On the Recognition of Four-Directional Orthogonal Ray Graphs*

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Abstract. Orthogonal ray graphs are the intersection graphs of horizontal and vertical rays (i.e. half-lines) in the plane. If the rays can have any possible orientation (left/right/up/down) then the graph is a 4-*directional orthogonal ray graph (4-DORG)*. Otherwise, if all rays are only pointing into the positive x and y directions, the intersection graph is a *2-DORG*. Similarly, for *3-DORGs*, the horizontal rays can have any direction but the vertical ones can only have the positive direction. The recognition problem of 2-DORGs, which are a nice subclass of bipartite comparability graphs, is known to be polynomial, while the recognition problems for 3-DORGs and 4-DORGs are open. Recently it has been shown that the recognition of unit grid intersection graphs, a superclass of 4-DORGs, is NP-complete. In this paper we prove that the recognition problem of 4-DORGs is polynomial, given a partition $\{L, R, U, D\}$ of the vertices of G (which corresponds to the four possible ray directions). For the proof, given the graph G, we first construct two cliques G_1, G_2 with both directed and undirected edges. Then we successively augment these two graphs, constructing eventually a graph $\widetilde{G}$ with both directed and undirected edges, such that G has a 4-DORG representation if and only if $\widetilde{G}$ has a transitive orientation respecting its directed edges. As a crucial tool for our analysis we introduce the notion of an S-*orientation* of a graph, which extends the notion of a transitive orientation. We expect that our proof ideas will be useful also in other situations. Using an independent approach we show that, given a permutation π of the vertices of U (π is the order of y-coordinates of ray endpoints for U), while the partition $\{L, R\}$ of $V \setminus U$ is not given, we can still efficiently check whether G has a 3-DORG representation.

1 Introduction

Segment graphs, i.e. the intersection graphs of segments in the plane, have been the subject of wide spread research activities (see e.g. [2, 12]). More tractable

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subclasses of segment graphs are obtained by restricting the number of directions for the segments to some fixed positive integer k [4, 11]. These graphs are called *k-directional segment graphs*. For the easiest case of $k = 2$ directions, segments can be assumed to be parallel to the x- and y-axis. If intersections of parallel segments are forbidden, then 2-directional segment graphs are bipartite and the corresponding class of graphs is also known as *grid intersection graphs* (GIG), see [9]. The recognition of GIGs is NP-complete [10].

Since segment graphs are a fairly complex class, it is natural to study the subclass of *ray intersection graphs* [1]. Again, the number of directions can be restricted by an integer k, which yields the class of *k-directional ray intersection graphs*. Particularly interesting is the case where all rays are parallel to the x- or y-axis. The resulting class is the class of *orthogonal ray graphs*, which the subject of this paper. A *k-directional orthogonal ray graph*, for short a k-DORG ($k \in \{2, 3, 4\}$), is an orthogonal ray graph with rays in k directions. If $k = 2$ we assume that all rays point in the positive x- and the positive y-direction, if $k = 3$ we additionally allow the negative x-direction.

The class of 2-DORGs was introduced in [19], where it is shown that the class of 2-DORGs coincides with the class of bipartite graphs whose complements are circular arc graphs, i.e. intersection graphs of arcs on a circle. This characterization implies the existence of a polynomial recognition algorithm (see [13]), as well as a characterization based on forbidden subgraphs [5]. Alternatively, 2-DORGs can also be characterized as the comparability graphs of ordered sets of height two and interval dimension two. This yields another polynomial recognition algorithm (see e.g. [7]), and due to the classification of 3-interval irreducible posets ([6], [21, sec 3.7]) a complete description of minimally forbidden subgraphs. In a very nice recent contribution on 2-DORGs [20], a clever solution has been presented for the jump number problem for the corresponding class of posets and shows a close connection between this problem and a hitting set problem for axis aligned rectangles in the plane.

4-DORGs in VLSI design. In [18] 4-DORGs were introduced as a mathematical model for defective nano-crossbars in PLA (programmable logic arrays) design. A nano-crossbar is a rectangular circuit board with $m \times n$ orthogonally crossing wires. Fabrication defects may lead to disconnected wires. The bipartite intersection graph that models the surviving crossbar is an orthogonal ray graph.

We briefly mention two problems for 4-DORGs that are tackled in [18]. One of them is that of finding, in a nano-crossbar with disconnected wire defects, a maximal surviving square (perfect) crossbar, which translates into finding a maximal k such that the balanced complete bipartite graph $K_{k,k}$ is a subgraph of the orthogonal ray graph modeling the crossbar. This *balanced biclique problem* is NP-complete for general bipartite graphs but turns out to be polynomially solvable on 4-DORGs [18]. The other problem, posed in [16], asks how difficult it is to find a subgraph that would model a given logic mapping and is shown in [18] to be NP-hard.

