

# Lifetimes and HQE

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## Abstract

Kolya Uraltsev was one of the inventors of the Heavy Quark Expansion (HQE), that describes inclusive weak decays of hadrons containing heavy quarks and in particular lifetimes. Besides giving a pedagogic introduction into the subject, we review the development and the current status of the HQE, which just recently passed several non-trivial experimental tests with an unprecedented precision. In view of many new experimental results for lifetimes of heavy hadrons, we also update several theory predictions:  $\tau(B^+)/\tau(B_d) = 1.04_{-0.01}^{+0.05} \pm 0.02 \pm 0.01$ ,  $\tau(B_s)/\tau(B_d) = 1.001 \pm 0.002$ ,  $\tau(\Lambda_b)/\tau(B_d) = 0.935 \pm 0.054$  and  $\bar{\tau}(\Xi_b^0)/\bar{\tau}(\Xi_b^+) = 0.95 \pm 0.06$ . The theoretical precision is currently strongly limited by the unknown size of the non-perturbative matrix elements of four-quark operators, which could be determined with lattice simulations.

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# 1 Introduction

Lifetimes are among the most fundamental properties of elementary particles. In this work we consider lifetimes of hadrons containing heavy quarks, which decay via the weak interaction. Their masses and lifetimes read (according to PDG [1] and HFAG [2])

## *B*-mesons

	$B_d = (\bar{b}d)$	$B^+ = (\bar{b}u)$	$B_s = (\bar{b}s)$	$B_c^+ = (\bar{b}c)$
Mass (GeV)	5.27955(26)	5.27925(26)	5.3667(4)	6.2745(18)
Lifetime (ps)	1.519(5)	1.638(4)	1.512(7)	0.500(13)
$\tau(X)/\tau(B_d)$	1	$1.076 \pm 0.004$	$0.995 \pm 0.006$	$0.329 \pm 0.009$

(1.1)

## *b*-baryons

	$\Lambda_b = (udb)$	$\Xi_b^0 = (usb)$	$\Xi_b^- = (dsb)$	$\Omega_b^- = (ssb)$
Mass (GeV)	5.6194(6)	5.7918(5)	5.79772(55)	6.071(40)
Lifetime (ps)	1.451(13)	1.477(32)	1.599(46)	$1.54 \begin{smallmatrix} +26 \\ -22 \end{smallmatrix}$
$\tau(X)/\tau(B_d)$	$0.955 \pm 0.009$	$0.972 \pm 0.021$	$1.053 \pm 0.030$	$1.01 \begin{smallmatrix} +17 \\ -14 \end{smallmatrix}$

(1.2)

The masses and the lifetimes of the  $\Xi_b^0$ ,  $\Xi_b^-$  and the  $\Omega_b^-$  have been measured by the LHCb Collaboration [3, 4, 5] just after the first version of this article appeared on the arXiv. We have given above these new values instead of the HFAG and PDG averages. Alternative lifetime averages were, e.g., obtained in [6].

## *D*-mesons

	$D^0 = (\bar{u}c)$	$D^+ = (\bar{d}c)$	$D_s^+ = (\bar{s}c)$
Mass (GeV)	1.86491(17)	1.8695(4)	1.9690(14)
Lifetime (ps)	0.4101(15)	1.040(7)	0.500(7)
$\tau(X)/\tau(D^0)$	1	$2.536 \pm 0.017$	$1.219 \pm 0.017$

(1.3)

## *c*-baryons

	$\Lambda_c = (udc)$	$\Xi_c^+ = (usc)$	$\Xi_c^0 = (dsc)$	$\Omega_c = (ssc)$
Mass (GeV)	2.28646(14)	$2.4676 \begin{smallmatrix} +4 \\ -10 \end{smallmatrix}$	$2.47109 \begin{smallmatrix} +35 \\ -100 \end{smallmatrix}$	$2.6952 \begin{smallmatrix} +18 \\ -16 \end{smallmatrix}$
Lifetime (ps)	0.200(6)	0.442(26)	$0.112 \begin{smallmatrix} +13 \\ -10 \end{smallmatrix}$	0.069(12)
$\tau(X)/\tau(D^0)$	$0.488 \pm 0.015$	1.08(6)	0.27(3)	$0.17 \pm 0.03$

(1.4)

One of the first observations to make is the fact that all lifetimes are of the same order of magnitude, they are all in the pico-second range and they differ at most by a factor of 25. Looking exclusively at hadrons containing one *b*-quark (and no *c*-quark), one even finds that all lifetimes are equal within about 10%. This clearly calls for a theoretical explanation.

In this review we will discuss the theoretical framework describing decay rates of inclusive

decays of hadrons containing a heavy quark, the so-called **Heavy Quark Expansion**. A special case of such observables are the lifetimes of hadrons, which are given by the inverse of the total decay rates. Kolya Uraltsev was one of the main pioneers in the development of the HQE, which has its roots back in the 1970s. When I began my career, Kolya’s theory was already a kind of textbook knowledge and my PhD and my first scientific papers were devoted to the calculation of higher order QCD corrections within the framework of the HQE. Thus it was very inspiring to meet Kolya personally at one of my first international conferences, which was held in 2000 in Durham, where we discussed the so-called “missing charm puzzle” [7] and the decay rate difference  $\Delta\Gamma_s$  of  $B_s$ -mesons [8]. I benefited a lot from many follow-up encounters with Kolya, e.g., at CERN, in Portoroz and in Siegen. At the end of 2012 I was working with a student from Munich [9] on  $D$ -meson lifetimes and in that respect Kolya was sending me several long emails regarding the history of lifetime predictions, which clearly influenced this review.

Many of the most convincing precision tests of the HQE have just been performed recently. In the beginning of 2012  $\Delta\Gamma_s$  has been measured for the first time, i.e., with a statistical significance of five standard deviations, in accordance with the HQE prediction. The long standing puzzle concerning the lifetime of the  $\Lambda_b$ -baryon - for many years a very strong challenge for the HQE - seems to have been settled experimentally, with the latest results just appearing in 2014. It is a real tragedy that Kolya did not have more time to celebrate the successes of the theory, to which he contributed so much.

We start in Section 2 with a very basic and pedagogic introduction into lifetimes of weakly decaying particles. Readers with some familiarity to weak decays can skip Section 2.1, which covers some simple but instructive estimates for the decays of the muon, the tauon, the charm-quark and the bottom-quark. In Section 2.2 we discuss the structure of the HQE in detail and in Section 2.3 we give a brief review of the discussed observables. In Section 3 we investigate the history of the HQE and we highlight Kolya’s contribution, while we discuss the status quo in experiment and theory in Section 4. Here we also give some numerical updates of theory predictions for lifetime ratios. In Section 5 we give an outlook on what has to be done in order to improve further the theoretical accuracy and we conclude.

## 2 Basic considerations about lifetimes

We start the discussion of the theoretical framework for describing heavy hadron lifetimes, with some simple estimates that are based on the prime example for a weak decay, the decay of a muon. Later on we show how this naive picture has to be generalised for the case of hadrons containing a heavy quark.

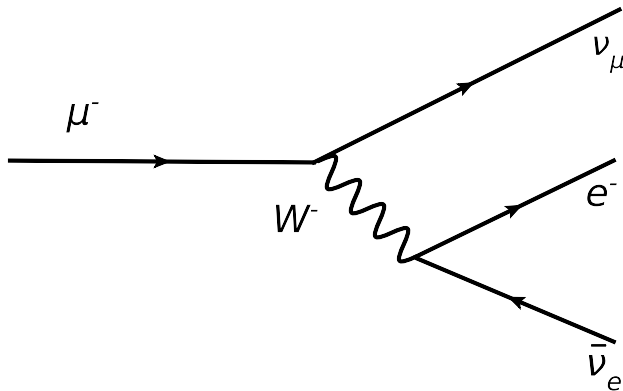
### 2.1 Naive estimates

#### 2.1.1 The muon decay

The muon decay  $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e$  represents the most simple weak decay, because there are no QCD effects involved.<sup>1</sup> This process is given by the following Feynman diagram.

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<sup>1</sup>This statements holds to a high accuracy. QCD effects arise for the first time at the two loop order.



Hence the total decay rate of the muon reads (see, e.g., [10] for an early reference)

$$\Gamma_{\mu \rightarrow \nu_\mu + e + \bar{\nu}_e} = \frac{G_F^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e}{m_\mu}\right) = \frac{G_F^2 m_\mu^5}{192\pi^3} c_{3,\mu}. \quad (2.5)$$

$f$  denotes the phase space factor for one massive particle in the final state. It is given by

$$f(x) = 1 - 8x^2 + 8x^6 - x^8 - 24x^4 \ln(x). \quad (2.6)$$

The coefficient  $c_{3,\mu}$  is introduced here to be consistent with our later notation. The result in Eq.(2.5) is already very instructive, since we get now for the measurable lifetime of the muon

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5 f\left(\frac{m_e}{m_\mu}\right)}. \quad (2.7)$$

Thus the lifetime of a weakly decaying particle is proportional to the inverse of the fifth power of the mass of the decaying particle. Using the measured values [1] for  $G_F = 1.1663787(6) \cdot 10^{-5} \text{ GeV}^{-2}$ ,  $m_e = 0.510998928(11) \text{ MeV}$  and  $m_\mu = 0.1056583715(35) \text{ GeV}$  we predict<sup>2</sup> the lifetime of the muon to be

$$\tau_\mu^{Theo.} = 2.18776 \cdot 10^{-6} \text{ s}, \quad (2.8)$$

which is in excellent agreement with the measured value [1] of

$$\tau_\mu^{Exp.} = 2.1969811(22) \cdot 10^{-6} \text{ s}. \quad (2.9)$$

The remaining tiny difference (the prediction is about 0.4% smaller than the experimental value) is due to higher order electro-weak corrections. These corrections are crucial for a high precision determination of the Fermi constant. The dominant contribution is given by the 1-loop QED correction, calculated already in the 1950s [11, 12]:

$$c_{3,\mu} = f\left(\frac{m_e}{m_\mu}\right) \left[ 1 + \frac{\alpha}{4\pi} 2 \left( \frac{25}{4} - \pi^2 \right) \right]. \quad (2.10)$$

Taking this effect into account ( $\alpha = 1/137.035999074(44)$  [1]) we predict

$$\tau_\mu^{Theo.} = 2.19699 \cdot 10^{-6} \text{ s}, \quad (2.11)$$

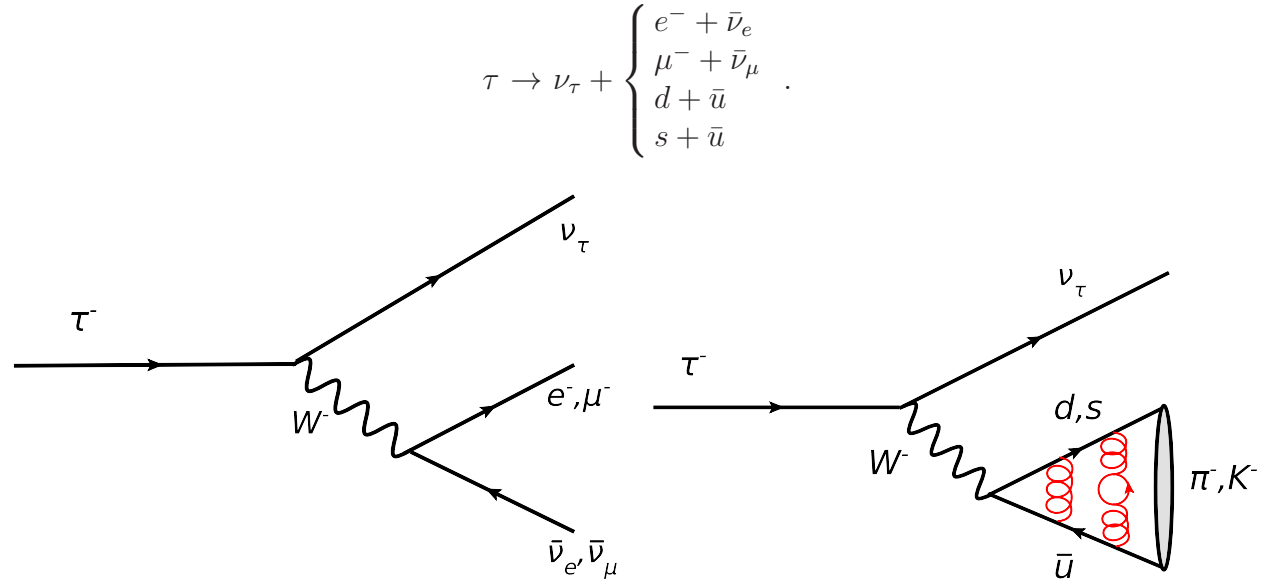
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<sup>2</sup>This is of course not really correct, because the measured muon lifetime was used to determine the Fermi constant, but for pedagogical reasons we assume that the Fermi constant is known from somewhere else.

which is almost identical to the measured value given in Eq.(2.9). The complete 2-loop QED corrections have been determined in [13], a review of loop-corrections to the muon decay is given in [14] and two very recent higher order calculations can be found in, e.g., [15, 16]. The phase space factor is almost negligible for the muon decay - it reads  $f(m_e/m_\mu) = 0.999813 = 1 - 0.000187051$  - but it will turn out to be quite sizable for a decay of a b-quark into a charm quark.

### 2.1.2 The tau decay

Moving to the tau lepton, we have now two leptonic decay channels as well as decays into quarks:



Heavier quarks, like charm- or bottom-quarks cannot be created, because the lightest meson containing such quarks ( $D^0 = c\bar{u}$ ;  $M_{D^0} \approx 1.86$  GeV) is heavier than the tau lepton ( $m_\tau = 1.77682(16)$  GeV). Thus the total decay rate of the tau lepton reads

$$\begin{aligned} \Gamma_\tau &= \frac{G_F^2 m_\tau^5}{192\pi^3} \left[ f\left(\frac{m_e}{m_\tau}\right) + f\left(\frac{m_\mu}{m_\tau}\right) + N_c |V_{ud}|^2 g\left(\frac{m_u}{m_\tau}, \frac{m_d}{m_\tau}\right) + N_c |V_{us}|^2 g\left(\frac{m_u}{m_\tau}, \frac{m_s}{m_\tau}\right) \right] \\ &=: \frac{G_F^2 m_\tau^5}{192\pi^3} c_{3,\tau} . \end{aligned} \quad (2.12)$$

The factor  $N_c = 3$  is a colour factor and  $g$  denotes a new phase space function, when there are two massive particles in the final state. If we neglect the phase space factors ( $f(m_e/m_\tau) = 1 - 7 \cdot 10^{-7}$ ;  $f(m_\mu/m_\tau) = 1 - 0.027$ ; ...) and if we use  $V_{ud}^2 + V_{us}^2 \approx 1$ , then we get  $c_{3,\tau} = 5$  and thus the simple approximate relation

$$\frac{\tau_\tau}{\tau_\mu} = \left(\frac{m_\mu}{m_\tau}\right)^5 \frac{1}{5} . \quad (2.13)$$

Using the experimental values for  $\tau_\mu$ ,  $m_\mu$  and  $m_\tau$  we predict

$$\tau_\tau^{Theo.} = 3.26707 \cdot 10^{-13} \text{ s} , \quad (2.14)$$

which is quite close to the experimental value of

$$\tau_{\tau}^{Exp.} = 2.906(1) \cdot 10^{-13} \text{ s} . \quad (2.15)$$

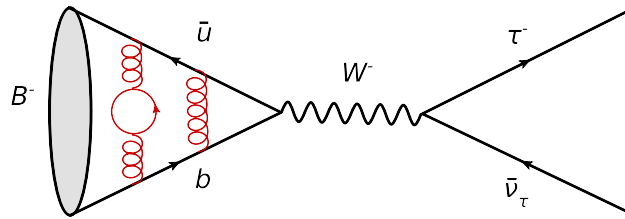
Now the theory prediction is about 12% larger than the measured value. This is mostly due to sizable QCD corrections, when there are quarks in the final state - which was not possible in the muon decay. These QCD corrections are currently calculated up to five loop accuracy [17], a review of higher order corrections can be found in [18].

Because of the pronounced and clean dependence on the strong coupling, tau decays can also be used for precision determinations of  $\alpha_s$ , see, e.g., the review [19]. This example shows already, that a proper treatment of QCD effects is mandatory for precision investigations of lifetimes. In the case of meson decays this will even be more important.

### 2.1.3 Meson decays

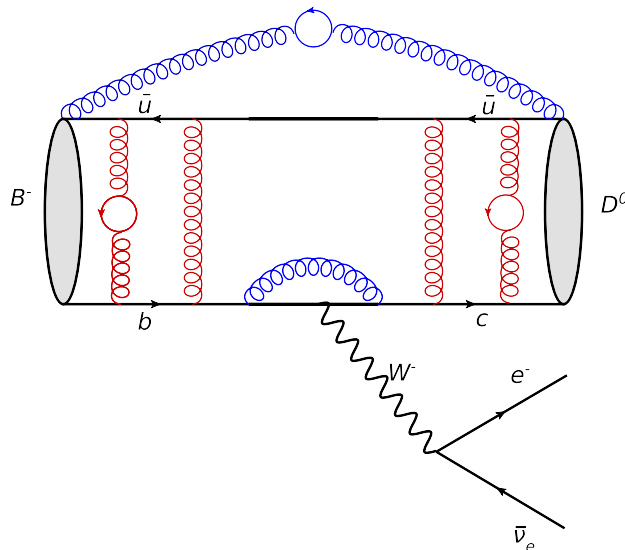
Studying weak decays of mesons - instead of leptons - one immediately comes to the problem of treating the QCD effects related to the formation of a bound state in the initial state as well as in the final states and there is of course also the possibility of having QCD effects between the initial and the final states. Meson decays can be classified according to their final states:

- **Leptonic decays** have only leptons in the final state, e.g.,  $B^- \rightarrow \tau^- \bar{\nu}_{\tau}$ .



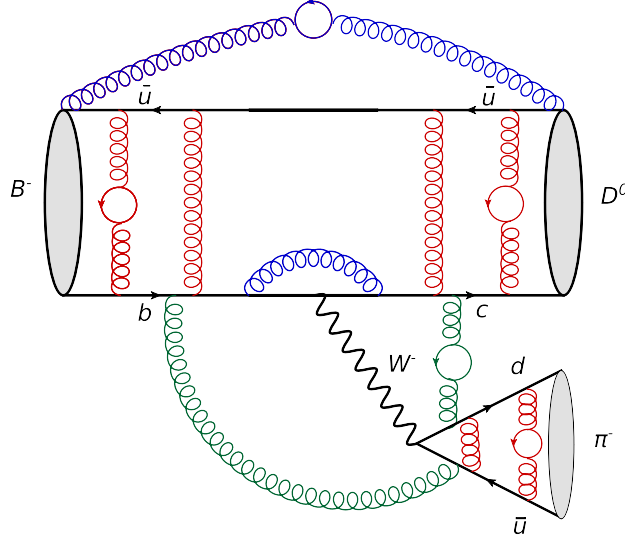
Such decays have the simplest hadronic structure. Gluons bind the quarks of the initial state into a hadron. All non-perturbative effects are described by so-called decay constants.

- **Semi-leptonic decays** have both leptons and hadrons in the final state, e.g.,  $B^- \rightarrow D^0 e^- \bar{\nu}_e$ .



Now the hadronic structure is more complicated. We have the binding of hadrons in the initial state and in the final states. Moreover there is the possibility of having strong interactions between the initial state and the final states. The non-perturbative physics is in this case described by so-called form factors.

