Logic and Semantic Theory in the High Middle Ages

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1 Introduction

Many people unfamiliar with the history of logic may think of the Middle Ages as a "Dark Ages" in logic, with little development beyond Aristotelian syllogistic and full of scholastic wrangling focused on uninteresting details. This could not be further from the case. Instead, the Middle Ages, especially at the end of the High Middle Ages and into the 14th century, was a period of vibrant activity in logic, in many different areas—the (re)birth of propositional logic, the development of interactive and dynamic reasoning, sophisticated semantic theories able to address robust paradoxes, and more. The period can be characterized by a focus on the applied aspects of logic, such as how it relates to linguistic problems, and how it is used in inter-personal contexts.

To attempt to survey the full chronological and biographical story of medieval logic from the end of antiquity until the birth of the Renaissance in the space of a chapter would be an impossible endeavor. Instead, this chapter focuses on four topics which are in certain respects uniquely and particularly medieval: (1) the analysis of properties of terms, specifically signification and supposition; (2) theories of consequences; (3) the study of *sophismata* and *insolubilia*; and (4) the mysterious *disputationes de obligationibus*. We treat each of these in turn, giving short thematic overviews of their rise and development. First, we provide a short historical comment on the introduction of the Aristotelian corpus into the medieval West and the effects that this had on the development of logic in the Middle Ages.

2 The reception of Aristotle

In the second decade of the 6th century, Anicius Manlius Severinus Boethius decided to embark upon the project of translating from Greek into Latin all of the Aristotelian and Platonic works that he could obtain, beginning with Aristotle's logical corpus. In 523, he was sentenced to prison for treason and by his death, around 526, he had completed an admirable percentage of his task, translating Aristotle's *Categories*, *On Interpretation*, the *Prior Analytics*, the *Topics*, the *Sophistical Refutations*, and most likely the *Posterior Analytics*, as well as Porphyry's *Introduction* (to the *Categories*). He also wrote commentaries on the

Introduction, the *Categories*, and *On Interpretation*, and his own textbooks on logic, including the unique little treatise on hypothetical syllogisms (syllogisms with molecular, i.e., non-categorical, premises). These translations, the first of Aristotle into Latin, were to shape the path of medieval western philosophy in a tremendous fashion.

After Boethius's death, many of his translations of Aristotle were lost. Only the translations of *On Interpretation* and the *Categories* remained. These two books, along with Porphyry's *Introduction* formed the basic logical and grammatical corpus for the next six centuries. It wasn't until the 1120s that his other translations were rediscovered.[1] In the 1150s, James of Venice translated the *Posterior Analytics* (if Boethius had translated the *Posterior Analytics*, this translation has never been found), and also re-translated the *Sophistical Refutations*. The newly completed Aristotelian logical corpus was available in western Europe by the latter half of the 12th century, though it wasn't until the birth of the universities in the early 13th century that they began to circulate widely (the exception being the *Sophistical Refutations* which was seized upon almost immediately). These works established their place as canonical texts in logic and natural philosophy, and their effect on the development of these fields was quickly seen. The material in these new texts provided medieval logicians not only with a stronger logical basis to work from, but also proved to be a "jumping off" point for many novel—and in some cases radically *non*-Aristotelian—developments. Aristotle long remained the authority on logic, but though most medieval logicians give the required nod to the Philosopher, they certainly felt free to explore and develop their own ideas.

3 Properties of terms

In the Middle Ages, logic was a linguistic science; it arose from a desire to understand how language is used (properly) in order to assist in textual exegesis. As a result, from an early period the study of logic was closely connected to the study of grammar (indeed, these two studies, along with rhetoric, made up the *trivium*, the branch of learning that was the core of the undergraduate's curriculum in the early universities). William of Sherwood (1190–1249) in his *Introduction to Logic* (c.1240) explains that "Logic is principally concerned with the syllogism, the understanding of which requires an understanding of the proposition; and, because every proposition is made up of terms, an understanding of the term is necessary" [26, p. 21]. Hence if we wish to become proficient in logic we must first become masters of terms, and more specifically, their properties, the two most important of which being signification and supposition. The period from 1175 to 1250 marked the height of what is known as *terminist logic*, because of its emphasis on terms and their properties. In this section, we discuss the ideas of signification and supposition found in late terminist writers such as William of Sherwood, Peter of Spain (mid 13th C), and Lambert of Lagny (mid 13th C), which reflect the most mature and settled views on the issues.

[1] Dod notes that "how and where these translations... were found is not known" [9, p. 46], nor indeed how and where they were lost in the first place.

3.1 Signification

According to Lambert of Lagny in his treatise on the *Properties of Terms*, the signification of a word is the "concept on which an utterance is imposed by the will of the person instituting the term" [23, p. 104]. As such, signification is one of the constitutive parts of a word's meaning. Four things are required for signification:

- A thing
- A concept of the thing
- An utterance (which may be mental, spoken, or written)
- A union of the utterance and the concept (effected by the will's imposition)

A thing is any extra-mental thing, whether it be an Aristotelian substance (e.g., Socrates the man), an Aristotelian accident (e.g., the whiteness which is in Socrates), or an activity (e.g., the running which Socrates is doing right now). On the medieval view, these extra-mental things are presented to the soul by means of a concept. A term gains signification when it is used in an utterance and the utterance is connected to a concept by the will of the speaker (or thinker or writer). The concept imposed upon the term in the speaker's utterance is the signification of the term. The signification of a sentence is then built out of the signification of its terms, in a compositional fashion.

