GALOIS GROUPS OF LOCAL FIELDS, LIE ALGEBRAS AND RAMIFICATION

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ABSTRACT. Suppose K is a local field with finite residue field of characteristic $p\neq 2$ and $K_{<p}(M)$ is its maximal p-extension such that $\mathrm{Gal}(K_{<p}(M)/K)$ has period p^M and nilpotent class $<p$. If $\mathrm{char}K=0$ we assume that K contains a primitive p^M-th root of unity. The paper contains an overview of methods and results describing the structure of this Galois group together with its filtration by ramification subgroups.

INTRODUCTION

Everywhere in the paper p is a prime number. For any profinite group Γ and $s\in\mathbb{N}$, $C_s(\Gamma)$ denotes the closure of the subgroup of commutators of order s.

Let K be a complete discrete valuation field with a finite residue field $k\simeq\mathbb{F}_{p^{N_0}}$, $N_0\in\mathbb{N}$. Let K_{sep} be a separable closure of K and $\Gamma_K=\mathrm{Gal}(K_{sep}/K)$. Denote by $K(p)$ the maximal p-extension of K in K_{sep}. Then $\Gamma_K(p)=\mathrm{Gal}(K(p)/K)$ is a profinite p-group. As a matter of fact, the major information about Γ_K comes from the knowledge of the structure of $\Gamma_K(p)$. This structure is very well-known and is related to the following three cases (ζ_p is a primitive p-th root of unity) [17]:

— $\mathrm{char}\,K=p$;

— $\mathrm{char}\,K=0$, $\zeta_p\notin K$;

— $\mathrm{char}\,K=0$, $\zeta_p\in K$.

In all these cases the maximal abelian quotient of period p of $\Gamma_K(p)$ is isomorphic to K^*/K^{*p}. Therefore, $\Gamma_K(p)$ has infinitely many generators in the first case, has $[K:\mathbb{Q}_p]+1$ generators in the second case and $[K:\mathbb{Q}_p]+2$ generators in the third case. In the first two cases $\Gamma_K(p)$ is free and in the last case it has one relation of a very special form, cf. [17, 23, 24].

The above results can't be considered as completely satisfactory because they do not essentially reflect the appearance of $\Gamma_K(p)$ as a Galois group of an algrebraic extension of local fields. In other words, let LF be the category of couples (K, K_{sep}) where the morphisms are compatible continuous morphisms of local fields and let PGr be the category

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of profinite groups. Then the functor $(K, K_{sep}) \mapsto \Gamma_K(p)$ (as well as the functor $(K, K_{sep}) \mapsto \Gamma_K$) is not fully faithful.

The situation can be cardinally improved by taking into account a natural additional structure on $\Gamma_K(p)$ and Γ_K given by the decreasing filtration of ramification subgroups. The ramification filtration $\{\Gamma_K(p)^{(v)}\}_{v\geqslant 0}$ of $\Gamma_K(p)$ (as well as the appropriate filtration $\{\Gamma_K^{(v)}\}_{v\geqslant 0}$ of Γ_K) has many non-trivial properties. For example, it is left-continuous at any $v_0 \in \mathbb{Q}$, $v_0 > 1$, i.e. $\bigcap_{v<v_0} \Gamma_K(p)^{(v)} = \Gamma_K(p)^{(v_0)}$, but is not right-continuous, i.e. the closure of $\bigcup_{v>v_0} \Gamma_K(p)^{(v)}$ is not equal to $\Gamma_K(p)^{(v_0)}$. Another example [14, 15], for any $v_1, v_2 < v_0$, $(\Gamma_K(p)^{(v_1)}, \Gamma_K(p)^{(v_2)}) \not\subset \Gamma_K(p)^{(v_0)}$ and $(\Gamma_K(p)^{(v_1)})^p \not\subset \Gamma_K(p)^{(v_0)}$ and in some sense the groups $\Gamma_K(p)/\Gamma_K(p)^{(v_0)}$ have no "simple" relations [5].

The significance of study of ramification filtration was very well-understood long ago, e.g. cf. Shafarevich's Introduction to [17]. (The author also had interesting discussions on this subject in the IAS with A.Weil, P.Deligne and F.Pop.) As a matter of fact, the knowledge of ramification filtration is equivalent to the knowledge of the original field K due to the following local analogue of the Grothendieck Conjecture.

Theorem 0.1. *The functor* $(K, K_{sep}) \mapsto (\Gamma_K(p), \{\Gamma_K(p)^{(v)}\}_{v\geqslant 0})$ *from* LF *to the category of profinite* p*-groups with filtration is fully faithful.*

This result was first proved in the mixed characteristic case in the context of the whole Galois group Γ_K by Mochizuki [21] as a spectacular application of p-adic Hodge-Tate theory. The case of arbitrary characteristic was established by the author by different method in [7] under the assumption $p \neq 2$. Note that the characteristic p case was obtained via the explicit description of ramification filtration modulo the subgroup of third commutators from [2]. Then the mixed characteristic case was deduced from it via the Fontaine-Wintenberger field-of-norms functor. In paper [11] we removed the restriction $p \neq 2$ and reproved the statement in the context of the pro-p-group $\Gamma_K(p)$.

The study of ramification filtration in full generality seems not to be a realistically stated problem: it is not clear how to specify subgroups of a given profinite p-group. If we replace $\Gamma_K(p)$ by its maximal abelian quotient $\Gamma_K(p)^{ab}$ then the appropriate ramification filtration is very well-known but reflects very weak information about the original filtration of $\Gamma_K(p)$. This can be seen from class field theory where we have the reciprocity map $K^* \longrightarrow \Gamma_K^{ab}$ and the ramification subgroups appear as the images of the subgroups of principal units of K^*. In particular, we can observe only integral breaks of our filtration.

As a matter of fact, the ramification subgroups can be described on the abelian level without class field theory. The reason is that cyclic extensions of K can be studied via much more elementary tools: we can use the Witt-Artin-Schreier theory in the characteristic p case and the Kummer theory in the mixed characteristic case. Trying to develop this approach to the case of nilpotent Galois groups we developed in [1, 2] a nilpotent analogue of the Witt-Artin-Schreier theory. This theory allows us to describe quite efficiently p-extensions of fields of characteristic p with Galois p-groups of nilpotent class $< p$. Such groups arise from Lie algebras due to the classical equivalence of the categories of p-groups and Lie $\mathbb{F}_p$-algebras of nilpotent class $< p$, [20]. In [1, 2, 4] we applied our theory to local fields $\mathcal{K} = k((t_0))$, where $k \simeq \mathbb{F}_{p^{N_0}}$, and constructed explicitly the sets of generators of the appropriate ramification subgroups. This result demonstrates the advantage of our techniques: it is stated in terms of extensions of scalars of involved Lie algebras but this operation does not exist in group theory.

A generalization of our approach to local fields K of mixed characteristic was sketched earlier by the author in [6]. This approach allowed us to work with the groups $\Gamma_K/\Gamma_K^{p^M}C_p(\Gamma_K)$ under the assumption that a primitive p^M-th root of unity $\zeta_{p^M} \in K$. At that time we obtained explicit constructions of our theory only modulo subgroup of third commutators. Recently, we can treat the general case. First results are related to the case $M = 1$ and can be found in [11] (we discuss them also in Subsection 3.6 of this paper). The case of arbitrary M as well as the case of higher dimensional local fields will be considered in upcoming papers. In the case of local fields it would be very interesting to relate our theory to constructions of "nilpotent class field theory" from [19].

Note that the main constructions of the nilpotent Artin-Schreier theory do not suggest that the basic field is local. They can be applied also to global fields but it is not clear what sort of applications we can expect in this direction.

On the other hand, we can't expect the existence of an easy "nilpotent Kummer theory" for global fields. According to anabelian philosophy, for global fields E, the quotient of $\Gamma_E(p)$ by the subgroup of third commutators should already reflect all basic properties of the field E.

1. Nilpotent Artin-Schreier theory

In this section we discuss basic constructions of nilpotent Artin-Schreier theory. The main reference for this theory is [2]. We shall call this version contravariant and introduce also its covariant analogue, cf. Subsection 1.2 below. Everywhere M is a fixed natural number.

1.1. **Lifts modulo p^M, $M \in \mathbb{N}$.** Suppose K is a field of characteristic p and K_{sep} is a separable closure of K. Let $\{x_i\}_{i\in I}$ be a p-basis for K. This means that the elements $x_i \bmod K^{*p}$, $i \in I$, form a basis of the $\mathbb{F}_p$-module K^*/K^{*p}. Note that if E is any subfield of K_{sep} containing K then $\{x_i\}_{i\in I}$ can be taken also as a p-basis for E.

Let W_M be the functor of Witt vectors of length M. For a field $K \subset E \subset K_{sep}$, define $O_M(E)$ as the subalgebra in $W_M(E)$ generated over $W_M(\sigma^{M-1}E)$ by the Teichmuller representatives $[x_i]\in W_M(K)\subset W_M(E)$ of all x_i. Then $O_M(E)$ is a lift of E modulo p^M: it is a flat $W_M(\mathbb{F}_p)$-algebra such that $O_M(E)/pO_M(E)=E$. The system of lifts $O_M(E)$ essentially depends on the original choice of a p-basis in K. If σ is the absolute Frobenius (i.e. the morphism of p-th powers) then $W_M(\sigma)$ induces a σ-linear morphism on $O_M(E)$ and we usually denote it again by σ. Note that $O_M(E)|_{\sigma=\mathrm{id}}=W_M(\mathbb{F}_p)$, if E is normal over K then the Galois group $\mathrm{Gal}(E/K)$ acts on $O_M(E)$ and the invariants of this action coincide with $O_M(K)$.