- Non-leptonic decays have only hadrons in the final state, e.g.,  $B^- \rightarrow D^0 \pi^-$ .



These are the most complicated decays and they can only be treated by making additional assumptions that allow then for a factorisation.

Related to this considerations we introduce the notation for two classes of decays - *inclusive* and *exclusive* decays. In exclusive modes every final state hadron is identified. This is in principle what experiments can do well, while theory has the problem to describe the peculiar QCD binding effects in the hadronic states. An example from above would be the decay  $B^- \rightarrow D^0 \pi^-$ , where one explicitly detects the  $D^0$  and  $\pi^-$  in the final state. The corresponding inclusive decay is  $b \rightarrow c \bar{u} d$ . In inclusive modes we only care about the quark content of the final state, this is clearly theoretically easier, while experiments have the problem of summing up all decays that belong to a certain inclusive decay mode. Another example that will appear later on, is the inclusive  $b \rightarrow c \bar{c} s$  transition; corresponding exclusive decays are in this case

$$B_d^0 \rightarrow D^{*-} D_s^{*+}, D^- D_s^{*+}, D^{*-} D_s^+, D^- D_s^+, J/\Psi K_S, \dots$$

To get a feeling for the arising branching fractions we list the theory value [20] for  $b \rightarrow c \bar{c} s$ , with some measured [1] exclusive branching ratios.

$$\text{Br}(b \rightarrow c \bar{c} s) = (23 \pm 2)\% , \quad (2.16)$$

$$\text{Br}(D^{*-} D_s^{*+}) = (1.77 \pm 0.14)\% , \quad (2.17)$$

$$\text{Br}(D^{*-} D_s^+) = (8.0 \pm 1.1) \cdot 10^{-3} , \quad (2.18)$$

$$\text{Br}(D^- D_s^{*+}) = (7.4 \pm 1.6) \cdot 10^{-3} , \quad (2.19)$$

$$\text{Br}(D^- D_s^+) = (7.2 \pm 0.8) \cdot 10^{-3} , \quad (2.20)$$

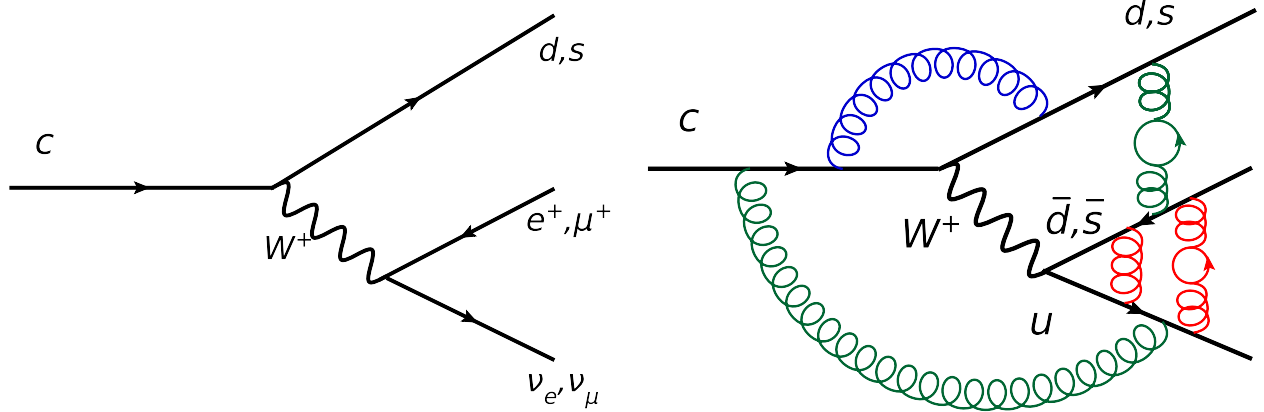
$$\text{Br}(J/\Psi K_S) = (8.73 \pm 0.32) \cdot 10^{-4} . \quad (2.21)$$

Here one can already guess that quite some number of exclusive decay channels has to be summed up in order to obtain the inclusive branching ratio.

#### 2.1.4 Charm-quark decay

Before trying to investigate the complicated meson decays, let us look at the decay of free  $c$ - and  $b$ -quarks. Later on we will show that the free quark decay is the leading term in a systematic expansion in the inverse of the heavy (decaying) quark mass - the HQE.

A charm quark can decay weakly into a strange- or a down-quark and a  $W^+$ -boson, which then further decays either into leptons (semi-leptonic decay) or into quarks (non-leptonic decay).



Calculating the total inclusive decay rate of a charm-quark we get

$$\Gamma_c = \frac{G_F^2 m_c^5}{192 \pi^3} |V_{cs}|^2 c_{3,c} , \quad (2.22)$$



with

$$\begin{aligned}
c_{3,c} = & g\left(\frac{m_s}{m_c}, \frac{m_e}{m_c}\right) + g\left(\frac{m_s}{m_c}, \frac{m_\mu}{m_c}\right) + N_c|V_{ud}|^2 h\left(\frac{m_s}{m_c}, \frac{m_u}{m_c}, \frac{m_d}{m_c}\right) + N_c|V_{us}|^2 h\left(\frac{m_s}{m_c}, \frac{m_u}{m_c}, \frac{m_s}{m_c}\right) \\
& + \left|\frac{V_{cd}}{V_{cs}}\right|^2 \left\{ g\left(\frac{m_d}{m_c}, \frac{m_e}{m_c}\right) + g\left(\frac{m_d}{m_c}, \frac{m_\mu}{m_c}\right) + N_c|V_{ud}|^2 h\left(\frac{m_d}{m_c}, \frac{m_u}{m_c}, \frac{m_d}{m_c}\right) \right. \\
& \left. + N_c|V_{us}|^2 h\left(\frac{m_d}{m_c}, \frac{m_u}{m_c}, \frac{m_s}{m_c}\right) \right\}. \tag{2.23}
\end{aligned}$$

$h$  denotes a new phase space function, when there are three massive particles in the final state. If we set all phase space factors to one ( $f(m_s/m_c) = f(0.0935/1.471) = 1 - 0.03, \dots$  with  $m_s = 93.5(2.5)$  MeV [1]) and use  $|V_{ud}|^2 + |V_{us}|^2 \approx 1 \approx |V_{cd}|^2 + |V_{cs}|^2$ , then we get  $|V_{cs}|^2 c_{3,c} = 5$ , similar to the  $\tau$  decay. In that case we predict a charm lifetime of

$$\tau_c = \begin{cases} 0.84 \text{ ps} \\ 1.70 \text{ ps} \end{cases} \quad \text{for } m_c = \begin{cases} 1.471 \text{ GeV (Pole-scheme)} \\ 1.277(26) \text{ GeV } (\overline{MS} - \text{scheme}) \end{cases}. \tag{2.24}$$

These predictions lie roughly in the ball-bark of the experimental numbers for  $D$ -meson lifetimes, but at this stage some comments are appropriate:

- Predictions of the lifetimes of free quarks have a huge parametric dependence on the definition of the quark mass ( $\propto m_q^5$ ). This is the reason, why typically only lifetime ratios (the dominant  $m_q^5$  dependence as well as CKM factors and some sub-leading non-perturbative corrections cancel) are determined theoretically. We show in this introduction for pedagogical reasons the numerical results of the theory predictions of lifetimes and not only ratios. In our case the value obtained with the  $\overline{MS}$ -scheme for the charm quark mass is about a factor of 2 larger than the one obtained with the pole-scheme. In LO-QCD the definition of the quark mass is completely arbitrary and we have these huge uncertainties. If we calculate everything consistently in NLO-QCD, the treatment of the quark masses has to be defined within the calculation, leading to a considerably weaker dependence of the final result on the quark mass definition. Bigi, Shifman, Uraltsev and Vainshtein have shown in 1994 [21] that the pole mass scheme is always affected by infra-red renormalons, see also the paper of Beneke and Braun [22] that appeared on the same day on the arXiv and the review in this issue [23]. Thus short-distance definitions of the quark mass, like the  $\overline{MS}$ -mass [24] seem to be better suited than the pole mass. More recent suggestions for quark mass concepts are the kinetic mass from Bigi, Shifman, Uraltsev and Vainshtein [25, 26] introduced in 1994, the potential subtracted mass from Beneke [27] and the  $\Upsilon(1s)$ -scheme from Hoang, Ligeti and Manohar [28, 29], both introduced in 1998. In [20] we compared the above quark mass schemes for inclusive non-leptonic decay rates and found similar numerical results for the different short distance masses. Thus we rely in this review - for simplicity - on predictions based on the  $\overline{MS}$ -mass scheme and we discard the pole mass, even if we give several times predictions based on this mass scheme for comparison. Concerning the concrete numerical values for the quark masses we also take the same numbers as in [20]. In that work relations between different quark mass schemes were strictly used at NLO-QCD accuracy (higher terms were discarded), therefore the num-

bers differ slightly from the PDG [1]-values, which would result in

$$\tau_c = \begin{cases} 0.44 \text{ ps} \\ 1.71 \text{ ps} \end{cases} \quad \text{for } m_c = \begin{cases} 1.67(7) \text{ GeV} & (\text{Pole-scheme}) \\ 1.275(25) \text{ GeV} & (\overline{MS} \text{ - scheme}) \end{cases} . \quad (2.25)$$

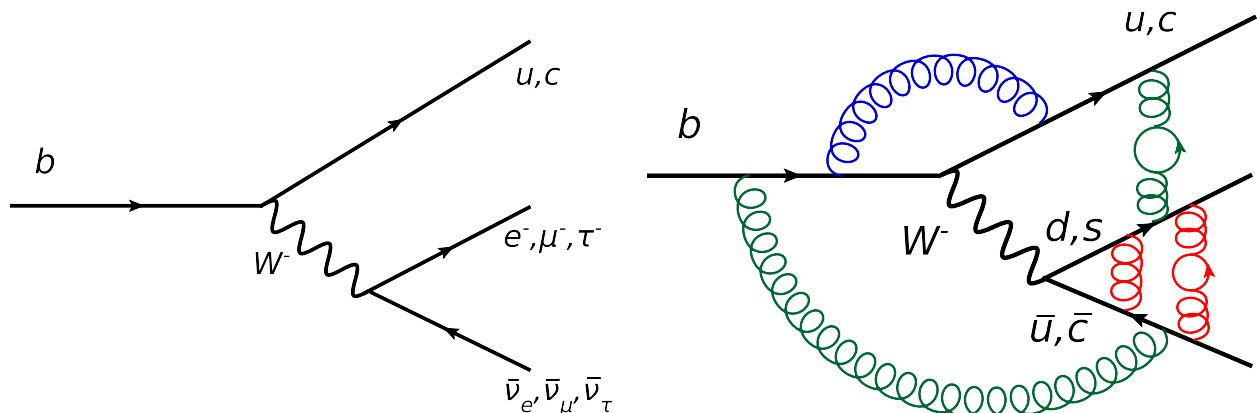
Since our final lifetime predictions are only known up to NLO accuracy and we expand every expression consistently up to order  $\alpha_s$ , we will stay with the parameters used in [20].

- Taking only the decay of the  $c$ -quark into account, one obtains the same lifetimes for all charm-mesons, which is clearly a very bad approximation, taking the large spread of lifetimes of different  $D$ -mesons into account, see Eq.(1.3). Below we will see that in the case of charmed mesons a very sizable contribution comes from non-spectator effects where also the valence quark of the  $D$ -meson is involved in the decay.
- Perturbative QCD corrections will turn out to be very important, because  $\alpha_s(m_c)$  is quite large.
- In the above expressions we neglected, e.g., annihilation decays like  $D^+ \rightarrow l^+ \nu_l$ , which have very small branching ratios [1] (the corresponding Feynman diagrams have the same topology as the decay  $B^- \rightarrow \tau^- \bar{\nu}_\tau$ , that was mentioned earlier). In the case of  $D_s^+$  meson the branching ratio into  $\tau^+ \nu_\tau$  will, however, be sizable [1] and has to be taken into account.

$$\text{Br}(D_s^+ \rightarrow \tau^+ \nu_\tau) = (5.43 \pm 0.31)\% . \quad (2.26)$$

In the framework of the HQE the non-spectator effects will turn out to be suppressed by  $1/m_c$  and since  $m_c$  is not very large, the suppression is also not expected to be very pronounced. This will change in the case of  $B$ -mesons. Because of the larger value of the  $b$ -quark mass, one expects a better description of the meson decay in terms of the simple  $b$ -quark decay.

### 2.1.5 Bottom-quark decay



Calculating the total inclusive decay rate of a  $b$ -quark we get

$$\Gamma_b = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 c_{3,b} , \quad (2.27)$$

with

$$\begin{aligned}
c_{3,b} = & \left\{ g\left(\frac{m_c}{m_b}, \frac{m_e}{m_b}\right) + g\left(\frac{m_c}{m_b}, \frac{m_\mu}{m_b}\right) + g\left(\frac{m_c}{m_b}, \frac{m_\tau}{m_b}\right) \right. \\
& + N_c |V_{ud}|^2 h\left(\frac{m_c}{m_b}, \frac{m_u}{m_b}, \frac{m_d}{m_b}\right) + N_c |V_{us}|^2 h\left(\frac{m_c}{m_b}, \frac{m_u}{m_b}, \frac{m_s}{m_b}\right) \\
& \left. + N_c |V_{cd}|^2 h\left(\frac{m_c}{m_b}, \frac{m_c}{m_b}, \frac{m_d}{m_b}\right) + N_c |V_{cs}|^2 h\left(\frac{m_c}{m_b}, \frac{m_c}{m_b}, \frac{m_s}{m_b}\right) \right\} \\
& + \left| \frac{V_{ub}}{V_{cb}} \right|^2 \left\{ g\left(\frac{m_u}{m_b}, \frac{m_e}{m_b}\right) + g\left(\frac{m_u}{m_b}, \frac{m_\mu}{m_b}\right) + g\left(\frac{m_u}{m_b}, \frac{m_\tau}{m_b}\right) \right. \\
& + N_c |V_{ud}|^2 h\left(\frac{m_u}{m_b}, \frac{m_u}{m_b}, \frac{m_d}{m_b}\right) + N_c |V_{us}|^2 h\left(\frac{m_u}{m_b}, \frac{m_u}{m_b}, \frac{m_s}{m_b}\right) \\
& \left. + N_c |V_{cd}|^2 h\left(\frac{m_u}{m_b}, \frac{m_c}{m_b}, \frac{m_d}{m_b}\right) + N_c |V_{cs}|^2 h\left(\frac{m_u}{m_b}, \frac{m_c}{m_b}, \frac{m_s}{m_b}\right) \right\}.
\end{aligned} \tag{2.28}$$

In this formula penguin induced decays have been neglected, they will enhance the decay rate by several per cent, see [20]. More important will, however, be the QCD corrections. To proceed further we can neglect the masses of all final state particles, except for the charm quark and for the tau lepton. In addition we can neglect the contributions proportional to  $|V_{ub}|^2$  since  $|V_{ub}/V_{cb}|^2 \approx 0.01$ . Using further  $|V_{ud}|^2 + |V_{us}|^2 \approx 1 \approx |V_{cd}|^2 + |V_{cs}|^2$ , we get the following simplified formula

$$c_{3,b} = \left[ (N_c + 2) f\left(\frac{m_c}{m_b}\right) + g\left(\frac{m_c}{m_b}, \frac{m_\tau}{m_b}\right) + N_c g\left(\frac{m_c}{m_b}, \frac{m_c}{m_b}\right) \right]. \tag{2.29}$$

If we have charm quarks in the final states, then the phase space functions show a huge dependence on the numerical value of the charm quark mass (values taken from [20])

$$f\left(\frac{m_c}{m_b}\right) = \begin{cases} 0.484 \\ 0.518 \\ 0.666 \end{cases} \quad \text{for } \begin{cases} m_c^{\text{Pole}} = 1.471 \text{ GeV}, m_b^{\text{Pole}} = 4.650 \text{ GeV} \\ \bar{m}_c(\bar{m}_b) = 1.277 \text{ GeV}, \bar{m}_b(\bar{m}_b) = 4.248 \text{ GeV} \\ \bar{m}_c(\bar{m}_b) = 0.997 \text{ GeV}, \bar{m}_b(\bar{m}_b) = 4.248 \text{ GeV} \end{cases}. \tag{2.30}$$

The big spread in the values for the space functions clearly shows again that the definition of the quark mass is a critical issue for a precise determination of lifetimes. The value for the pole quark mass is only shown to visualise the strong mass dependence. As discussed above short-distance masses like the  $\overline{\text{MS}}$ -mass are theoretically better suited. Later on we will argue further for using  $\bar{m}_c(\bar{m}_b)$  and  $\bar{m}_b(\bar{m}_b)$  - so both masses at the scale  $m_b$  -, which was suggested in [30], in order to sum up large logarithms of the form  $\alpha_s^n (m_c/m_b)^2 \log^n (m_c/m_b)^2$  to all orders. Thus only the result using  $\bar{m}_c(\bar{m}_b)$  and  $\bar{m}_b(\bar{m}_b)$  should be considered as the theory prediction, while the additional numbers are just given for completeness.

The phase space function for two identical particles in the final states reads [31, 32, 33, 34] (see [35] for the general case of three different masses)

$$g(x) = \sqrt{1 - 4x^2} (1 - 14x^2 - 2x^4 - 12x^6) + 24x^4 (1 - x^4) \log \frac{1 + \sqrt{1 - 4x^2}}{1 - \sqrt{1 - 4x^2}}, \tag{2.31}$$

with  $x = m_c/m_b$ . Thus we get in total for all the phase space contributions

$$c_{3,b} = \begin{cases} 9 \\ 2.97 \\ 3.25 \\ 4.66 \end{cases} \quad \text{for} \quad \begin{cases} m_c = 0, \\ m_c^{\text{Pole}}, m_b^{\text{Pole}} \\ \bar{m}_c(\bar{m}_c), \bar{m}_b(\bar{m}_b) \\ \bar{m}_c(\bar{m}_b), \bar{m}_b(\bar{m}_b) \end{cases} . \quad (2.32)$$

The phase space effects are now quite dramatic. For the total  $b$ -quark lifetime we predict (with  $V_{cb} = 0.04151_{-0.00115}^{+0.00056}$  from [36], for similar results see [37].)

$$\tau_b = 2.60 \text{ ps} \quad \text{for} \quad \bar{m}_c(\bar{m}_b), \bar{m}_b(\bar{m}_b) . \quad (2.33)$$

This number is about 70% larger than the experimental number for the  $B$ -meson lifetimes. There are in principle two sources for that discrepancy: first we neglected several CKM-suppressed decays, which are however not phase space suppressed as well as penguin decays. An inclusion of these decays will enhance the total decay rate roughly by about 10% and thus reduce the lifetime prediction by about 10%. Second, there are large QCD effects, that will be discussed in the next subsection; including them will bring our theory prediction very close to the experimental number. For completeness we show also the lifetime predictions, for different (theoretically less motivated) values of the quark masses.

$$\tau_b = \begin{cases} 0.90 \text{ ps} \\ 1.42 \text{ ps} \\ 2.59 \text{ ps} \\ 3.72 \text{ ps} \end{cases} \quad \text{for} \quad \begin{cases} m_c = 0, m_b^{\text{Pole}} \\ m_c = 0, \bar{m}_b(\bar{m}_b) \\ m_c^{\text{Pole}}, m_b^{\text{Pole}} \\ \bar{m}_c(\bar{m}_c), \bar{m}_b(\bar{m}_b) \end{cases} . \quad (2.34)$$

By accident a neglect of the charm quark mass can lead to predictions that are very close to experiment. As argued above, only the value in Eq.(2.33) should be considered as the theory prediction for the  $b$ -quark lifetime and not the ones in Eq.(2.34). Next we introduce the missing, but necessary concepts for making reliable predictions for the lifetimes of heavy hadrons.