4-DORGs and UGIGs. A *unit grid intersection graph* (UGIG) is a GIG that admits an orthogonal segment representation with all segments of equal (unit)

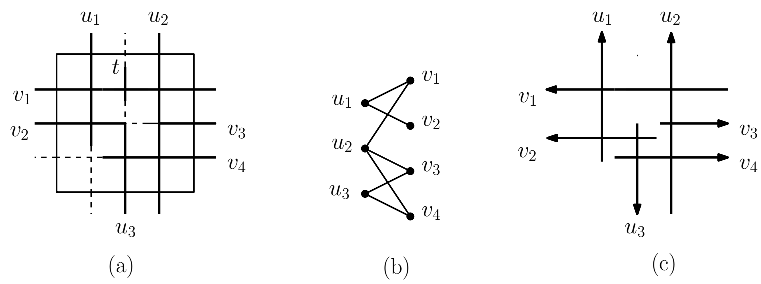


Fig. 1. (a) A nano-wire crossbar with disconnected wire defects, (b) the bipartite graph modeling this crossbar, and (c) a 4-DORG representation of this graph. Note that vertex t is not present, since the corresponding wire is not connected to the crossbar boundary, hence with the remaining circuit.

length. Every 4-DORG is a GIG. This can be seen by intersecting the ray representation with a rectangle R, that contains all intersections between the rays in the interior. To see that every 4-DORG is a UGIG, we first fix an appropriate length for the segments, e.g. the length d of the diagonal of R. If we only keep the initial part of length d from each ray we get a UGIG representation. Essentially this construction was already used in [18].

Unit grid intersection graphs were considered in [15]. There it is shown that UGIG contains P_6-free bipartite graphs, interval bigraphs and bipartite permutation graphs. Actually, these classes are already contained in 2-DORG. Another contribution of [15] is to provide an example showing that the inclusion of UGIG in GIG is proper. In [17] it is shown that interval bigraphs belong to UGIG. Hardness of Hamiltonian cycle for inputs from UGIG and hardness of graph isomorphism for inputs from GIG have been shown in [22]. Very recently it was shown that the recognition of UGIGs is NP-complete [14]. With this last result we find 4-DORG nested between 2-DORG and UGIG with easy and hard recognition, respectively. This fact was central for our motivation to attack the open recognition problem for 4-DORGs [19].

Our contribution. In this paper we prove that, given a graph G along with a partition $\{L, R, U, D\}$ of its vertices, it can be efficiently checked whether G has a 4-DORG representation such that the vertices of L (resp. the vertices of R, U, D) correspond to the rays pointing leftwards (resp. rightwards, upwards, downwards). To obtain our result, we first construct two cliques G_1, G_2 that have both directed and undirected edges. Then we iteratively augment G_1 and G_2, constructing eventually a graph $\widetilde{G}$ with both directed and undirected edges. As we prove, the input graph G has a 4-DORG representation if and only if $\widetilde{G}$ has a transitive orientation respecting its directed edges. As a crucial tool for our results, we introduce the notion of an *S-orientation* of an arbitrary graph, which extends the notion of a transitive orientation. By setting $D = \emptyset$, our results trivially imply that, given a partition $\{L, R, U\}$ of the vertices of G, it can be efficiently checked whether G has a 3-DORG representation according to this partition. With an independent approach, we show that if we are given a permutation π of the vertices of U (which represents the order of y-coordinates of ray-endpoints for the set U) but the partition $\{L, R\}$ of $V \setminus U$ is unknown, then we can still efficiently check whether G has a 3-DORG representation. The method we use to prove this result can be viewed as a particular *partition refinement technique*. Such techniques have various applications in string sorting, automaton minimization, and graph algorithms (see [8] for an overview).

Notation. We consider in this article simple undirected and directed graphs. For a graph G, we denote its vertex and edge set by $V(G)$ and $E(G)$, respectively. In an undirected graph G, the edge between vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. The set $N(v) = \{u \in V : uv \in E\}$ is called the *neighborhood* of the vertex v of G. If the graph G is directed, we denote by $\langle uv \rangle$ the oriented arc from u to v. If G is the complete graph (i.e. a clique), we call an orientation λ of all (resp. of some) edges of G a (*partial*) *tournament* of G. If in addition λ is transitive, then we call it a (partial) transitive tournament. Given two matrices A and B of size $n \times n$ each, we call by $O(\mathrm{MM}(n))$ the time needed by the fastest known algorithm for multiplying A and B; currently this can be done in $O(n^{2.376})$ time [3].

Let G be a 4-DORG. Then, in a 4-DORG representation of G, every ray is completely determined by one point on the plane and the direction of the ray. We call this point the *endpoint* of this ray. Given a 4-DORG G along with a 4-DORG representation of it, we may not distinguish in the following between a vertex of G and the corresponding ray in the representation, whenever it is clear from the context. Furthermore, for any vertex u of G we will denote by u_x and u_y the x-coordinate and the y-coordinate of the endpoint of the ray of u in the representation, respectively.

2 4-Directional Orthogonal Ray Graphs

In this section we investigate some fundamental properties of 4-DORGs and their representations, which will then be used for our recognition algorithm. The next observation on a 4-DORG representation is crucial for the rest of the section.