A (continuous) automorphism $\psi\in\mathrm{Aut}(E)$ generally can't be extended to $\mathrm{Aut}O_M(E)$ if ψ changes the original p-basis. But the morphism $\sigma^{M-1}\psi$ admits "almost a lift" $\sigma^{M-1}O_M(E)\longrightarrow O_M(E)$ given by the following composition

$$\sigma^{M-1}O_M(E)\subset W_M(\sigma^{M-1}E)\xrightarrow{W_M(\sigma^{M-1}\psi)}W_M(\sigma^{M-1}E)\subset O_M(E).$$

The existence of such lift allowed us to extend the modulo p methods from [1] to the modulo p^M situation in [2, 4].

1.2. **Covariant and contravariant nilpotent Artin-Schreier theories.** Suppose L is a Lie algebra over $W_M(\mathbb{F}_p)$. For $s\in\mathbb{N}$, let $C_s(L)$ be an ideal of s-th commutators in L, e.g. $C_2(L)$, resp., $C_3(L)$, is generated by the comutators $[l_1,l_2]$, resp. $[[l_1,l_2].l_3]$, where all $l_i\in L$. The algebra L has nilpotent class $<p$ if $C_p(L)=0$.

The basic ingredient of our theory is the equivalence of the categories of p-groups of nilpotent class $<p$ and the category of Lie $\mathbb{Z}_p$-algebras of the same nilpotent class. This equivalence can be described on the level of objects killed by p^M as follows.

Suppose L is a Lie $W_M(\mathbb{F}_p)$-algebra of nilpotent class $<p$. If A is envelopping algebra for L and J is the augmentation ideal in A then there is a natural embedding of L into A/J^p (and L can be recovered as a submodule of the module of primitive elements modulo J^p in A, cf. [1] Section 1). The Campbell-Hausdorff formula is the map $L\times L\longrightarrow L$,

$$(l_1,l_2)\mapsto l_1\circ l_2=l_1+l_2+\frac{1}{2}[l_1,l_2]+\dots$$

such that in $A\,\mathrm{mod}\,J^p$ we have $\widetilde{\exp}(l_1)\widetilde{\exp}(l_2)=\widetilde{\exp}(l_1\circ l_2)$, where $\widetilde{\exp}(x)=\sum_{0\leqslant i<p}x^i/i!$ is the truncated exponential. The set L can be provided with the composition law $(l_1,l_2)\mapsto l_1\circ l_2$ which gives a group structure on L. We denote this group by $G(L)$. Clearly, this group has period p^M. Then the correspondence $L\mapsto G(L)$ is the above mentioned equivalence of the categories of p-groups of period p^M and Lie $W_M(\mathbb{F}_p)$-algebras.

Here and below we shall use the notation $L_K:=L\otimes_{W_M(\mathbb{F}_p)}O_M(K)$ and $L_{K_{sep}}=L\otimes_{W_M(\mathbb{F}_p)}O_M(K_{sep})$. Then Γ_K and the absolute Frobenius σ act through the second factor on $L_{K_{sep}}$, $L_{K_{sep}}|_{\sigma=\mathrm{id}}=L$ and $(L_{K_{sep}})^{\Gamma_K}=L_K$. The covariant nilpotent Artin-Schreier theory states that for any $e \in G(L_K)$, the set $F(e) = \{f \in G(L_{K_{sep}}) \mid \sigma(f) = e \circ f\}$ is not empty and the map $g \mapsto (-f) \circ g(f)$ is a group homomorphism $\pi_f(e) : \Gamma_K \longrightarrow G(L)$. The correspondence $e \mapsto \pi_f(e)$ has the following properties:

a) if $f' \in F(e)$ then $f' = f \circ c$, where $c \in G(L)$, and $\pi_f(e)$ and $\pi_{f'}(e)$ are conjugated via c;

b) for any $\pi \in \mathrm{Hom}(\Gamma_K, G(L))$, there are $e \in G(L_K)$ and $f \in F(e)$ such that $\pi_f(e) = \pi$;

c) for appropriate elements $e, e' \in G(L_K)$ and $f, f' \in G(L_{K_{sep}})$, we have $\pi_f(e) = \pi_{f'}(e')$ iff there is an $x \in G(\mathcal{L}_K)$ such that $f' = x \circ f$ and (therefore) $e' = \sigma(x) \circ e \circ (-x)$; e and e' are called R-equivalent via $x \in G(L_K)$.

According to above properties a)-c), the correspondence $e \mapsto \pi_f(e)$ establishes an identification of the set of all R-equivalent elements in $G(L_K)$ and the set of all conjugacy classes of $\mathrm{Hom}(\Gamma_K, G(L))$.

The above theory can be proved in a similar way to its contravariant version established in [2]. In the contravariant theory for any $e \in G(L_K)$, the set $\{f \in G(L_{K_{sep}}) \mid \sigma(f) = f \circ e\}$ is not empty, the correspondence $g \mapsto g(f) \circ (-f)$ establishes a group homomorphism from Γ_K^0 to $G(L_K)$, where Γ_K^0 coincides with Γ_K as a set but has the opposite group law $(g_1 g_2)^0 = g_2 g_1$. (Equivalently, if $a \in K_{sep}$ then $(g_1 g_2)a = g_2(g_1 a)$.) We have also the properties similar to above properties a)-c) but in c) there should be $f' = f \circ x$ and $e' = x \circ e \circ (-\sigma x)$.

The both (covariant and contravariant) theories admit a pro-finite version where L becomes a profinite $W_M(\mathbb{F}_p)$-Lie algebra and the set $\mathrm{Hom}(\Gamma_K, G(L))$ is the set of all continuous group morphisms.

1.3. **Identification** η_0**.** Suppose $\mathcal{K} = k((t_0))$ where t_0 is a fixed uniformiser in $\mathcal{K}$ and $k \simeq \mathbb{F}_{p^{N_0}}$ with $N_0 \in \mathbb{N}$. Then $\{t_0\}$ is a p-basis for $\mathcal{K}$, and we have the appropriate system of lifts $O_M(\mathcal{E})$ modulo p^M for all subfields $\mathcal{K} \subset \mathcal{E} \subset \mathcal{K}_{sep}$. In addition, fix an element $\alpha_0 \in W(k)$ such that $\mathrm{Tr}(\alpha_0) = 1$, where Tr is the trace map for the field extension $W(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \supset \mathbb{Q}_p$.

Let $\mathbb{Z}^+(p) = \{a \in \mathbb{N} \mid (a, p) = 1\}$ and $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$.

For $M \in \mathbb{N}$, let $\widetilde{\mathcal{L}}_M$ be a profinite free Lie $\mathbb{Z}/p^M$-algebra with the (topological) module of generators $\mathcal{K}^*/\mathcal{K}^{*p^M}$ and $\mathcal{L}_M = \widetilde{\mathcal{L}}_M / C_p(\widetilde{\mathcal{L}}_M)$. From time to time we drop the subscript M off to simplify the notation.

Let $\mathcal{L} = \mathcal{L}_M$. Then $\mathcal{L}_k := \mathcal{L} \otimes W_M(k)$ has the generators

$$\{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}$$

due to the following identifications (where $t = [t_0]$ is the Teichmuller representative of t_0):

$$\mathcal{K}^*/\mathcal{K}^{*p^M} \otimes_{W_M(\mathbb{F}_p)} W_M(k) =$$

$$\mathrm{Hom}_{W_M(\mathbb{F}_p)}(O_M(\mathcal{K})/(\sigma-\mathrm{id})O_M(\mathcal{K}),W_M(k))=$$
$$\mathrm{Hom}_{W_M(\mathbb{F}_p)}((W_M(\mathbb{F}_p)\alpha_0)\oplus_{a\in\mathbb{Z}^+(p)}(W_M(k)t^{-a}),W_M(k))=$$
$$W_M(k)D_0\times\prod_{\substack{a\in\mathbb{Z}^+(p)\\ n\in\mathbb{Z}\,\mathrm{mod}\,N_0}}W_M(k)D_{an}$$

Note that the first identification uses the Witt pairing, D_0 appears from $t_0\otimes 1\in\mathcal{K}^*/\mathcal{K}^{*p^M}\otimes W_M(k)$ and for all $a\in\mathbb{Z}^+(p)$ and $w\in W_M(k)$, $D_{an}(wt^{-a})=\sigma^n w$.

For any $n\in\mathbb{Z}/N_0$, set $D_{0n}=t\otimes(\sigma^n\alpha_0)=(\sigma^n\alpha_0)D_0$.

Let $e_0=\sum_{a\in\mathbb{Z}^0(p)}t^{-a}D_{a0}\in\mathcal{L}_K$, choose $f_0\in F(e_0)$ and set $\eta_0=\pi_{f_0}(e_0)$. Then η_0 is a surjective homomorphism from $\Gamma_{\mathcal{K}}$ to $G(\mathcal{L})$ and it induces a group isomorphism $\Gamma_{\mathcal{K}}/\Gamma_{\mathcal{K}}^{p^M}C_p(\Gamma_{\mathcal{K}})\simeq G(\mathcal{L})$. Note that the construction of η_0 depends up to conjugacy only on the original choice of the uniformizer t_0 and the element $\alpha_0\,\mathrm{mod}\,p^M\in W_M(k)$. On the level of maximal abelian quotients of period p^M, η_0 induces the isomorphism of local class field theory $\Gamma_{\mathcal{K}}^{ab}\otimes_{\mathbb{Z}_p}W_M(\mathbb{F}_p)\simeq\mathcal{K}^*/\mathcal{K}^{*p^M}$.