## 2.2 The structure of the HQE

### 2.2.1 The effective Hamiltonian

Above we tried to make clear, that for any numerical reliable quantitative estimate of meson decays, QCD effects have to be taken properly into account. To do so, weak decays of heavy quarks are not described within the full standard model, but with the help of an effective Hamiltonian. We start here simply with the explicit form of the effective Hamiltonian and refer the interested reader to some excellent reviews by Buchalla, Buras and Lautenbacher [38], by Buras [39], by Buchalla [40] and a recent one by Grozin [41]. The effective Hamiltonian reads

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \left[ \sum_{q=u,c} V_c^q (C_1 Q_1^q + C_2 Q_2^q) - V_p \sum_{j=3} C_j Q_j \right] . \quad (2.35)$$

Without QCD corrections only the operator  $Q_2$  arises and the *Wilson coefficient*  $C_2$  is equal to one,  $C_2 = 1$ .  $Q_2$  has a current-current structure:

$$Q_2 = c_\alpha \gamma_\mu (1 - \gamma_5) \bar{b}_\alpha \times d_\beta \gamma^\mu (1 - \gamma_5) \bar{u}_\beta . \quad (2.36)$$

$\alpha$  and  $\beta$  denote colour indices. The  $V$ s describe different combinations of CKM elements. With the inclusion of QCD one gets additional operators.  $Q_1$  has the same Dirac structure as  $Q_2$ , but it has a different colour structure,  $Q_3, \dots, Q_6$  arise from QCD penguin diagrams etc.. Due to renormalisation all Wilson coefficients become scale dependent functions. In LO-QCD we get<sup>3</sup>  $C_2(4.248 \text{ GeV}) = 1.1$  and  $C_1(4.248 \text{ GeV}) = -0.24$  and the penguin coefficients are below 5%, with the exception of  $C_8$ , the coefficient of the chromo-magnetic operator. With this operator product expansion (OPE) a separation of the scales was achieved. The high energy physics is described by the Wilson coefficients, they can be calculated in perturbation theory. The low energy physics is described by the matrix elements of the operators  $Q_i$ . Moreover large logarithms of the form  $\alpha_s(m_b) \ln(m_b^2/M_W^2)$ , which spoil the perturbative expansion in the full standard model, are now summed up to all orders. For semi-leptonic decays like  $b \rightarrow cl^- \bar{\nu}_l$  the Wilson coefficient  $C_2$  is simply 1, while the remaining ones vanish.

### 2.2.2 The free quark decay with the effective Hamiltonian

Now we can calculate the free quark decay starting from the effective Hamiltonian instead of the full standard model. If we again neglect penguins, we get in leading logarithmic approximation the same structure as in Eq.(2.28),

$$c_{3,b}^{\text{LO-QCD}} = c_{3,b}^{ce\bar{\nu}_e} + c_{3,b}^{c\mu\bar{\nu}_\mu} + c_{3,b}^{c\bar{u}d} + c_{3,b}^{c\bar{u}s} + c_{3,b}^{c\tau\bar{\nu}_\tau} + c_{3,b}^{c\bar{c}s} + c_{3,b}^{c\bar{c}d} \dots \quad (2.37)$$

$$= \left[ (2 + \mathcal{N}_a(\mu)) f\left(\frac{m_c}{m_b}\right) + g\left(\frac{m_c}{m_b}, \frac{m_\tau}{m_b}\right) + \mathcal{N}_a(\mu) g\left(\frac{m_c}{m_b}, \frac{m_c}{m_b}\right) \right], \quad (2.38)$$

with changing the colour factor  $N_c = 3$  - stemming from QCD - into

$$\mathcal{N}_a(\mu) = 3C_1^2(\mu) + 3C_2^2(\mu) + 2C_1(\mu)C_2(\mu) \approx 3.3 \quad (\text{LO}, \mu = 4.248 \text{ GeV}). \quad (2.39)$$

This effect enhances the total decay rate by about 10% and thus brings down (if also the sub-leading decays are included) the prediction for the lifetime of the  $b$ -quark to about

$$\tau_b \approx 2.10 \text{ ps} \quad \text{for } \bar{m}_c(\bar{m}_b), \bar{m}_b(\bar{m}_b) \quad . \quad (2.40)$$

Going to next-to-leading logarithmic accuracy we have to use the Wilson coefficients of the effective Hamiltonian to NLO accuracy and we have to determine one-loop QCD corrections within the effective theory. These NLO-QCD corrections turned out to be very important for the inclusive  $b$ -quark decays. For massless final state quarks the calculation was done in 1991 [42]:

$$c_{3,b} = c_{3,b}^{\text{LO-QCD}} + 8 \frac{\alpha_s}{4\pi} \left[ \left( \frac{25}{4} - \pi^2 \right) + 2 (C_1^2 + C_2^2) \left( \frac{31}{4} - \pi^2 \right) - \frac{4}{3} C_1 C_2 \left( \frac{7}{4} + \pi^2 \right) \right] \quad . \quad (2.41)$$

The first QCD corrections in Eq.(2.41) stems from semi-leptonic decays. It can be guessed from the correction to the muon decay in Eq.(2.10) by decomposing the factor 8 in Eq.(2.41) as  $8 = 3 \cdot C_F \cdot 2$ : 3 comes from the three leptons  $e^-, \mu^-, \tau^-$ ,  $C_F$  is a QCD colour factor and 2 belongs to the correction in Eq.(2.10). The second and the third term in Eq.(2.41) stem from non-leptonic decays.

It turned out, however, that effects of the charm quark mass are crucial, see, e.g., the estimate

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<sup>3</sup>We use as an input for the strong coupling  $a_s(M_Z) = 0.1184$ .

in [43]. NLO-QCD corrections with full mass dependence were determined for  $b \rightarrow cl\bar{\nu}$  already in 1983 [44], for  $b \rightarrow c\bar{u}d$  in 1994 [45], for  $b \rightarrow c\bar{c}s$  in 1995 [46], for  $b \rightarrow$  no charm in 1997 [47] and for  $b \rightarrow sg$  in 2000 [48, 49]. Since there were several misprints in [46]-leading to IR divergent expressions -, the corresponding calculation was redone in [20] and the numerical result was updated.<sup>4</sup> With the results in [20] we predict (using  $\bar{m}_c(\bar{m}_b)$  and  $\bar{m}_b(\bar{m}_b)$ )

$$c_{3,b} = \begin{cases} 9 & (m_c = 0 = \alpha_s) \\ 5.29 \pm 0.35 & (\text{LO} - \text{QCD}) \\ 6.88 \pm 0.74 & (\text{NLO} - \text{QCD}) \end{cases} . \quad (2.42)$$

Comparing this result with Eq.(2.32) one finds a huge phase space suppression, which reduces the value of  $C_{3,b}$  from 9 in the mass less case to about 4.7 when including charm quark mass effect. Switching on in addition QCD effects  $c_{3,b}$  is enhanced back to a value of about 6.9. The LO  $b \rightarrow c$  transitions contribute about 70% to this value, the full NLO-QCD corrections about 24% and the  $b \rightarrow u$  and penguin contributions about 6% [20].

For the total lifetime we predict thus

$$\tau_b = (1.65 \pm 0.24) \text{ ps} , \quad (2.43)$$

which is our final number for the lifetime of a free  $b$ -quark. This number is now very close to the experimental numbers in Eq.(1.1), unfortunately the uncertainty is still quite large. To reduce this, a calculation at the NNLO order would be necessary. Such an endeavour seems to be doable nowadays. The dominant Wilson coefficients  $C_1$  and  $C_2$  are known at NNLO accuracy [50] and the two loop corrections in the effective theory have been determined e.g. in [51, 52, 53, 54, 55, 56] for semi-leptonic decays and partly in [57] for non-leptonic decays. It is amusing to note, that a naive treatment with vanishing charm quark masses and neglecting the sizable QCD-effects, see Eq.(2.33), yields by accident a similar result as in Eq.(2.43). The same holds also for the semi leptonic branching ratio, where a naive treatment ( $m_c = 0 = \alpha_s$ ) gives

$$B_{sl} = \frac{\Gamma(b \rightarrow ce^-\bar{\nu}_e)}{\Gamma_{\text{tot}}} = \frac{1}{9} = 11.1\% , \quad (2.44)$$

while the full treatment (following [20]) gives

$$B_{sl} = (11.6 \pm 0.8)\% . \quad (2.45)$$

This number agrees well with recent measurements [1, 58]

$$\begin{aligned} B_{sl}(B_d) &= (10.33 \pm 0.28)\% , \\ B_{sl}(B^+) &= (10.99 \pm 0.28)\% , \\ B_{sl}(B_s) &= (10.61 \pm 0.89)\% . \end{aligned} \quad (2.46)$$

### 2.2.3 The HQE

Now we are ready to derive the heavy quark expansion for inclusive decays.<sup>5</sup> The decay rate of the transition of a B-meson to an inclusive final state  $X$  can be expressed as a phase

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<sup>4</sup>The authors of [46] left particle physics and it was not possible to obtain the correct analytic expressions. The numerical results in [46] were, however, correct.

<sup>5</sup>We delay almost all referencing related to the creation of the HQE to Section 3.

space integral over the square of the matrix element of the effective Hamiltonian sandwiched between the initial  $B$ -meson<sup>6</sup> state and the final state  $X$ . Summing over all final states  $X$  with the same quark quantum numbers we obtain

$$\Gamma(B \rightarrow X) = \frac{1}{2m_B} \sum_X \int_{\text{PS}} (2\pi)^4 \delta^{(4)}(p_B - p_X) |\langle X | \mathcal{H}_{eff} | B \rangle|^2. \quad (2.47)$$

If we consider, e.g., a decay into three particles, i.e.  $B \rightarrow 1 + 2 + 3$ , then the phase space integral reads

$$\int_{\text{PS}} = \prod_{i=1}^3 \left[ \frac{d^3 p_i}{(2\pi)^3 2E_i} \right] \quad (2.48)$$

and  $p_X = p_1 + p_2 + p_3$ . With the help of the optical theorem the total decay rate in Eq.(2.47) can be rewritten as

$$\Gamma(B \rightarrow X) = \frac{1}{2m_B} \langle B | \mathcal{T} | B \rangle, \quad (2.49)$$

with the transition operator

$$\mathcal{T} = \text{Im } i \int d^4 x T [\mathcal{H}_{eff}(x) \mathcal{H}_{eff}(0)], \quad (2.50)$$

consisting of a non-local double insertion of the effective Hamiltonian.

A second operator-product-expansion, exploiting the large value of the  $b$ -quark mass  $m_b$ , yields for  $\mathcal{T}$

$$\mathcal{T} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[ c_{3,b} \bar{b}b + \frac{c_{5,b}}{m_b^2} \bar{b} g_s \sigma_{\mu\nu} G^{\mu\nu} b + 2 \frac{c_{6,b}}{m_b^3} (\bar{b}q)_\Gamma (\bar{q}b)_\Gamma + \dots \right] \quad (2.51)$$

and thus for the decay rate

$$\Gamma = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[ c_{3,b} \frac{\langle B | \bar{b}b | B \rangle}{2M_B} + \frac{c_{5,b}}{m_b^2} \frac{\langle B | \bar{b} g_s \sigma_{\mu\nu} G^{\mu\nu} b | B \rangle}{2M_B} + \frac{c_{6,b}}{m_b^3} \frac{\langle B | (\bar{b}q)_\Gamma (\bar{q}b)_\Gamma | B \rangle}{M_B} + \dots \right]. \quad (2.52)$$

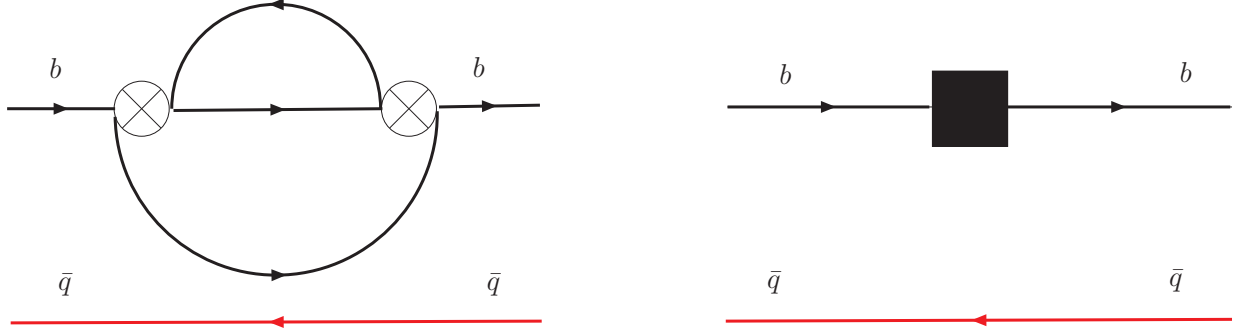
The individual contributions in Eq.(2.52) have the following origin and interpretation:

### Leading term in Eq.(2.52):

To get the first term we contracted all quark lines, except the beauty-quark lines, in the product of the two effective Hamiltonians. This leads to the following two-loop diagram on the l.h.s., where the circles with the crosses denote the  $\Delta B = 1$ -operators from the effective Hamiltonian.

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<sup>6</sup>The replacements one has to do when considering a  $D$ -meson decay are either trivial or we explicitly comment on them.



Performing the loop integrations in this diagram we get the Wilson coefficient  $c_{3,b}$  that contains all the loop functions and the dimension-three operator  $\bar{b}b$ , which is denoted by the black square in the diagram on the r.h.s. . This has been done already in Eq.(2.38), Eq.(2.41) and Eq.(2.42).

A crucial finding for the HQE was the fact, that the matrix element of the dimension-three operator  $\bar{b}b$  can also be expanded in the inverse of the  $b$ -quark mass. According to the Heavy Quark Effective Theory (HQET) we get<sup>7</sup>

$$\frac{\langle B|\bar{b}b|B\rangle}{2M_B} = 1 - \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right), \quad (2.53)$$

with the matrix element of the kinetic operator  $\mu_\pi^2$  and the matrix element of the chromomagnetic operator  $\mu_G^2$ , defined in the  $B$ -rest frame as<sup>8</sup>

$$\mu_\pi^2 = \frac{\langle B|\bar{b}(i\vec{D})^2b|B\rangle}{2M_B} + \mathcal{O}\left(\frac{1}{m_b}\right), \quad (2.54)$$

$$\mu_G^2 = \frac{\langle B|\bar{b}\frac{g_s}{2}\sigma_{\mu\nu}G^{\mu\nu}b|B\rangle}{2M_B} + \mathcal{O}\left(\frac{1}{m_b}\right). \quad (2.55)$$

With the above definitions for the non-perturbative matrix-elements the expression for the total decay rate in Eq.(2.52) becomes

$$\Gamma = \frac{G_F^2 m_b^5}{192\pi^3} V_{cb}^2 \left\{ c_{3,b} \left[ 1 - \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right) \right] + 2c_{5,b} \left[ \frac{\mu_G^2}{m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right) \right] + \frac{c_{6,b}}{m_b^3} \frac{\langle B|(\bar{b}q)_\Gamma(\bar{q}b)_\Gamma|B\rangle}{M_B} + \dots \right\}. \quad (2.56)$$

The leading term in Eq.(2.56) describes simply the decay of a free quark. Since here the spectator-quark (red) is not involved in the decay process at all, this contribution will be the same for all different  $b$ -hadrons, thus predicting the same lifetime for all  $b$ -hadrons.

<sup>7</sup>We use here the conventional relativistic normalisation  $\langle B|B\rangle = 2EV$ , where  $E$  denotes the energy of the meson and  $V$  the space volume. In the original literature sometimes different normalisations have been used, which can lead to confusion.

<sup>8</sup>We use here  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ . In the original literature sometimes the notation  $i\sigma G := i\gamma_\mu\gamma_\nu G^{\mu\nu}$  was used, which differs by a factor of  $i$  from our definition of  $\sigma$ .



The first corrections are already suppressed by two powers of the heavy  $b$ -quark mass - we have no corrections of order  $1/m_b$ ! This non-trivial result explains, why our description in terms of the free  $b$ -quark decay was so close to the experimental values of the lifetimes of  $B$ -mesons.

In the case of  $D$ -mesons the expansion parameter  $1/m_c$  is not small and the higher order terms of the HQE will lead to sizable corrections. The leading term  $c_{3,c}$  for charm-quark decays gives at the scale  $\mu = M_W$  for vanishing quark mass  $c_{3,c} = 5$ . At the scale  $\mu = \bar{m}_c(\bar{m}_c)$  and realistic values of final states masses we get

$$c_{3,c} = \begin{cases} 5 & (m_s = 0 = \alpha_s) \\ 6.29 \pm 0.72 & (\text{LO} - \text{QCD}) \\ 11.61 \pm 1.55 & (\text{NLO} - \text{QCD}) \end{cases} . \quad (2.57)$$

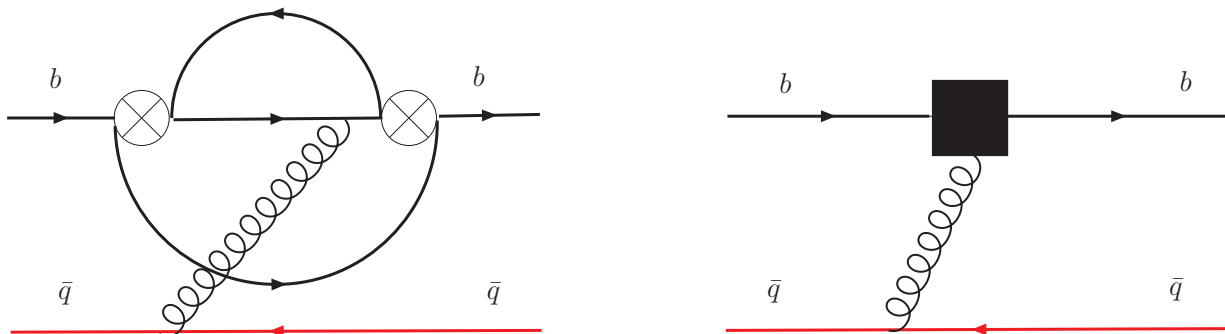
Here we have a large QCD enhancement of more than a factor of two, while phase space effects seem to be negligible.

The  $1/m_b^2$ -corrections in Eq.(2.56) have two sources: first the expansion in Eq.(2.53) and the second one - denoted by the term proportional to  $c_{5,b}$  - will be discussed below.

Concerning the different  $1/m_b^3$ -corrections, indicated in Eq.(2.56), we will see that the first two terms of the expansion in Eq.(2.52) are triggered by a two-loop diagram, while the third term is given by a one-loop diagram. This will motivate, why the  $1/m_b^3$ -corrections proportional to  $c_{3,b}$  and  $c_{5,b}$  can be neglected in comparison to the  $1/m_b^3$ -corrections proportional to  $c_{6,b}$ ; the former ones will, however, be important for precision determination of semi-leptonic decay rates<sup>9</sup>.

### Second term in Eq.(2.52):

To get the second term in Eq.(2.52) we couple in addition a gluon to the vacuum. This is denoted by the diagram below, where a gluon is emitted from one of the internal quarks of the two-loop diagram. Doing so, we obtain the so-called chromo-magnetic operator  $\bar{b}g_s\sigma_{\mu\nu}G^{\mu\nu}b$ , which already appeared in the expansion in Eq.(2.53).