Observation 1 *Let $G = (V, E)$ be a graph that admits a 4-DORG representation, in which L (resp. R, U, D) is the set of leftwards (resp. rightwards, upwards, downwards) oriented rays. If $u \in U$ and $v \in R$ (resp. $v \in L$), then $uv \in E$ if and only if $u_x > v_x$ (resp. $u_x < v_x$) and $u_y < v_y$. Similarly, if $u \in D$ and $v \in R$ (resp. $v \in L$), then $uv \in E$ if and only if $u_x > v_x$ (resp. $u_x < v_x$) and $u_y > v_y$.*

For the remainder of the section, let $G = (V, E)$ be an arbitrary input graph with vertex partition $V = L \cup R \cup U \cup D$, such that $E \subseteq (L \cup R) \times (U \cup D)$.

The oriented cliques G_1 and G_2. In order to decide whether the input graph $G = (V, E)$ admits a 4-DORG representation, in which L (resp. R, U, D) is the set of leftwards (resp. rightwards, upwards, downwards) oriented rays, we first construct two auxiliary cliques G_1 and G_2 with $|V|$ vertices each. We partition the vertices of G_1 (resp. G_2) into the sets L_x, R_x, U_x, D_x (resp. L_y, R_y, U_y, D_y).

The intuition behind this notation for the vertices of G_1 and G_2 is that, if G has a 4-DORG representation with respect to the partition $\{L, R, U, D\}$, then each of these vertices of G_1 (resp. G_2) corresponds to the x-coordinate (resp. y-coordinate) of the endpoint of a ray of G in this representation.

We can now define some orientation of the edges of G_1 and G_2. The intuition behind these orientations comes from Observation 1: if the input graph G is a 4-DORG, then it admits a 4-DORG representation such that, for every $u \in U \cup D$ and $v \in L \cup R$, we have that $u_x > v_x$ (resp. $u_y > v_y$) in this representation if and only if $\langle u_x v_x \rangle$ (resp. $\langle u_y v_y \rangle$) is an oriented edge of the clique G_1 (resp. G_2). That is, since all x-coordinates (resp. y-coordinates) of the endpoints of the rays in a 4-DORG representation can be linearly ordered, these orientations of the edges of G_1 (resp. G_2) build a transitive tournament.

Therefore, the input graph G admits a 4-DORG representation if and only if some edges of G_1, G_2 are forced to have specific orientations in these transitive tournaments of G_1 and G_2, while some pairs of edges of G_1, G_2 are not allowed to have a specific *pair* of orientations in these tournaments. Motivated by this, we introduce in the next two definitions the notions of *type-1-mandatory* orientations and of *forbidden pairs* of orientations, which will be crucial for our analysis in the remainder of Section 2.

Definition 1 (type-1-mandatory orientations). *Let $u \in U \cup D$ and $v \in L \cup R$, such that $uv \in E$. If $u \in U$ and $v \in R$ (resp. $v \in L$) then the orientations $\langle u_x v_x \rangle$ (resp. $\langle v_x u_x \rangle$) and $\langle v_y u_y \rangle$ of G_1 and G_2 are called* type-1-mandatory*. If $u \in D$ and $v \in R$ (resp. $v \in L$) then the orientations $\langle u_x v_x \rangle$ (resp. $\langle v_x u_x \rangle$) and $\langle u_y v_y \rangle$ of G_1 and G_2 are called* type-1-mandatory*. The set of all type-1-mandatory orientations of G_1 and G_2 is denoted by M_1.*

Definition 2 (forbidden pairs of orientations). *Let $u \in U \cup D$ and $v \in R \cup L$, such that $uv \notin E$. If $u \in U$ and $v \in R$ (resp. $v \in L$) then the pair $\{\langle u_x v_x \rangle, \langle v_y u_y \rangle\}$ (resp. the pair $\{\langle v_x u_x \rangle, \langle v_y u_y \rangle\}$) of orientations of G_1 and G_2 is called* forbidden*. If $u \in D$ and $v \in R$ (resp. $v \in L$) then the pair $\{\langle u_x v_x \rangle, \langle u_y v_y \rangle\}$ (resp. the pair $\{\langle v_x u_x \rangle, \langle u_y v_y \rangle\}$) of orientations of G_1 and G_2 is called* forbidden*.*

For simplicity of notation in the remainder of the paper, we introduce in the next definition the notion of *optional edges.*

Definition 3 (optional edges). *Let $\{\langle pq \rangle, \langle ab \rangle\}$ be a pair of forbidden orientations of G_1 and G_2. Then each of the (undirected)* edges *pq and ab is called* optional *edges.*

The augmented oriented cliques G_1^* and G_2^*. We iteratively augment the cliques G_1 and G_2 into the two larger cliques G_1^* and G_2^*, respectively, as follows. For every *optional* edge pq of G_1 (resp. of G_2), where $p \in U_x \cup D_x$ and $q \in L_x \cup R_x$ (resp. $p \in U_y \cup D_y$ and $q \in L_y \cup R_y$), we add two vertices $r_{p,q}$ and $r_{q,p}$ and we add all needed edges to make the resulting graph G_1^* (resp. G_2^*) a clique. Note that, if the initial graph G has n vertices and m non-edges (i.e. $\binom{n}{2} - m$ edges), then G_1^* and G_2^* are cliques with $n+2m$ vertices each. We now introduce the notion of *type-2-mandatory* orientations of G_1^* and G_2^*.