Signification served as the basis for one of the primary divisions of terms recognized by medieval grammarians and logicians, into categorematic and syncategorematic (or, significative and consignificative). A categorematic term is one which signifies or has signification on its own, apart from any other word. Nouns, proper names, and verbs are, for the most part, categorematic terms. (Different grammatical forms of the being verb *est* 'is' are an exception). On the other hand, a syncategorematic term is one which does not signify on its own, but only in conjunction with another word or in the context of a sentence. Now, clearly, these "definitions" are not sufficient to uniquely identify all word as either categorematic or syncategorematic. As a result, most medieval authors illustrated the definitions of categorematic and syncategorematic terms ostensively, and different authors highlighted different syncategorematic terms. Logical connectives ('and', 'or', 'not', 'if') and quantifiers ('all', 'some', 'none') were all recognized as syncategorematic terms, but the list of syncategorematic terms is longer than the list of what are modernly recognized as the so-called "logical constants", and many of them are perhaps surprising. For example, William of Sherwood in his *Syncategoremata* (mid 13th C) [27] discusses the following syncategorematic terms: *omnis* 'every', *totum* 'whole', *decem* 'ten', *infinita* 'infinite', *qualislibet* 'of whatever sort', *uterque* 'each', *nullus* 'none', *nihil* 'nothing', *neutrum* 'neither', *praeter* 'except', *solus* 'single', *tantum* 'only', *est* 'is', *non* 'not', *necessario* 'necessarily', *contingenter* 'contingently', *incipit* 'begins', *desinit* 'cease', *si* 'if', *nisi* 'if not', *quin* 'without', *et* 'and', and *vel* 'or'. Some of these words, such as *infinita*, can be used both categorematically and syncategorematically, and this fact gave rise to certain logical puzzles, or *sophismata*.

If a term has only one signification imposed upon it, then it is univocal; if it has more than one signification imposed on it, then it it is equivocal, such as 'bat'

(the athletic equipment) and 'bat' (the flying mammal), in English. Equivocal terms must be distinguished from each other, in order to avoid the fallacy of *quaternio terminorum* 'four terms', a type of equivocation found in syllogisms. But univocal terms must also be carefully attended to, for even though they will always have the same signification no matter the context, these terms can still be used to stand for or refer to different things, depending on the context of the sentence in which they appear. This notion of a term "standing for", or referring to, different objects is called *suppositio* 'supposition'. While a (categorematic) term can signify in isolation from other terms, it will supposit for things only in the context of a complete sentence, in the same way that syncategorematic terms only have signification in the context of a complete sentence.

3.2 Supposition

Signification gives only an indirect way of referring to things, via the concepts which are signified. In order to refer directly to things, and not just the concepts they fall under, the mechanism of supposition is used. Lambert of Lagny defines the supposition of a term as the "acceptance of a term for itself, for its signified thing, for some suppositum contained under its signified thing, or for more than one suppositum contained under its signified thing" [23, p. 106]. For example, the univocal term 'man' signifies the concept of man, but it can supposit for the word 'man', the concept man, or individual or multiple men, depending on the grammatical context in which it appears.

Every author writing on supposition divided supposition into a different number of types, and distinguished them in different ways, but three main divisions were recognized, between simple or natural supposition, material supposition, and formal or personal supposition. A word that supposits simply stands for what it signifies; for example, in "Man is a species", 'man' supposits simply. A word that supposits materially stands for the word itself; for example, in "Man is a three-letter word", 'man' supposits materially. Finally, a word that supposits formally or personally stands for actual things which fall under the concept signified; for example, in "A man is running", 'man' supposits personally. Personal supposition is the most important type of supposition, since it allows us to talk about individuals.

Both simple and material supposition are not further subdivided. Personal supposition, the most common type, has many different subtypes. Of these, the ones that are most interesting are the types of supposition had by terms which, either by their nature or due to some added word such as a universal quantifier, can apply equally well to more than one thing. These two types of supposition are called strong mobile supposition and weak immobile supposition.[2] A common term has strong mobile supposition when it is preceded by a universal affirmative or universal negative quantifier (e.g., *omnis* 'every' or *nullus* 'no, none'). It has weak immobile supposition "when it is interpreted necessarily for more than one suppositum contained under it but not for all, and a descent cannot be made under it" [23, p. 112]. For example, from the sentence "All men are running", it is possible to descend to the singular sentences "Socrates is running", "Aristotle is running", "Plato is running", and

[2]The mobility referred to in their names is a reference to the possibility (or lack thereof) of "descending to" or "ascending from" singulars, see examples below.

so on for all (currently existing) men. Similarly, from the conjunction of the singulars "Socrates is running", "Aristotle is running", "Plato is running", etc., it is possible to ascend to the universal "All men are running". Such ascent and descent is not possible when the universal claim is prefixed by a negation, e.g., "Not all men are running". Likewise, when the subject term has been modified by an exceptive or an exclusive, it is not possible to descend. For example, it is not possible to descend from "Every man except Socrates is running" to a singular because, e.g., "Aristotle except Socrates is running" is ungrammatical. Likewise, from "Only men are running", we cannot conclude of any individual man that *he* is running, since the addition of the exclusive term 'only' changes the supposition of 'man'.