<sup>9</sup>Kolya made substantial contributions to these higher order terms, which will be discussed somewhere else in these book. For our purpose of investigating lifetimes they can, however, be safely neglected, because there the hadronic uncertainties are still considerably larger.

Since this operator is of dimension five, the corresponding contribution is - as seen before - suppressed by two powers of the heavy quark mass, compared to the leading term. The corresponding Wilson coefficient  $c_{5,b}$  reads, e.g., for the semi-leptonic decay  $b \rightarrow ce^- \bar{\nu}_e$ <sup>10</sup> and the non-leptonic decays  $b \rightarrow c\bar{u}d$  and  $b \rightarrow c\bar{c}s$

$$c_{5,b}^{ce\bar{\nu}_e} = -(1-z)^4 \left[ 1 + \frac{\alpha_s}{4\pi} \dots \right] , \quad (2.58)$$

$$c_{5,b}^{c\bar{u}d} = -|V_{ud}|^2 (1-z)^3 \left[ \mathcal{N}_a(\mu) (1-z) + 8C_1 C_2 + \frac{\alpha_s}{4\pi} \dots \right] , \quad (2.59)$$

$$c_{5,b}^{c\bar{c}s} = -|V_{cs}|^2 \left\{ \mathcal{N}_a(\mu) \left[ \sqrt{1-4z}(1-2z)(1-4z-6z^2) + 24z^4 \log \left( \frac{1+\sqrt{1-4z}}{1-\sqrt{1-4z}} \right) \right] + 8C_1 C_2 \left[ \sqrt{1-4z} \left( 1 + \frac{z}{2} + 3z^2 \right) - 3z(1-2z^2) \log \left( \frac{1+\sqrt{1-4z}}{1-\sqrt{1-4z}} \right) \right] + \frac{\alpha_s}{4\pi} \dots \right\} , \quad (2.60)$$

with the quark mass ratio  $z = (m_c/m_b)^2$ . For vanishing charm-quark masses and  $V_{ud} \approx 1$  we get  $c_{5,b}^{c\bar{u}d} = -3$  at the scale  $\mu = M_W$ , which reduces in LO-QCD to about  $-1.2$  at the scale  $\mu = m_b$ .

For the total decay rate we have to sum up all possible quark level-decays

$$c_{5,b} = c_{5,b}^{ce\bar{\nu}_e} + c_{5,b}^{c\mu\bar{\nu}_\mu} + c_{5,b}^{c\tau\bar{\nu}_\tau} + c_{5,b}^{c\bar{u}d} + c_{5,b}^{c\bar{c}s} + \dots . \quad (2.61)$$

Neglecting penguin contributions we get numerically

$$c_{5,b} = \begin{cases} \approx -9 & (m_c = 0 = \alpha_s) \\ -3.8 \pm 0.3 & (\bar{m}_c(\bar{m}_c), \alpha_s(m_b)) \end{cases} , \quad (2.62)$$

For  $c_{5,b}$  both QCD effects as well as phase space effects are quite pronounced. The overall coefficient of the matrix element of the chromo-magnetic operator  $\mu_G^2$  normalised to  $2m_b^2$  in Eq.(2.56) is given by  $c_{3,b} + 4c_{5,b}$ , which is sometimes denoted as  $c_{G,b}$ . For semi-leptonic decays like  $b \rightarrow ce^- \bar{\nu}_e$ , it reads<sup>11</sup>

$$c_{G,b}^{ce\bar{\nu}_e} = c_{3,b}^{ce\bar{\nu}_e} + 4c_{5,b}^{ce\bar{\nu}_e} = (-3) \left[ 1 - \frac{8}{3}z + 8z^2 - 8z^3 + \frac{5}{3}z^4 + 4z^2 \ln(z) \right] . \quad (2.63)$$

For the sum of all inclusive decays we get

$$c_{G,b} = \begin{cases} -27 = -3c_3 & (m_c = 0 = \alpha_s) \\ -7.9 \approx -1.1c_3 & (\bar{m}_c(\bar{m}_c), \alpha_s(m_b)) \end{cases} , \quad (2.64)$$

leading to the following form of the total decay rate

$$\Gamma = \frac{G_F^2 m_b^5}{192\pi^3} V_{cb}^2 \left[ c_{3,b} - c_{3,b} \frac{\mu_\pi^2}{2m_b^2} + c_{G,b} \frac{\mu_G^2}{2m_b^2} + \frac{c_{6,b}}{m_b^3} \frac{\langle B | (\bar{b}q)_\Gamma (\bar{q}b)_\Gamma | B \rangle}{M_B} + \dots \right] . \quad (2.65)$$

<sup>10</sup>The result in Eq.(94) of the review [59] has an additional factor 6 in  $c_{5,b}^{ce\bar{\nu}_e}$ .

<sup>11</sup>We differ here slightly from Eq.(7) of [60], who have a different sign in the coefficients of  $z^2$  and  $z^3$ . We agree, however, with the corresponding result in [35].

Both  $1/m_b^2$ -corrections are reducing the decay rate and their overall coefficients are of similar size as  $c_{3,b}$ . To estimate more precisely the numerical effect of the  $1/m_b^2$  corrections, we still need the values of  $\mu_\pi^2$  and  $\mu_G^2$ . Current values [61, 62] of these parameters read for the case of  $B_d$  and  $B^+$ -mesons

$$\mu_\pi^2(B) = (0.414 \pm 0.078) \text{ GeV}^2 , \quad (2.66)$$

$$\mu_G^2(B) \approx \frac{3}{4} (M_{B^*}^2 - M_B^2) \approx (0.35 \pm 0.07) \text{ GeV}^2 . \quad (2.67)$$

For  $B_s$ -mesons only small differences compared to  $B_d$  and  $B^+$ -mesons are predicted [63]

$$\mu_\pi^2(B_s) - \mu_\pi^2(B_d) \approx (0.08 \dots 0.10) \text{ GeV}^2 , \quad (2.68)$$

$$\frac{\mu_G^2(B_s)}{\mu_G^2(B_d)} \approx 1.07 \pm 0.03 , \quad (2.69)$$

while sizable differences are expected [63] for  $\Lambda_b$ -baryons.

$$\mu_\pi^2(\Lambda_b) - \mu_\pi^2(B_d) \approx (0.1 \pm 0.1) \text{ GeV}^2 , \quad (2.70)$$

$$\mu_G^2(\Lambda_b) = 0 . \quad (2.71)$$

Inserting these values in Eq.(2.65) we find that the  $1/m_b^2$ -corrections are decreasing the decay rate slightly ( $m_b = \bar{m}_b(\bar{m}_b) = 4.248 \text{ GeV}$ ):

	$B_d$	$B^+$	$B_s$	$\Lambda_d$	
$-\frac{\mu_\pi^2}{2m_b^2}$	-0.011	-0.011	-0.014	-0.014	(2.72)
$\frac{c_{G,b}}{c_{3,b}} \frac{\mu_\pi^2}{2m_b^2}$	-0.011	-0.011	-0.011	0.00	

The kinetic and the chromo-magnetic operator each reduce the decay rate by about 1%, except for the case of the  $\Lambda_b$ -baryon, where the chromo-magnetic operator vanishes. The  $1/m_b^2$ -corrections exhibit now also a small sensitivity to the spectator-quark. Different values for the lifetimes of  $b$ -hadrons can arise due to different values of the non-perturbative parameters  $\mu_G^2$  and  $\mu_\pi^2$ , the corresponding numerical effect will, however, be small.

$X :$	$B^+$	$B_s$	$\Lambda_d$	
$\frac{\mu_\pi^2(X) - \mu_\pi^2(B_d)}{2m_b^2}$	$0.000 \pm 0.000$	$0.002 \pm 0.000$	$0.003 \pm 0.003$	(2.73)
$\frac{c_{G,b}}{c_{3,b}} \frac{\mu_G^2(X) - \mu_G^2(B_d)}{2m_b^2}$	$0.000 \pm 0.000$	$0.000 \dots 0.001$	$-0.011 \pm 0.003$	

Thus we find that the  $1/m_b^2$ -corrections give no difference in the lifetimes of  $B^+$ - and  $B_d$ -mesons, they enhance the  $B_s$ -lifetime by about 3 per mille, compared to the  $B_d$ -lifetime and they reduce the  $\Lambda_b$ -lifetime by about 1% compared to the  $B_d$ -lifetime.

To get an idea of the size of these corrections in the charm-system, we first investigate the Wilson coefficient  $c_5$ .

$$c_{5,c} = \begin{cases} \approx -5 & (m_c = 0 = \alpha_s) \\ -1.7 \pm 0.3 & (\bar{m}_c(\bar{m}_c), \alpha_s(m_b)) \end{cases} , \quad (2.74)$$

At the scale  $\mu = m_c$  the non-leptonic contribution to  $c_5$  is getting smaller than in the bottom case and it even changes sign. For the coefficient  $c_G$  we find

$$c_{G,c} = \begin{cases} \approx -15 = -3 c_{3,c} & (m_c = 0 = \alpha_s) \\ 4.15 \pm 1.48 = (0.37 \pm 0.13) c_{3,c} & (\bar{m}_c(\bar{m}_c), \alpha_s(m_b)) \end{cases} . \quad (2.75)$$

We see for that for the charm case the overall coefficient of the chromo-magnetic operator has now a positive sign and the relative size is less than in the bottom case. For  $D^0$ - and  $D^+$ -mesons the value of the chromo-magnetic operator reads

$$\mu_G^2(D) \approx \frac{3}{4} (M_{D^*}^2 - M_D^2) \approx 0.41 \text{ GeV}^2 , \quad (2.76)$$

which is of similar size as in the  $B$ -system. Normalising this value to the charm quark mass  $m_c = \bar{m}_c(\bar{m}_c) = 1.277 \text{ GeV}$ , we get however a bigger contribution compared to the bottom case and also a different sign.

$$c_{G,c} \frac{\mu_G^2(D)}{2m_c^2} \approx +0.05 c_{3,c} . \quad (2.77)$$

Now the second order corrections are non-negligible, with a typical size of about + 5% of the total decay rate. Concerning lifetime differences of  $D$ -mesons, we find no visible effect due to the chromo-magnetic operator [9]

$$\frac{\mu_G^2(D^+)}{\mu_G^2(D^0)} \approx 0.993 , \quad (2.78)$$

$$\frac{\mu_G^2(D_s^+)}{\mu_G^2(D^0)} \approx 1.012 \pm 0.003 . \quad (2.79)$$

For the kinetic operator a sizable SU(3) flavour breaking was found by Bigi, Mannel and Uraltsev [63]

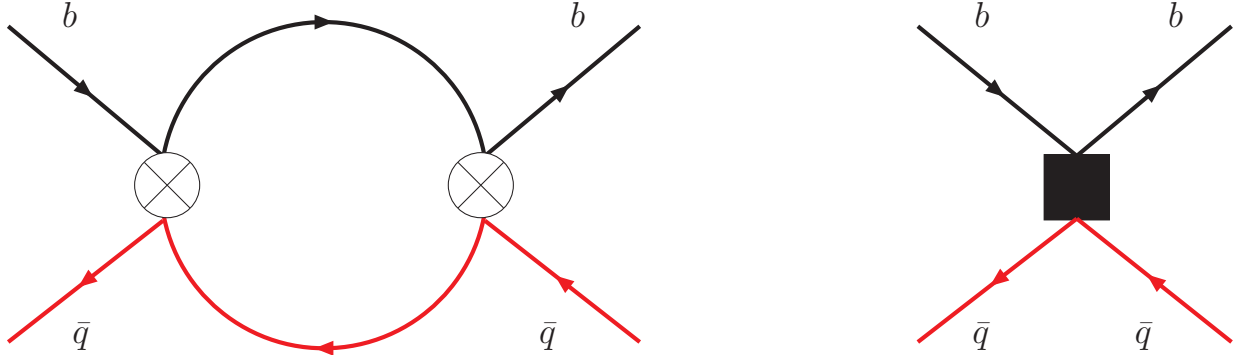
$$\mu_\pi^2(D_s^+) - \mu_\pi^2(D^0) \approx 0.1 \text{ GeV}^2 , \quad (2.80)$$

leading to an reduction of the  $D_s^+$ -lifetime of the order of 3% compared to the  $D^0$ -lifetime

$$\frac{\mu_\pi^2(D_s^+) - \mu_\pi^2(D^0)}{2m_c^2} \approx 0.03 . \quad (2.81)$$

### Third term in Eq.(2.52):

The next term is obtained by only contracting two quark lines in the product of the two effective Hamiltonian in Eq.(2.50). The  $b$ -quark and the spectator quark of the considered hadron are not contracted. For  $B_d$ -mesons ( $q = d$ ) and  $B_s$ -mesons ( $q = s$ ) we get the following so-called *weak annihilation* diagram.

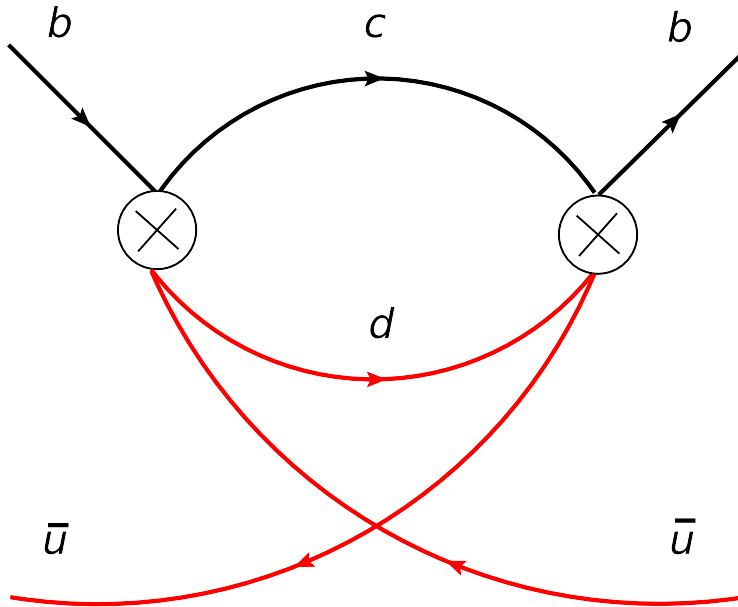


Performing the loop integration on the diagram on the l.h.s. we get the Wilson coefficient  $c_6$  and dimension six four-quark operators  $(\bar{b}q)_\Gamma(\bar{q}b)_\Gamma$ , with Dirac structures  $\Gamma$ . The corresponding matrix elements of these  $\Delta B = 0$  operators are typically written as

$$\langle B | (\bar{b}q)_\Gamma (\bar{q}b)_\Gamma | B \rangle = c_\Gamma f_B^2 M_B B_\Gamma, \quad (2.82)$$

with the bag parameter  $B_\Gamma$ , the decay constant  $f_B$  and a numerical factor  $c_\Gamma$  that contains some colour factors and sometimes also ratios of masses.

For the case of the  $B^+$ -meson we get a similar diagram, with the only difference that now the external spectator-quark lines are crossed, this is the so-called *Pauli interference* diagram.



There are two very interesting things to note. First this is now a one-loop diagram. Although being suppressed by three powers of the  $b$ -quark mass it is enhanced by a phase space factor of  $16\pi^2$  compared to the leading two-loop diagrams. Second, now we are really sensitive to the flavour of the spectator-quark, because in principle, each different spectator quark gives a different contribution<sup>12</sup>. These observations are responsible for the fact that lifetime differences in the system of heavy hadrons are almost entirely due to the contribution of weak annihilation and Pauli interference diagrams.

<sup>12</sup>This difference is, however, negligible, if one considers, e.g.,  $B_s$  vs.  $B_d$ .

In the case of the  $B_d$  meson four different four-quark operators arise

$$Q^q = \bar{b}\gamma_\mu(1 - \gamma_5)q \times \bar{q}\gamma^\mu(1 - \gamma_5)b, \quad Q_S^q = \bar{b}(1 - \gamma_5)q \times \bar{q}(1 - \gamma_5)b, \quad (2.83)$$

$$T^q = \bar{b}\gamma_\mu(1 - \gamma_5)T^a q \times \bar{q}\gamma^\mu(1 - \gamma_5)T^a b, \quad T_S^q = \bar{b}(1 - \gamma_5)T^a q \times \bar{q}(1 - \gamma_5)T^a b, \quad , \quad (2.84)$$

with  $q = d$  for the case of  $B_d$ -mesons.  $Q$  denotes colour singlet operators and  $T$  colour octet operators. For historic reasons the matrix elements of these operator are typically expressed as

$$\frac{\langle B_d|Q^d|B_d\rangle}{M_{B_d}} = f_B^2 B_1 M_{B_d}, \quad \frac{\langle B_d|Q_S^d|B_d\rangle}{M_{B_d}} = f_B^2 B_2 M_{B_d}, \quad (2.85)$$

$$\frac{\langle B_d|T^d|B_d\rangle}{M_{B_d}} = f_B^2 \epsilon_1 M_{B_d}, \quad \frac{\langle B_d|T_S^d|B_d\rangle}{M_{B_d}} = f_B^2 \epsilon_2 M_{B_d}. \quad (2.86)$$

The bag parameters  $B_{1,2}$  are expected to be of order one in vacuum insertion approximation, while the  $\epsilon_{1,2}$  vanish in that limit. We will discuss below several estimates of  $B_i$  and  $\epsilon_i$ . Decay constants can be determined with lattice-QCD, see, e.g., the reviews of FLAG [64] or with QCD sum rules, see, e.g., the recent determination in [65]. Later on, we will see, however, that the Wilson coefficients of  $B_1$  and  $B_2$  are affected by sizable numerical cancellations, enhancing hence the relative contribution of the colour suppressed  $\epsilon_1$  and  $\epsilon_2$ . The corresponding Wilson coefficients of the four operators can be written as

$$c_6^{Q^d} = 16\pi^2 [ |V_{ud}|^2 F^u + |V_{cd}|^2 F^c ], \quad c_6^{Q_S^d} = 16\pi^2 [ |V_{ud}|^2 F_S^u + |V_{cd}|^2 F_S^c ], \quad (2.87)$$

$$c_6^{T^d} = 16\pi^2 [ |V_{ud}|^2 G^u + |V_{cd}|^2 G^c ], \quad c_6^{T_S^d} = 16\pi^2 [ |V_{ud}|^2 G_S^u + |V_{cd}|^2 G_S^c ]. \quad (2.88)$$

$F^q$  describes an internal  $c\bar{q}$  loop in the above weak annihilation diagram. The functions  $F$  and  $G$  are typically split up in contributions proportional to  $C_2^2$ ,  $C_1 C_2$  and  $C_1^2$ .

$$F^u = C_1^2 F_{11}^u + C_1 C_2 F_{12}^u + C_2^2 F_{22}^u, \quad (2.89)$$

$$F_S^u = \dots \quad (2.90)$$

Next, each of the  $F_{ij}^q$  can be expanded in the strong coupling

$$F_{ij}^u = F_{ij}^{u,(0)} + \frac{\alpha_s}{4\pi} F_{ij}^{u,(1)} + \dots, \quad (2.91)$$

$$F_{S,ij}^u = \dots \quad (2.92)$$

As an example we give the following LO results

$$F_{11}^{u,(0)} = -3(1-z)^2 \left(1 + \frac{z}{2}\right), \quad F_{S,11}^{u,(0)} = 3(1-z)^2 (1+2z), \quad (2.93)$$

$$F_{12}^{u,(0)} = -2(1-z)^2 \left(1 + \frac{z}{2}\right), \quad F_{S,12}^{u,(0)} = 2(1-z)^2 (1+2z), \quad (2.94)$$

$$F_{22}^{u,(0)} = -\frac{1}{3}(1-z)^2 \left(1 + \frac{z}{2}\right), \quad F_{S,22}^{u,(0)} = \frac{1}{3}(1-z)^2 (1+2z), \quad (2.95)$$

$$G_{22}^{u,(0)} = -2(1-z)^2 \left(1 + \frac{z}{2}\right), \quad G_{S,22}^{u,(0)} = 2(1-z)^2 (1+2z), \quad (2.96)$$

with  $z = m_c^2/m_b^2$ .