1.4. **Why Campbell-Hausdorff?** In this Subsection it will be explained that in our theory, we are, essentially, forced to use the Campbell-Hausdorff composition law.

Assume for simplicity, that $M=1$ and $\mathcal{K}=\mathbb{F}_p((t_0))$. Let $\mathcal{K}(p)$ be the maximal p-extension of $\mathcal{K}$ and $\Gamma_{\mathcal{K}}(p)=\mathrm{Gal}(\mathcal{K}(p)/\mathcal{K})$. For $s\in N$ and $a_1,a_2,\dots,a_s,\dots\in\mathbb{Z}^0(p)$, consider the elements $T_{a_1\dots a_s}\in\mathcal{K}(p)$ such that:

$$T_{a_1}^p-T_{a_1}=t_0^{-a_1},$$
$$T_{a_1a_2}^p-T_{a_1a_2}=t_0^{-a_1}T_{a_2}$$
$$\dots\dots\dots$$
$$T_{a_1\dots a_s}^p-T_{a_1\dots a_s}=t_0^{-a_1}T_{a_2\dots a_s}$$
$$\dots\dots\dots$$

Then the system $\{T_{a_1\dots a_s}\mid s\geqslant 0,a_i\in\mathbb{Z}^0(p)\}$ is linearly independent over $\mathcal{K}$ and if $\mathcal{M}=\bigoplus_{\substack{a_1,\dots,a_s\\ s\geqslant 0}}\mathbb{F}_pT_{a_1\dots a_s}$ then $\mathcal{K}(p)=\mathcal{M}\otimes_{\mathbb{F}_p}\mathcal{K}$ and $\Gamma_{\mathcal{K}}(p)$ acts on $\mathcal{M}$ via a natural embedding $\Gamma_{\mathcal{K}}(p)\hookrightarrow\mathrm{GL}_{\mathbb{F}_p}(\mathcal{M})$. This construction would have given us an efficient approach to an explicit construction of the maximal p-extension $\mathcal{K}(p)$ if we could describe explicitly the image of $\Gamma_{\mathcal{K}}(p)$ in $\mathrm{GL}_{\mathbb{F}_p}(\mathcal{M})$.

Analyze the situation at different levels $s\geqslant 1$.

• **1st level.** Here all equations are independent and we can introduce a minimal system of generators τ_a, $a\in\mathbb{Z}^0(p)$, of $\Gamma_{\mathcal{K}}(p)$ with their explicit action via $\tau_a:T_{a_1}\mapsto T_{a_1}+\delta(a,a_1)$ at this level. (Here and below δ is the Kronecker symbol.)

• **2nd level.** Here the roots $T_{a_1a_2}$ are not (algebraically) independent. For example, the following identity

$$(T_{a_1}T_{a_2})^p = (T_{a_1}+t_0^{-a_1})(T_{a_2}+t_0^{-a_2}) = T_{a_1}T_{a_2}+t_0^{-a_1}T_{a_2}+t_0^{-a_2}T_{a_1}+t_0^{-(a_1+a_2)}$$

implies under the assumption $(a_1+a_2, p) = 1$ (and after a suitable choice of involved roots of Artin-Schreier equations) that

$$T_{a_1}T_{a_2} = T_{a_1a_2} + T_{a_2a_1} + T_{a_1+a_2}.$$

The presence of the term $T_{a_1+a_2}$ creates a problem: $\tau_{a_1+a_2}$ should act non-trivially on either $T_{a_1a_2}$ or $T_{a_2a_1}$ but they both do not depend on the index a_1+a_2. The situation can be resolved by a slight correction of involved equations. Namely, let $T_{a_1a_2}$ be such that

$$T_{a_1a_2}^p - T_{a_1a_2} = t_0^{-a_1}T_{a_2} + \eta(a_1,a_2)t_0^{-(a_1+a_2)}$$

where the constants $\eta(a_1,a_2) \in k$, $a_1, a_2 \in \mathbb{Z}^0(p)$, satisfy the relations

$$\eta(a_1,a_2) + \eta(a_2,a_1) = 1. \tag{1.1}$$

With the above correction, the elements $T_{a_1a_2}$, $a_1, a_2 \in \mathbb{Z}^0(p)$, can be chosen in such a way that we have the following:

— relations: $T_{a_1}T_{a_2} = T_{a_1a_2} + T_{a_2a_1}$;

— Galois action: $\tau_a(T_{a_1a_2}) = T_{a_1a_2} + T_{a_1}\delta(a_2,a) + \eta(a_1,a_2)\delta(a_1,a_2,a)$.

Relation (1.1) will look more natural if we introduce the constants on the first level via $\eta(a) = 1$, $a \in \mathbb{Z}^0(p)$. Then (1.1) can be rewritten as $\eta(a_1)\eta(a_2) = \eta(a_1,a_2) + \eta(a_2,a_1)$. These relations can be satisfied only if $p \neq 2$ and the simplest choice is $\eta(a_1,a_2) = 1/2$ for all a_1, a_2.

The above picture can be generalized to higher levels as follows.

• ***s*-th level,** $s < p$. Here we have:

- the equations: $T_{a_1\dots a_s}^p = T_{a_1\dots a_s} + \eta(a_1)t^{-a_1}T_{a_2\dots a_s} + \dots$

$$+\eta(a_1,\dots,a_{s-1})t_0^{-(a_1+\dots+a_{s-1})}T_{a_s} + \eta(a_1,\dots,a_s)t_0^{-(a_1+\dots+a_s)}$$

-the relations: $T_{a_1\dots a_k}T_{b_1\dots b_l} = \sum T_{\text{insertions of } a\text{'s into } b\text{'s}}$, where $k+l < p$;

- the Galois action: $\tau_a(T_{a_1\dots a_s}) = T_{a_1\dots a_s} + T_{a_1\dots a_{s-1}}\delta(a,a_s)\eta(a_s) +$

$$\dots + T_{a_1}\delta(a,a_2,\dots,a_s)\eta(a_2,\dots,a_s) + \delta(a,a_1,\dots,a_s)\eta(a_1,\dots,a_s)$$

- the constants: if $k+l < p$ then

$$\eta(a_1,\dots,a_k)\eta(b_1,\dots,b_l) = \sum \eta(\text{insertions of } a\text{'s into } b\text{'s})$$

with their simplest choice $\eta(a_1,\dots,a_s) = 1/s!$

Remark. An insertion of the ordered collection $a_1, \dots, a_k$ into the ordered collection $b_1, \dots, b_l$ is the ordered collection $c_1, \dots, c_{k+l}$ such that

— $\{1, \dots, k+l\} = \{i_1, \dots, i_k\} \coprod \{j_1, \dots, j_l\}$;

— $i_1 < \dots < i_k$ and $j_1 < \dots < j_l$;

— $a_1 = c_{i_1}, \dots,\ a_k = c_{i_k}$ and $b_1 = c_{j_1}, \dots,\ b_l = c_{j_l}$.

The following formalism allows us to present the above information on all levels $1 \leqslant s < p$ in the following compact way.

Let $\widetilde{\mathcal{A}}$ be a pro-finite associative $\mathbb{F}_p$-algebra with the set of free generators $\{D_a \mid a \in \mathbb{Z}^0(p)\}$. Introduce the elements of the appropriate extensions of scalars of $\mathcal{A}$

$$E = 1 + \sum_{\substack{1 \leqslant s < p \\ a_i \in \mathbb{Z}^0(p)}} \eta(a_1, \dots, a_s) t_0^{-(a_1 + \dots + a_s)} D_{a_1} \dots D_{a_s}$$

$$= \widetilde{\exp}\Big(\sum_{a \in \mathbb{Z}^0(p)} t_0^{-a} D_a\Big) \in \mathcal{A}_{\mathcal{K}}\,,$$

$$\mathcal{F} = 1 + \sum_{\substack{1 \leqslant s < p \\ a_i \in \mathbb{Z}^0(p)}} \eta(a_1, \dots, a_s) T_{a_1 \dots a_s} D_{a_1} \dots D_{a_s} \in \mathcal{A}_{\mathcal{K}_{sep}}$$

Define the diagonal map as the morphism of $\mathbb{F}_p$-algebras

$$\Delta : \mathcal{A} \operatorname{mod} \deg\, p \longrightarrow \mathcal{A} \otimes \mathcal{A} \operatorname{mod}\, \deg\, p$$

such that for any $a \in \mathbb{Z}^0(p)$, $D_a \mapsto D_a \otimes 1 + 1 \otimes D_a$. Then we have the following properties:

— $\Delta(E) \equiv E \otimes E \operatorname{mod} \deg p$; $\Delta(\mathcal{F}) \equiv \mathcal{F} \otimes \mathcal{F} \operatorname{mod} \deg p\,$;

— $\sigma(\mathcal{F}) \equiv E\mathcal{F} \operatorname{mod} \deg p$; $\tau_a(\mathcal{F}) \equiv \mathcal{F}\, \widetilde{\exp}(D_a) \operatorname{mod} \deg p\,$.

Now we can verify the existence of $f \in \mathcal{L}_{\mathcal{K}_{sep}}$ such that $\mathcal{F} = \widetilde{\exp}(f)$ modulo deg p, and recover the basic relations $\sigma(f) = (\sum_a t_0^{-a} D_a) \circ f$ and $\tau_a(f) = f \circ D_a$, $a \in \mathbb{Z}^0(p)$, of our nilpotent Artin-Schreier theory.