Definition 4 (type-2-mandatory orientations). *For every optional edge pq of G_1^*, the orientations $\langle pr_{p,q}\rangle$ and $\langle qr_{q,p}\rangle$ of G_1^* are called* type-2-mandatory *orientations of G_1^*. For every optional edge pq of G_2^*, the orientations $\langle r_{p,q}p\rangle$ and $\langle r_{q,p}q\rangle$ of G_2^* are called* type-2-mandatory *orientations of G_2^*. The set of all type-2-mandatory orientations of G_1^* and G_2^* is denoted by M_2.*

The coupling of G_1^* and G_2^* into the oriented clique G^*. Now we iteratively construct the clique G^* from the cliques G_1^* and G_2^*, as follows. Initially G^* is the union of G_1^* and G_2^*, together with all needed edges such that G^* is a clique. Then, for every pair $\{\langle pq\rangle, \langle ab\rangle\}$ of *forbidden orientations* of G_1^* and G_2^* (where $pq \in E(G_1)$ and $ab \in E(G_2)$, cf. Definition 2), we *merge* in G^* the vertices $r_{b,a}$ and $r_{p,q}$, i.e. we have $r_{b,a} = r_{p,q}$ in G^*. Recall that each of the cliques G_1^* and G_2^* has $n+2m$ vertices. Therefore, since G_1^* and G_2^* have m pairs $\{\langle pq\rangle, \langle ab\rangle\}$ of forbidden orientations, the resulting clique G^* has $2n+3m$ vertices. We now introduce the notion of *type-3-mandatory* orientations of G^*.

Definition 5 (type-3-mandatory orientations). *For every pair $\{\langle pq\rangle, \langle ab\rangle\}$ of forbidden orientations of G_1^* and G_2^*, the orientation $\langle r_{q,p}r_{a,b}\rangle$ is called a* type-3-mandatory *orientation of G^*. The set of all type-3-mandatory orientations of G^* is denoted by M_3.*

Whenever the orientation of an edge uv of G^* is type-1 (resp. type-2, type-3)-mandatory, we may say for simplicity that the *edge uv* (instead of its *orientation*) is type-1 (resp. type-2, type-3)-mandatory. An example for the construction of G^* from G_1^* and G_2^* is illustrated in Figure 2, where it is shown how two optional edges $pq \in E(G_1^*)$ and $ab \in E(G_2^*)$ are joined together in G^*, where $\{\langle pq\rangle, \langle ab\rangle\}$ is a pair of forbidden orientations of G_1^* and G_2^*. For simplicity of the presentation, only the optional edges pq and ab, the type-2-mandatory edges $pr_{p,q}$, $qr_{q,p}$, $ar_{a,b}$, $br_{b,a}$, and the edges $r_{p,q}r_{q,p}$ and $r_{a,b}r_{b,a}$ are shown in Figure 2. Furthermore, the type-2-mandatory orientations $\langle pr_{p,q}\rangle$, $\langle qr_{q,p}\rangle$, $\langle r_{a,b}a\rangle$, and $\langle r_{b,a}b\rangle$, as well as the type-3-mandatory orientation $\langle r_{q,p}r_{a,b}\rangle$, are drawn with double arrows in Figure 2 for better visibility.

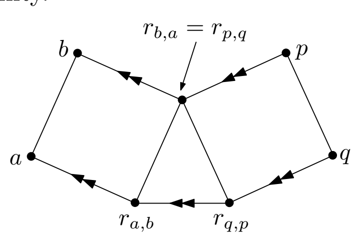


Fig. 2. An example of joining in G^* the pair of optional edges $\{pq, ab\}$, where $pq \in E(G_1)$ and $ab \in E(G_2)$.

In the next theorem we provide a characterization of 4-DORGs in terms of a transitive tournament λ^* of the clique G^*. The main novelty of the characterization of Theorem 1 is that it does not rely on the *forbidden pairs* of orientations. This characterization will be used in Section 4, in order to provide our main result of the paper, namely the recognition of 4-DORGs with respect to the vertex partition $\{L, R, U, D\}$.

Theorem 1. *The next two conditions are equivalent:*

1. *The graph $G = (V, E)$ with n vertices has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$.*
2. *There exists a transitive tournament λ^* of G^*, such that $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$, and in addition:*
   (a) let pq be an optional edge of G_1^* and $pw \notin M_2$ be an incident edge of pq in G_1^*; then $\langle wr_{p,q}\rangle \in \lambda^*$ implies that $\langle wp\rangle \in \lambda^*$,
   (b) let pq be an optional edge of G_2^* and $pw \notin M_2$ be an incident edge of pq in G_2^*; then $\langle r_{p,q}w\rangle \in \lambda^*$ implies that $\langle pw\rangle \in \lambda^*$,
   (c) let pq be an optional edge of G_1^* (resp. G_2^*), where $p \in U_x \cup D_x$ (resp. $p \in U_y \cup D_y$); then we have:
      (i) either $\langle pq\rangle, \langle r_{p,q}q\rangle, \langle r_{p,q}r_{q,p}\rangle \in \lambda^*$ or $\langle qp\rangle$, $\langle qr_{p,q}\rangle$, $\langle r_{q,p}r_{p,q}\rangle \in \lambda^*$,
      (ii) for any incident optional edge pq' of G_1^* (resp. G_2^*), either $\langle pq\rangle, \langle r_{p,q'}q\rangle \in \lambda^*$ or $\langle qp\rangle, \langle qr_{p,q'}\rangle \in \lambda^*$,
      (iii) for any incident optional edge $p'q$ of G_1^* (resp. G_2^*), either $\langle r_{p,q}q\rangle, \langle r_{p,q}r_{q,p'}\rangle \in \lambda^*$ or $\langle qr_{p,q}\rangle, \langle r_{q,p'}r_{p,q}\rangle \in \lambda^*$.