Putting everything together we arrive at the following expression for the decay rate of a  $B_d$ -meson

$$\begin{aligned}\Gamma_{B_d} &= \frac{G_F^2 m_b^5}{192\pi^3} V_{cb}^2 \left[ c_3 - c_3 \frac{\mu_\pi^2}{2m_b^2} + c_G \frac{\mu_G^2}{2m_b^2} + \frac{16\pi^2 f_B^2 M_{B_d}}{m_b^3} \tilde{c}_6^{B_d} + \mathcal{O}\left(\frac{1}{m_b^3}, \frac{16\pi^2}{m_b^4}\right) \right] \\ &\approx \frac{G_F^2 m_b^5}{192\pi^3} V_{cb}^2 \left[ c_3 - 0.01c_3 - 0.01c_3 + \frac{16\pi^2 f_B^2 M_{B_d}}{m_b^3} \tilde{c}_6^{B_d} + \mathcal{O}\left(\frac{1}{m_b^3}, \frac{16\pi^2}{m_b^4}\right) \right],\end{aligned}\tag{2.97}$$

with

$$\begin{aligned}\tilde{c}_6^{B_d} &= |V_{ud}|^2 (F^u B_1 + F_S^u B_2 + G^u \epsilon_1 + G_S^u \epsilon_2) \\ &\quad + |V_{cd}|^2 (F^c B_1 + F_S^c B_2 + G^c \epsilon_1 + G_S^c \epsilon_2) .\end{aligned}\tag{2.98}$$

The size of the third contribution in Eq.(2.97) is governed by size of  $\tilde{c}_6$  and its pre-factor. The pre-factor gives

$$\frac{16\pi^2 f_{B_d}^2 M_{B_d}}{m_b^3} \approx 0.395 \approx 0.05 c_3 ,\tag{2.99}$$

where we used  $f_{B_d} = (190.5 \pm 4.2)$  MeV [64] for the decay constant. If  $\tilde{c}_6$  is of order 1, we would expect corrections of the order of 5% to the total decay rate, which are larger than the formally leading  $1/m_b^2$ -corrections. The LO-QCD expression for  $\tilde{c}_6^{B_d}$  can be written as

$$\begin{aligned}\tilde{c}_6^{B_d} &= |V_{ud}|^2 (1-z)^2 \left\{ \left( 3C_1^2 + 2C_1 C_2 + \frac{1}{3}C_2^2 \right) \left[ (B_2 - B_1) + \frac{z}{2}(4B_2 - B_1) \right] \right. \\ &\quad \left. + 2C_2^2 \left[ (\epsilon_2 - \epsilon_1) + \frac{z}{2}(4\epsilon_2 - \epsilon_1) \right] \right\} .\end{aligned}\tag{2.100}$$

However, in Eq.(2.100) several cancellations are arising. In the first line there is a strong cancellation among the bag parameters  $B_1$  and  $B_2$ . In vacuum insertion approximation  $B_1 - B_2$  is zero and the next term proportional to  $4B_2 - B_1$  is suppressed by  $z \approx 0.055$ . Using the latest lattice determination of these parameters [66] - dating back to 2001 -

$$B_1 = 1.10 \pm 0.20 , \quad B_2 = 0.79 \pm 0.10 , \quad \epsilon_1 = -0.02 \pm 0.02 , \quad \epsilon_2 = 0.03 \pm 0.01\tag{2.101}$$

one finds  $B_1 - B_2 \in [0.01, 0.61]$  and  $(4B_2 - B_1)z/2 \in [0.07, 0.12]$ , so the second contribution is slightly suppressed compared to the first one. Moreover there is an additional cancellation among the  $\Delta B = 1$  Wilson coefficients. Without QCD the combination  $3C_1^2 + 2C_1 C_2 + \frac{1}{3}C_2^2$  is equal to  $1/3$ , in LO-QCD this combination is reduced to about  $0.05 \pm 0.05$  at the scale of  $m_b$  (varying the renormalisation scale between  $m_b/2$  and  $2m_b$ ). Hence  $B_1$  and  $B_2$  give a contribution between 0 and 0.07 to  $\tilde{c}_6^{B_d}$ , leading thus at most to a correction of about 4 per mille to the total decay rate. This statement depends, however, crucially on the numerical values of the bag parameters, where we are lacking a state-of-the-art determination.

There is no corresponding cancellation in the coefficients related to the colour-suppressed bag parameters  $\epsilon_{1,2}$ . According to [66]  $\epsilon_2 - \epsilon_1 \in [0.02, 0.08]$ , leading to a correction of at most 1.0% to the decay rate. Relying on the lattice determination in [66] we find that the colour-suppressed operators can be numerically more important than the colour allowed operators and the total decay rate of the  $B_d$ -meson can be enhanced by the weak annihilation at most by about 1.4%. The status at NLO-QCD will be discussed below.

The Pauli interference contribution to the  $B^+$ -decay rate gives

$$\tilde{c}_6^{B^+} = (1 - z)^2 [(C_1^2 + 6C_1C_2 + C_2^2) B_1 + 6(C_1^2 + C_2^2) \epsilon_1] . \quad (2.102)$$

The contribution of the colour-allowed operator is slightly suppressed by the  $\Delta B = 1$  Wilson coefficients. Without QCD the bag parameter  $B_1$  has a pre-factor of one, which changes in LO-QCD to about -0.3. Taking again the lattice values for the bag parameter from [66], we expect Pauli interference contributions proportional to  $B_1$  to be of the order of about  $-1.8\%$  of the total decay rate. In the coefficient of  $\epsilon_1$  no cancellation is arising and we expect (using again [66]) this contribution to be between 0 and  $-1.5\%$  of the total decay rate. All in all Pauli interference seems to reduce the total  $B^+$ -decay rate by about 1.8% to 3.3%. The status at NLO-QCD will again be discussed below.

In the charm system the pre-factor of the coefficient  $c_6$  reads

$$\frac{16\pi^2 f_D^2 M_D}{m_c^3} \approx \begin{cases} 6.2 \approx 0.6 c_3 & \text{for } D^0, D^+ \\ 9.2 \approx 0.8 c_3 & \text{for } D_s^+ \end{cases} , \quad (2.103)$$

where we used  $f_{D^0} = (209.2 \pm 3.3)$  MeV and  $f_{D_s^+} = (248.3 \pm 2.7)$  MeV [64] for the decay constants. Depending on the strength of the cancellation among the  $\Delta C = 1$  Wilson coefficients and the bag parameters, large corrections seem to be possible now: In the case of the weak annihilation the cancellation of the  $\Delta C = 1$  Wilson coefficients seems to be even more pronounced than at the scale  $m_b$ . Thus a knowledge of the colour-suppressed operators is inalienable. In the case of Pauli interference no cancellation occurs and we get values for the coefficient of  $B_1$ , that are smaller than  $-1$  and we get a sizable, but smaller contribution from the colour-suppressed operators. Unfortunately there is no lattice determination of the  $\Delta C = 0$  matrix elements available, so we cannot make any final, profound statements about the status in the charm system. Numerical results for the NLO-QCD case will also be discussed below.

#### Fourth term in Eq.(2.52):

If one takes in the calculation of the weak annihilation and Pauli interference diagrams also small momenta and masses of the spectator quark into account, one gets corrections that are suppressed by four powers of  $m_b$  compared to the free-quark decay. These dimension seven terms are either given by four-quark operators times the small mass of the spectator quark or by a four quark operator with an additional derivative. Examples are the following  $\Delta B = 0$



operators

$$P_1 = \frac{m_{d,s}}{m_b} \bar{b}_i (1 - \gamma_5) d_i \times \bar{d}_j (1 - \gamma_5) b_j, \quad (2.104)$$

$$P_2 = \frac{m_{d,s}}{m_b} \bar{b}_i (1 + \gamma_5) d_i \times \bar{d}_j (1 + \gamma_5) b_j, \quad (2.105)$$

$$P_3 = \frac{1}{m_b^2} \bar{b}_i \overleftarrow{D}_\rho \gamma_\mu (1 - \gamma_5) D^\rho d_i \times \bar{d}_j \gamma^\mu (1 - \gamma_5) b_j, \quad (2.106)$$

$$P_4 = \frac{1}{m_b^2} \bar{b}_i \overleftarrow{D}_\rho (1 - \gamma_5) D^\rho d_i \times \bar{d}_j (1 + \gamma_5) b_j. \quad (2.107)$$

These operators have currently only been estimated within vacuum insertion approximation. However, for the corresponding operators appearing in the decay rate difference of neutral  $B$ -meson first studies with QCD sum rules have been performed [67, 68].

Putting everything together we arrive at the *Heavy-Quark Expansion* of decay rates of heavy hadrons

$$\Gamma = \Gamma_0 + \frac{\Lambda^2}{m_b^2} \Gamma_2 + \frac{\Lambda^3}{m_b^3} \Gamma_3 + \frac{\Lambda^4}{m_b^4} \Gamma_4 + \dots, \quad (2.108)$$

where the expansion parameter is denoted by  $\Lambda/m_b$ . From the above explanations it is clear that  $\Lambda$  is not simply given by  $\Lambda_{QCD}$  - the pole of the strong coupling constant - as stated often in the literature. Very naively one expects  $\Lambda$  to be of the order of  $\Lambda_{QCD}$ , because both denote non-perturbative effects. The actual value of  $\Lambda$ , has, however, to be determined by an explicit calculation for each order of the expansion separately. At order  $1/m_b^2$  one finds that  $\Lambda$  is of the order of  $\mu_\pi$  or  $\mu_G$ , so roughly below 1 GeV. For the third order  $\Lambda^3$  is given by  $16\pi^2 f_B^2 M_B$  times a numerical suppression factor, leading to values of  $\Lambda$  larger than 1 GeV. Moreover, each of the coefficients  $\Gamma_j$ , which is a product of a perturbatively calculable Wilson coefficient and a non-perturbative matrix element, can be expanded in the strong coupling

$$\Gamma_j = \Gamma_j^{(0)} + \frac{\alpha_s(\mu)}{4\pi} \Gamma_j^{(1)} + \frac{\alpha_s^2(\mu)}{(4\pi)^2} \Gamma_j^{(2)} + \dots. \quad (2.109)$$

Before we apply this framework to experimental observables, we would like to make some comments of caution.

A possible drawback of this approach might be that the expansion in the inverse heavy quark mass does not converge well enough — advocated under the labelling *violation of quark hadron duality*. There is a considerable amount of literature about theoretical attempts to prove or to disprove duality, but all of these attempts have to rely on strong model assumptions.

Kolya published some general investigations of quark hadron duality violation in [69, 70] and some investigations within the two dimensional 't Hooft model [71, 72], that indicated the validity of quark hadron duality. Other investigations in that direction were e.g. performed by Grinstein and Lebed in 1997 [73] and 1998 [74] and by Grinstein in 2001 [75, 76]. In our opinion the best way of tackling this question is to confront precise HQE-based predictions with precise experimental data. An especially well suited candidate for this problem is the decay  $b \rightarrow c\bar{c}s$ , which is CKM dominant, but phase space suppressed. The actual expansion parameter of the HQE is in this case not  $1/m_b$  but  $1/(m_b\sqrt{1-4z})$ ; so violations of duality

should be more pronounced. Thus a perfect observable for testing the HQE is the decay rate difference  $\Delta\Gamma_s$  of the neutral  $B_s$  mesons, which is governed by the  $b \rightarrow c\bar{c}s$  transition. The first measurement of this quantity in 2012 and several follow-up measurements are in perfect agreement with the HQE prediction and exclude thus huge violations of quark hadron duality, see [77] and the discussion below.

### 2.3 Overview of observables

In this section we give a brief overview of observables, whose experimental values can be compared with HQE predictions. As we have discussed above, the general expression for the lifetime ratio of two heavy hadrons  $H_1$  and  $H_2$  reads

$$\begin{aligned} \frac{\tau(H_1)}{\tau(H_2)} &= \frac{\Gamma_2}{\Gamma_1} = 1 + \frac{\Gamma_2 - \Gamma_1}{\Gamma_1} = 1 + \frac{\mu_\pi^2(H_1) - \mu_\pi^2(H_2)}{2m_b^2} + \frac{c_G \mu_G^2(H_2) - \mu_G^2(H_1)}{c_3} \\ &\quad + \frac{c_6(H_2) \langle H_2|Q|H_2 \rangle}{c_3 m_b^3 M_B} - \frac{c_6(H_1) \langle H_1|Q|H_1 \rangle}{c_3 m_b^3 M_B} + \mathcal{O}\left(\frac{\Lambda^4}{m_b^4}\right), \end{aligned} \quad (2.110)$$

where we have used the HQE expression for  $\Gamma_1$  and expanded the ratio consistently in  $1/m_b$ . Another possibility would be to use the experimental value for the lifetime  $\tau_1$  of the hadron  $H_1$  and the relation  $\Gamma_1 = 1/\tau_1$  to express the decay rate  $\Gamma_1$ . This gives

$$\begin{aligned} \frac{\tau(H_1)}{\tau(H_2)} &= 1 + \frac{G_F^2 m_b^3}{384\pi^3} V_{cb}^2 \tau_1 \left[ c_3 (\mu_\pi^2(H_1) - \mu_\pi^2(H_2)) + c_G (\mu_G^2(H_2) - \mu_G^2(H_1)) \right] \\ &\quad + \frac{G_F^2 m_b^2}{192\pi^3} V_{cb}^2 \tau_1 \left[ \frac{c_6(H_2) \langle H_2|Q|H_2 \rangle - c_6(H_1) \langle H_1|Q|H_1 \rangle}{M_B} + \mathcal{O}\left(\frac{\Lambda}{m_b}\right) \right]. \end{aligned} \quad (2.111)$$

Both methods yield similar numerical results. The relative difference of them is given by the deviation of the  $b$ -lifetime prediction in Eq.(2.43) from the measured lifetime:

$$\delta = \frac{1.65 \text{ ps}}{1.519 \text{ ps}} = 1.086. \quad (2.112)$$

Switching between the two methods will change the relative size of the HQE-corrections by 9%. This intrinsic uncertainty has to be kept in mind for error estimates; it could be reduced by an NNLO-QCD calculation of  $c_3$ .

We will discuss the following classes of lifetime ratios:

- In the case of B-mesons, there are two well-measured ratios

$$\frac{\tau(B_s)}{\tau(B_d)}, \quad \frac{\tau(B^+)}{\tau(B_d)}. \quad (2.113)$$

We have an almost perfect cancellation in the first ratio, therefore this clean ratio can be used to search for new physics effects, see, e.g., [78, 79]. The second ratio is dominated by Pauli interference.

- Concerning b-baryons, we expect some visible  $1/m_b^2$ - and  $1/m_b^3$ -corrections. Until recently only the  $\Lambda_b$  lifetime was studied experimentally. In 2014 also more precise numbers for the  $\Xi_b$ -baryons became available [3, 4] and we can study now ratios like

$$\frac{\tau(\Lambda_b)}{\tau(B_d)}, \quad \frac{\tau(\Xi_b^+)}{\tau(\Xi_b^0)}. \quad (2.114)$$

- The  $B_c$ -meson is quite different from the above discussion, because now both constituent quarks have large decay rates and we have simultaneously an expansion in  $1/m_b$  and in  $1/m_c$ .

$$\tau(B_c). \quad (2.115)$$

- The ratio of D-meson lifetimes is similar to the ones of  $B$ -mesons. The big issue is here simply if the HQE shows any convergence at all in the ratios

$$\frac{\tau(D_s^+)}{\tau(D^0)}, \quad \frac{\tau(D^+)}{\tau(D^0)}. \quad (2.116)$$

Decay rate differences  $\Delta\Gamma$  of neutral mesons can be determined by a very similar HQE approach as discussed above, see, e.g., [77] for an introduction into mixing. The general expressions for the mixing contribution  $\Gamma_{12}$  starts at order  $1/m_b^3$  and it can be written as

$$\Gamma_{12}^q = \left(\frac{\Lambda}{m_b}\right)^3 \Gamma_3 + \left(\frac{\Lambda}{m_b}\right)^4 \Gamma_4 + \dots \quad (2.117)$$

In the mixing sector we get the following observables:

- In the neutral  $B$ -meson system  $\Delta\Gamma_q$  denotes the difference of the total decay rates of the heavy (H) mesons eigenstate and the light (L) eigenstate. They are extracted from  $\Gamma_{12}$  via the relations

$$\Delta\Gamma_d = \Gamma_L^d - \Gamma_H^d = 2|\Gamma_{12}^d| \cos \phi_d, \quad \Delta\Gamma_s = \Gamma_L^s - \Gamma_H^s = 2|\Gamma_{12}^s| \cos \phi_s, \quad (2.118)$$

with the mixing phase defined as  $\phi_q = \arg(-M_{12}^q/\Gamma_{12}^q)$ . Related quantities, that also rely on the HQE for  $\Gamma_{12}$  are the so-called semi-leptonic asymmetries

$$a_{sl}^d = \left| \frac{\Gamma_{12}^d}{M_{12}^d} \right| \sin \phi_d, \quad a_{sl}^s = \left| \frac{\Gamma_{12}^s}{M_{12}^s} \right| \sin \phi_d, \quad (2.119)$$

that were already discussed in 1987 by Bigi, Khoze, Uraltsev and Sanda [80] and even earlier in [81, 82, 83].

- In the case of neutral D-mesons the expression of the decay rate difference  $\Delta\Gamma_D$  in terms of  $\Gamma_{12}$  and  $M_{12}$  is more complicated, than in the case of  $B$ -mesons. Here, typically the quantity  $y$  is discussed

$$y = \frac{\Delta\Gamma_D}{2\Gamma_D}. \quad (2.120)$$

Before comparing recent data with HQE predictions, we will do some historical investigations of the origin of the HQE.

### 3 A brief history of lifetimes and the HQE

We give here a brief history of the theoretical investigations of lifetimes of heavy hadrons and the heavy quark expansion. We do not discuss the development of the Heavy Quark Effective Theory (HQET), which happened in the late 1980s and early 1990s. We also concentrate on total decay rates, thus leaving out many of the important contributions to the theory of semi-leptonic decays.

Heavy hadrons were discovered as  $J/\psi$ -states in 1974 [84, 85]. At about that time the first investigations of weak decays of heavy hadrons started. We structure the theoretical development in three periods: pioneering studies, systematic studies and precision studies. It is of course quite arbitrary, where the exact borders between these periods are drawn.

#### 3.1 Pioneering studies

Here we summarise the first investigations of heavy meson decays, without having a systematic expansion at hand.

