Overcoming volumetric locking in three-dimensional material point analysis

William M. Coombs[1,*], Lei Wang[2] and Charles E. Augarde[1]

[1] Department of Engineering, Durham University
Lower Mountjoy, South Road, Durham, DH1 3LE (UK)
w.m.coombs@durham.ac.uk

[2] Department of Computer Science, University of Oxford
Wolfson Building, Parks Road, Oxford, OX1 3QD (UK)

Abstract The material point method is ideally suited to modelling large deformation problems in three dimensions, especially in cases where the finite element method struggles due to mesh distortion. However, when the method is used to analyse problems with near-incompressible material behaviour, such as in geotechnical engineering using models with isochoric plastic flow, it suffers from severe volumetric locking. This causes the method to over predict the forces for a given displacement and induces spurious stress oscillations through the problem domain. Several methods have been proposed in the finite element literature but few of these have been applied to the material point method. In this paper we present a way to avoid volumetric locking for three-dimensional material point analyses with simplex elements (linear tetrahedra) using an *F* bar patch approach. Not only does the technique avoid the over-stiff behaviour associated with volumetric locking but it also reduces the stress oscillations in the method, which are often attributed to cell-crossing instabilities. The formulation is validated against two three-dimensional benchmark problems.

Keywords: *material point method, volumetric locking, simplex elements, F bar method*

1 Introduction

Volumetric locking occurs in numerical analysis when the number of volumetric constraints, often a consequence of the constitutive behavior of the material being modelled, excessively constrains the movement of an element's nodes. This is often an issue for Material Point Methods (MPMs) due to the relatively high material point/node ratio (to reduce integration errors) and the types of materials modelled using the method (isochoric plasticity is common). Despite this, the issue of volumetric locking in MPMs has received relatively little attention to date. Notable exceptions are the papers of Mast *et al.* (2012) and Coombs *et al.* (2018), however these works introduce additional complexity into the underlying element formulation and can only be applied to quadrilateral/hexahedral elements, respectively. The $\bar{F}$ approach of Coombs et al. (2018)[1] makes different assumptions about the deformation gradient variation across the element; the deviatoric component varies in the conventional manner whereas the volumetric component of the deformation gradient is constant within each element. This reduces the volumetric constraint on the element and removes the over-stiff behavior associated with volumetric locking. Although many MPMs are based on structured quadrilateral/hexahedral background grids, several real geotechnical problems require the use of unstructured tetrahedra, especially in the area of soil-structure interaction (such as the installation of piles and seabed ploughing for offshore renewable energy cable installation). However, it is not possible to apply the $\bar{F}$ approach to simplex elements due the fact that they can only represent a constant strain field within each element. This paper briefly outlines a material point formulation that eliminates volumetric locking for simplex element-based background grids[2] using the $\bar{F}$ patch approach of de Souza Neto *et al*. (2005) and demonstrates the performance of the method.

2 Material Point Continuum Formulation

This paper is restricted to quasi-static analysis and adopts an updated Lagrangian continuum formulation, solved using an implicit material point approach based on Charlton *et al*. (2017); see Coombs *et al*. (2020) for details of the numerical implementation and associated AMPLE code. Within this formulation, the updated Lagrangian weak statement of equilibrium can be expressed as

$$\int_{\varphi_t(\Omega)} \big(\sigma_{ij}(\nabla_x \eta)_{ij} - b_i\eta_i\big)dv - \int_{\varphi_t(\partial\Omega)} (t_i\eta_i)ds = 0, \tag{1}$$

where φ_t is the motion of the material body with domain, Ω, which is subjected to tractions, t_i, on its boundary, $\partial\Omega$, and body forces, b_i, acting over its volume, v, which generate a Cauchy stress, σ_{ij}, through the body. η_i are a set of admissible virtual displacements. The corresponding Galerkin weak statement of equilibrium over each background element, E, is

$$\int_{\varphi_t(E)} \big[\nabla_x S_{vp}\big]^T\{\sigma\}\,dv - \int_{\varphi_t(E)} \big[S_{vp}\big]^T\{b\}\,dv - \int_{\varphi_t(\partial E)} \big[S_{vp}\big]^T\{t\}\,ds = 0 \tag{2}$$

where $[S_{vp}]$ are the basis functions linking the nodes (or vertices, v) and the material points, p, and $\{\sigma\}$, $\{t\}$ and $\{b\}$ are the vector forms of the Cauchy stress, tractions and

[1] The $\bar{F}$ material point approach of Coombs *et al*. (2018) is essentially the $\bar{F}$ method of de Souza Neto *et al*. (1996) applied to the MPM with minor modifications to account for the differences between the methods.