The basic mechanism of supposition explains how terms supposit in simple present-tensed assertoric sentences. When the tense of a sentence is past or future, or the sentence has been modified by a modal operator such as "necessarily", "is known", "is thought of", medieval logicians appealed to the notions of *ampliation* and *restriction*. Lambert defines restriction as "a lessening of the scope of a common term as a consequence of which the common term is interpreted for fewer supposita than its actual supposition requires" [23, p. 134]. One way that a common term can be restricted is through the addition of an adjective. For example, we can restrict the supposition of "man" by adding to it the adjective "white"; "white man" supposits for fewer things than "man" unmodified.

If the modifying word or phrase does not restrict the term's supposition but rather expands it, then we are dealing with the opposite of restriction, which is called ampliation. Ampliation is an "extension of the scope of a common term as a consequence of which the common term can be interpreted for more supposita than its actual supposition requires" [23, p. 137]. As an example, Lambert offers "A man is able to be Antichrist" [23, p. 137]; in this sentence, 'man' is ampliated to stand not only for existing men but for future men. Ampliation can be caused by the addition of predicates (such as 'possible'), verbs ('can'), adverbs ('potentially'), or participles ('praised'). Lambert divides ampliation into two types, ampliation "by means of supposita" and ampliation "by means of times". The former is caused by "verbs whose corresponding action is related to the subject and said of the subject but is not in the subject—such as 'can', 'is thought', 'is praised'" [23, p. 138]. A term ampliated in this way stands for both existent and nonexistent things. Ampliation by reason of times is caused by modifiers which "cause a term to be extended to all the differences of time" [23, p. 138]. Examples of this kind of modifier are temporal operators such as 'always', modal operators such as 'necessarily' and 'possibly', and changes in the tense of the verb.

Theories of supposition are closely tied to considerations of temporal and alethic modalities (such as 'necessary' and 'possible') [45]; we cannot go further into these issues here due to reasons of space.

4 Theories of consequences

The central notion in logic, both then and now, is the concept of "following from" or, more technically "logical consequence". The question of which propositions follow from other propositions, at heart, what all of logic is about. There is

a distinct lack of any general theory of logical consequence in Aristotle. The portion of logical consequence which is studied and discussed in the Aristotelian corpus—the syllogistic—is but a fraction of the field and moving from the study of syllogisms to the study of logical consequence in general is one of the highest accomplishments of the medieval logicians.

What do we mean by '(logical) consequence'? Modern logic distinguishes different logical relationships between two propositions. There is the relationship of implication, which holds between the antecedent and the consequent of a true conditional. There is the relationship of entailment, which holds between two propositions when one cannot be true without the other one also being true. And then there is the relationship of inference or derivation, which is the relationship which arises from the process of moving from one proposition (the premise) to another (the conclusion) in an argument. While there are certainly good reasons for distinguishing these three relationships, these distinctions were not always made in medieval logic. The Latin word *consequentia* literally means 'follows with', and it was used indiscriminately to cover implication, inference, and entailment. One consequence of this is that the primary evaluative notion was not validity but "goodness". Whereas a conditional is either true or false, and not valid or invalid, and an argument is either valid or invalid, and not true or false, both of these can be *bona consequentiae* 'good consequences' or *bene sequiter* 'follow correctly'.

The word *consequentia* was introduced into Latin by Boethius as a literal translation of Greek ἀκολουχῆσις. Some scholars have argued that the roots of theories of logical consequence are to be found not in the syllogistic but instead in the theory of topical inferences [11]. As a result, Boethius's translation of the *Topics* and his commentary on the same were influential. Another important work of his was the short, non-Aristotelian treatise "On Hypothetical Syllogisms". Despite the name, this text does not focus solely on "if... then..." statements; rather, Boethius (and others after him) used "hypothetical" in contrast with "categorical". Any molecular proposition, i.e., one formed out of other propositions by means of negation, conjunction, disjunction, or implication, was considered "hypothetical", and Boethius's treatise on syllogisms or arguments using such propositions was a first step towards modern propositional logic.[3]

The study of logical consequence began in earnest in the 12th century. The concept proved difficult to properly define and classify. In the 12th century, the most sophisticated attempt was produced by Peter Abelard (c.1079–1142), a brilliant logician who produced not one, but two comprehensive theories of logical consequence. The first, in the *Dialectica* (c.1116), focuses on the treatment of topics and of hypothetical syllogisms, following Boethius and Aristotle. The other, later, discussion occurs in a fragment of his commentary on Boethius's commentary on the *Topics*, which is a part of the *Logica Ingredientibus* (before 1120). Of the two discussions, that in the *Dialectica* is more complete and clear. Unfortunately, it also turns out to be inconsistent [19].