- According to Kolya (see, e.g., [63])<sup>13</sup> the first time, that heavy flavour hadrons have been described asymptotically by a free quark decay was in 1973 by Nikolaev [86]. The charm-quark decay as the dominant contribution to  $D$ -meson decays was considered, e.g., in 1974/5 by Gaillard, Lee and Rosner [87], by Kingsley, Treiman, Wilczek and Zee [88], by Ellis, Gaillard and Nanopoulos [89] and by Altarelli, Cabibbo and Maiani [90]. In [89] the total lifetime of the charm meson was calculated to be about  $0.5 \text{ ps}^{-1}$ , by taking only the LO-QCD value of  $c_3$  with vanishing internal quark masses into account.
- Pauli interference was introduced in 1979 by Guberina, Nussinov, Peccei and Rückl [91]. Without having any systematic expansion at hand these authors found

$$\frac{\tau(D^+)^{\text{PI}1979}}{\tau(D^0)} = \frac{c_-^2 + 2c_+^2 + 2}{4c_+^4 + 2} = \frac{\mathcal{N}_a + 2}{\mathcal{N}_a + 2 + (C_1^2 + 6C_1C_2 + C_2^2)}. \quad (3.121)$$

This result can be obtained from our formulae by the following modifications:

- For the  $D^0$  decay rate only  $\Gamma_0$ , i.e., only the free quark decay, is taken into account in LO-QCD and with vanishing internal quark masses, i.e., no  $1/m_c^2$ - and  $1/m_c^3$ -corrections are considered.
- For the  $D^+$  decay rate only  $\Gamma_0$  and the Pauli interference in  $\Gamma_3$  are taken into account in LO-QCD and with vanishing internal quark masses. Since at that time no systematic expansion was available, the contributions were simply added. This corresponds to making the following replacements in our formulae:  $(4\pi f_D)^2 \approx (2.63 \text{ GeV})^2 \rightarrow m_c^2$  and  $M_D \approx m_c$ , which is of course very crude and more importantly not really justified. In addition the bag parameters were used in vacuum insertion approximation, i.e.,  $B = 1$  and  $\epsilon = 0$ .

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<sup>13</sup>In an email from 4.11.2012 Kolya wrote to me: *The present generation may not appreciate how nontrivial (or even heretic) such a proposition could sound that time! It was the era of traditional hadron physics where descriptions like Veneziano model or Regge theory were assumed to underlie hadrons, and their common (indisputable) feature was soft interactions leading to exponential suppression of any form factor...*

With modern inputs Eq.(3.121) gives a value of about 1.5, while the authors obtained with input parameters from 1979 and without using the renormalisation group for the  $\Delta C = 1$  Wilson coefficients a ratio of about 10. It is also quite interesting to note Fig. 1 of [91], which presents the leading  $\bar{c}c$ -term, weak annihilation and Pauli interference. Further studies of Pauli interference were done slightly later in, e.g., [92].

- Weak annihilation suffers from chirality suppression, thus it was proposed in 1979 by Bander, Silverman and Soni [93] and also by Fritzsche and Minkowski [94] and by Bernreuther and Nachtmann and Stech [95] to consider gluon emission from the ingoing quark lines in order to explain the large lifetime ratio in the charm system, see Fig. 1. This yields a large contribution proportional to  $f_D^2/\langle E_{\bar{q}}^2 \rangle$ , where  $f_D \approx 200$  MeV is

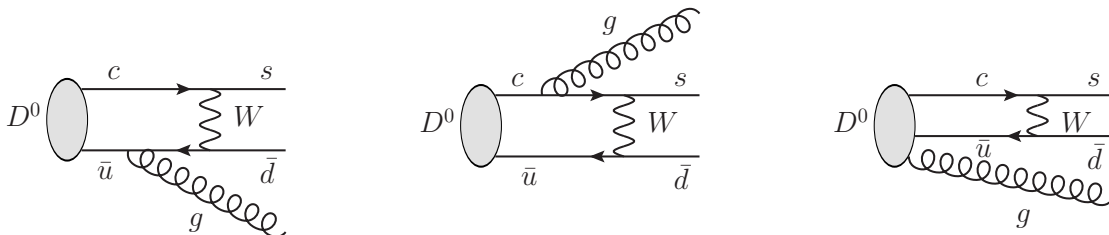


Figure 1: Gluon emission from the weak annihilation diagram.

the  $D$  meson decay constant and  $\langle E_{\bar{q}} \rangle$  denotes the average energy of the initial anti-quark. Thus the one-gluon emission weak annihilation seems to be not suppressed at all, compared to the leading free-quark decay. In [95], the authors additionally included the Cabibbo-suppressed weak annihilation of  $D^+$  and obtained for the effects of weak annihilation in  $D^0$  and  $D^+$

$$\frac{\tau(D^+)^{\text{WA 1980}}}{\tau(D^0)} \approx 5.6 - 6.9. \quad (3.122)$$

One should keep in mind, that Pauli interference, which is now known to be the dominant effect, is still neglected here. Comparing with the experimental numbers in the Introduction, one sees what a severe overestimation these early analyses, that did not allow for any power-counting, were. If the arguments of [93, 94, 95] were correct, then no systematic HQE would be possible - we come back to this point below.

- More systematic studies and further investigations of the Pauli interference effect can be found in [96, 97, 98]. The following formula - Eq.(3.123) - was first derived by Shifman and Voloshin and presented several years later in the review of Khoze and Shifman from 1983 [96]. It was pointed out in February 1984 by Bilic that in the original version there was a sign error, which was corrected in the same year [97] by Shifman and Voloshin and shortly afterwards by Bilic, Guberina, Trampetic [98].

$$\begin{aligned} \Gamma(D^+) = \frac{G_F^2}{2M_D} \langle D^+ | & \frac{m_c^5}{64\pi^3} \frac{2C_+^2 + C_-^2}{3} \bar{c}c + \frac{m_c^2}{2\pi} \left[ (C_+^2 + C_-^2) (\bar{c}\Gamma_\mu T^A d)(\bar{d}\Gamma_\mu T^A c) \right. \\ & \left. + \frac{2C_+^2 - C_-^2}{3} (\bar{c}\Gamma_\mu d)(\bar{d}\Gamma_\mu c) \right] | D^+ \rangle. \end{aligned} \quad (3.123)$$

We have rewritten the original expression in the colour-singlet and colour-octet basis commonly used today for  $\Delta C = 0$  operators. In order to compare easier with our formulae we can switch from the  $C_+$ ,  $C_-$ -basis to the  $C_1$ ,  $C_2$ -basis

$$2C_+^2 + C_-^2 = \mathcal{N}_a, \quad (3.124)$$

$$C_+^2 + C_-^2 = 2(C_1^2 + C_2^2), \quad (3.125)$$

$$2C_+^2 - C_-^2 = C_1^2 + 6C_1C_2 + C_2^2. \quad (3.126)$$

Neglecting weak annihilation, the total decay rate for  $D^0$  is given by the first term in Eq.(3.123). For the bag parameters vacuum insertion approximation is used. In early analyses the lifetime ratios were generally underestimated

$$\frac{\tau(D^+)^{\text{HQE 1984}}}{\tau(D^0)} \approx 1.5, \quad (3.127)$$

which was mainly due to a too small estimate for the decay constant  $f_D \approx 160 - 170$  MeV. The present value [64] of  $f_D = 209.2$  MeV yields  $\tau(D^+)/\tau(D^0) \approx 2.2$ , which drastically improves the consistency with experiments. To some extent Eq.(3.123) given in [96, 97] can be seen as a starting point for a systematic expansion in the inverse of the heavy quark mass.

- In 1986 [99] Shifman and Voloshin considered for the first time the effects of hybrid renormalisation, coming thus much closer to the present state of theory predictions for the ratio of  $D^+$  and  $D^0$  lifetimes. Moreover they predicted [99]

$$\frac{\tau(B_s)^{\text{HQE 1986}}}{\tau(B_d)} \approx 1, \quad \frac{\tau(B^+)^{\text{HQE 1986}}}{\tau(B_d)} \approx 1.1, \quad \frac{\tau(\Lambda_b)^{\text{HQE 1986}}}{\tau(B_d)} \approx 0.96, \quad (3.128)$$

which is amazingly close to current experimental values.

In [97, 99], it was also argued, that  $\tau(D_s^+) \approx \tau(D^0)$ , which contradicted the experimental situation at that time. In 1986 it was further shown by Guberina, Rückl and Trampetic [100] that the HQE was able to correctly reproduce the hierarchy of lifetimes in the charm sector

$$\text{HQE 1986: } \tau(D^+) > \tau(D^0) > \tau(\Xi_c^+) > \tau(\Lambda_c^+) > \tau(\Xi_c^0) > \tau(\Omega_c^0). \quad (3.129)$$

In 1986 Khoze, Shifman, Uraltsev and Voloshin [101] refined the analysis of [96, 97], by taking into account weak annihilation and  $B$ -mixing. In particular they found that the decay rate difference in the neutral  $B_s$ -system may be sizable

$$\frac{\Delta\Gamma_s^{\text{HQE 1986}}}{\Gamma_s} \approx 0.07 \left( \frac{f_{B_s}}{130 \text{ MeV}} \right)^2 \approx 0.22, \quad (3.130)$$

where we inserted the most recent FLAG-average [64] for the decay constant. The authors of [101] emphasised also that the weak annihilation effects suggested in [93, 94, 95] formally leads to huge corrections in the  $1/m_q$ -expansion, which spoils a systematic expansion. This problem somehow stopped<sup>14</sup> further work in that direction until the issue was settled in January 1992 by Bigi and Uraltsev [103].

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<sup>14</sup>Blok and Shifman stated in 1992 [102]: ”Probably for this reason the problem of pre-asymptotic corrections in the inclusive widths has been abandoned for many years.”.

## 3.2 Systematic studies

Here we describe the development of the HQE in its current form.

- For inclusive semi-leptonic decays, where the above issue was not severe, it was shown already in 1990 by Chay, Georgi and Grinstein [104], that in an expansion in inverse powers of the heavy quark mass no  $1/m_q$ -corrections are appearing and therefore a consistent, systematic expansion seemed to be in reach for these decays<sup>15</sup>.
- In 1992 Bigi and Uraltsev [103] explained the apparent contradiction between the  $1/m_q$  scaling of the HQE and the  $f_D^2/\langle E_{\bar{q}} \rangle^2$  enhanced gluon bremsstrahlung of [93, 94, 95]. They showed, that these power-enhanced terms cancel in fully inclusive rates between different cuts as indicated in Figure 2 and pre-asymptotic effects hence scale with  $1/m_c^3$ , consistently with the HQE. This seminal work opened now the way for the HQE in its

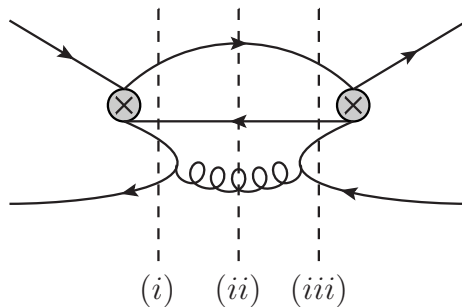


Figure 2: Different cuts contributing to the weak annihilation. The  $f_D^2/\langle E_{\bar{u}} \rangle^2$  enhanced term due to the cut (ii) considered in [93, 94, 95] is cancelled by interference effects (i) and (iii), such that the fully inclusive rate experiences the correct  $1/m_c^3$  scaling behaviour predicted by the HQE.

current form. The explicit proof of the cancellation of all power-enhanced terms was done in 1998 by Beneke, Buchalla, Greub, Lenz and Nierste [107], in the context of the calculation of  $\Gamma_3^{(1)}$  for  $\Delta\Gamma_s$ .

- The HQE in its current form was written down in July 1992 in [108] by Bigi, Uraltsev and Vainshtein for semi-leptonic and non-leptonic decays with one heavy quark in the final state. By working out the expansion in Eq.(2.53) the absence of  $1/m_q$ -corrections was shown also for the non-leptonic decays. In addition  $\Gamma_2^{(0)}$  was determined with the inclusion of charm mass effects, i.e., the values of the Wilson coefficient  $c_5$  given in Eq.(2.58) and Eq.(2.59). In the original paper there are some misprints, that were partly corrected<sup>16</sup> in an erratum. The full set of correct formulae was given end of 1992 by Bigi, Blok, Shifman, Uraltsev and Vainshtein in [109]. In these two papers [108, 109] a different normalisation was used for the physical meson states than we did in Eq.(2.53).

At about the same time Blok and Shifman investigated the rule of discarding terms of

<sup>15</sup>The famous Luke's Theorem [105] was proven in the context of the HQET. This theorem can be considered as a generalisation of the Ademollo-Gatto theorem from 1964 [106].

<sup>16</sup>In Eq.(4) of the erratum the factors  $m_Q^2$  should be in the denominator instead of the numerator.

order  $1/N_c$  in inclusive  $b \rightarrow c\bar{u}d$ - and  $c \rightarrow s\bar{d}u$ -decays [102], as well as in the  $b \rightarrow c\bar{c}s$ -decay [32]. In that respect they also determined the  $1/m_b^2$ -corrections for inclusive non-leptonic decays. More precisely they determined the contribution of  $c_{5,b}^{c\bar{u}d}$  proportional to  $C_1C_2$  - see Eq.(2.59) - in [102] and the contribution of  $c_{5,b}^{c\bar{u}d}$  proportional to  $C_1C_2$  - see Eq.(2.60)- in [32].

The complete formulae for the case of two heavy particles in the final state with identical masses, e.g.,  $b \rightarrow c\bar{c}s$  - see Eq.(2.60) - are given in December 1993 by Bigi, Blok, Shifman and Vainshtein and in January 1994 in a book contribution of Bigi, Blok, Shifman, Uraltsev and Vainshtein from 1994 [34]. In these papers now the same normalisation for the meson states as in Eq.(2.53) is used. The case for two arbitrary masses was studied by Falk, Ligeti, Neubert and Nir [35] in 1994.

- Now the door was open for many phenomenological investigations, which led also to several challenges for the new theory tool:
  - Inclusive non-leptonic decays were considered by Palmer and Stech in May 1993 [110]. It turned out that the theory prediction for the decay  $b \rightarrow c\bar{c}s$  did not fit to the data. Related investigations of the *missing charm puzzle* and the inclusive semi-leptonic branching ratio were done in November 1993 by Bigi, Blok, Shifman and Vainshtein [33] (*The baffling semi-leptonic branching ratio*). In May 1994 it was suggested by Dunietz, Falk and Wise [111] (*Inconclusive inclusive nonleptonic B decays*) that this discrepancy points towards a violation of local quark hadron duality in the decay  $b \rightarrow c\bar{c}s$  - a suggestion, which is now ruled out by the 2012 measurement of  $\Delta\Gamma_s$ , which is in perfect agreement with the HQE prediction, see below. Moreover the current theory prediction for the semi-leptonic branching ratio in Eq.(2.45) agrees well with the experimental numbers given in Eq.(2.46), although there is still some space for deviations.
  - An early extraction of  $V_{cb}$ ,  $m_c$  and  $m_b$  was done in 1993 by Luke and Savage [60] and by Bigi and Uraltsev [112] and further in 1995 by Falk, Luke and Savage [113]. This kind of studies form a big industry now, see, e.g., the review about the determination of  $V_{cb}$  and  $V_{ub}$  by Kowalewski and Mannel in the PDG [1].
  - Bigi and Uraltsev applied the HQE to charm lifetimes in [112] and also some aspects in [114]. For the  $D_s^+$  meson they found

$$\frac{\tau(D_s^+)^{\text{HQE1994}}}{\tau(D^0)} = 0.9 - 1.3, \quad (3.131)$$

where the uncertainty dominantly arises from the weak annihilation.

- Lifetimes of  $b$ -hadrons were also further studied. In that respect the expressions for  $\Gamma_3^{(0)}$  with charm quark mass dependence were presented by Kolya Uraltsev [115] in 1996 <sup>17</sup> and slightly later by Neubert and Sachrajda [116]. This mass dependence turned out to be important. Moreover the inclusion of colour-suppressed four quark operators was found to be crucial. Neubert and Sachrajda [116] gave a very

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<sup>17</sup>In an email dated from 4.11.2012, Kolya claimed that these results were known since a long time: *Effects of the internal quark masses were in fact considered; the expressions were at hand, and plugging numbers were so a simple matter that this was not even noted specially. The expressions (they are given, for instance, in arXiv:hep-ph/9602324) were taken from the same mid-1980s notes I mentioned above.*



nice and comprehensive review of the status quo in 1996 for the different  $b$ -hadron lifetime predictions in LO-QCD. At that time the measured  $\Lambda_b$ -lifetime was in conflict with early HQE predictions, that predicted a value of around 1.5 ps, see Eq.(3.128). The old data [117, 118, 119, 120] pointed, however, more to values around 1.0 – 1.3 ps.

Year	Exp	Decay	$\tau(\Lambda_b)$ [ps]	$\tau(\Lambda_b)/\tau(B_d)$
1998	OPAL	$\Lambda_c l$	$1.29 \pm 0.25$	$0.85 \pm 0.16$
1997	ALEPH	$\Lambda_c l$	$1.21 \pm 0.11$	$0.80 \pm 0.07$
1995	ALEPH	$\Lambda_c l$	$1.02 \pm 0.24$	$0.67 \pm 0.16$
1992	ALEPH	$\Lambda_c l$	$1.12 \pm 0.37$	$0.74 \pm 0.24$

(3.132)

Neubert and Sachrajda concluded that this points - if the experimental values stay - either to anomalously large matrix elements (they will be discussed below) or to a violation of quark-hadron duality. The latter attitude was quite popular at that time, see, e.g., the paper by Altarelli, Martinelli, Petrarca and Rapuano from 1996 [121] or the paper from Cheng from 1997 [122] or the work from Ito, Matsuda and Matsui also from 1997 [123]. Nowadays we know that the  $\Lambda_b$ -lifetime was a purely experimental problem and the measured values are in good agreement with the HQE estimates. These estimates suffer, however, from sizable hadronic uncertainties, which could be reduced by a state of the art lattice calculation.

- The lifetime of the  $B_c$ -meson, where both the  $b$ - and the  $c$ -quark decay weakly was studied systematically in 1996 by Beneke and Buchalla [124].

### 3.3 Precision studies

Some motivation and some topics of precision studies can be found already in the recommendations given in the seminal 1992 paper [108]: ” *The general procedure outlined above can be improved in four respects:*

- (i) *Some of the numerical predictions stated above were tentative since not all the relevant calculations have been performed yet. Since the “missing” computations involve perturbation theory this presents “merely” a technical delay.*
- (ii) *The real accuracy obtainable in this approach can be determined by calculating terms of order  $1/m_Q^4$  and estimating the size of the relevant matrix elements.*
- (iii) *...*

Item (i) concerns the field of determining higher order QCD-corrections. After being involved in several NLO-QCD calculations within the HQE, I of course disagree with the use of the word “*merely*” above. Besides being a tedious task, these efforts had also a conceptual value, since they provided an explicit proof of the arguments for a cancellation of singularities due to quark thresholds given by Bigi and Uraltsev [103]. Item (ii) suggests the discussion of higher order terms in HQE, which has been done currently for many observables, see below. Another crucial topic, that was, however, not emphasised in [108], is the non-perturbative determination of the arising matrix elements.

### 3.3.1 NLO-QCD

For semi-leptonic decays the NLO QCD corrections in  $\Gamma_2^{(1)}$  proportional to  $\mu_\pi^2$  were determined in 2007 by Becher, Boos and Lunghi [125] and confirmed in 2012 [126]. The corresponding corrections proportional to  $\mu_G^2$  were calculated very recently by Alberti, Gambino and Nandi [127].