2. Ramification filtration in $\mathcal{L} = \mathcal{L}_{M+1}$

In this Section we describe and illustrate the main trick used in papers [1, 2, 4]. This trick allowed us to find explicit generators of ramification subgroups under the identification η_0 from Subsection 1.3. Remind that we work over $\mathcal{K} = k((t_0))$, where $k \simeq \mathbb{F}_{p^{N_0}}$, $N_0 \in \mathbb{N}$.

2.1. **Auxiliary field $\mathcal{K}'=\mathcal{K}(r^*,N)$, [1, 2, 4].** The field $\mathcal{K}'$ is a totally ramified extension of $\mathcal{K}$ in $\mathcal{K}_{sep}$. It depends on two parameters: $r^*\in\mathbb{Q}$ such that $r^*>0$ and $v_p(r^*)=0$, and $N\in\mathbb{N}$ such that if $q=p^N$ then $b^*:=r^*(q-1)\in\mathbb{N}$. Note that for a given r^*, there are infinitely many ways to choose N, in particular, we can always assume that N is sufficiently large.

By definition, $[\mathcal{K}':\mathcal{K}]=q$ and the Herbrand function $\varphi_{\mathcal{K}'/\mathcal{K}}$ has only one edge point (r^*,r^*). It can be proved that $\mathcal{K}'=k((t_0'))$, where $t_0=t_0'^qE(-1,t_0'^{b^*})$. Here for $w\in W(k)$,

$$E(w,X)=\exp(wX+\sigma(w)X^p/p+\dots+\sigma^n(w)X^{p^n}/p^n+\dots)\in\mathbb{Z}_p[[X]]$$

is the Shafarevich version of the Artin-Hasse exponential.

Note that if $r^*\notin\mathbb{N}$, $\mathcal{K}'/\mathcal{K}$ is neither Galois nor a p-extension.

2.2. **The criterion.** Consider the following lifts modulo p^{M+1} with respect to the p-basis $\{t_0\}$ of $\mathcal{K}$

$$O_{M+1}(\mathcal{K})=W_{M+1}(k)((t))=W_{M+1}(\sigma^M\mathcal{K})[t]$$

$$O_{M+1}(\mathcal{K}_{sep})=W_{M+1}(\sigma^M\mathcal{K}_{sep})[t]\subset W_{M+1}(\mathcal{K}_{sep}).$$

Remind that $t=[t_0]\in O_{M+1}(\mathcal{K})$ is the Teichmüller representative of t_0 in $W_{M+1}(\mathcal{K})$.

For $\mathcal{K}'=\mathcal{K}(r^*,N)$ and its uniformiser t_0' from Subsection 2.1 consider the appropriate lifts $O'_{M+1}(\mathcal{K}')$ and $O'_{M+1}(\mathcal{K}'_{sep})$. If $t'=[t_0']$ then t and t' can be related one-to-another in $W_{M+1}(\mathcal{K}')$ via

$$t^{p^M}=t'^{p^Mq}\exp(-p^Mt'^{b^*}-\dots-pt'^{p^{M-1}b^*})E(-1,t'^{p^Mb^*})\,.$$

This implies the following relations between the lifts for $\mathcal{K}$ and $\mathcal{K}'$

$$\sigma^MO_{M+1}(\mathcal{K})\subset W_{M+1}(\sigma^M\mathcal{K})\subset O'_{M+1}(\mathcal{K}')$$

$$\sigma^MO_{M+1}(\mathcal{K}_{sep})\subset W_{M+1}(\sigma^M\mathcal{K}_{sep})\subset O'_{M+1}(\mathcal{K}'_{sep})$$

As earlier, take $e_0=\sum_{a\in\mathbb{Z}^0(p)}t^{-a}D_{a0}\in\mathcal{L}_{\mathcal{K}}$, $f_0\in\mathcal{L}_{\mathcal{K}_{sep}}$ such that $\sigma f_0=e_0\circ f_0$ and consider $\pi_{f_0}(e_0):\Gamma_{\mathcal{K}}\longrightarrow G(\mathcal{L})$. Similarly, let $e_0'=\sum_{a\in\mathbb{Z}^0(p)}t'^{-a}D_{a,-N}$, choose $f_0'\in\mathcal{L}_{\mathcal{K}_{sep}}$ such that $\sigma f_0'=e_0'\circ f_0'$ and consider $\pi_{f_0'}(e_0'):\Gamma_{\mathcal{K}'}\longrightarrow G(\mathcal{L})$.

For $Y\in\mathcal{L}_{\mathcal{K}_{sep}}$ and an ideal $\mathcal{I}$ in $\mathcal{L}$, define the field of definition of $Y\,\mathrm{mod}\,\mathcal{I}_{\mathcal{K}_{sep}}$ over $\mathcal{K}$ as $\mathcal{K}(Y\,\mathrm{mod}\,\mathcal{I}_{\mathcal{K}_{sep}}):=\mathcal{K}_{sep}^{\mathcal{H}}$, where $\mathcal{H}=\{g\in\Gamma_{\mathcal{K}}\mid g(Y)\equiv Y\,\mathrm{mod}\,\mathcal{I}_{\mathcal{K}_{sep}}\}$.

For any finite field extension $\mathcal{E}/\mathcal{K}$ in $\mathcal{K}_{sep}$ define its biggest ramification number $v(\mathcal{E}/\mathcal{K})=\max\{v\mid\Gamma_{\mathcal{K}}^{(v)}\text{acts non-trivially on }\mathcal{E}\}$.

For $v_0\in\mathbb{Q}_{>0}$, let the ideal $\mathcal{L}^{(v_0)}$ of $\mathcal{L}$ be such that $G(\mathcal{L}^{(v_0)})=\eta_0(\Gamma_{\mathcal{K}}^{(v_0)})$. Let $f_M=\sigma^Mf_0$ and $f_M'=\sigma^Mf_0'$. Our method from [1, 2, 4] is based on the following criterion.

Proposition 2.1. *Let* $X \in \mathcal{L}_{\mathcal{K}_{sep}}$ *be such that* $f_M = X \circ \sigma^N(f'_M)$. *Suppose* $v_0, r^* \in \mathbb{Q}_{>0}$, $v_p(r^*) = 0$ *and* $r^* < v_0$. *Then* $\mathcal{L}^{(v_0)}$ *is the minimal ideal in the family of all ideals* $\mathcal{I}$ *of* $\mathcal{L}$ *such that*

$$v(\mathcal{K}'(X \operatorname{mod} \mathcal{I}_{\mathcal{K}_{sep}})/\mathcal{K}') \leqslant v_0 q - b^*.$$

The proof is quite formal and is based on the following properties of upper ramification numbers. If $v = v(\mathcal{K}(f \operatorname{mod} \mathcal{I}_{\mathcal{K}_{sep}})/\mathcal{K})$ then:

a) $v(\mathcal{K}'(f' \operatorname{mod} \mathcal{I}_{\mathcal{K}'_{sep}})/\mathcal{K}') = v$;

b) $v(\mathcal{K}'(f' \operatorname{mod} \mathcal{I}_{\mathcal{K}'_{sep}})/\mathcal{K}) = \varphi_{\mathcal{K}'/\mathcal{K}}(v)$;

c) if $v > r^*$ then $\varphi_{\mathcal{K}'/\mathcal{K}}(v) < v$.

2.3. **Illustration of the criterion.** The criterion from Proposition 2.1 was applied in [1, 2, 4] to describe the structure of $\eta_0(\Gamma_{\mathcal{K}}^{(v)})$ by induction by proceeding from the situation modulo p^M to the situation modulo p^{M+1} and from the situation modulo $C_s(\mathcal{L})$ to the situation modulo $C_{s+1}(\mathcal{L})$, where $2 \leqslant s < p$.

Typically, for an ideal $I \subset \mathcal{L}$, we used the knowledge of the structure of $\mathcal{L}^{(v_0)} \operatorname{mod} I$ to prove (after choosing r^* sufficiently close to v_0) that $X \in \mathcal{L}_{\mathcal{K}'} \operatorname{mod} (\mathcal{L}_{\mathcal{K}'}^{(v_0)} + I_{\mathcal{K}'})$. Then we could apply our Criterion to $\mathcal{L}^{(v_0)} \operatorname{mod} J$ for an appropriate (slightly smaller than I) ideal J because $X \operatorname{mod} J_{\mathcal{K}_{sep}}$ satisfied over $\mathcal{L}_{\mathcal{K}'}$ just an Artin-Schreier equation of degree p. Notice that $f_0 \operatorname{mod} J_{\mathcal{K}_{sep}}$ and $f'_0 \operatorname{mod} J_{\mathcal{K}_{sep}}$ satisfy very complicated relations over $\mathcal{L}_{K'}$. We give below two examples to illustrate how our method works in more explicit but similar situations.

2.3.1. *First example.* Suppose $M \geqslant 0$ and $F \in O_{M+1}(\mathcal{K}_{sep})$ is such that $F - \sigma^{N_0} F = t^{-a}$, $a \in \mathbb{Z}^0(p)$.