Through the course of the next two centuries, logicians continued to wrestle

[3]It is, however, *not* modern propositional logic, since the notion of implication or conditional which he uses is neither the modern material conditional nor strict implication, but is instead based on the idea of subjects and predicates either containing each other or being repugnant to (contradictory with) each other. As a result, the logical theory in "On Hypothetical Syllogisms" validates many theorems which are not acceptable with either material or strict implication.

with the concept of consequence. In the early part of the 14th century treatises devoted solely to the notion of consequence begin to appear, with the earliest being written by Walter Burley (c.1275–1344) and William of Ockham (c.1285–1347). These treatises define different types of consequences (e.g., formal and material, simple and 'as of now', etc.), what it means for a consequence to be good, or for one proposition to follow from another, and list rules of inferences which preserve the goodness of a consequence. These rules of inference mirror modern propositional logic very closely. For example, in Burley's *De puritate artis logicae*, which appeared in both longer and shorter versions, the following rules all appear, where P, Q, and R are all atomic propositions [3, p. 312–313], [6]:[4]

- $P \to Q \vdash \neg\Diamond(P \wedge \neg Q)$
- $P \to Q \vdash (Q \to R) \to (P \to R)$
- $P \to Q \vdash (R \to P) \to (R \to Q)$
- $P \to Q, Q \to R, \ldots, T \to U \vdash P \to U$
- $P \to Q, (P \wedge Q) \to R \vdash P \to R$
- $P \to Q, (Q \wedge R) \to S \vdash (P \wedge R) \to S$
- $P \to Q \vdash \neg Q \to \neg P$
- $\neg(P \wedge Q) \dashv\vdash \neg P \vee \neg Q$
- $\neg(P \vee Q) \dashv\vdash \neg P \wedge \neg Q$

The 14th century saw the gradual codification of two main views on the nature of logical consequence, the English and the continental, with Robert Fland (c.1350), John of Holland (1360s), Richard Lavenham (d.1399), and Ralph Strode (d.1387) as canonical examples on one side of the channel and Jean Buridan (c.1300–a.1358), Albert of Saxony (1316–1390), and Marsilius of Inghen (1340–1396) on the other side.

4.1 The English tradition

The English tradition (which was taken up in Italy at the end of the 14th century and into the 15th [17]) is characterized by an overtly epistemic definition of (formal) consequence in terms of the containment of the consequent in the antecedent. For example, Strode gives the following definition:

> A consequence said to be formally valid is one of which if it is understood to be as is adequately signified through the antecedent then it is understood to be just as is adequately signified through the consequent. For if someone understands you to be a man then he understands you to be an animal [40].

[4]Note that the medieval logicians did not use symbolic notation (with the exception of using letters to stand for arbitrary terms in discussions of syllogistic). Nevertheless, these are accurate and faithful representations of the rules which are found in Burley.

This echoes ideas found in Abelard's views on consequences, which stress a tight connection between the antecedent and consequent, or between the premises and conclusion. It is not sufficient that a consequent be merely "accidentally" truth-preserving; there must be something more that binds the propositions together.[5] Such a definition certainly entails necessary truth-preservation, but it goes beyond it. In stressing the epistemic aspects of consequence, Strode, and others in the English tradition, are emphasizing one of the hallmarks of the medieval approach to logic, namely, the emphasis on the epistemic context of logic and the idea that logic is an applied science which must be evaluated in the context of its applications.

4.2 The continental tradition

In contrast, the defining marks of validity in the continental tradition are modality and signification. On this side of the channel, Jean Buridan's *Tractatus de consequentiis* (1330s) [14] provides a canonical example. The first chapter of this treatise is devoted to the definition of consequence. Buridan begins by presenting a general definition which he then revises on the basis of objections and counterexamples. The first general definition is:

> Many people say that of two propositions, the one which cannot be true while the other is not true is the antecedent, and the one which cannot not be true while the other is true is the consequent, so that every proposition is antecedent to another proposition when it cannot be true without the other being true [14, p. 21].

However innocuous this definition might see, it is problematic for Buridan, and others who follow suit, for whom the relationships of antecedent and consequent are not between propositions in the modern sense of the term—abstract entities which are necessarily existing—but rather between specific tokens of propositions, spoken, written, or mental, which only have truth values when they exist, and do not otherwise. But proposition tokens have specific properties which interfere with this definition. Buridan points out that

Every man is running, therefore some man is running. (1)

is a valid consequence, but it does not satisfy the definition given, because it is possible for the antecedent to be true without the consequent, if someone formed "Every man is running" without forming "Some man is running", in which case it would be possible for the former to be true without the latter.

His second revision is to supplement the definition with the following clause "when they are formed together [14, p. 21]", but even this is not sufficient, for consider the following:

No proposition is negative, therefore no donkey runs. (2)

On the second definition, this would be a consequence, since there is no circumstance under which the antecedent is true, so there is no circumstance under

[5] In this respect, the medieval approach to logic shares many methodological characteristics with logicians who pursue relevance logic projects nowadays.

which the antecedent is true without the consequent. The problem with the sentence is that its contrapositive:

Some donkey runs, therefore some proposition is negative. (3)

is not a valid consequence, and Buridan wishes to maintain contraposition as a sound rule of inference [14, p. 22]. The final revision that he gives does away with reference to truth-value altogether, and is defined in terms of signification:

> Some proposition is antecedent to another which is related to it in such a way that it is impossible for things to be in whatever way the first signifies them to be without their being in whatever way the other signifies them to be, when these propositions are formed together.[6]

The problem with a proposition such as "No proposition is negative" is that it is a *self-refuting* proposition; it cannot be formed without its very formation making it false. Self-refuting propositions cause problems for theories of truth and consequence, and turn up as central players in treatises on *insolubilia* and *sophismata*, to which we turn next.