As discussed above, the NLO QCD corrections in  $\Gamma_3^{(1)}$  were crucial for proofing the consistency of the HQE. They were determined for  $\Delta\Gamma_s$  in 1998 by Beneke, Buchalla, Greub, Lenz and Nierste [107]. In this case the diagrams in Fig.(3) appear. The same authors as well as the

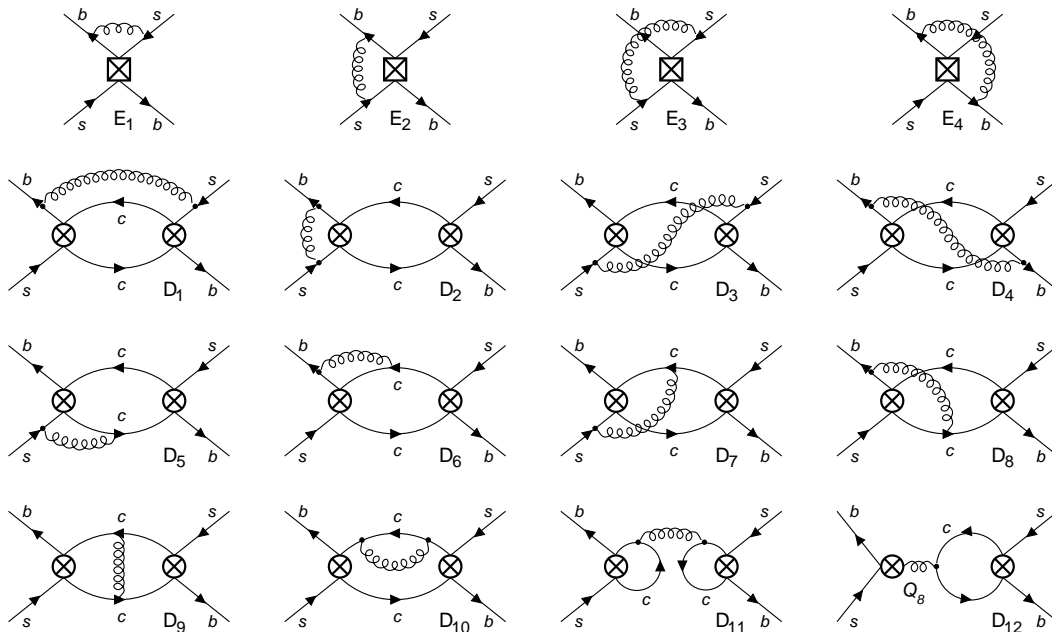


Figure 3: NLO-QCD diagrams contributing to  $\Delta\Gamma_s$ .

Rome group - Franco, Lubicz, Mescia and Tarantino - calculated the NLO-QCD corrections for  $\tau(B^+)/\tau(B_d)$  in 2002 [30, 128]. The Rome group included also the NLO-corrections for  $\tau(B_s)/\tau(B_d)$  and partly for  $\tau(\Lambda_d)/\tau(B_d)$  [128] - here some penguin diagrams are still missing. Some dominant NLO-corrections for  $\tau(B_s)/\tau(B_d)$  have already been determined in 1998 by Keum and Nierste [129]. In [30] it was also shown that the use of  $\bar{m}_c(\bar{m}_b)$  automatically sums up logarithms of the form  $\alpha_s^n z \log^n z$  to all orders. For  $\Delta\Gamma_d$  and the semi-leptonic asymmetries  $a_{sl}^d$  and  $a_{sl}^s$ ,  $\Gamma_3^{(1)}$  was determined in 2003 by Beneke, Buchalla, Lenz and Nierste [130] and by Ciuchini, Franco, Lubicz, Mescia and Tarantino [131]. For the decay rate difference of neutral  $D$ -mesons the above formulae were rewritten in 2010 by Bobrowski, Lenz, Riedl and Rohrwild [132] and for  $D$ -meson lifetime ratios some missing contributions were calculated in 2013 by Lenz and Rauh [9].

A pioneering studies of some integrals that appear in NNLO-QCD for  $\Delta\Gamma_s$  has been performed in 2012 by Asatrian, Hovhannisyan and Yeghiazaryan [133].

### 3.3.2 Higher order terms in the OPE

For semi-leptonic decays  $1/m_b$  corrections to the kinetic and chromo-magnetic operator were studied in 1994 by Bigi, Shifman, Uraltsev and Vainshtein [25]. Similar contributions to semi-leptonic decays were studied in 1995 by Blok, Dikeman and Shifman [134] and 1996 by

Kremm and Kapustin [135]. Even higher corrections -  $\Gamma_4^{(0)}$  and  $\Gamma_5^{(0)}$  - to the semi-leptonic decay width were investigated in 2010 by Mannel, Turczyk and Uraltsev [136].

$1/m_q$ -corrections to weak annihilation and Pauli interference, i.e.,  $\Gamma_4^{(0)}$  were determined for  $\Delta\Gamma_s$  in 1996 by Beneke, Buchalla and Dunietz [137] and they turned out to be sizable. The corresponding corrections for  $\Delta\Gamma_d$  were calculated in 2001 by Dighe, Hurth, Kim and Yoshikawa [138] and for  $b$ -lifetimes in by Gabbiani, Onishchenko and Petrov in 2003 [139] and 2004 [140] (in the latter one also  $1/m_q^2$ -corrections were investigated) and by Lenz and Nierste in 2003 [141]. Badin, Gabbiani and Petrov studied also  $\Gamma_5^{(0)}$  for  $\Delta\Gamma_s$  in 2007 [142]. In  $\Gamma_5^{(0)}$  several completely unknown matrix elements are arising. Moreover the Wilson coefficients have very small numerical values. Thus we are not including these corrections in our estimates.

One can also try to apply the methods of the HQE to  $D$ -mixing. First efforts in that direction were made in 1992 by Georgi [143] and by Ohl, Ricciardi and Simmons [144]. It turns out that the leading term,  $\Gamma_3$ , suffers from a severe GIM cancellation and thus the HQE leads to very small predictions for  $D$ -mixing. One idea to circumvent this severe cancellation was to consider higher orders in the HQE, in particular  $\Gamma_6$  and  $\Gamma_9$ . Bigi and Uraltsev have shown in 2000 [145] how in  $\Gamma_6$  and  $\Gamma_9$  the  $1/m_c$ -suppression could be overcompensated by a lifting of the GIM-suppression. They concluded that values of  $x$  and  $y$  of up to 1% are not excluded within the HQE.

### 3.3.3 Non-perturbative parameters

Early studies of  $\mu_\pi^2$  have been done, e.g., in 1993 by Bigi, Shifman, Uraltsev and Vainshtein [146]. In 1994 [147] some ideas how to extract this quantity from experiment were developed by Bigi, Grozin, Shifman, Uraltsev and Vainshtein. The same quantity has also been determined with QCD sum rules in 1993 by Ball and Braun [148]. A kind of contradicting result was obtained in 1996 by Neubert [149] with the same method. Calculations within lattice QCD were, e.g., performed by Kronfeld and Simone in 2000 [150]. The most recent value for  $\mu_\pi^2$  for  $B$ -mesons comes from a fit of semi-leptonic decays by Gambino and Schwanda in 2013 [61], somehow confirming the first QCD sum rule calculation:

$\mu_\pi^2$	
$(0.52 \pm 0.12) \text{ GeV}^2$	QCD – SR 1993
$(0.10 \pm 0.05) \text{ GeV}^2$	QCD – SR 1996
$(0.45 \pm 0.12) \text{ GeV}^2$	Lattice 2000
$(0.414 \pm 0.078) \text{ GeV}^2$	Fit 2013

(3.133)

$\mu_G^2$  can in principle be determined from experiment - see Eq.(2.67) -, for  $B$ -mesons it was further investigated by Kolya in 2001 [62]. The differences of  $\mu_G^2$  and  $\mu_\pi^2$ , if one considers instead of the lightest  $B$ -mesons,  $B_s$ -mesons or  $\Lambda_b$ -baryons were studied by Bigi, Mannel and Uraltsev in 2011 [63].

The new results seem also to confirm the bound

$$\mu_\pi^2 > \mu_G^2 \tag{3.134}$$

that was derived by Bigi, Shifman, Uraltsev and Vainshtein [146, 25] and by Voloshin [151]. Kapustin, Ligeti, Wise and Grinstein [152] claimed that the above bound will be weakened due to perturbative corrections. A study of Kolya Uraltsev [153] came, however, to a different

conclusion.

Matrix elements of four-quark operators relevant for lifetime ratios are almost unknown. For  $\tau(B^+)/\tau(B_d)$  the latest lattice calculation was performed by Becirevic in 2001 and only published as proceedings [66]. Unfortunately these parameters, see Eq.(2.101) have never been updated. The same matrix elements can also be used for the case of  $\tau(B_s)/\tau(B_d)$ . There is also an earlier lattice study from Di Pierro and Sachrajda from 1999 [154], as well as two QCD sum rule studies from Baek, Lee, Liu and Song in 1997 [155] and one year later from Cheng and Yang [156].

$B_1$	$B_2$	$\epsilon_1$	$\epsilon_2$	
$1.01 \pm 0.01$	$0.99 \pm 0.01$	$-0.08 \pm 0.02$	$-0.01 \pm 0.03$	1997 QCD – SR
$1.06 \pm 0.08$	$1.01 \pm 0.06$	$-0.01 \pm 0.03$	$-0.01 \pm 0.02$	1998 Lattice
$0.96 \pm 0.04$	$0.95 \pm 0.02$	$-0.14 \pm 0.01$	$-0.08 \pm 0.01$	1998 QCD – SR
$1.10 \pm 0.20$	$0.79 \pm 0.10$	$-0.02 \pm 0.02$	$0.03 \pm 0.01$	2001 Lattice

(3.135)

Comparing these numbers, the authors [155, 156] of the QCD sum rule evaluation seem to have very aggressive error estimates. Because of the very pronounced cancellations in Eq.(2.100) precise values for these bag parameters are crucial for an investigation of the lifetime ratio  $\tau(B^+)/\tau(B_d)$ . To some extent our definition of the bag parameters given in Eq.(2.85) and Eq.(2.86) above was a little too simplistic. In reality we are considering the isospin breaking combinations

$$\frac{\langle B_d | Q^d - Q^u | B_d \rangle}{M_{B_d}} = f_B^2 B_1 M_{B_d}, \quad \frac{\langle B_d | Q_S^d - Q_S^u | B_d \rangle}{M_{B_d}} = f_B^2 B_2 M_{B_d}, \quad (3.136)$$

$$\frac{\langle B_d | T^d - T^u | B_d \rangle}{M_{B_d}} = f_B^2 \epsilon_1 M_{B_d}, \quad \frac{\langle B_d | T_S^d - T_S^u | B_d \rangle}{M_{B_d}} = f_B^2 \epsilon_2 M_{B_d}. \quad (3.137)$$

This definition leads to the cancellation of unwanted penguin contractions, see Fig. 4 and enables thus in principle very precise calculations. For the  $\Lambda_b$ -lifetime our knowledge of the matrix elements is even worse. There is only an exploratory lattice study from Di Pierro, Sachrajda and Michael available, dating back to 1999 [157]. Here also any update would be extremely welcome. In that case two matrix elements are arising that are parameterised by  $L_1$  and  $L_2$

$$\frac{\langle \Lambda_b | Q^q | \Lambda_b \rangle}{M_{\Lambda_b}} = f_B^2 M_B L_1, \quad (3.138)$$

$$\frac{\langle \Lambda_b | T^q | \Lambda_b \rangle}{M_{\Lambda_b}} = f_B^2 M_B L_2, \quad (3.139)$$

where the operators were defined in Eq.(2.83) and Eq.(2.84). The numerical values obtained in [157] are shown in Eq.(3.144). In elder works the colour re-arranged operator was investigated instead of the colour octett operator  $T^q$ . There the following definition was used

$$\frac{\langle \Lambda_b | \bar{b}_\alpha \gamma^\mu (1 - \gamma_5) q_\alpha \cdot \bar{q}_\beta \gamma_\mu (1 - \gamma_5) b_\beta \rangle}{M_{\Lambda_b}} = -\frac{f_B^2 M_B}{6} r, \quad (3.140)$$

$$\frac{\langle \Lambda_b | \bar{b}_\alpha \gamma^\mu (1 - \gamma_5) q_\beta \cdot \bar{q}_\beta \gamma_\mu (1 - \gamma_5) b_\alpha \rangle}{M_{\Lambda_b}} = \tilde{B} \frac{f_B^2 M_B}{6} r. \quad (3.141)$$

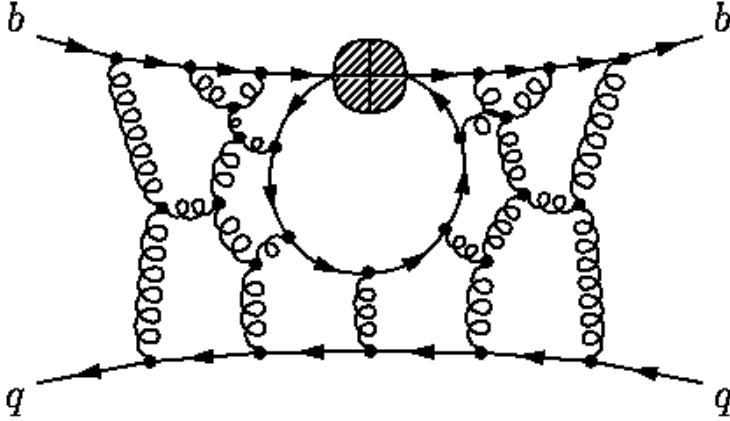


Figure 4: Penguin contractions that cancel in lifetime ratios like  $\tau(B^+)/\tau(B_d)$  and  $\tau(\Xi_b^0)/\tau(\Xi_b^+)$ . They will, however, give a contribution to  $\tau(\Lambda)/\tau(B_d)$ . Since these contributions introduce a mixing of operators of different dimensionality, they are difficult to handle.

The two parameter sets are related by<sup>18</sup>

$$r = -6L_1, \quad \tilde{B} = -\frac{1}{3} - 2\frac{L_2}{L_1}. \quad (3.142)$$

Because of the long standing discrepancy between experiment and theory for the  $\Lambda_b$ -lifetime people of course tried different methods to determine the missing matrix elements: Rosner related in 1996 the four-quark matrix elements to results from spectroscopy [159] and found<sup>19</sup>

$$r = \frac{4 M_{\Sigma_b^*} - M_{\Sigma_b}}{3 M_{B^*} - M_B}. \quad (3.143)$$

At that time the values of the masses of the baryons were almost unknown, which resulted in quite rough and large estimates, yielding  $r \approx 1.6$ . This situation changed completely now and we will use the method of Rosner with new experimental numbers for the baryon masses<sup>20</sup> to update the  $\Lambda_b$  lifetime below. Colangelo and de Fazio applied in 1996 [160] the method of QCD sum rules and obtained relatively small numbers for  $r$ . Huang, Liu and Zhu managed in 1999 [161], however, to obtain with the same method much larger numbers, that also lead to a lifetime of the  $\Lambda_b$ -baryon, that was compatible with the measurements at that time. Even earlier (1979) estimates within the bag model and the non-relativistic quark model for charmed hadrons from Guberina, Nussinov, Peccei and Rückl [91] pointed towards smaller

<sup>18</sup>In [157] the parameters  $r$  and  $\tilde{B}$  were interchanged in Eq.(3.142), while the correct relation was given in [158].

<sup>19</sup>Neubert and Sachrajda [116] quoted in 1996 this formula as  $r = 4/3(M_{\Sigma_b^*}^2 - M_{\Sigma_b}^2)/(M_{B^*}^2 - M_B^2)$ , which gives values that are about 10% larger than Eq.(3.143). Cheng [122] quoted in 1997 the same formula as given in Eq.(3.143).

<sup>20</sup>PDG [1] gives:  $M_{\Sigma_b^*} - M_{\Sigma_b} = 21.2(2.0)$  MeV,  $M_{\Sigma_b^*} = 5832.1(2.0)$  MeV,  $M_{\Sigma_b} = 5811.3(2.0)$  MeV and  $M_{B^*} = 5325.20(0.40)$  MeV.

values of  $r$ . All in all currently the following numerical values are available:

$L_1 = -\frac{r}{6}$	$L_2$ or $\frac{r}{9}$	$r$	$\tilde{B}$	
-0.103(10)	0.069(7)	0.62(6)	1	2014 Spectroscopy update
-0.22(4)	0.17(2)	1.32(24)	1.21(34)	1999 Exploratory Lattice
-0.22(5)	0.14(3)	1.3(3)	1	1999 QCD – SR v1
-0.60(15)	0.40(10)	3.6(9)	1	1999 QCD – SR v2
-0.033(17)	0.022(11)	0.2(1)	1	1996 QCD – SR
$\approx -0.03$	$\approx 0.02$	$\approx 0.2$	1	1979 Bag model
$\approx -0.08$	$\approx 0.06$	$\approx 0.5$	1	1979 NRQM

(3.144)

In [157] the two parameters  $L_1$  and  $L_2$  were calculated, else only  $r$  was determined. In the latter case we assumed  $\tilde{B} = 1$  (valence quark approximation) in order to determine  $L_2$ . Comparing all these numbers we find that two studies obtain values of  $r$  larger than one. One is the exploratory lattice calculation. This method could in principle give a reliable value, if an up-to-date study would be made. The second one is the QCD sum rule estimate from Huang, Liu and Zhu in 1999 [161]. In principle this is a reliable method, if it is applied properly. The calculation in [161] seems to be, however, in contradiction with the one from Colangelo and de Fazio [160]. In 1996 also the method from Rosner [159] gave values for  $r$  larger than one. This changed with new precise measurements of the  $\Sigma_b^{(*)}$ -masses. Now Rosner’s methods gives a small value in accordance with the QCD sum rule estimate from Colangelo and de Fazio [160] and with the early estimates from [91]. We will vary  $r$  between 0.2 (Colangelo and de Fazio) and 1.32 ( Di Pierro, Sachrajda and Michael) with a central value of 0.62 (Rosner). The unclear situation with the matrix elements resulted in a broad range of different theory predictions and as long as the experimental values for the  $\Lambda_b$ -lifetime were low - the HFAG average from 2003 was

$$\frac{\tau(\Lambda_b)^{\text{HFAG 2003}}}{\tau(B_d)} = 0.798 \pm 0.034 \quad (3.145)$$

- there was a tendency to use preferably the larger values for  $r$  in order to see, how far one can “stretch” the HQE. Estimates from that time [162, 140, 128, 163, 164, 157, 161, 160] read

Year	Author	$\tau(\Lambda_b)/\tau(B_d)$
2007	Tarantino	$0.88 \pm 0.05$
2004	Petrov et al.	$0.86 \pm 0.05$
2002	Rome	$0.90 \pm 0.05$
2000	Körner, Melic	0.81...0.92
1999	Guberina, Melic, Stefanic	0.90
1999	diPierro, Sachrajda, Michael	$0.92 \pm 0.02$
1999	Huang, Liu, Zhu	$0.83 \pm 0.04$
1996	Colangelo, deFazio	$> 0.94$

(3.146)

Nowadays it is clear that the low  $\Lambda_b$ -lifetime was a purely experimental issue. On the other hand the precise HQE prediction is still unknown, because we have no reliable calculation of the hadronic matrix elements at hand.

Finally we need matrix elements of dimension six and dimension seven operators that are arising in mixing quantities. The status of the dimension six operators for mixing is considerably more advanced than for the lifetime case; it is discussed in detail in the FLAG

review [64]. For the numerically important dimension seven contributions vacuum insertion approximation is used and first studies with QCD sum rules have been performed by Mannel, Pecjak and Pivovarov [67, 68].

## 4 Status Quo of lifetimes and the HQE

In this final section we update several of the lifetime predictions and compare them with the most recent data, obtained many times at the LHC experiments.