If $M = 0$ then F is a root of the Artin-Schreier equation $F - F^{q_0} = t^{-a}$ with $q_0 = p^{N_0}$ and directly from the definition of ramification subgroups it follows that $v(\mathcal{K}(F)/\mathcal{K}) = a$. The case of arbitrary M corresponds to the Witt theory. Here the left-hand side $F - \sigma^{N_0} F$ is already a Witt vector of length $M+1$, and careful calculations with components of Witt vectors give that $v(\mathcal{K}(F)/\mathcal{K}) = p^M a$. Our criterion allows us to obtain this result in a much more easier way.

Take the field $\mathcal{K}' = \mathcal{K}(r^*, N)$ where r^* and N (recall that $q = p^N$) are such that

$$ap^M > r^* > ap^{M-1} q/(q-1). \tag{2.1}$$

Consider the appropriate lift $O'_{M+1}(\mathcal{K}'_{sep})$ and let $F' \in O'_{M+1}(\mathcal{K}'_{sep})$ be such that $F' - \sigma^{N_0} F' = t'^{-a}$.

Set $F_M = \sigma^M F$ and $F'_M = \sigma^M F'$.

Clearly, $\mathcal{K}(F_M) = \mathcal{K}(F)$ and $\mathcal{K}'(F'_M) = \mathcal{K}'(F')$.

According to restrictions (2.1) we have

$$t^{-ap^M} = t'^{-aqp^M} \exp(ap^M t'^{b^*} + \dots + apt'^{p^{M-1} b^*}) E(a, t'^{p^M b^*})$$

$$= t'^{-aqp^M} + ap^M t'^{-ap^M q + b^*} + f_0,$$

where $f_0 \in t' W_M(k)[[t']] \subset O'_{M+1}(\mathcal{K}')$. Therefore,

$$F_M = \sigma^N F'_M + p^M X + \sum_{i \geqslant 0} \sigma^{iN_0}(f_0),$$

where $X - \sigma^{N_0} X = at'^{-ap^M q + b^*}$.

In this situation an analogue of our Criterion states that

$$v(\mathcal{K}(F_M)/\mathcal{K}) = ap^M \quad \Leftrightarrow v(\mathcal{K}'(X)/\mathcal{K}') = ap^M q - b^*$$

But the right-hand side of this assertion corresponds to the case $M = 0$ and was explained in the beginning of this section.

2.3.2. *Second example.* Consider the following modulo p situation, i.e. the situation where $M = 0$. (The appropriate case of arbitrary M can be considered similarly.)

Let $F, G \in \mathcal{K}_{sep}$ be such that

$$F - \sigma^{N_0} F = t_0^{-a}, \quad G - \sigma^{N_0} G = t_0^{-b} \sigma^{n_0}(F) \tag{2.2}$$

with $a, b \in \mathbb{Z}^0(p)$ and $0 \leqslant n_0 < N_0$. Let

$$A = \max\left\{a + bp^{-n_0}, ap^{-N_0+n_0} + b\right\}$$

Prove that $v(\mathcal{K}(F,G)/\mathcal{K}) = A$ if either $a \neq b$ or $2n_0 \neq N_0$.

Take $\mathcal{K}' = \mathcal{K}'(r^*, N)$, where r^* and $q = p^N$ satisfy the following restrictions

$$\frac{qA}{2(q-1)} < r^* < \frac{qA}{q-1}, \quad r^* > \frac{q}{q-1}\max\{a, ap^{-N_0} + b\}$$

(We can take $r^* \in (A/2, A)$ such that $r^* > \max\{a, b\}$ and then choose sufficiently large N to satisfy these conditions.) Consider $F', G' \in \mathcal{K}'_{sep}$ such that

$$F' - \sigma^{N_0} F' = t_0'^{-a}, \quad G' - \sigma^{N_0} G' = t_0'^{-b} \sigma^{n_0}(F')$$

Notice that

$$t_0^{-a} = t_0'^{-aq} + at_0'^{-aq+b^*} + o_1$$

where $o_1 \in t_0'^{-aq+2b^*} k[[t'_0]]$. Therefore,

$$F = F'^q + T_F + \sum_{i \geqslant 0} \sigma^{iN_0} o_1, \tag{2.3}$$

where $T_F - \sigma^{N_0} T_F = at_0'^{-aq+b^*}$. We can choose F' in such a way that (use that $-aq+b^* > 0$) $T_F = at_0'^{-aq+b^*} + o_2$, where $o_2 \in t_0'^{p^{N_0}(-aq+b^*)} k[[t'_0]]$.

As earlier,

$$t_0^{-b} = t_0'^{-bq} + bt_0'^{-bq+b^*} + o_3 \tag{2.4}$$

with $o_3 \in t_0'^{-bq+2b^*} k[[t'_0]]$. Then (2.3) and (2.4) imply that

$$t_0^{-b} \sigma^{n_0} F = (t_0'^{-bq} \sigma^{n_0} F')^q + at_0'^{-bq + p^{n_0}(-aq+b^*)} +$$

$$bt_0'^{-bq+b^*}(\sigma^{n_0}F')^q + o_3(\sigma^{n_0}F')^q + o_4$$

where $o_4 \in t_0'k[[t_0']]$. Therefore, $G = G'^q + T_G + \sum_{i\geqslant 0}\sigma^{iN_0}o_4$, where

$$T_G - \sigma^{N_0}T_G = at'^{-bq+p^{n_0}(-aq+b^*)}$$
$$+bt'^{-bq+b^*}(\sigma^{n_0}F')^q + o_3(\sigma^{n_0}F')^q$$

An appropriate analogue of our criterion gives that

$$v(\mathcal{K}(F,G)/\mathcal{K}) = A \quad \Leftrightarrow \quad v(\mathcal{K}'(T_F,T_G)/\mathcal{K}') = qA - b^*$$

First of all, $T_F \in k[[t_0']]$ and, therefore, we should prove that $v(\mathcal{K}'(T_G)/\mathcal{K}') = qA - b^*$. Notice that

$$F'^{qp^{n_0}} = (\sigma^{N_0}F')^{qp^{n_0-N_0}} = F'^{qp^{n_0-N_0}} - t_0'^{-aqp^{n_0-N_0}}.$$

Let $O_{\mathcal{K}'(F')}$ and $\mathrm{m}_{\mathcal{K}'(F')}$ be the valuation ring and, resp., the maximal ideal for $\mathcal{K}'(F')$. Clearly, $t_0'^a\sigma^{N_0}F' \in O_{\mathcal{K}'(F')}$ and, therefore,

$$t_0'^{-bq+b^*}F'^{qp^{n_0-N_0}} \in \mathrm{m}_{\mathcal{K}'(F')}$$

(use that $b^* > q(ap^{n_0-2N_0}+b)$) and $o_3(\sigma^{n_0}F')^q \in \mathrm{m}_{\mathcal{K}'(F')}$ (use that $2b^* - (ap^{n_0-N_0}+b) > 0$).

This implies that $\mathcal{K}'(T_G) \subset \mathcal{K}'(T_G,F') = \mathcal{K}'(T,F')$, where

$$T - \sigma^{N_0}T = a\sigma^{n_0}\left(t'^{-(a+bp^{-n_0})q+b^*}\right) - bt_0'^{-(ap^{n_0-N_0}+b)q+b^*},$$

and $v(\mathcal{K}'(T_G)/\mathcal{K}') = v(\mathcal{K}'(T,F')/\mathcal{K}')$.

If either $a \neq b$ or $2n_0 \neq N_0$ the right-hand side of this equation is not trivial and (use that $v(\mathcal{K}'(F')/\mathcal{K}') = a < A$) $v(\mathcal{K}'(T,F')/\mathcal{K}') = v(\mathcal{K}'(T)/\mathcal{K}') = qA - b^*$.

2.4. **Ramification subgroups modulo $\Gamma_{\mathcal{K}}^{p^M}C_p(\Gamma_{\mathcal{K}})$.** As earlier, let $\mathcal{L} = \mathcal{L}_M$ and for $v \geqslant 0$, let $\mathcal{L}^{(v)} = \eta_0(\Gamma_{\mathcal{K}}^{(v)}) \subset \mathcal{L}$. The ideal $\mathcal{L}^{(v)}$ was described in [4] as follows.

For $\gamma \geqslant 0$ and $N \in \mathbb{Z}$, introduce the elements $\mathcal{F}^0_{\gamma,-N} \in \mathcal{L}_k$ via

$$\mathcal{F}^0_{\gamma,-N} = \sum_{\substack{1\leqslant s<p\\ a_i,n_i}} a_1p^{n_1}\eta(n_1,\dots,n_s)[\dots[D_{a_1\bar{n}_1},D_{a_2\bar{n}_2}],\dots,D_{a_s\bar{n}_s}]$$

Here:

— all $a_i \in \mathbb{Z}^0(p)$, $n_i \in \mathbb{Z}$, $n_1 \geqslant 0$, $n_1 \geqslant n_2 \geqslant \dots \geqslant n_s \geqslant -N$, $\bar{n}_s = n_s \bmod N_0$;

— $a_1p^{n_1} + a_2p^{n_2} + \dots + a_sp^{n_s} = \gamma$;

— if $n_1 = \dots = n_{s_1} > \dots > n_{s_{r-1}+1} = \dots = n_{s_r}$ then $\eta(n_1,\dots,n_s) = (s_1!\dots(s_r-s_{r-1})!)^{-1}$.

Let $\mathcal{L}(v)_N$ be the minimal ideal of $\mathcal{L}$ such that $\mathcal{L}(v)_N \otimes W_M(k)$ contains all $\mathcal{F}^0_{\gamma,-N}$ with $\gamma \geqslant v$. Then there is an $N^*_M(v) \in \mathbb{N}$ and an ideal $\mathcal{L}(v)$ of $\mathcal{L}$ such that for all $N \geqslant N^*_M(v)$, $\mathcal{L}(v)_N = \mathcal{L}(v)$.

