$ and $\underline{\mathcal{L}}^*$. As a matter of fact, the cases $j_1 = 0$ or $j_2 = p-1$ are just the existence of a maximal etale subobject and a maximal multiplicative quotient. In the case a), the appropriate statements in $\underline{\mathcal{L}}_{cr}^*$ are just easy exercises or one can use very general approach from Subsection 1.5 of [11]. $\square$

**Remark.** An analogue of property a) for $\underline{\mathrm{CM}\Gamma}_{1,K}^{st}$ is false because there are appropriate non-trivial extensions in the category $\underline{\mathcal{L}}^*$. Nevertheless, there is a chance to have such analogue in smaller category $\underline{\mathrm{CM}\Gamma}_{1,K}^{0,st}$ of the objects $\widetilde{\mathcal{CV}}^{ft}(\mathcal{L})$ such that $\mathcal{L} \in \underline{\mathcal{L}}^{ft}[1] := \{\mathcal{L} \in \underline{\mathcal{L}}^{ft} \mid p\mathcal{L} = 0\}$. A partial evidence for this is given in Subsection 5.5 below.

## 5. Generalization of the Shafarevich conjecture

As earlier, $p$ is a fixed prime number, $p > 2$. Suppose $k = \bar{\mathbb{F}}_p$, $K = W(k)[1/p]$, $\underline{\mathrm{M}\Gamma}_K$ and $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ are the categories of $\mathbb{Z}_p$-modules with continuous action of $\Gamma_K$ and, resp., $\Gamma_{\mathbb{Q}}$. Choose an extension of the $p$-adic valuation to $\bar{\mathbb{Q}}$ and use it to identify $\Gamma_K$ with a subgroup of $\Gamma_{\mathbb{Q}}$.

**5.1. The category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$.** The objects of the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$ are the pairs $H_{\mathbb{Q}} = (H, \widetilde{H}_{cr})$ such that

- $H \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ is unramified outside of $p$;

• $\widetilde{H}_{cr} = (H_{cr}, H^0, j) \in \underline{\mathrm{CM}\Gamma}_K^{cr}$ — the full subcategory in $\underline{\mathrm{CM}\Gamma}_K$ of the objects of the form $\widetilde{CV}^{ft}(\mathcal{L})$, where $\mathcal{L} \in \underline{\mathcal{L}}_{cr}^{ft}$;

• $H|_{\Gamma_K} = H_{cr}$.

• morphisms in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$ are compatible morphisms of Galois modules.

Clearly, $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$ is a special pre-abelian category.

Let $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}[1]$ be the subcategory in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$ consisting of all $H_{\mathbb{Q}} = (H, \widetilde{H}_{cr})$ such that $pH = 0$. Denote by $\mathbb{Q}_{cr}(p)$ the field-of-definition of all $H \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ such that $H_{\mathbb{Q}} = (H, \widetilde{H}_{cr}) \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}[1]$. In other words, $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}_{cr}(p))$ iff $\tau$ acts trivially on the first components $H$ of all $H_{\mathbb{Q}} \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}[1]$.

Let $\mathcal{H}_{\mathbb{Q}} = \{(H^{(n)}, \widetilde{H}_{cr}^{(n)})\}_{n\geqslant 0}$ be a $p$-divisible group in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$. Then $\mathcal{H} = \{H^{(n)}\}_{n\geqslant 0}$ is a $p$-divisible group in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$.

**Proposition 5.1.** *If $\mathbb{Q}_{cr}(p)$ is totally ramified at $p$ then there are $p$-divisible groups $\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \dots \supset \mathcal{H}_{p-1} \supset \mathcal{H}_p = 0$ in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ such that for all $0 \leqslant i < p$, $\mathcal{H}_i/\mathcal{H}_{i+1}$ is the product of several copies of the Tate twist $\mathbb{Q}_p/\mathbb{Z}_p(i)$ of the trivial $p$-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$.*

*Proof.* We have $\mathrm{Gal}(\mathbb{Q}_{cr}(p)/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}_{cr}(p)K/K)$. Therefore, we can apply local results about Galois modules from $\underline{\mathrm{CM}\Gamma}_{1,K}^{cr}$ to the objects of the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}[1]$. In particular, the tamely ramified part of $\mathrm{Gal}(\mathbb{Q}_{cr}(p)K/K)$ comes from $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ where $\zeta_p$ is a primitive $p$-th root of unity. (Indeed, it is a quotient of prime to $p$ order of the Galois group of the maximal abelian extension of $\mathbb{Q}$ unramified outside of $p$.) Therefore, any simple subquotient of $(H^{(1)}, \widetilde{H}_{cr}^{(1)}) \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$ comes from simple subquotients $\mathcal{F}_j$, $0 \leqslant j < p$, of $\widetilde{H}_{cr}^{(1)}$ (cf. Subsection 4.3 for the definition of $\mathcal{F}_j$). It remains to apply Corollary 4.4 and Theorem A.1 from Appendix. □