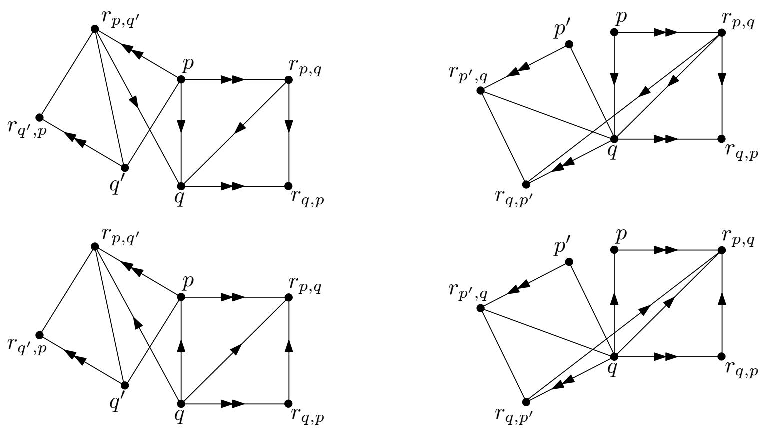
Furthermore, as we can prove, given a transitive tournament λ^* of G^* as in Theorem 1, a 4-DORG representation of G can be computed in $O(n^2)$ time. An example of the orientations of condition 2(c) in Theorem 1 (for the case of G_1^*) is shown in Figure 3. For simplicity of the presentation, although G_1^* is a clique, we show in Figure 3 only the edges that are needed to illustrate Theorem 1.

3 *S*-orientations of graphs

In this section we introduce a new way of augmenting an *arbitrary* graph G by adding a new vertex and some new edges to G. This type of augmentation process is done with respect to a particular edge $e_i = x_i y_i$ of the graph G, and is called the *deactivation* of e_i in G. In order to do so, we first introduce the crucial notion of an *S-orientation* of a graph G (cf. Definition 7), which extends the classical notion of a transitive orientation. For the remainder of this section, G denotes an arbitrary graph, and not the input graph discussed in Section 2.

Definition 6. *Let $G = (V, E)$ be a graph and let (x_i, y_i), $1 \le i \le k$, be k* ordered *pairs of vertices of G, where $x_i y_i \in E$. Let V_{out}, V_{in} be two* disjoint *vertex subsets of G, where $\{x_i : 1 \le i \le k\} \subseteq V_{out} \cup V_{in}$. For every $i = 1, 2, \ldots, k$:*

- *a* special neighborhood *of x_i is a vertex subset $S(x_i) \subseteq \left(N(x_i) \cap \left(\bigcap_{x_j = x_i} N(y_j)\right)\right) \setminus \{x_j : 1 \le j \le k\}$,*
- *the* forced neighborhood orientation *of x_i is:*
  - *the set $F(x_i) = \{\langle x_i z\rangle : z \in S(x_i)\}$ of oriented edges of G, if $x_i \in V_{out}$,*
  - *the set $F(x_i) = \{\langle z x_i\rangle : z \in S(x_i)\}$ of oriented edges of G, if $x_i \in V_{in}$.*



(a) (b)

Fig. 3. An example of the orientations of the clique G_1^* in the transitive tournament λ^*, where $p \in U_x \cup D_x$ (cf. condition 2(c) in Theorem 1): (a) both possible orientations where the optional edges pq and pq' are incident and (b) both possible orientations where the optional edges pq and $p'q$ are incident. In both (a) and (b), the orientations of the type-2-mandatory edges are drawn with double arrows. The case for G_2 is the same, except that the orientation of the type-2-mandatory edges is the opposite.

Definition 7. *Let $G = (V, E)$ be a graph. For every $i = 1, 2, \ldots, k$ let $S(x_i)$ be a special neighborhood in G. Let T be a transitive orientation of G. Then T is an* S-orientation *of G on the special neighborhoods $S(x_i)$, $1 \leq i \leq k$, if for every $i = 1, 2, \ldots, k$:*

1. *$F(x_i) \subseteq T$ and*
2. *for every $z \in S(x_i)$, $\langle x_i y_i \rangle \in T$ if and only if $\langle z y_i \rangle \in T$.*