Theorem 2.2. *For any $v \geqslant 0$, $\mathcal{L}^{(v)} = \mathcal{L}(v)$.*

This statement was obtained in the contravariant setting in [4] and uses the elements $\mathcal{F}_{\gamma,-N}$ given by the same formula (as for $\mathcal{F}^0_{\gamma,-N}$) but with the factor $(-1)^{s-1}$. Indeed, the contravariant version of Theorem 2.2 appears by replacing the Lie bracket $[l_1, l_2]$ in $\mathcal{L}$ by the bracket $[l_1, l_2]^0 = [l_2, l_1]$. Therefore, $[\dots[D_1, D_2], \dots, D_s]$ should be replaced by $[D_s, \dots, [D_2, D_1]\dots] = (-1)^{s-1}[\dots[D_1, D_2], \dots, D_s]$.

Remark. For the ideal $\mathcal{L}^{(v)} \bmod C_2(\mathcal{L})$ we have the generators coming from $\mathcal{F}^0_{\gamma,-N}$ taken modulo the ideal of second commutators. Such generators are non-zero only if γ is integral. Therefore, $\mathcal{L}^{(v)} \otimes W(k)$ is generated on the abelian level by the images of $p^n D_{am}$, where $m \in \mathbb{Z}/N_0$ and $p^n a \geqslant v$.

The proof of Theorem 2.2 is quite technical and it would be nice to put it into a more substantial context. This could be done on the basis of its following interpretation.

Assume for simplicity that $M = 0$. Choose r^*, N as in Subsection 2.1. We can assume that $N \equiv 0 \bmod N_0$.

If we replace t_0 by t_0^q then the identification η_0 will be not changed. Indeed, $e_0 = \sum_{a \in \mathbb{Z}^0(p)} t_0^{-a} D_{a0}$ is replaced by $\iota(e_0) = \sum_{a \in \mathbb{Z}^0(p)} t_0^{-aq} D_{a0} = (\sigma c) \circ e_0 \circ (-c)$, where $c = (\sigma^{N-1} e_0) \circ \dots \circ (\sigma e_0) \circ e_0$.

When proving above Theorem 2.2 in [4] we actually established that $\mathcal{L}^{(v_0)}$ appears as the minimal ideal $\mathcal{I}$ of $\mathcal{L}$ such that the replacement (deformation) $d(1) : t_0 \mapsto t_0^q E(-1, t_0^{b^*})$ does not affect the identification $\eta_0 \bmod \mathcal{I}$. The same holds also for the one-parameter deformation $\mathcal{D}(u) : t_0 \mapsto t_0^q E(-u, t_0^{b^*})$ with parameter u. In terms of the nilpotent Artin-Schreier theory this means the existence of $c = c(u) \in \mathcal{L}_{\mathcal{K}[u]}$ such that

$$\mathcal{D}(u)(e_0) \equiv (\sigma c) \circ e_0 \circ (-c) \bmod \mathcal{I}_{\mathcal{K}[u]}$$

(we assume that $\sigma(u) = u$). This condition is satisfied on the linear level (i.e. for the coefficients of u) if and only if the above elements $\mathcal{F}_{\gamma,-N}$, where $\gamma \geqslant v$ and $N \geqslant N^*_M(v)$, belong to $\mathcal{I}_k$. As a matter of fact, the main difficulty we resolved in [1, 4] was that on the level of higher powers u^i, $i > 1$, we do not obtain new conditions. We do expect to obtain this fact in an easier way by more substantial use of ideas of deformation theory.

3. The mixed characteristic case

In this Section we sketch main ideas which allowed us to apply the above characteristic p results to the mixed characteristic case.

Suppose K is a finite etension of $\mathbb{Q}_p$ with the residue field k. We fix a choice of uniformising element π_0 in K and assume that K contains a primitive p^M-th root of unity ζ_M.

3.1. **The field of norms functor** [26]**.** Let $\widetilde{K}$ be a composite of the field extensions $K_n = K(\pi_n)$, where $n \geqslant 0$, $K_0 = K$, and $\pi_{n+1}^p = \pi_n$. The field of norms functor $\mathcal{X}$ provides us with:

1) a complete discrete valuation field $\mathcal{K} = \mathcal{X}(\widetilde{K})$ of characteristic p. The residue field of $\mathcal{K}$ can be canonically identified with k, and $\mathcal{K}$ has a fixed uniformizer t_0: by definition, $\mathcal{K}^* = \varprojlim K_n^*$, where the connecting morphisms are induced by the norm maps, and $t_0 = \varprojlim \pi_n$;

2) if E is an algebraic extension of $\widetilde{K}$, then $\mathcal{X}(E)$ is separable over $\mathcal{K}$, and the correspondence $E \mapsto \mathcal{X}(E)$ gives equivalence of the category of algebraic extensions of $\widetilde{K}$ and the category of separable extensions of $\mathcal{K}$. In particular, $\mathcal{X}$ gives the identification of $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}_{\mathrm{sep}}/\mathcal{K})$ with $\Gamma_{\widetilde{K}} \subset \Gamma_K$;

3) the above identification $\Gamma_{\widetilde{K}} = \Gamma_{\mathcal{K}}$ is compatible with the ramification filtrations in Γ_K and $\Gamma_{\mathcal{K}}$; this means that if $\varphi_{\widetilde{K}/K} = \lim_{n\to\infty} \varphi_{K_n/K}$ then for any $x \geqslant 0$, $\Gamma_{\mathcal{K}}^{(x)} = \Gamma_K^{(y)} \cap \Gamma_{\widetilde{K}}$ with $y = \varphi_{\widetilde{K}/K}(x)$.

3.2. **Three questions.** Let

$$K_{<p}(M) = \bar{K}^{\Gamma_K^{p^M} C_p(\Gamma_K)} \quad , \quad \mathcal{K}_{<p}(M) = \mathcal{K}_{\mathrm{sep}}^{\Gamma_{\mathcal{K}}^{p^M} C_p(\Gamma_{\mathcal{K}})}.$$

Then $\mathcal{K}_{<p}(M) \supset \mathcal{X}(K_{<p}(M)\widetilde{K}) \supset \mathcal{K}$, and there is a subgroup $\mathcal{H}_M$ of $\Gamma_{\mathcal{K}}(M) := \mathrm{Gal}(\mathcal{K}_{<p}(M)/\mathcal{K})$, such that

$$\mathrm{Gal}(\mathcal{X}(K_{<p}(M)\widetilde{K})/\mathcal{K}) = \Gamma_{\mathcal{K}}(M)/\mathcal{H}_M.$$

Under the identification η_0 from Subsection 1.3 we have $\mathcal{H}_M \simeq G(\mathcal{J}_M)$, where $\mathcal{J}_M$ is an ideal of the Lie algebra $\mathcal{L}_M$.

Question A. *What is the ideal* $\mathcal{J}_M$*?*

Remind that the Lie algebra $\mathcal{L}_M \otimes W(k)$ has a system of generators $\{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\} \cup \{D_0\}$. So, more precisely,

What are explicit generators of the ideal $\mathcal{J}_M$*?*

It is easy to see that $K_{<p}(M) \cap \widetilde{K} := K_M = K(\pi_M)$. Therefore, for $\Gamma_K(M) := \mathrm{Gal}(K_{<p}(M)/K)$, we have the following exact sequence of p-groups

$$1 \longrightarrow \Gamma_{\mathcal{K}}(M)/\mathcal{H}_M \longrightarrow \Gamma_K(M) \longrightarrow \langle\tau_0\rangle^{\mathbb{Z}/p^M} \longrightarrow 1,$$

where $\tau_0 \in \mathrm{Gal}(K_M/K)$ is defined by the relation $\tau_0(\pi_M) = \zeta_M\pi_M$. Using the equivalence of the category of Lie $\mathbb{Z}/p^M$-algebras of nilpotent class $< p$ and of the category of p-groups of the same nilpotent class, cf. Subsection 1.2, we can rewrite the above exact sequence of p-groups as the following exact sequence of Lie $\mathbb{Z}/p^M$-algebras

$$0 \longrightarrow \mathcal{L}_M/\mathcal{J}_M \longrightarrow L_M \longrightarrow (\mathbb{Z}/p^M)\tau_0 \longrightarrow 0$$

Here L_M is the Lie $\mathbb{Z}/p^M$-algebra such that $G(L_M)=\Gamma_K(M)$. This sequence of Lie algebras splits in the category of $\mathbb{Z}/p^M$-modules and, therefore, can be given by a class of differentiations $\mathrm{ad}\hat\tau_0$ of $\mathcal{L}_M$, where $\hat\tau_0$ is a lift of τ_0 to an automorphism of L_M.

Question B. *What are the differentiations* $\mathrm{ad}\hat\tau_0$?

More precisely,

Find the elements $\mathrm{ad}(\hat\tau_0)(D_{a0})$, $a\in\mathbb{Z}^0(p)$.

As we have mentioned in Subsection 3.1 the ramification filtrations in $\Gamma_{\mathcal{K}}$ and Γ_K are compatible.