**5.2. The category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}$.** The objects of the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}$ are the pairs $H_{\mathbb{Q}} = (H, \widetilde{H}_{st})$ such that

• $H \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ is unramified outside of $p$;

• $\widetilde{H}_{st} = (H_{st}, H^0, j) \in \underline{\mathrm{CM}\Gamma}_K^{0,st}$ — the full subcategory in $\underline{\mathrm{CM}\Gamma}_K$ consisting of $\widetilde{CV}^{ft}(\mathcal{L})$ such that $\mathcal{L} \in \underline{\mathcal{L}}^{ft}$;

• $H|_{\Gamma_K} = H_{st}$;

• morphisms in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}$ are compatible morphisms of Galois modules.

Clearly, $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}$ is a special pre-abelian category.

Let $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}[1]$ be the subcategory in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}$ consisting of all $H_{\mathbb{Q}} = (H, \widetilde{H}_{st})$ such that $pH = 0$. Denote by $\mathbb{Q}_{st}(p)$ the field-of-definition of all $H \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ such that $H_{\mathbb{Q}} = (H, \widetilde{H}_{st}) \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,st}[1]$.

Let $\mathcal{H}_{\mathbb{Q}} = \{(H^{(n)}, \widetilde{H}_{cr}^{(n)})\}_{n\geqslant 0}$ be a $p$-divisible group in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{p,cr}$. Similarly to Proposition 5.1 we obtain the following property.

**Proposition 5.2.** *If $\mathbb{Q}_{st}(p)$ is totally ramified at $p$ then there are $p$-divisible groups $\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2$ in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ such that $\mathcal{H}_0/\mathcal{H}_1$ is the product of several copies of $\mathbb{Q}_p/\mathbb{Z}_p$, $\mathcal{H}_2$ is the product of several copies of $(\mathbb{Q}_p/\mathbb{Z}_p)(p-1)$ and all simple subquotients in $\mathcal{H}_1/\mathcal{H}_2$ come from objects $\mathcal{F}_j = (\mathbb{F}_p(j), 0, 0)$ with $1 \leqslant j \leqslant p-2$.*

**5.3. General criterion.** Suppose $X/\mathbb{Q}$ is a projective variety, $p$ is a prime number and $N \in \mathbb{N}$. Then $V = H_{et}^N(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$ is a finite dimensional $\mathbb{Q}_p$-vector space with continuous $\Gamma_{\mathbb{Q}}$-action.

**Proposition 5.3.** *Suppose there is a filtration of $\mathbb{Q}_p[\Gamma_{\mathbb{Q}}]$-modules*

$$V = V_0 \supset V_1 \supset \cdots \supset V_N \supset V_{N+1} = 0$$

*such that for all $i$, $V_i/V_{i+1} \simeq \mathbb{Q}_p(i)^{s_i}$, where $s_i \geqslant 0$ and $\mathbb{Q}_p(i)$ is the Tate twist. Then $h^{a,b}(X_{\mathbb{C}}) = 0$ if $a+b = N$ and $a \neq b$.*

*Proof.* The relation between the etale and de Rham cohomology of $X$ implies that $h^{a,b}(X_{\mathbb{C}}) = s_a$. Choose a prime $l \neq p$ such that the variety $X$ has good reduction modulo $l$. Then the corresponding Frobenius $\sigma_l$ acts on $V$ with eigenvalues $\lambda$ such that $|\lambda| = l^{N/2}$. But for any $i$, $\sigma_l$ acts on $\mathbb{Q}_p(i)$ via the multiplication by $l^i$. Therefore, $h^{a,b}(Y_{\mathbb{C}})$ with $a+b = N$ can be different from 0 only if $l^a = l^{N/2}$. ☐