Definition 8. *Let $G = (V, E)$ be a graph. For every $i = 1, 2, \ldots, k$ let $S(x_i)$ be a special neighborhood in G. Let T be an S-orientation of G on the sets $S(x_i)$, $1 \leq i \leq k$. Then T is* consistent *if, for every $i = 1, 2, \ldots, k$, it satisfies the following conditions, whenever $zw \in E$, where $z \in S(x_i)$ and $w \in (N(x_i) \cap N(y_i)) \setminus S(x_i)$:*

- *if $x_i \in V_{out}$, then $\langle wz \rangle \in T$ implies that $\langle w x_i \rangle \in T$,*
- *if $x_i \in V_{in}$, then $\langle zw \rangle \in T$ implies that $\langle x_i w \rangle \in T$.*

In the next definition we introduce the notion of *deactivating* an edge $e_i = x_i y_i$ of a graph G, where $S(x_i)$ is a special neighborhood in G. In order to deactivate edge e_i of G, we augment appropriately the graph G, obtaining a new graph $\widetilde{G}(e_i)$ that has one new vertex.

Definition 9. *Let $G = (V, E)$ be a graph and let $S(x_i)$ be a special neighborhood in G. The graph $\widetilde{G}(e_i)$ obtained by* deactivating *the edge $e_i = x_i y_i$ (with respect to S_i) is defined as follows:*

1. *$V(\widetilde{G}(e_i)) = V \cup \{a_i\}$ (i.e. add a new vertex a_i to G),*
2. *$E(\widetilde{G}(e_i)) = E \cup \{z a_i : z \in N(x_i) \setminus S(x_i)\}$.*

Algorithm 1 Recognition of 4-DORGs

Input: An undirected graph $G = (V, E)$ with a vertex partition $V = L \cup R \cup U \cup D$
Output: A 4-DORG representation for G, or the announcement that G is not a 4-DORG graph

$n \leftarrow |V|$; $\quad m \leftarrow \binom{n}{2} - |E|$ $\{m$ is the number of non-edges in $G\}$
Construct from G the clique G_1 with vertex set $L_x \cup R_x \cup U_x \cup D_x$ and the clique G_2 with vertex set $L_y \cup R_y \cup U_y \cup D_y$
Construct the set M_1 of type-1-mandatory orientations in G_1 and G_2
Construct the m forbidden pairs of orientations of G_1 and G_2
Construct from G_1, G_2 the augmented cliques G_1^*, G_2^* and the set M_2 of type-2-mandatory orientations
Construct from G_1^*, G_2^* the clique G^* and the set M_3 of type-3-mandatory orientations
for $i = 1$ to m **do**
  Let $p_i q_i \in E(G_1), a_i b_i \in E(G_2)$ be the optional edges in the ith pair of forbidden orientations, where $p_i \in U_x \cup D_x$, $q_i \in L_x \cup R_x$, $a_i \in U_y \cup D_y$, $b_i \in L_y \cup R_y$
  $(x_{2i-1}, y_{2i-1}) \leftarrow (p_i, q_i)$; $\quad (x_{2i}, y_{2i}) \leftarrow (q_i, r_{p_i, q_i})$
  $(x_{2m+2i-1}, y_{2m+2i-1}) \leftarrow (a_i, b_i)$; $\quad (x_{2m+2i}, y_{2m+2i}) \leftarrow (b_i, r_{a_i, b_i})$
  $S(x_i) \leftarrow \{r_{x_j, y_i} : x_j = x_i\}$
Construct the graph $\widetilde{G}^*$ by iteratively deactivating all edges $x_i y_i$, $1 \leq i \leq 4m$
if $\widetilde{G}^*$ has a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$ **then**
  return the 4-DORG representation of G computed by Theorem 1
else
  return "G is not a 4-DORG graph with respect to the partition $\{L, R, U, D\}$"

After deactivating the edge e_k of G, obtaining the graph $\widetilde{G}(e_k)$, we can continue by sequentially deactivating the edges $e_{k-1}, e_{k-2}, \ldots, e_1$, obtaining eventually the graph $\widetilde{G}$.

Theorem 2. *Let $G = (V, E)$ be a graph and $S(x_i)$, $1 \leq i \leq k$, be a set of k special neighborhoods in G. Let M_0 be an arbitrary set of edge orientations of G, and let $\widetilde{G}$ be the graph obtained after deactivating all edges $e_i = x_i y_i$, where $1 \leq i \leq k$.*

- *If G has a consistent S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$, then $\widetilde{G}$ has a transitive orientation $\widetilde{T}$ such that $M_0 \cup F(x_i) \subseteq \widetilde{T}$ for every $i = 1, 2, \ldots, k$.*
- *If $\widetilde{G}$ has a transitive orientation $\widetilde{T}$ such that $M_0 \cup F(x_i) \subseteq \widetilde{T}$ for every $i = 1, 2, \ldots, k$, then G has an S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$.*