[2] That is, background grids comprised of three-noded linear triangles or four-noded linear tetrahedra.

body forces, respectively. Note that the spatial gradients of the basis functions, $[\nabla S_{vp}]$, are with respect to the deformed coordinates[3], x_i.

The key idea behind the $\bar{F}$ approach is to modify the volumetric component of the deformation gradient to reduce the volumetric constraint on the element. For linear quadrilateral/hexahedral elements this is straightforward as the conventional deformation gradient varies across the element and therefore the volumetric component can be taken as a constant over each element. However, for linear triangles/tetrahedra the deformation gradient is already constant over each element so this approach is not possible. The $\bar{F}$ patch method overcomes this issue by defining a number of patches, or collections of elements, and assuming that the volumetric component of the deformation gradient is constant over each patch. The $\bar{F}$ patch incremental deformation gradient (see de Souza Neto *et al*. (2005)) is given by

$$\Delta\bar{F}_{ij} = \left(\frac{v_{patch}}{\tilde{V}_{patch}\left(\det\left(\Delta F_{ij}\right)\right)}\right)^{\frac{1}{n_D}} \Delta F_{ij}\,, \tag{4}$$

where n_D is the number of dimensions, v_{patch} and $\tilde{V}_{patch}$ are the volumes of the patch in the deformed configuration and at the start of the load step, respectively, and ΔF_{ij} is the conventional deformation gradient increment of the material point determined via the spatial gradient of the basis functions and the incremental nodal displacements. Changing the definition of the deformation gradient via Eq. (4) introduces an additional component to the stiffness of the material point's *parent* element and also introduces coupling terms between the elements within a patch. This increases the bandwidth of the global stiffness matrix; in this paper each tetrahedral element will influence seven other elements (eight elements per patch).

3 Numerical Implementation

The $\bar{F}$ patch method requires the definition of patches of elements but the definition of these patches is arbitrary. In this paper the patches are formed by first constructing a background mesh comprised of 10-noded quadratic tetrahedra (or 6-noded triangles for two-dimensional analysis) and then splitting each these elements into eight 4-noded linear tetrahedra. This means that the bandwidth of the mesh is the same as that of the original quadratic mesh and the process of converting to linear elements does not introduce any additional nodes. During the analysis, if the material points have moved such that an element within a patch no longer contains material points, it is removed from the patch for the load step under consideration. Note that if the element is repopulated by a material point during a later stage of the analysis, it would be reintroduced into its original patch. The use of an unstructured background mesh increases the complexity (and

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[3] For many MPMs this requires the derivatives of the basis functions to be mapped from the configuration at the start of the load step to the current configuration using the procedure explained in Charlton *et al*. (2017).

potentially cost) of determining the element location of each material point. In this paper we adopt a modified version of the Walking-in-Triangulation (WiT) procedure of Devillers *et al*. (2002) which, for an example problem containing 31,000 material points and elements, reduces the searching time to 0.054% of a naïve searching method based on local element positions (see Wang et al. (2020) for details).

4 Numerical Examples

The first numerical example presented in this paper is the analysis of a 1m rigid square *smooth* footing bearing onto a three-dimensional domain. The weightless soil had a Young's modulus of 10MPa, a Poisson's ratio of 0.48 and a perfect plasticity von Mises yield strength of 0.85kPa. Due to symmetry only a quarter of the footing was modelled and the resulting 5 by 5m domain, as shown to the right of Fig. 1, was discretized by a background grid of 4,176 elements with 11 material pointe per element. Roller boundary conditions were imposed on the four vertical sides and the base of the domain and a vertical displacement of 2mm was imposed on the footing area over 20 equal displacement-controlled increments. A moving mesh strategy (Beuth, 2012) was adopted so that the background mesh remained coincident with the imposed boundary conditions.