**5.4. Crystalline case.** Suppose $X$ has everywhere good reduction. Consider $\mathbb{Q}_5[\Gamma_{\mathbb{Q}}]$-module $V = H_{et}^4(X_{\bar{Q}}, \mathbb{Q}_5)$. All subquotients of $V$ come from appropriate filtered modules associated with de Rham cohomology of $X$ via Breuil's functor. Therefore, any finite subquotient $H$ of $V$ appears as the first component of an appropriate object $H_{\mathbb{Q}} = (H, \widetilde{H}_{cr})$ of the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{5,cr}$. In particular, if $T$ is a Galois invariant lattice in $V$ then the 5-divisible group $\{T/5^n\}_{n\geqslant 0}$ appears as the first component of the appropriate 5-divisible group in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{5,cr}$. Therefore, part b) of Theorem 0.2 is implied by the following Proposition. (This Proposition was stated without proof in the end of [7].)

**Lemma 5.4.** *Modulo GRH (Generalised Riemann Hypothesis) the field $\mathbb{Q}_{cr}(5)$ is totally ramified at 5.*

*Proof.* Because the higher ramification subgroup $\Gamma_K^{(1)}$ acts trivially on $\mathbb{Q}_{cr}(5)$ the normalized discriminant of any subfield $L$ in $\mathbb{Q}_{cr}(5)$, which is finite over $\mathbb{Q}$, is less than $25 < d_{340}^*$, cf. [25]. Here for $N \in \mathbb{N}$, $d_N^*$ is the Odlyzko estimate for the normalized discriminant of algebraic number fields of given degree $N$ under GRH. Therefore, $[\mathbb{Q}_{cr}(5) : \mathbb{Q}] < 340$.

The maximal abelian extension of $\mathbb{Q}$ in $\mathbb{Q}_{cr}(5)$ equals $\mathbb{Q}(\zeta_{25})$, where $\zeta_{25}$ is a 25-th primitive root of unity. Let $L_2$ be the maximal abelian extension of $L_1$ inside $\mathbb{Q}_{cr}(5)$. The class number $h(L_1) = 1$ implies that $L_2$ is totally ramified at 5 over $\mathbb{Q}$ and $L_2/L_1$ is a 5-extension. Because the total degree is less than 340 we have $[L_2 : L_1] \leqslant 5$ and one can see that $L_2 = L_1(\sqrt[5]{2+\sqrt{5}})$. Then the maximal upper ramification number of this field extension is 1 and the maximal lower number is 8, therefore, the normalized discriminant of $L_2$ equals 21.6288... $< d^*_{160}$, cf. [25]. This implies that $h(L_2) = 1$, the maximal abelian extension $L_3$ of $L_2$ inside $\mathbb{Q}_{cr}(5)$ is totally ramified at 5 and $L_3/L_2$ is a 5-extension. But $[L_3 : L_2] \leqslant 3$ implies that $L_3 = L_2$. $\square$

**Remark.** a) For part a) of Theorem 0.2 take $V = H^3(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_5)$. Then the corresponding ramification estimates are better, cf. Subsection 4.3. As a result, one can use unconditional Odlyzko estimates to find that all modulo 5 subquotients of $V$ are defined over $\mathbb{Q}(\zeta_5, \sqrt[5]{\zeta_5 + \zeta_5^{-1}})$, cf. [7], Subsection 7.5.1, where this field was denoted by $\mathbb{Q}(5, 3)$.

b) Unconditional Odlyzko estimates are still sufficient to prove that $h^2(X_{\mathbb{C}}) = h^{1,1}(X_{\mathbb{C}})$, when $X$ has everywhere good reduction and is defined over $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{5})$, cf. Section 7 of [7].

**5.5. Semi-stable case.** Suppose $X$ has semi-stable reduction modulo 3 and good reduction modulo all primes $l \neq 3$. Consider $V = H^2_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_3)$ and proceed similarly to Subsection 5.4. We need the following lemma.

**Lemma 5.5.** $\mathbb{Q}_{st}(3)$ *is totally ramified at 3.*

*Proof.* For a complete proof cf. [11], Lemma 5.2. Note that the upper estimate for the normalized discriminant of $\mathbb{Q}_{st}(3)/\mathbb{Q}$ is $3^{3-1/3} < d_{238}$, therefore $[\mathbb{Q}_{st}(3) : \mathbb{Q}] < 238$. Because $K_1 = \mathbb{Q}(\sqrt[3]{3}, \exp(2\pi i/9))$ is contained in $\mathbb{Q}_{st}(3)$, the group $\mathrm{Gal}(\mathbb{Q}_{st}(3)/\mathbb{Q})$ is soluble. Then the proof that $\mathbb{Q}_{st}(3) = K_1$ requires calculations with fundamental units inside $K_1$. Namely, we need that:

- the class number of $K_1$ is 1;
- for any unit $u \in K_1^* \setminus K_1^{*3}$, $\sqrt[3]{u} \notin \mathbb{Q}_{st}(3)$.