The ramification filtration of $\mathrm{Gal}(K_M/K)=\mathbb{Z}/p^M\tau_0$ has a very simple structure. Let e_K be the absolute ramification index of K and for $s\in\mathbb{Z}$, $s\geqslant 0$, $v_s=e_Kp/(p-1)+se_K$. Then

if $0\leqslant v\leqslant v_0$, then $(\mathbb{Z}/p^M\tau_0)^{(v)}$ is generated by τ_0;

if $v_s<v\leqslant v_{s+1}, 0\le s<M$, then $(\mathbb{Z}/p^M\tau_0)^{(v)}$ is generated by $p^s\tau_0$;

if $v>v_M$, then $(\mathbb{Z}/p^M\tau_0)^{(v)}=0$.

Therefore, we shall obtain a description of the ramification filtration $\Gamma_K(M)^{(v)}$ of $\Gamma_K(M)$ by answering the following question.

Question C. *How to construct "good" lifts* $\widehat{p^s\tau_0}\in L_M^{(v_s)}$, $0\le s\le M$?

Below we announce partial results related to above questions.

3.3. **The ideal $\mathcal{J}_M$.** Consider the decreasing central filtration by the commutator subgroups $\{C_s(\Gamma_K(M))\}_{s\ge 2}$. This filtration corresponds to the following decreasing central filtration of ideals of $\mathcal{L}_M$

$$J_1:=\mathcal{L}_M\supset J_2\supset\dots\supset J_p=\mathcal{J}_M$$

We can treat $\mathcal{L}_M$ as a free pro-finite object in the category of Lie $\mathbb{Z}/p^M$-algebras of nilpotent class $<p$. Its module of generators is $\mathcal{K}^*/\mathcal{K}^{*p^M}$. In this Subsection we announce an explicit description of the filtration $\{J_s\}_{2\leqslant s\leqslant p}$. Particularly, in the case $s=p$ we obtain an answer to above question A.

Let $\mathcal{U}$ be the submodule of $\mathcal{K}^*/\mathcal{K}^{*p^M}$ generated by the images of principal units. $\mathcal{U}$ can be identified with a submodule in the power series ring $W_M(k_0)[[t]]$ via the correspondences

$$E(w,t_0^a)\,\mathrm{mod}\,\mathcal{K}^{*p^M}\mapsto wt^a\mathrm{mod}p^MW(k)[[t]],$$

where $w\in W(k)$, $a\in\mathbb{Z}^+(p)=\{a\in\mathbb{N}\mid(a,p)=1\}$ and $E(w,X)$ is the Shafarevich function from Subsection 2.1.

Let $H_0\in W(k)[[t]]$ be a power series such that $\zeta_M=H_0(\pi_0)$. Let $\widetilde{H}_0=H_0\,\mathrm{mod}\,p^M\in W_M(k)[[t]]$. Then there is a unique $S\in W_M(k)[[t]]$, such that $\widetilde{H}_0^{p^M}=E(1,S)$. Note that the differential $dS=0$, in particular, $S\in W_M(k_0)[[t^p]]$, and therefore, $S\mathcal{U}\subset\mathcal{U}$ under the above identification of $\mathcal{U}$ with the submodule $\underset{a\in\mathbb{Z}^+(p)}{\oplus} W_M(k)\,t^a$ in $W_M(k)[[\,t\,]]$.

For $s\geq 1$ define a decreasing filtration of $\mathcal{K}^*/\mathcal{K}^{*p^M}$ as follows:

$\left(\mathcal{K}^*/\mathcal{K}^{*p^M}\right)^{(1)}=\mathcal{K}^*/\mathcal{K}^{*p^M}$ and $\left(\mathcal{K}^*/\mathcal{K}^{*p^M}\right)^{(s)}=S^{s-1}\mathcal{U}$, if $s\geq 2$.

This filtration determines a decreasing filtration of ideals $\{\mathcal{L}_M(s)\}_{s\geqslant 1}$ in $\mathcal{L}_M$ which can be characterized as the minimal central filtration of $\mathcal{L}_M$ such that for all s, $(\mathcal{K}^*/\mathcal{K}^{*p^M})^{(s)}\subset\mathcal{L}_M(s)$.

Theorem 3.1. *For $1\leqslant s\leqslant p$, $J_s=\mathcal{L}_M(s)$.*

Remark. a) The element $\widetilde{H}_0^{p^M}-1$ appears as the denominator in the Brückner-Vostokov explicit formula for the Hilbert symbol and can be replaced in that formula by S, cf. [3]; this element S can be considered naturally as an element of Fontaine's crystalline ring of p-adic periods and coincides with the p-adic period of the multiplicative p-divisible group.

b) A first non-abelian case of the above theorem corresponds to $s=2$ and is equivalent to the formula

$$S\mathcal{U}=\mathrm{Ker}(\Gamma_{\mathcal{K}}(M)^{ab}\longrightarrow\Gamma_K(M)^{ab}).$$

By Laubie's theorem [18] the functor $\mathcal{X}$ is compatible with the reciprocity maps of class field theories of $\mathcal{K}$ and K cf. also [12]. Therefore,

$$S\mathcal{U}=\mathrm{Ker}(\mathcal{N}:\mathcal{K}^*/\mathcal{K}^{*p^M}\longrightarrow K^*/K^{*p^M}), \tag{3.1}$$

where $\mathcal{N}$ is induced by the projection $\mathcal{K}^*=\varprojlim K_n^*\longrightarrow K_0^*=K^*$. The right-hand side of (3.1) can be interpreted as the set of all $(w_a)_{a\in\mathbb{Z}^+(p)}\in W(k)^{\mathbb{Z}^+(p)}\,\mathrm{mod}p^M$ such that $\prod_{a\in\mathbb{Z}^+(p)}E(w_a,\pi_0^a)\in K^{*p^M}$. So, formula (3.1) can be deduced from the Brückner-Vostokov explicit reciprocity law.

3.4. **Differentiation** $\mathrm{ad}\hat{\tau}_0\in\mathrm{Diff}(\mathcal{L}_M)$**.** It can be proved that there are only finitely many different ideals $\mathcal{L}_M^{(v)}\,\mathrm{mod}\,\mathcal{J}_M$. Therefore, we can fix sufficiently large natural number N_1 (which depends only on N_0, e_K and M), set $\mathcal{F}_\gamma^0:=\mathcal{F}_{\gamma,-N_1}^0$ and use these elements to describe the ramification filtration $\{\mathcal{L}_M^{(v)}\mathrm{mod}\mathcal{J}_M\}_{v\geq 0}$.

The elements $\mathcal{F}_\gamma^0$, $\gamma>0$, can be given modulo the ideal of third commutators $C_3(\mathcal{L}_{M\,k})$ as follows :

if $\gamma=ap^l\in\mathbb{N}$, where $a\in\mathbb{Z}^+(p)$ and $l\in\mathbb{Z}_{\geqslant 0}$, then

$$\mathcal{F}_\gamma^0=ap^lD_{a\bar{l}}+\sum_{s,n,a_1,a_2}\eta(n)a_1p^s[D_{a_1\bar{s}},D_{a_2\bar{s}-\bar{n}}];$$

if $\gamma\notin\mathbb{N}$, then

$$\mathcal{F}(\gamma)=\sum_{s,n,a_1,a_2}\eta(n)a_1p^s[D_{a_1,\bar{s}},D_{a_2,\bar{s}-\bar{n}}].$$

In the above sums $0 \leqslant s < M$, $0 \leqslant n < N_1$, $a_1, a_2 \in \mathbb{Z}^0(p)$, $p^s(a_1 + a_2p^{-n}) = \gamma$, $\eta(n) = 1$ if $n \neq 0$ and $\eta(0) = 1/2$.

Let $S \in W_M(k)[[t]]$ be the element introduced in Subsection 3.3. Remind that $S = \sigma S'$, where $S' \in W_M(k_0)[[t]]$. For $l \geqslant 1$, let $\alpha_l \in W_M(k)$ be such that

$$S - pS' = \sum_{l\geqslant 1} \alpha_l t^l.$$

Note that

a) if $l < e_K p/(p-1)$ then $\alpha_l = 0$;

b) for any $l \geqslant 1$, we have $l\alpha_l = 0$.

Theorem 3.2. *There is a lift $\hat{\tau}_0 \in L_M$ of τ_0 such that*

$$\mathrm{ad}\hat{\tau}_0(D_0) = \sum_{\substack{l\geqslant 1\\ 0\leqslant n<N_0}} \sigma^n(\alpha_l \mathcal{F}_l^0) \mathrm{mod} C_3(\mathcal{L}_M)$$

and, for $a \in \mathbb{Z}^+(p)$,

$$\mathrm{ad}\hat{\tau}_0(D_{a0}) = \sum_{\substack{f\in\mathbb{Z}\\ l\geqslant 1}} \sigma^{-f}(\alpha_l \mathcal{F}^0_{l+ap^f}) \mathrm{mod} C_3(L_{Mk}).$$

3.5. **Good lifts $\widehat{p^s\tau_0}$, $0 \leq s < M$.** If $s = 0$, let $\hat{\tau}_0 \in L_M$ be a lift from Theorem 3.2. Define the lifts $\widehat{p^s\tau_0}$ by induction on s as follows

$$\widehat{p^s\tau_0} = p(\widehat{p^{s-1}\tau_0}) + \frac{1}{2} \sum_{\substack{l\geqslant 1\\ 0\leqslant n<N_0}} \alpha_l \sum_{\substack{a_1,a_2\in\mathbb{Z}^+(p)\\ a_1+a_2=\frac{p^s e_K}{p-1}}} a_1[D_{a_1n}, D_{a_2n}].$$

Theorem 3.3. *A lift $\hat{\tau}_0$ from Theorem 3.2 can be chosen in such a way that all $\widehat{p^s\tau_0}$, $0 \leq s < M$, are "good" modulo $C_3(L_{Mk})$, i.e. $\widehat{p^s\tau_0} \in L_M^{\varphi(v_s)} \mathrm{mod}\, C_3(L_{Mk})$, where $v_s = e_K p^s/(p-1)$ and φ is the Herbrand function of the extension $K_{<p}(M)/K$.*

Theorem 3.3 gives a complete description of the ramification filtration of $\Gamma_K(M)$ modulo $C_3(\Gamma_K(M))$. This result can be compared with the description of the filtration $\Gamma_K(1)^{(v)} \cap C_2(\Gamma_K(1))$ modulo $C_3(\Gamma_K(1))$ in [27].