4 Efficient Recognition of 4-DORGs

In this section we complete our analysis in Sections 2 and 3 and we present our 4-DORG recognition algorithm (cf. Algorithm 1). Let $G = (V, E)$ be an arbitrary input graph that is given along with a vertex partition $V = L \cup R \cup U \cup D$, such that $E \subseteq (L \cup R) \times (U \cup D)$. Assume that G has n vertices and m non-edges (i.e. $\binom{n}{2} - m$ edges). First we construct from G the cliques G_1, G_2, then we construct the augmented cliques G_1^*, G_2^*, and finally we combine G_1^* and G_2^* to produce the clique G^* (cf. Section 2). Then, for a specific choice of $4m$ ordered pairs (x_i, y_i) of vertices, where $1 \leq i \leq 4m$ (cf. Algorithm 1), and for particular sets $S(x_i)$ and neighborhood orientations $F(x_i)$, $1 \leq i \leq 4m$ (cf. Definitions 6 and 7), we iteratively deactivate the edges $x_i y_i$, $1 \leq i \leq 4m$ (cf. Section 3), constructing thus the graph $\widetilde{G}^*$. Then, we can prove that for a specific partial orientation of the graph $\widetilde{G}^*$, $\widetilde{G}^*$ has a transitive orientation that extends this partial orientation if and only if the input graph G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. The proof of correctness of Algorithm 1 and the timing analysis are given in the next theorem.

Theorem 3. *Let $G = (V, E)$ be a graph with n vertices, given along with a vertex partition $V = L \cup R \cup U \cup D$, such that $E \subseteq (L \cup R) \times (U \cup D)$. Then Algorithm 1 constructs in $O(\mathrm{MM}(n^2))$ time a 4-DORG representation for G with respect to this vertex partition, or correctly announces that G does not have a 4-DORG representation.*

5 Recognizing 3-DORGs with partial representation restrictions

In this section we consider a bipartite graph $G = (A, B, E)$, where $|A| = m$ and $|B| = n$, given along with an ordering $\pi = (v_1, v_2, \ldots, v_m)$ of the vertices of A. The question we address is the following: "Does G admit a 3-DORG representation where A (resp. B) is the set of rays oriented upwards (resp. horizontal, i.e. either leftwards or rightwards), such that, whenever $1 \leq i < j \leq m$, the y-coordinate of the endpoint of $v_i \in A$ is greater than that of $v_j \in A$?" Our approach uses the adjacency relations in G to recursively construct an x-coordinate ordering of the endpoints of the rays in the set A. If during the process we do not reach a contradiction, we eventually construct a 3-DORG representation for G, otherwise we conclude that such a representation does not exist.

Definition 10. *Let P_1, P_2 be two ordered partitions of the same base set S. Then P_1 and P_2 are* compatible *if there exists an ordered partition R of S which is refining and order preserving for both P_1 and P_2. A linear order L* respects *an ordered partition P of S, if L and P are compatible.*

Here we provide the main ideas and an overview of our algorithm. We start with the trivial partition of the set A (consisting of a single set including all elements of A). During the algorithm we process each vertex of $V = A \cup B$ once, and each time we process a new vertex we refine the current partition of the vertices of A, where the final partition of A implies an x-coordinate ordering of the rays of A. In particular, the algorithm proceeds in $|A| = m$ phases, where during phase i we process vertex $v_i \in A$ (the sequence of the vertices in A is

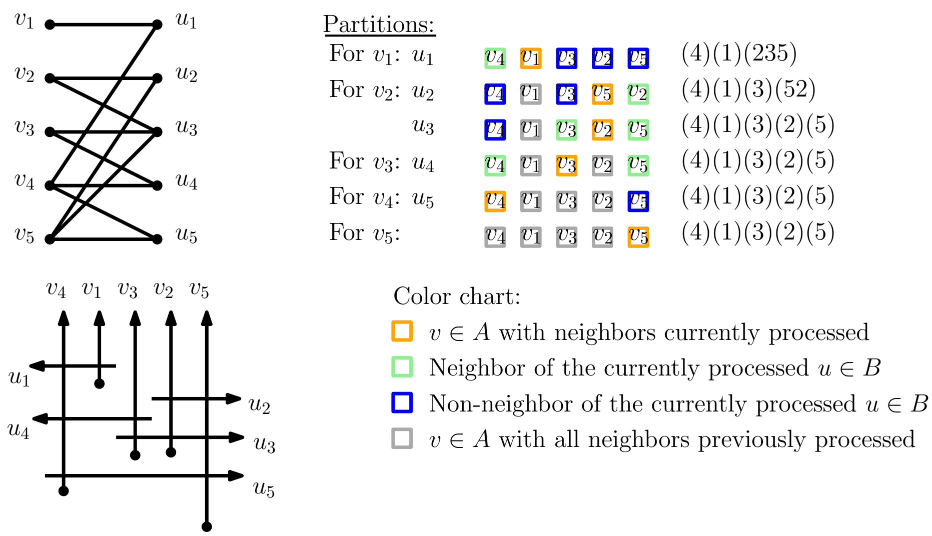


Fig. 4. Construction of a 3-DORG representation. Top left of the figure: the bipartite graph G with the given vertex ordering $\pi = (v_1, v_2, v_3, v_4, v_5)$. Top-right: the chain of partition refinements. Bottom left: The 3-DORG representation of G as read from the partition chain.

according to the given ordering π). During phase i, we process sequentially every neighbor $u \in N(v_i) \subseteq B$ that has not been processed in any previous phase $j < i$.