The both properties were verified via the computing package SAGE, cf. www.sagemath.org and Appendix B of [11]. $\square$

Note that under the condition that $\mathbb{Q}_{st}(p)$ is totally ramified at $p$ we can identify $\underline{\mathrm{M}\Gamma}^{p,st}_{\mathbb{Q}}$ with full subcategories in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$ and in $\underline{\mathrm{CM}\Gamma}^{0,st}_{K}$.

Apply Proposition 5.2. Theorem 0.1 will be proved if we show that the 3-divisible group $\mathcal{H}_1/\mathcal{H}_2$ is the product of 3-divisible groups $(\mathbb{Q}_3/\mathbb{Z}_3)(1)$.

This would follow from $\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}^{3,st}_{\mathbb{Q}}[1]}(\mathcal{F}_1, \mathcal{F}_1) = 0$. Most natural way is to verify this in the category $\underline{\mathrm{CM}\Gamma}^{0,st}_{1,K}$ or, equivalently, in the category

$\underline{\mathcal{L}}^{ft}[1] = \{\mathcal{L} \in \underline{\mathcal{L}}^{ft} \mid p\mathcal{L} = 0\}$. Here we come again to the principal difference between the theories of torsion crystalline and torsion semi-stable Galois modules. In the semi-stable case we can efficiently work only with the Galois modules obtained from the objects of $\underline{\mathcal{L}}^t[1] = \{\mathcal{L} \in \underline{\mathcal{L}}^t \mid p\mathcal{L} = 0\}$ (which can be identified with $\underline{\mathcal{L}}^*$), and this category is strictly bigger than $\underline{\mathcal{L}}^{ft}[1]$. (In the crystalline case $\underline{\mathcal{L}}^t_{cr} = \underline{\mathcal{L}}^{ft}_{cr}$, cf. Subsection 4.2.1.)

If $\mathcal{L}_1 \in \underline{\mathcal{L}}^t$ is such that $\mathcal{V}^t(\mathcal{L}_1) = \mathcal{F}_1$, then $\mathrm{Ext}_{\underline{\mathcal{L}}^t[1]}(\mathcal{L}_1, \mathcal{L}_1) \neq 0$ and the question about $\mathrm{Ext}_{\underline{\mathcal{L}}^{ft}[1]}(\mathcal{L}_1, \mathcal{L}_1) = 0$ is still open. We did not resolve this issue in [11] but treated it in the following simpler situation. Remind that all points of the Galois modules coming from $\underline{\mathrm{M}\Gamma}^{3,st}_{\mathbb{Q}}$ are defined over relatively small field $K\mathbb{Q}_{st}(3)$. This allows us to replace the category $\underline{\mathcal{L}}^t[1]$ by a smaller one $\underline{\mathcal{L}}^t_{\mathbb{Q}}[1]$, where $\mathrm{Ext}_{\underline{\mathcal{L}}^t_{\mathbb{Q}}[1]}(\mathcal{L}_1, \mathcal{L}_1) = \mathbb{Z}/3\mathbb{Z}$ is generated by one object $\mathcal{L}_{11}$ (we use in [11], Subsection 5.4, slightly different notation). Then we prove that any object of $\underline{\mathcal{L}}^t_{\mathbb{Q}}[1]$, which has only subquotients of the form $\mathcal{L}_1$, is isomorphic to the product of several copies of $\mathcal{L}_1$ and $\mathcal{L}_{11}$.