5 *Insolubilia* and *sophismata*

The *Sophistici Elenchi* was one of the first of the new Aristotelian works translated in the middle of the 12th century to gain a wide readership [9, p. 69]. The study of fallacies and sophistical reasoning held the same draw in the Middle Ages as it did in ancient Greece and in modern times: It is not sufficient to know how to reason properly (as is taught via the syllogism in the *Prior Analytics*); in order to win in a dispute, one must also be able to recognize when one's opponent is reasoning improperly. This gave rise to the study of *insolubiliae* 'insoluble sentences' and *sophismata* 'sophisms'.

The medieval genre of *sophismata*-literature developed in the 12th century and was firmly established in both grammatical and logical contexts by the end of that century [2, p. 518]. In the context of logic, a sophisma or insoluble is a problematic sentence, a sentence whose analysis either leads to an apparent contradiction, or for which two equally plausible analyses can be given, one for its truth and one for its falsity.[7] Treatises on sophisms generally followed a similar framework:

1. The sophism is stated, sometimes along with a *casus*, a hypothesis about how the world is, or extra information about how the sophism should be analyzed.

2. An argument for its truth and an argument for its falsity are presented.

[6] Buridan is not entirely content with this definition either, but this has to do with his theory of truth, rather than any problem with the definition itself, and we do not have the space to go into these problems here. But cf. [14] and [16].

[7] In this section, we group *insolubilia* and *sophismata* in our discussion, even though historians of logic will sometimes try to bifurcate the two (cf., e.g., [37] and [30]. Both of the terms are somewhat wider in scope than modern 'paradox', which implies some sort of logical contradiction. Not all insolubles are in fact unsolvable; rather, they are so named because they are difficult to solve. And not all sophisms involve the use of sophistical (i.e., fallacious) reasoning.

3. There is a claim about the truth value of the *sophisma*.

4. The apparent contradiction is resolved by explaining why the arguments supporting the opposite solution are wrong.

The result is a sentence which has a definite truth value under the *casus* (if one is given).

Many sophisms dealt with paradoxes that arise from logical predicates, such as truth, necessity, validity, etc. Medieval logicians recognized the importance of the task of providing non-trivial and non-*ad hoc* resolutions to these insolubles and sophisms. The most productive era in the theory of insolubles was from 1320 to 1350. During this period, many treatises on *insolubilia* and *sophismata* were written, and discussions of insoluble sentences appeared in other, non-dedicated works, too. Some of the most important authors writing on the topic during this period include Thomas Bradwardine (c.1295–1349), Richard Kilvington (d.1361), Roger Swyneshed (d.1365), William Heytesbury (c.1310–1372), John Wyclif (c.1330–1384), and Peter of Ailly (1351–1420). In the remainder of this section, we discuss (1) the liar paradox and related insolubles, (2) sophisms relating to validity and logical consequence, and (3) other types of sophisms and insolubles.

5.1 The liar paradox

The most famous insoluble is the liar paradox:

This sentence is false. (4)

The earliest known medieval formulation of the paradox is in Adam of Balsham's *Ars disserendi* (1132). However, Adam "says nothing whatever to indicate that he was aware of the very special problems they pose, that they were current topics of philosophical discussion in his day, or how one might go about trying to answer those questions" [34, p. 25]. It wasn't until the later part of the 12th century that the problematic aspects of the liar sentence (and related sentences) were taken up in earnest. Over the course of the next two centuries, many attempts to solve the paradox were provided. These solutions can be divided into the following five families: (1) classification under the fallacy *secundum quid et simpliciter*; (2) *transcasus* theories, (3) distinguishing between the *actus exercitus* 'exercised act' and the *actus significatus* 'signified act'; (4) *restrictio* theories; and (5) *casatio* theories.

Secundum quid et simpliciter. This is Aristotle's solution. In chapter 25 of the *Sophistical Refutations*, Aristotle makes a distinction between "arguments which depend upon an expression that is valid of a particular thing, or in a particular respect, or place, or manner, or relation, and not valid absolutely" [1], that is, between expressions which are valid *secundum quid* 'according to something' and those which are valid *simpliciter* 'simply' (or 'absolutely'). He goes on to say:

> Is it possible for the same man at the same time to be a keeper and a breaker of his oath?... If a man keeps his oath in this particular instance or in this particular respect, is he bound also to be a keeper of oaths absolutely, but he who swears that he will break his oath,

> and then breaks it, keeps this particular oath only; he is not a keeper of his oath... The argument is similar, also, as regards the problem whether the same man can at the same time say what is both false and true: but it appears to be a troublesome question because it is not easy to see in which of the two connexions the word 'absolutely' is to be rendered—with 'true' or with 'false'. There is, however, nothing to prevent it from being false absolutely, though true in some particular respect or relation, i.e. being true in some things, though not 'true' absolutely [1].

How this solves the paradox is not entirely clear; on this view, the Liar sentence can apparently be solved both ways. Many medieval logicians who adopted the Aristotelian reply argued that the Liar is false *simpliciter*, and true *secundum quid*; however, it was left unspecified with respect to what *quid* it is true.

Transcasus. The Latin word *transcasus* has no straightforward translation into English. It is a literal translation of Greek μεταπτωσις 'a change, transferring'. Conceptually, it is related to the Stoic notion of μεταπιπτωντα, from the same root, which are propositions whose truth value change over time [37]. In *transcasus*, it is not that the truth value of the liar sentence changes, but rather, what the sentence refers to (and hence how its truth value should be evaluated). On such solutions, when someone says "I am speaking a falsehood", the sentence is not self-referential but instead refers to what that person said immediately prior. If he didn't say anything before, then the liar sentence is just false.