3.6. **The modulo p case.** In this Subsection we give an overview of the results from [11] related to the modulo p aspect of problems discussed in this paper.

Let $M = 1$ and $c_0 := e_K p/(p-1) (= v_1)$. We can simply drop off M from all above notation instead of substituting $M = 1$.

The ideals $\mathcal{L}(s)$, $1 \leqslant s \leqslant p$, can be described now quite explicitly as follows. Define a weight filtration on $\mathcal{L}$ by setting for $a \in \mathbb{Z}^0(p)$ and $n \in \mathbb{Z}/N_0$, $\mathrm{wt}(D_{an}) = s \in \mathbb{N}$ if $(s-1)c_0 \leqslant a < sc_0$. Then for all s, $\mathcal{L}(s) = \{l \in \mathcal{L} \mid \mathrm{wt}(l) \geqslant s\}$, cf. Section 3 of [11].

The field-of-norms $\mathcal{K}$ admits a standard embedding into $R_0 = \mathrm{Frac}\, R$, where R is Fontaine's ring [26]. This also identifies R_0 with the completion of $\mathcal{K}_{sep}$ and, therefore, we have a natural embedding of $\mathcal{K}_{<p}$ into R_0. In particular, f_0 can be considered as an element of $\mathcal{L}_{\mathcal{R}_0}$.

Choose a continuous automorphism h_0 of $\mathcal{K}$ such that

$$h_0(t) \equiv \tau_0(t) \,\mathrm{mod}\, t^{c_0(p-1)}\mathrm{m}_R,$$

where m_R is the maximal ideal in R. The formalism of nilpotent Artin-Schreier theory allows us to describe efficiently the lifts $\hat{h}_0 \in \mathrm{Aut}\mathcal{K}_{<p}$ of h_0, [2]. This can be done by specifying the image $\hat{h}_0(f_0)$ in the form

$$\hat{h}_0(f_0) = c(\hat{h}_0) \circ (\mathrm{Ad}\hat{h}_0 \otimes 1) f_0$$

where $c(\hat{h}_0) \in \mathcal{L}_{\mathcal{K}}$ and $\mathrm{Ad}\hat{h}_0$ is the conjugation of $G(\mathcal{L})$ via $\hat{h}_0$.

It can be proved then that the lifts $\hat{\tau}_0 \in \mathrm{Aut} K_{<p}$ satisfy

$$\hat{\tau}_0(f_0) \equiv \hat{h}_0(f_0) \,\mathrm{mod}\, t^{c_0(p-1)}\mathcal{M}_{R_0},$$

where $\mathcal{M}_{R_0} = \sum_{1\leqslant s<p} t^{-c_0 s}\mathcal{L}(s)_{\mathrm{m}_R} + \mathcal{L}(p)_{R_0}$, and are uniquely determined by this conditions. Therefore, the lifts $\hat{\tau}_0$ can be uniquely described via the morphisms $\mathrm{Ad}\hat{\tau}_0$ and the elements $c(\hat{h}_0) \,\mathrm{mod}\, t^{c_0(p-1)}\mathcal{M}_{\mathcal{K}}$, where $\mathcal{M}_{\mathcal{K}} = \sum_{1\leqslant s<p} t^{-c_0 s}\mathcal{L}(s)_{\mathrm{m}_{\mathcal{K}}} + \mathcal{L}(p)_{\mathcal{K}}$.

The above elements $c(\hat{h}_0)$ satisfy complicated relations but the whole situation can be linearized as follows.

In [11] we proved that the action of the group $\langle \hat{h}_0 \rangle^{\mathbb{Z}/p}$ on f_0 comes from the action of the additive group scheme $\mathbb{G}_{a,\mathbb{F}_p} = \mathrm{Spec}\mathbb{F}_p[u]$ on $\mathcal{M}_{\mathcal{K}_{<p}}/t^{c_0(p-1)}\mathcal{M}_{\mathcal{K}_{<p}}$, where $\mathcal{M}_{\mathcal{K}_{<p}}$ is defined similarly to $\mathcal{M}_{R_0}$. This implies that if $c[u] = c_0 + c_1 u + \dots + c_{p-1}u^{p-1} \in \mathcal{L}_{\mathcal{K}}[u]$ is the polynomial with coefficients in $\mathcal{L}_{\mathcal{K}}$ such that for $0 \leqslant k < p$, $c[u]|_{u=k} = c(\hat{h}_0^k)$ then its residue modulo $t^{c_0(p-1)}\mathcal{M}_{\mathcal{K}}$ is well-defined and can be uniquely recovered from its first coefficient $c_1 \,\mathrm{mod}\, t^{c_0(p-1)}\mathcal{M}_{\mathcal{K}}$.

Now we can state the main results from [11].

3.6.1. There is a bijection

$$\hat{\tau}_0 \mapsto (c_1,\, \{\mathrm{ad}\hat{\tau}_0(D_{a0}) \mid a \in \mathbb{Z}^0(p)\})$$

of the set of lifts $\hat{\tau}_0$ and solutions $(c_1, \{V_a \mid a \in \mathbb{Z}^0(p)\})$, where $c_1 \in \mathcal{L}_{\mathcal{K}}$, $V_0 \in \alpha_0\mathcal{L}$, $V_a \in \mathcal{L}_k$ with $a \in \mathbb{Z}^+(p)$, of the following equation

$$\sigma c_1 - c_1 + \sum_{a\in\mathbb{Z}^0(p)} t^{-a}V_a =$$

$$\sum_{k,l\geqslant 1} \frac{1}{k!}\, t^{l-(a_1+\dots+a_k)}\alpha_l[\dots[a_1 D_{a_1 0}, D_{a_2 0}], \dots, D_{a_k 0}]$$

$$-\sum_{k\geqslant 2} \frac{1}{k!} t^{-(a_1+\dots+a_k)}[\dots[V_{a_1}, D_{a_2 0}], \dots, D_{a_k 0}]$$

$$-\sum_{k\geqslant 1}\frac{1}{k!}\,t^{-(a_2+\dots+a_k)}[\dots[\sigma c_1, D_{a_20}],\dots,D_{a_k0}]$$

(Remind that $\alpha_l\in k$ were defined in Subsection 3.4 and they equal 0 if $l\neq c_0, c_0+p, c_0+2p,\dots$.)

3.6.2. A lift $\hat\tau_0$ is "good", i.e. $\hat\tau_0\in L^{(c_0)}$ iff all $V_a\in\mathcal{L}_k^{(c_0)}$ and

$$c_1\equiv-\sum_{\substack{\gamma>0\\ l\geqslant 1}}\sum_{0\leqslant i<N^*}\sigma^i(\alpha_l\mathcal{F}^0_{\gamma,-i}t^{-\gamma+l})\bmod(\mathcal{L}_{\mathcal{K}}^{(c_0)}+t^{c_0(p-1)}\mathcal{M}_{\mathcal{K}})$$

3.6.3. Let the operators F_0 and G_0 on $\mathcal{L}_k$ be such that for any $l\in\mathcal{L}_k$,

$$F_0(l)=\sum_{k\geqslant 1}\frac{1}{k!}[\dots[l,\underbrace{D_{00}],\dots,D_{00}}_{k-1\text{ times}}],\ \ G_0(l)=\sum_{k\geqslant 0}\frac{1}{k!}[\dots[l,\underbrace{D_{00}],\dots,D_{00}}_{k\text{ times}}]$$

Then the correspondence

$$(c_1^0,V_0)\mapsto(c_1^0-\sum_{\substack{0\leqslant i<N^*\\ l\geqslant 1}}\sigma^i(\alpha_l\mathcal{F}^0_{l,-i}),V_0)$$

establishes a bijection between the set of all good lifts $\hat\tau_0$ and the set of all (x,y) such that $x\in\mathcal{L}_k^{(c_0)}$, $y\in\alpha_0\mathcal{L}^{(c_0)}$, and

$$(G_0\sigma-\mathrm{id})(x)+F_0(y)=\sum_{l\geqslant 1}\sigma^{N^*}\left(\alpha_l\mathcal{F}^0_{l,-N^*}\right)\tag{3.2}$$

3.6.4. For any above solution (x,y) of (3.2) we have

$$y\equiv\alpha_0\sum_{l\geqslant 1}\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_l\mathcal{F}^0_{l,-N^*})\mathrm{mod}\alpha_0[D_0,\mathcal{L}^{(c_0)}]$$

This implies the existence of a good lift $\hat\tau_0$ such that the (only) relation in the Lie algebra L_k (recall that $G(L)=\mathrm{Gal}(K_{<p}/K)$) appears in the form

$$\mathrm{ad}\hat\tau_0(D_0)=\sum_{l\geqslant 1}\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_l\mathcal{F}^0_{l,-N^*})\,.$$

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