The reaction force versus displacement response of the standard MPM (sMPM) and the $\bar{F}$ patch MPM are shown in Fig. 1, where it is clear that the sMPM suffers from volumetric locking and does not reach a limit load. The $\bar{F}$ patch MPM prediction is consistent with the perfect plasticity material assumption and reaches a limit load after approximately 1mm of vertical displacement.

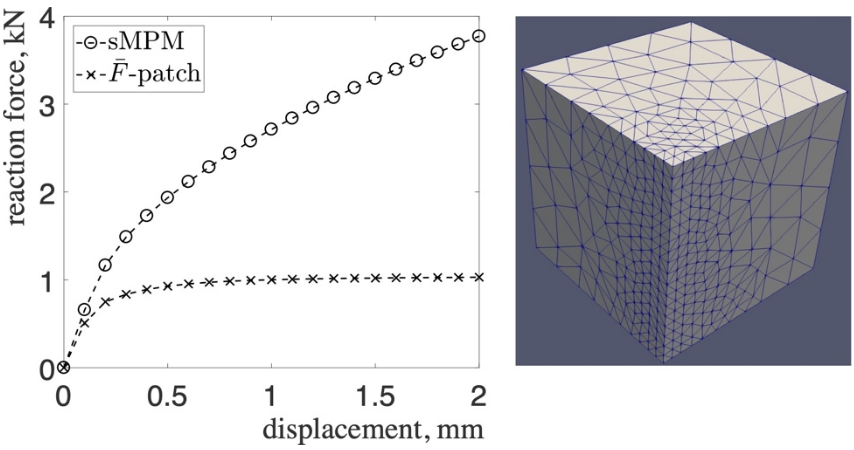

Fig. 1 Footing analysis: reaction force-displacement response and background mesh

Fig. 2 shows the magnitude of the Cauchy stress for (a) the sMPM and (b) the $\bar{F}$ patch MPM. The sMPM shows the spurious checkerboarding artifact of volumetric locking whereas the proposed $\bar{F}$ patch MPM produces a physically more realistic response showing stress concentration under the imposed displacement. The checkerboarding of the sMPM is caused by the inability of the method to deform in a natural way whilst imposing the volumetric constraints, caused by the isochoric plastic flow at each yielded material point, on each element. The $\bar{F}$ patch MPM replaces these constraints by a single volumetric constraint on each patch of eight elements.

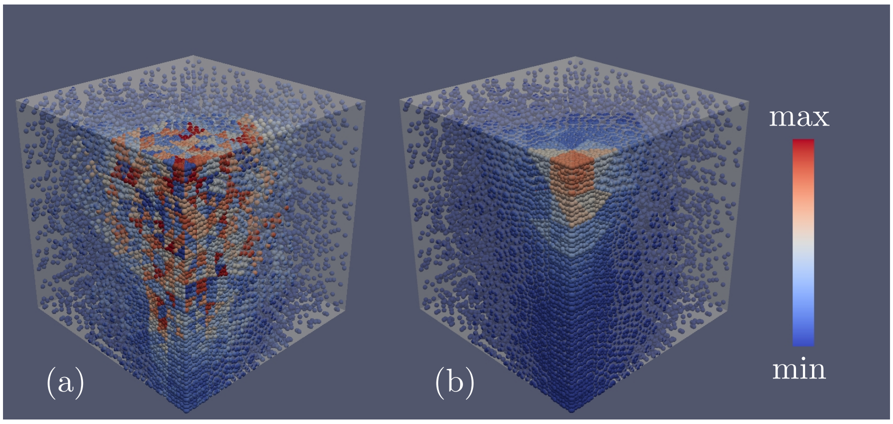

Fig. 2 Cauchy stress magnitude, $\|\sigma_{ij}\|_2$, for the (a) sMPM and (b) $\bar{\mathrm{F}}$ patch MPM.