The *actus exercitus* and the *actus significatus*. This solution takes advantage of the fact that the liar sentence (as usually formulated in medieval treatises) involves assertion: I say "I am saying something false", or Plato says "Plato is saying something false", or similar. When such an assertion is made, it is possible to distinguish between what the speaker says he is doing (signified act) and what he is actually doing (exercised act). This view, which is not well understood, is espoused by Johannes Duns Scotus in his *Questiones super libro elenchorum* (c.1295), who says that the exercised act of the liar is "speaking the truth" and the signified act of the liar is "speaking a falsehood" [10]. Because the liar sentence expresses something which is not true, it is false.

Restrictio. Restriction solutions are the most straightforward: By restricting the allowed grammatical/syntactic forms to disbar self-referential sentences, it is possible to rule the liar paradox as without truth-value because it is self-referential (ungrammatical). In [32], Spade discusses 71 different texts dealing with the liar paradox. Fourteen of these texts espouse some type of restriction theory, either explicitly or implicitly. These include a number of mid- to late-14th century anonymous treatises, as well as treatises by well-known logicians such as Walter Burley, in his *Insolubilia* (before 1320) [31] and William of Ockham, in his *Summa logicae* (1324–1327) [24] and his *Tractatus super libros elenchorum* (before 1328) [25].

Restriction solutions exist across a broad spectrum, ranging from very weak, forbidding only a small amount of self-reference, to very strong, forbidding all self-reference. On such strong restriction theories, not only does it turn out that the liar has no truth value, but so also such insolubles as the linked liars:

Plato: What Socrates says is false (5)

Socrates: What Plato says is false (6)

as well as seemingly non-paradoxical sentences which just happen to be self-referential, such as:

This sentence has five words. (7)

Cassatio. The Latin word *cassatio* means to make null or void, or to cancel. On the view of the cassators, when you are uttering an insoluble, you are saying nothing; the paradoxicality of the sentence cancels out any meaning it might have had. Therefore an insoluble like the liar has the same truth value as the empty utterance: none. This solution was favored in the early period, and died out by the 1220s, though it continued to be mentioned in later catalogs of types of solutions [37, §2.5].

In addition to the liar and the linked liars, other liar-like insolubles were also considered. For example, suppose that Plato promises to give everyone who tells the truth a penny. Socrates then announces "You won't give me a penny". Or similarly, Plato is guarding a bridge, and will let only those who tell the truth cross; anyone who tells a lie will be thrown into the water. Socrates approaches and "You will throw me from the bridge". Both of these present the same problems for analysis as the liar paradox, though it is clear that they cannot be solved in similar ways (for example, restriction strategies make no sense here, since there is no self- or cross-reference).[8]

5.2 Paradoxes of validity

Earlier we mentioned some of the problems that arise in the analysis of propositions such as "No proposition is negative". In chapter 8 of his *Sophismata* [5], Buridan considers the related inferences "Every proposition is affirmative, therefore no proposition is negative" and "No proposition is negative, therefore some proposition is negative". These inferences are problematic, because, on the one hand, the antecedent is either the contrary or the contradictory of the consequent, so any time it is true the consequent will have to be false, according to the rules in the Square of Opposition; and on the other hand, "No proposition is negative" is itself a negative proposition, and any time that it exists, some negative proposition will exist, and thus some proposition will be negative. Nevertheless, it is not impossible that there be no negative propositions; as Buridan points out, "'Every proposition is affirmative' would be true if God annihilated all negatives, and then the consequent [of the first inference] would not be true, for it would not be" [5, p. 953]. This analysis leads Buridan to make an interesting distinction: He concludes that such a proposition "is possible, although it cannot be true" [5, p. 956], that is, he distinguishes between being "possible" and being "possibly-true"; a proposition can be one without being the other.

Other paradoxes of validity include self-referential propositions, but unlike the liar sentence they involve logical predicates other than truth. For example, Pseudo-Scotus offers a counterexample to Buridan's definition of logical consequence in terms of necessary truth preservation (cf. §4.2 above) [48]:

God exists, hence this argument is invalid. (8)

[8] Both of these examples are adapted from Peter of Ailly's *Insolubilia* [20].

Under standard medieval metaphysical and ontological assumptions, "God exists" is a necessary truth.[9] Suppose, then that (8) is valid. The antecedent is not only true, but necessarily true. However, if the argument is valid, then the consequent is false, since it asserts that the argument is invalid. But then the argument is not truth-preserving, and so cannot be valid. But if the argument is invalid, then it is necessarily invalid, and as a result the consequent is necessary. But a necessary truth following from a necessary truth is necessarily a valid inference, and thus (8) is valid.

Similar paradoxes can be leveled against Buridan's final definition, in terms of signification, too. For example, the definition is adequate for the counterexample that it was designed to obviate, but would have problems dealing with the following conference:

The consequent of (9) does not signify as is the case,
therefore, the consequent of (9) does not signify as is the case. (9)

While this precise example is not found, similar ideas are treated by Roger Swyneshed in his *Insolubilia* (1330x1335) [33].

5.3 Other classes of *insolubilia* and *sophismata*

In this section we briefly catalog other common types of *insolubilia* and *sophismata*.