### 4.1 $B$ -meson lifetimes

The most recent theory expressions for  $\tau(B^+)/\tau(B_s)$  and  $\tau(B_s)/\tau(B_d)$  are given in [165] (based on the calculations in [30, 128, 140, 66]). For the charged  $B$ -meson we get the updated relation (including  $\alpha_s$ -corrections and  $1/m_b$ -corrections)

$$\begin{aligned} \frac{\tau(B^+)}{\tau(B_d)}^{\text{HQE 2014}} &= 1 + 0.03 \left( \frac{f_{B_d}}{190.5 \text{ MeV}} \right)^2 [(1.0 \pm 0.2)B_1 + (0.1 \pm 0.1)B_2 \\ &\quad - (17.8 \pm 0.9)\epsilon_1 + (3.9 \pm 0.2)\epsilon_2 - 0.26] \\ &= 1.04_{-0.01}^{+0.05} \pm 0.02 \pm 0.01. \end{aligned} \quad (4.147)$$

Here we have used the lattice values for the bag parameters from [66]. Using all the values for the bag parameters quoted in Eq.(3.135), the central value of our prediction for  $\tau(B^+)/\tau(B_d)$  varies between 1.03 and 1.09. This is indicated by the first asymmetric error and clearly shows the urgent need for more profound calculations of these non-perturbative parameters. The second error in Eq.(4.147) stems from varying the matrix elements of [66] in their allowed range and the third error comes from the renormalisation scale dependence as well as the dependence on  $m_b$ .

Next we update also the prediction for the  $B_s$ -lifetime given in [165], by including also  $1/m_b^2$ -corrections discussed in Eq.(2.73).

$$\begin{aligned} \frac{\tau(B_s)}{\tau(B_d)}^{\text{HQE 2014}} &= 1.003 + 0.001 \left( \frac{f_{B_s}}{231 \text{ MeV}} \right)^2 [(0.77 \pm 0.10)B_1 + (1.0 \pm 0.13)B_2 \\ &\quad + (36 \pm 5)\epsilon_1 + (51 \pm 7)\epsilon_2] \\ &= 1.001 \pm 0.002. \end{aligned} \quad (4.148)$$

The values in Eq.(4.147) and Eq.(4.148) differ slightly from the ones in [165], because we have used updated lattice values for the decay constants<sup>21</sup> and we included the SU(3)-breaking of the  $1/m_b^2$ -correction - see Eq.(2.73) - for the  $B_s$ -lifetime, which was previously neglected. Comparing these predictions with the measurements given in Eq.(1.1), we find a perfect agreement for the  $B_s$ -lifetime, leaving thus only a little space for, e.g., hidden new  $B_s$ -decay channels, following, e.g., [78, 79]. There is a slight tension in  $\tau(B^+)/\tau(B_d)$ , which, however, could solely be due to the unknown values of the hadronic matrix elements. A value of, e.g.,

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<sup>21</sup>We have used  $f_{B_s} = 227.7 \text{ MeV}$  [64].

$\epsilon_1 = -0.092$  - and leaving everything else at the values given in Eq.(2.101) - would perfectly match the current experimental average from Eq.(1.1). Such a value of  $\epsilon_2$  is within the range of the QCD sum rule predictions [155, 156] shown in Eq.(3.135). Thus, for further investigations updated lattice values for the bag parameters  $B_1, B_2, \epsilon_1$  and  $\epsilon_2$  are indispensable.

The most recent experimental numbers for these lifetime ratios have been updated by the LHCb Collaboration in 2014 [166].

## 4.2 $b$ -baryon lifetimes

We discussed already the early stages of the long standing puzzle related to the lifetime of  $\Lambda_b$ -baryon. After 2003 one started to find contradicting experimental values [167, 168, 169, 170, 171, 172] - some of them still similarly low as the previous ones and others pointed more to a lifetime comparable to the one of the  $B_d$ -meson.

Year	Exp	Decay	$\tau(\Lambda_b)$ [ps]	$\tau(\Lambda_b)/\tau(B_d)$
2010	CDF	$J/\psi\Lambda$	$1.537 \pm 0.047$	$1.020 \pm 0.031$
2009	CDF	$\Lambda_c + \pi^-$	$1.401 \pm 0.058$	$0.922 \pm 0.038$
2007	D0	$\Lambda_c\mu\nu X$	$1.290 \pm 0.150$	$0.849 \pm 0.099$
2007	D0	$J/\psi\Lambda$	$1.218 \pm 0.137$	$0.802 \pm 0.090$
2006	CDF	$J/\psi\Lambda$	$1.593 \pm 0.089$	$1.049 \pm 0.059$
2004	D0	$J/\psi\Lambda$	$1.22 \pm 0.22$	$0.87 \pm 0.17$

(4.149)

The current HFAG average given in Eq.(1.2) clearly rules out now the small values of the  $\Lambda_b$ -lifetime. Updating the NLO-calculation from the Rome group [131] and including  $1/m_b$ -corrections from [140] we get for the current HQE prediction

$$\begin{aligned} \frac{\tau(\Lambda_b)}{\tau(B_d)}^{\text{HQE2014}} &= 1 - (0.8 \pm 0.5)\% \frac{1}{m_b^2} - (4.2 \pm 3.3)\% \frac{\Lambda_b}{m_b^3} - (0.0 \pm 0.5)\% \frac{B_d}{m_b^3} - (1.6 \pm 1.2)\% \frac{1}{m_b^4} \\ &= 0.935 \pm 0.054, \end{aligned} \tag{4.150}$$

where we have split up the corrections coming from the  $1/m_b^2$ -corrections discussed in Eq.(2.73), the  $1/m_b^3$ -corrections coming from the  $\Lambda_b$ -matrix elements, the  $1/m_b^3$ -corrections coming from the  $B_d$ -matrix elements and finally  $1/m_b^4$ -corrections studied in [140]. The number in Eq.(4.150) is smaller than some of the previous theory predictions because of several reasons: we have used updated, smaller lattice values for the decay constants, which gives a shift of about +0.01 in the lifetime ratio. Following our discussion of the dimension six matrix elements, we use three different determinations. Instead of using only the exploratory lattice one [157], we also take into account the QCD sum rule estimate of Colangelo and de Fazio [160] and the spectroscopy result of Rosner [159]. In 1996 Rosner's method gave a large value of the matrix element. New, precise measurements of the  $\Sigma_b^{(*)}$ -mass show, however, that the matrix element is much smaller than originally thought. This gives a third enhancement factor. To obtain the final number we also scaled the numerical value of the  $1/m_b^4$ -correction with the size of  $r$ . The current range of the theory prediction in Eq.(4.150) goes from 0.88 to 0.99. To reduce this large uncertainty, new lattice calculations are necessary. In these calculations also the penguin contractions from Fig.(4) have to be taken into account. More recent experimental studies of the  $\Lambda_b$ -lifetime further strengthen the case for a value of



the lifetime ratio close to one. The most recent and most precise measurement from LHCb gives [173]

$$\frac{\tau(\Lambda_b)^{\text{LHCb}}}{\tau(B_d)} = 0.974 \pm 0.006 \pm 0.004. \quad (4.151)$$

This results supersedes a previous LHCb measurement [174]. Combined with the world average for the  $B_d$ -lifetime one gets

$$\tau(\Lambda_b)^{\text{LHCb}} = 1.479 \pm 0.009 \pm 0.010 \text{ ps} . \quad (4.152)$$

Comparing the accuracy of these new measurements with the HFAG average given in Eq.(1.2) shows the dramatic experimental progress. LHCb has a further recent investigation of the  $\Lambda_b$ -lifetime [166] - based on different experimental techniques - and there is also a very new TeVatron (CDF) number available [175]

$$\tau(\Lambda_b)^{\text{CDF}} = 1.565 \pm 0.035 \pm 0.020 \text{ ps} . \quad (4.153)$$

All in all, now the new measurements of the  $\Lambda_b$ -lifetime are in nice agreement with the HQE result. This is now a very strong confirmation of the validity of the HQE and this makes also the motivation of many of the studies trying to explain the  $\Lambda_b$ -lifetime puzzle, e.g., [121, 122, 123], invalid.

In [30] it was shown that the lifetime ratio of the  $\Xi_b$ -baryons can be in principle be determined quite precisely, because here the above mentioned problems with penguin contractions do not arise, the diagrams from Fig. 4 cancel. Unfortunately there exists no non-perturbative determination of the matrix elements for  $\Xi_b$ -baryons. Cheng [122] suggested to use the relation

$$r_{\Xi_b} = \frac{4 M_{\Xi_b^*} - M_{\Xi_b}}{3 M_{B^*} - M_B} , \quad (4.154)$$

but there are no data available yet for the  $\Xi_b^*$ -mass. So, we are left with the possibility of assuming that the matrix elements for  $\Xi_b$  are equal to the ones of  $\Lambda_b$ . In that case we can give a rough estimate for the expected lifetime ratio - we update here a numerical estimate from 2008 [176]. In order to get rid of unwanted  $s \rightarrow u$ -transitions we define (following [30])

$$\frac{1}{\bar{\tau}(\Xi_b)} = \bar{\Gamma}(\Xi_b) = \Gamma(\Xi_b) - \Gamma(\Xi_b \rightarrow \Lambda_b + X) . \quad (4.155)$$

For a numerical estimate we scan over the the results for the  $\Lambda_b$ -matrix elements obtained on the lattice by the study of Di Pierro, Michael and Sachrajda [157], the QCD sum rule estimate of Colangelo and de Fazio [160] and the update of the spectroscopy method of Rosner [159]. Using also recent values for the remaining input parameters we obtain

$$\frac{\bar{\tau}(\Xi_b^0)^{\text{HQE 2014}}}{\bar{\tau}(\Xi_b^+)} = 0.95 \pm 0.04 \pm 0.01 \pm ??? , \quad (4.156)$$

where the first error comes from the range of the values used for  $r$ , the second denotes the remaining parametric uncertainty and ??? stands for some unknown systematic errors, which comes from the approximation of the  $\Xi_b$ -matrix elements by the  $\Lambda_b$ -matrix elements. We expect the size of these unknown systematic uncertainties not to exceed the error stemming

from  $r$ , thus leading to an estimated overall error of about  $\pm 0.06$ . As soon as  $\Xi_b$ -matrix elements are available the ratio in Eq.(4.156) can be determine more precisely than  $\tau(\Lambda_b)/\tau(B_d)$ . If we further approximate  $\bar{\tau}(\Xi_b^0) = \tau(\Lambda_b)$  - here similar cancellations are expected to arise as in  $\tau_{B_s}/\tau_{B_d}$  - , then we arrive at the following prediction

$$\frac{\tau(\Lambda_b)^{\text{HQE 2014}}}{\bar{\tau}(\Xi_b^+)} = 0.95 \pm 0.06 . \quad (4.157)$$

From the new measurements of the LHCb Collaboration [3, 4] (see also the CDF update [175]), we deduce

$$\frac{\tau(\Xi_b^0)^{\text{LHCb 2014}}}{\tau(\Xi_b^+)} = 0.92 \pm 0.03 , \quad (4.158)$$

$$\frac{\tau(\Xi_b^0)^{\text{LHCb 2014}}}{\tau(\Lambda_b)} = 1.006 \pm 0.021 , \quad (4.159)$$

$$\frac{\tau(\Lambda_b)^{\text{LHCb 2014}}}{\tau(\Xi_b^+)} = 0.918 \pm 0.028 , \quad (4.160)$$

which is in perfect agreement with the predictions above in Eq.(4.156) and Eq.(4.157), within the current uncertainties.

### 4.3 $D$ -meson lifetimes

In [9] the NLO-QCD corrections for the  $D$ -meson lifetimes were completed. Including  $1/m_c$ -corrections as well as some assumptions about the hadronic matrix elements one obtains

$$\frac{\tau(D^+)^{\text{HQE 2013}}}{\tau(D^0)} = 2.2 \pm 0.4^{\text{(hadronic)}}_{-0.07}^{+0.03(\text{scale})} , \quad (4.161)$$

$$\frac{\tau(D_s^+)^{\text{HQE 2013}}}{\tau(D^0)} = 1.19 \pm 0.12^{\text{(hadronic)}}_{-0.04}^{+0.04(\text{scale})} , \quad (4.162)$$

being very close to the experimental values shown in Eq.(1.3). Therefore this result seems to indicate that one might apply the HQE also to lifetimes of  $D$ -mesons, but definite conclusions cannot not be drawn without a reliable non-perturbative determination of the hadronic matrix elements, which is currently missing.

### 4.4 Mixing quantities

The current status of mixing quantities, both in the  $B$ - and the  $D$ -system, was very recently reviewed in [177]. The arising set of observables allows for model-independent searches for new physics effects in mixing, see e.g. [178, 179]. We discuss here only the decay rate differences  $\Delta\Gamma_s$ , because this provided one of the strongest proofs of the HQE. The HQE prediction - based on the NLO-QCD corrections [107, 130, 131, 180] and sub-leading HQE corrections [137, 138] gave in 2011 [165]

$$\Delta\Gamma_s^{\text{HQE 2011}} = (0.087 \pm 0.021) \text{ ps}^{-1} . \quad (4.163)$$

$\Delta\Gamma_s$  was measured for the first time in 2012 by the LHCb Collaboration [181]. The current average from HFAG [2] reads

$$\Delta\Gamma_s^{\text{Exp.}} = (0.091 \pm 0.09) \text{ ps}^{-1}, \quad (4.164)$$

it includes the measurements from LHCb [182, 183], ATLAS [184, 185], CMS [186], CDF [187] and D0 [188]. Experiment and theory agree perfectly for  $\Delta\Gamma_s$ , excluding thus huge violations of quark hadron duality. The new question is now: how precisely does the HQE work? The experimental uncertainty will be reduced in future, while the larger theory uncertainty is dominated from unknown matrix elements of dimension seven operators, see [180, 165]. Here a first lattice investigation or a continuation of the QCD sum rule study in [67, 68] would be very welcome.

## 5 Conclusion

We have started this review by giving a very basic introduction into lifetimes of weakly decaying particles, followed by a detailed discussion of the individual terms appearing in the HQE. Next we focused on the historical development of the theory, which we summarise briefly as: early investigations of the HQE are based on the work by Voloshin and Shifman [96, 97] in the early 1980s. A real systematic expansion was only possible after some conceptual issues have been solved in 1992 by Bigi and Uraltsev [103], which was proven in 1998 by Beneke et al. [107] in an explicit calculation. The HQE in its present form was developed in 1992 by Bigi, Uraltsev and Vainshtein [108] and about the same time by Blok and Shifman [102, 32]. For semi-leptonic decays the absence of  $1/m_q$ -corrections was already shown in 1990 by Chay, Georgi and Grinstein [104] and by Luke [105].

Since 1992 several discrepancies were arising, that shed some doubt on the validity of the HQE: inclusive non-leptonic decays (in particular predictions for the semi-leptonic branching ratio and the missing charm puzzle) and the  $\Lambda_b$ -lifetime were two prominent examples. We have discussed in detail, how all these issues were resolved. For the semi-leptonic branching ratio NLO-QCD corrections including finite charm-quark mass effect were crucial. The remaining small difference, see Eq.(2.45) vs. Eq.(2.46) is probably due to unknown NNLO-QCD effects. The problem of the  $\Lambda_b$ -lifetime was experimentally solved in the last months. One of the most convincing tests of the HQE was, however, the measurement of  $\Delta\Gamma_s$  from 2012 onwards - see Eq.(4.164) - in perfect agreement with the prediction stemming from early 2011 - see Eq.(4.163).

Thus, the theory in whose development Kolya played such a crucial role, has just now passed numerous non-trivial tests and its validity holds beyond any doubt. This makes also the motivation for looking for some modification of the HQE, see e.g. [111, 121, 122, 123] invalid. The new question is now: how precise is the HQE? This question is not only of academic interest, but it has practical consequences in searches for new physics. The quantification of a statistical significance of a possible discrepancy depends strongly on the intrinsic uncertainty of the HQE. Hence further studies in that direction are crucial. As a starting point for such

an endeavour we have updated several theory predictions for lifetime ratios

$$\frac{\tau(B^+)^{\text{HQE 2014}}}{\tau(B_d)} = 1.04^{+0.07}_{-0.03}, \quad (5.165)$$

$$\frac{\tau(B_s)^{\text{HQE 2014}}}{\tau(B_d)} = 1.001 \pm 0.002, \quad (5.166)$$

$$\frac{\tau(\Lambda_b)^{\text{HQE 2014}}}{\tau(B_d)} = 0.935 \pm 0.054, \quad (5.167)$$

$$\frac{\bar{\tau}(\Xi_b^0)^{\text{HQE 2014}}}{\bar{\tau}(\Xi_b^+)} = 0.95 \pm 0.06. \quad (5.168)$$

In order to see, how these predictions could be further improved, we compare for different observables what components of the theory prediction are currently known.

	$\frac{\tau(B^+)}{\tau(B_d)}$	$\Gamma_{12}$	$\frac{\tau(\Lambda_b)}{\tau(B_d)}$	$\frac{\tau(D^+)}{\tau(D_0)}$	$\frac{\bar{\tau}(\Xi_b^0)}{\bar{\tau}(\Xi_b^+)}$
$\Gamma_3^{(0)}$	+	+	+	+	+
$\Gamma_3^{(1)}$	+	+	0	+	+
$\Gamma_3^{(2)}$	-	-	-	-	-
$\langle \Gamma_3 \rangle$	0	+	0	-	-
$\Gamma_4^{(0)}$	+	+	+	+	+
$\Gamma_4^{(1)}$	-	-	-	-	-
$\langle \Gamma_4 \rangle$	-	0	-	-	-
$\Gamma_5^{(0)}$	-	+	+	-	+
$\langle \Gamma_5 \rangle$	-	-	-	-	-

For all these observables the LO-QCD term  $\Gamma_3^{(0)}$  and also the NLO-QCD corrections  $\Gamma_3^{(1)}$  are known. For the  $\Lambda_b$ -baryon, however, a part of the NLO-QCD calculation is still missing. NNLO-QCD corrections - denoted by  $\Gamma_3^{(2)}$  - have not been calculated for any of these observables; a first step for  $\Gamma_{12}$  has been done in [133]. The biggest problem are currently the non-perturbative matrix elements. Concerning the dimension 6 term  $\langle \Gamma_3 \rangle$  we have only for  $\Gamma_{12}$  several independent lattice calculations. For  $\tau(B^+)/\tau(B_d)$  the latest lattice number stems from 2001 [66], for  $\tau(\Lambda_b)/\tau(B_d)$  we have only an exploratory lattice study from 1999 [157] and for the  $D$ -meson lifetimes we have no lattice investigations at all. For the  $b$ -hadrons also several QCD sum rule determinations of these matrix elements are available [155, 156, 160, 161].

Concerning the power suppressed  $1/m_b$  corrections, we see that the LO-QCD term  $\Gamma_4^{(0)}$  is known for all observables and  $\Gamma_5^{(0)}$  is also known for some of the observables. The matrix elements of the dimension seven operators,  $\langle \Gamma_4 \rangle$ , have been determined by vacuum insertion approximation - a first step of a QCD sum rule calculation for  $\Delta\Gamma_s$  has been done in [68, 67]. For all lifetime ratios the uncertainty due to the unknown matrix elements of the dimension six operators is dominant. For  $\Delta\Gamma_s$  these operators have already been determined by several groups and thus the dominant uncertainty stems now from  $\Gamma_4$ . Here a full non-perturbative determination of the matrix elements of the dimension seven operators would be very desirable, as well as calculation of the corresponding NLO-QCD corrections, denoted by  $\Gamma_4^{(1)}$ .

Increasing the precision of the HQE will also help in shrinking the allowed space for new physics effects in tree-level decays [189], a topic that has also profound implications for other branches of flavour physics.

Kolya left us a very promising but also challenging legacy, which might in the end provide the way to identify new physics in the flavour sector.

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