Now come back to our 3-divisible group $\mathcal{H}_1/\mathcal{H}_2$ viewed as a 3-divisible group in the category $\underline{\mathrm{CM}\Gamma}^{0,st}_K$. If any subquotient of this group contains $H_{11} = \mathcal{C}\mathcal{V}^*(\mathcal{L}_{11})$ then we apply the devissage from Appendix A to deduce the existence of a 3-divisible group $\mathcal{H} = \{H^{(n)}\}_{n\geqslant 0}$ in $\underline{\mathrm{CM}\Gamma}^{0,st}_K$ such that $H^{(1)} = H_{11}$. The height of this 3-divisible group is 2 and it determines a 2-dimensional semi-stable $\mathbb{Q}_3[\Gamma_K]$-module. But the existence of such 2-dimensional representation contradicts to Theorem 6.1.1.2 from [14]. Therefore, all $\mathcal{L} \in \underline{\mathcal{L}}^{ft}[1]$ with simple subquotients isomorphic to $\mathcal{L}_1$ are just the products of copies of $\mathcal{L}_1$. In particular, $\mathrm{Ext}_{\underline{\mathcal{L}}^{ft}[1]}(\mathcal{L}_1, \mathcal{L}_1) = 0$ and $\mathcal{H}_2/\mathcal{H}_1$ is the product of copies of the trivial 3-divisible group $(\mathbb{Q}_3/\mathbb{Z}_3)(1)$.

**Remark.** The situation from Theorem 0.1 is quite exceptional. For semi-stable $\mathbb{Q}_p[\Gamma_K]$-modules with $p > 3$ and $2 \leqslant N < p$, the appropriate estimates for normalized discriminants of the fields-of-definition of their subquotients modulo $p$ are bigger than the appropriate Odlyzko estimates (even under GRH). In addition, any explicit calculations with elements of algebraic number fields of degree bigger, say, 100 are already very difficult (if possible).

## Appendix A. Formalism of pre-abelian categories

This is expository version of Appendix A of [11].

A pre-abelian category $\mathcal{C}$ is an additive category such that any its morphism has kernel and cokernel. A morphism $u$ of $\mathcal{C}$ is STRICT if the canonical map Coim $u :=$ Coker Ker $u \longrightarrow$ Im $u :=$ Ker Coker $u$ is isomorphism. Then $0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$ if $u$ is strict monomorphism, $v$ is strict epimorphism and $\mathrm{Coker}\,u = \mathrm{Ker}\,v$.

A pre-abelian category $\mathcal{C}$ is SPECIAL if the group of classes of equivalent short exact sequences is functorial in both arguments and there are standard 6-terms Hom − Ext exact sequences.

A typical example of special pre-abelian category is the category $\underline{\mathrm{FMod}}_R$ of free modules over a ring $R$ with filtration $F(H) \subset H$. The morphisms $\mathrm{Hom}_{\underline{\mathrm{FMod}}_R}((H, F(H)), (H_1, F(H_1))$ are morphisms $f : H \longrightarrow H_1$ of $R$-modules such that $f(F(H)) \subset F(H_1)$. The morphism $f$ is a strict monomorphism iff $H_1/f(H)$ has no $R$-torsion and $f$ is a strict epimorphism iff $f(H) = H_1$ and $f(F(H)) = F(H_1)$.

Denote by $\mathcal{C}(1)$ the full subcategory of killed by $p$ (i.e. such that $p\mathrm{id}_A = 0$) objects $A$ of a special pre-abelian category $\mathcal{C}$.

Mimicing Tate's definition [31] introduce the concept of a $p$-divisible object (or just $p$-divisible group if there is no risk of confusion) in $\mathcal{C}$.

Suppose $C = \{C^{(n)}\}_{n\geqslant 1}$ is a $p$-divisible group in $\mathcal{C}$. The following result provides us with very convenient devissage technique in $\mathcal{C}$. (For a complete proof of these statements see Theorems A.1 and A.2 of [11].)

**Theorem A.1.** a) *Suppose*

$$0 \longrightarrow D_1 \longrightarrow C^{(1)} \longrightarrow D_2 \longrightarrow 0$$

*is a short exact sequence in* $\mathcal{C}(1)$ *and* $\mathrm{Ext}_{\mathcal{C}(1)}(D_1, D_2) = 0$. *Then there is a short exact sequence of* $p$*-divisible groups*

$$0 \longrightarrow C_1 \longrightarrow C \longrightarrow C_2 \longrightarrow 0$$

*such that* $C_1^{(1)} = D_1$ *and* $C_2^{(1)} = D_2$.

b) *Suppose* $\mathrm{Ext}_{\mathcal{C}(1)}(C^{(1)}, C^{(1)}) = 0$ *then any* $p$*-divisible group* $D$ *in* $\mathcal{C}$ *such that* $D^{(1)} \simeq C^{(1)}$ *is isomorphic to* $C$.

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Department of Mathematical Sciences, Durham University, Science Laboratories, South Rd, Durham DH1 3LE, United Kingdom & Steklov Institute, Gubkina 8, Moscow 119991, Russia

*E-mail address*: victor.abrashkin@durham.ac.uk