The second numerical example of this paper is a much simpler problem – stretching of a cubic domain in one direction. The 2m domain had a Young's modulus of 10kPa, a Poisson's ratio of 0 and a von Mises perfect plasticity yield strength of 400Pa. The domain was modelled using five patches of eight tetrahedra (40 elements in total) and each element was discretized by 11 material points. The domain, as shown in Fig. 3(a), was stretched by an incremental displacement of 0.4m per load step on one face with a roller constraint placed on the opposite face. Two of the remaining four sides, one each at the x and z limits of the domain, where also restrained with a roller boundary condition. The moving mesh concept was used so that the background mesh remained coincident with the imposed boundary conditions throughout the analysis.

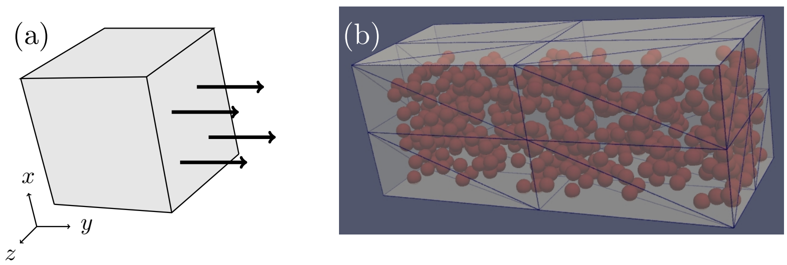

Fig. 3 Simple stretch: (a) problem definition and (b) deformed configuration with 6.8m displacement.

Fig. 4 shows the normal component of the Kichhoff stress in the direction of the stretching for all of the material points in the analysis over 11 load steps for both the standard (sMPM) and the $\overline{F}$ patch MPMs along with the analytical solution. It is clear that the $\overline{F}$ patch MPM significantly reduces the stress oscillation caused by material points moving between elements due to the coupling terms introduced via the patches. That is, in a similar way to domain-based MPMs, a material point has an influence on an element, and *vice-versa*, before it moves into the element if it is in the same patch.

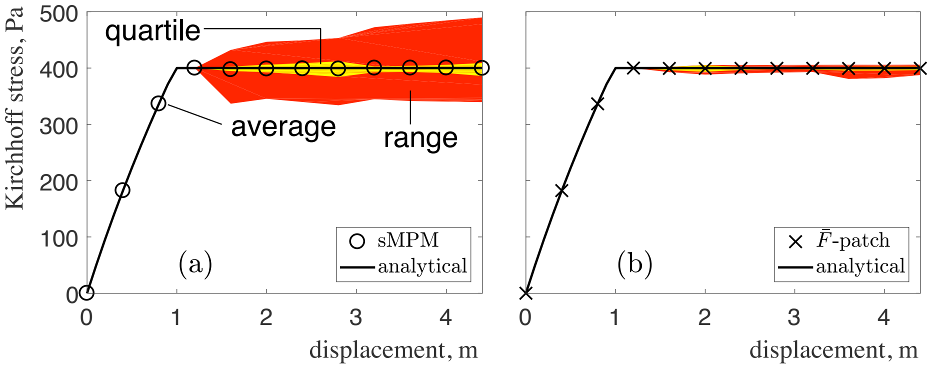

Fig. 4 Kirchhoff stress response for the (a) standard MPM and (b) $\overline{\mathrm{F}}$ patch MPM.

8 Conclusions

This paper has presented a $\overline{F}$ patch approach to avoid volumetric locking in simplex elements. Although the focused has been on three-dimensional analysis with an implicit material point formulation, the method is applicable to both implicit and explicit analysis for two and three-dimensional problems. The proposed method has two clear benefits: (i) it removes the over-stiff global response and (ii) significantly reduces the stress oscillations between elements of the standard MPM when modelling near-incompressible materials. Unfortunately, the bandwidth is increased compared to the standard MPM (in a similar way to changing from linear to quadratic elements), via coupling terms within each patch of elements, and the method requires a non-unique patch definition. However, it is the authors' opinion that the benefits offered by the method vastly outweigh this additional complexity.

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