(1) Sophisms which arise from *exponibilia* 'exponible [terms]'. An exponible term is one whose analysis requires breaking the sentence in which it appears down into a collection of sentences, each of which are simpler in form. A common example of a pair of exponible terms are *incipit* 'begins' and *desinit* 'ceases'. There are two ways that a sentence such as "Socrates begins to be white" can be analysed:

Socrates is not white at time t,
and t is the last moment at which he is not white. (10)

Socrates is not white at time t,
and t' is the first moment at which he is white. (11)

Confusing the two ways that such sentences can be expounded can result in sophisms. Such sophisms are discussed by William Heytesbury in his *Regulae solvendi sophismata* (1335) [12, 47] and Richard Kilvington in his *Sophismata* (before 1325) [15].

(2) Sophisms which arise from confusing the syncategorematic and categorematic uses of terms. The most common example is *Infinita est finita*, where *infinita* can be interpreted either categorematically or syncategorematically. This sentence is difficult to translate into English without losing the ambiguity; when *infinita* is used categorematically, it is taken as a substantive noun, 'the infinite',

[9] In fact, when it shows up in logical examples, it is almost always being used as a 'generic' logical truth, rather like $p \vee \neg p$. Nothing important turns on the fact that this proposition is about God.

and then the sentence says that 'the infinite is finite', which is false. But syncategorematically, it is taken adjectivally, and means that 'infinite are the finite', i.e., 'there are infinitely many finite things', which is true, e.g., numbers. Illicit shifts between the categorematic reading and the syncategorematic reading can lead to paralogisms and sophisms. Heytesbury discusses this in his treatise *De sensu composito et diviso* (before 1335) [13].

(3) Sophisms which arise from infelicities of presupposition and supposition. A typical example of these is "Socrates promises Plato a horse" (or in some cases, a penny), and yet for any given horse (or penny), it is not the case that Socrates has promised Plato *this* horse.

(4) Sophisms which arise from the re-imposition of terms, stipulating that they signify things other than their ordinary signification. We see examples of these in the next section.

Finally, we would be remiss in not mentioning (5) sophisms which illustrate that the more things change, the more things stay the same, or, more precisely, that medieval humor isn't all that different from modern humor, and that is the class of sophisms whose conclusion is *Tu es asinus* "You are an ass". One example of such is the paralogism: "This donkey is yours, this donkey is a father, therefore this donkey is your father", and if your father is a donkey, then you are one as well.

6 Obligational disputations

The final area of medieval logic that we cover in this chapter is the most peculiar and the most unfamiliar. While theories of meaning and reference, systems of logical consequence, and the study of paradoxes and sophistical reasoning are all part and parcel of the modern study of logic and philosophy of language, there is no such counterpart for the uniquely medieval genre of *disputationes de obligationibus*. The earliest treatises on these disputations are anonymous, and date from the first decades of the 13th century. In the following two centuries, scores of treatises on obligations were written, including ones by William of Sherwood, Nicholas of Paris (fl.1250), Walter Burley, Roger Swyneshed, Richard Kilvington, William of Ockham, Albert of Saxony, John of Wesel (1340/50s), Robert Fland, John of Holland, Richard Brinkley (fl.1365–1370), Richard Lavenham, Ralph Strode, Peter of Ailly, Peter of Candia (late 14th C), Peter of Mantua (d.1399), Paul of Venice (c.1369–1429), and Paul of Pergola (d.1455).

So what are these mysterious disputations, and why are they mysterious?

An *obligatio* is a dispute between two players, an Opponent and a Respondent. The Opponent puts forward propositions (one at a time, or in collections, depending on different author's rules), and the Respondent is obliged (hence the name) to follow certain rules in his responses to these propositions. These rules depend on the type of disputation; we give an example of one type below in §6.1 (which the reader can consult now to have a sense of what we are talking about). Early authors distinguish six types, or species, of *obligationes*: (1) *positio* 'positing', (2) *depositio* 'de-positing' (a type of denial), (3) *dubitatio* 'doubting', (4) *petitio* 'petition', (5) *impositio* 'imposition', and (6) *sit verum* 'let it be true'. Later authors argued that some of these types could be derived from the others and so reduced the number of species, generally to three [43, §4]. They are mysterious because their background and their purpose is unclear. Early texts

allude to Aristotle's three-fold division of disputations in Book VIII, chapter 4 of the *Topics* (cf., e.g., Walter Burley's *De obligationibus* (c.1302) [7]). There are indications in other texts that show that the medieval authors were interested in developing *obligationes* in the tradition of disputations as described in the *Topics*—not only in motivating the genre but also in discussions concerning what type of disputations *obligationes* are [43, §2]. Modern scholars have advanced many hypotheses about the purpose of these disputations. In our opinion, no single answer is going to tell the whole story. It is clear—certain texts tell us so explicitly [8]—that *obligationes* were used as training exercises for students. That there is a close connection between *obligationes* and *insolubilia*-literature is also clear given the use of *obligationes*-language in treatises on insolubles and sophisms [15, 18]. While the idea that *obligationes* were developed as a type of counterfactual reasoning is not in general tenable, it can be justified in some specific contexts [44].