For every $i = 1, 2, \ldots, m$ let $A_i = \{v_i, v_{i+1}, \ldots, v_m\}$ be the set of vertices of A that have not been processed before phase i. At the end of every phase i, we fix the position of vertex $v_i \in A$ in the final partition of A, and we ignore v_i in the subsequent phases (i.e. during the phases $j > i$ we consider only the restriction of the current partition to the vertices of A_{i+1}). Phase i starts with the partition of A_i that results at the end of phase $i-1$. For any vertex $u \in N(v_i)$ that we process during phase i, we check whether the current partition P of A_i is compatible with at least one of the ordered partitions $Q_1 = (N(u), A_i \setminus N(u))$ and $Q_2 = (A_i \setminus N(u), N(u))$. If not, then we conclude that G is not a 3-DORG with respect to the given ordering π of A. Otherwise we refine the current partition P into an ordered partition that is also a refinement of Q_1 (resp. Q_2). In the case where P is compatible with both Q_1 and Q_2, it does not matter if we compute a common refinement of P with Q_1 or Q_2. If we can execute all m phases of this algorithm without returning that a 3-DORG representation does not exist, then we can compute a 3-DORG representation of G in which the y-coordinates of the endpoints of the rays of A respect the ordering π. In this extended abstract this construction is illustrated in the example of Figure 4.

Theorem 4. *Given a bipartite graph $G = (V, E)$ with color classes A, B and an ordering π of A, we can decide in $O(|V|^2)$ time whether G admits a 3-DORG representation where A are the vertical rays and the y-coordinates of their endpoints respect the ordering π.*

References

1. S. Cabello, J. Cardinal, and S. Langerman. The clique problem in ray intersection graphs. In *ESA*, pages 241–252, 2012.
2. J. Chalopin and D. Gonçalves. Every planar graph is the intersection graph of segments in the plane: extended abstract. In *Proc STOC '09*, pages 631–638, 2009.
3. D. Coppersmith and S. Winograd. Matrix multiplication via arithmetic progressions. *Journal of Symbolic Computation*, 9(3):251–280, 1990.
4. V. Estivill-Castro, M. Noy, and J. Urrutia. On the chromatic number of tree graphs. *Discrete Mathematics*, 223(1-3):363–366, 2000.
5. T. Feder, P. Hell, and J. Huang. List homomorphisms and circular arc graphs. *Combinatorica*, 19:487–505, 1999.
6. S. Felsner. 3-interval irreducible partially ordered sets. *Order*, 11:12–5, 1994.
7. S. Felsner, M. Habib, and R. H. Möhring. On the interplay of interval dimension and dimension. *SIAM Journal on Discrete Mathematics*, 7:32–40, 1994.
8. M. Habib, C. Paul, and L. Viennot. Partition refinement techniques: An interesting algorithmic tool kit. *Int. J. Found. Comput. Sci.*, 10(2):147–170, 1999.
9. I.-A. Hartman, I. Newman, and R. Ziv. On grid intersection graphs. *Discrete Mathematics*, 87(1):41 – 52, 1991.
10. J. Kratochvíl. A special planar satisfiability problem and a consequence of its NP-completeness. *Discrete Applied Mathematics*, 52(3):233–252, 1994.
11. J. Kratochvíl and J. Matoušek. NP-hardness results for intersection graphs. *Commentationes Mathematicae Universitatis Carolinae*, 30(4):761–773, 1989.
12. J. Kratochvíl and J. Matoušek. Intersection graphs of segments. *J. Comb. Theory, Ser. B*, 62(2):289–315, 1994.
13. R. McConnell. Linear-time recognition of circular-arc graphs. In *Foundations of Computer Science, 2001. Proceedings. 42nd IEEE Symposium on*, pages 386–394, 2001.
14. I. Mustaţă and M. Pergel. Unit grid intersection graphs: recognition and properties, 2013. in preparation.
15. Y. Otachi, Y. Okamoto, and K. Yamazaki. Relationships between the class of unit grid intersection graphs and other classes of bipartite graphs. *Discrete Applied Mathematics*, 155(17):2383 – 2390, 2007.
16. W. Rao, A. Orailoglu, and R. Karri. Logic mapping in crossbar-based nanoarchitectures. *IEEE Design & Test of Computers*, 26(1):68–77, 2009.
17. D. Richerby. Interval bigraphs are unit grid intersection graphs. *Discrete Mathematics*, pages 1718–1719, 2009.
18. A. Shrestha, S. Tayu, and S. Ueno. Orthogonal ray graphs and nano-PLA design. In *Proc. IEEE ISCAS 2009*, pages 2930–2933, 2009.
19. A. M. S. Shrestha, S. Tayu, and S. Ueno. On orthogonal ray graphs. *Discrete Appl. Math.*, 158(15):1650–1659, Aug. 2010.
20. J. A. Soto and C. Telha. Jump number of two-directional orthogonal ray graphs. In *Proc. IPCO'11*, pages 389–403. Springer, 2011.
21. W. Trotter. *Combinatorics and Partially Ordered Sets.* The Johns Hopkins University Press, 1992.
22. R. Uehara. Simple geometrical intersection graphs. In *Proc. WALCOM'08*, pages 25–33. Springer, 2008.