The general procedure followed in the disputations did not vary drastically from author to author or type to type. The Respondent had three (in some cases, four) possible responses: concede, deny, or doubt (some authors also allowed him to draw distinctions in the case of ambiguous propositions). Which response was the correct response depended, in part, on whether the proposition was relevant (or pertinent) or irrelevant (or impertinent). In the tradition of Walter Burley, which came to be termed the *responsio vetus* 'the old response', a proposition was defined as relevant if it, or its negation, followed from the conjunction of all the propositions conceded along with the negations of all denied. On this definition, the set of relevant sentences potentially changed with each step of the disputation. Such a definition can also be found in the works of William of Sherwood, Ralph Strode, and Peter of Candia. This dynamic conception of relevance resulted in a number of consequences that later authors, particularly Roger Swyneshed, found problematic. Swyneshed, in what came to be termed the *responsio nova* 'new response', redefined relevance in his *Obligationes* (1330x1335) [35] into a static notion, where a proposition is relevant if it or its negation follows from the *positum* (the first proposition of the disputation), and is irrelevant otherwise. It is clear that on this definition, whether a proposition is relevant or not does not change with the course of the disputation. Swyneshed was followed in this redefinition by Robert Fland, Richard Lavenham, and John of Wesel, among others.

Regardless of which definition of relevance was used, the following general rules were accepted by everyone:

- A relevant proposition should be conceded if it follows, and denied if its negation follows.
- An irrelevant proposition should be conceded if it is (known to be) true, denied if it is (known to be) false, and doubted if it is neither (known to be) true nor (known to be) false.

6.1 *Positio*

Positio is the crown jewel of the *obligationes* regalia. It is the most prominently discussed, by both medieval and modern authors. *Positio* can be divided into multiple types. The first division is into possible and impossible *positio*; both divisions are further divided as to whether the proposition is simple or complex,

	Opponent	Respondent
1	φ.	I admit it.
2	$\neg\varphi \vee \psi$.	I grant it.
3	ψ	I grant it.

Figure 1: A simple example.

and then further as to whether the complex propositions are formed by conjunction ("conjoined *positio*") or disjunction ("indeterminate *positio*"). In any of these types of *positio*, it is also possible that a further stipulation is added, in which case the *positio* is called "dependent" [7, p. 378]. Most texts focus their discussion on possible *positio*.

The rules given above are, in slightly simplified form, Burley's rules for *positio*. These rules exhaustively cover all of the possibilities that the Respondent may face in the course of the disputation; how these rules play out in the context of actual disputations is made clear in Burley's examples, which are typical of 13th century developments. Due to issues of space, we cannot follow this up with a discussion of the later 14th century developments, but instead direct the reader to [43, §§4.2, 4.3].

We consider an example *positio* which appears, with slight variation, in many 13th-century treatises. It is fairly simple but illustrates Burley's rules nicely. Suppose that φ does not imply $\neg\psi$ and φ is known to be false; for example, let φ be 'The capitol of England is Paris' and ψ is 'It is raining'. Since φ is satisfiable (because if it were not, then it would imply $\neg\psi$), the Respondent should admit it when it is put forward as a *positum*. In the second round, Opponent asserts $\neg\varphi \vee \psi$. Now, either φ implies ψ, in which case the proposition follows from the *positum* and hence the Respondent should concede it, or ψ is independent of φ, and hence the proposition is irrelevant. In that case, we know that since φ is false, $\neg\varphi$ is true, so the disjunction is true, and true irrelevant propositions should be conceded. But then, ψ follows from the *positum* along with something correctly conceded, and hence when the Opponent asserts ψ, the Respondent must concede it too.

This example shows how, given a *positum* which is false, but not contradictory, the Opponent can force the Respondent to concede any other proposition consistent with it. The fact that this is possible is one of Swyneshed's primary motivations for revising the standard rules. Further formal properties, following from the assumption of a consistent *positum* (which is the definition of possible *positio*), include that no disputation requires the Respondent to concede φ in one round and to concede $\neg\varphi$ in another round (or to concede φ in one round and to deny it in another); the set of formulas conceded, along with the negations of those denied, will always be a consistent set; yet, it may be that the Respondent has to give different answers to the same propositions put forward at different times.

6.2 *Depositio.*

Depositio is just like *positio*, except that the Respondent is obliged to deny or reject the initial proposition (the *depositum*). A *depositio* with *depositum* φ will be completely symmetric to a *positio* with $\neg\varphi$ as the *positum*. Nevertheless,

early treatises on *obligationes*, such as that by Nicholas of Paris which dates from c.1230–50, still treat *depositio* at some length.

6.3 *Dubitatio.*

In *dubitatio*, the Respondent must *doubt* the statement that the Opponent puts forward (called the *dubitatum*). While *dubitatio* was discussed in 13th century texts, often at some length, later authors (both later medieval and modern authors) call *dubitatio* a trivial variant of *positio*, and thus spend little time discussing it. For example, Paul of Venice [28] reduces *dubitatio* to *positio* (in much the same way that he, and others, reduces *depositio* to *dubitatio*); Swyneshed, Lavenham [36], John of Wesel [29], Richard Brinkley [38], and John of Holland [21] do not mention *dubitatio* at all. However, such a trivializing view of *dubitatio* fails to recognize the higher-order aspects of the disputation, the mixing of both knowledge and truth, which result in a significantly more difficult type of disputation. Just as *positio* is only interesting when the *positum* is false, *dubitatio* is only interesting when the truth value of the *dubitatum* is known (whether it is true or false) [46, 42]. Thus, the complexity of this type of disputation partly arises from the interaction between knowledge, truth, and the obligations of the Respondent, as the Respondent in many cases is required to respond *dubio* 'I doubt it' to propositions that he actually knows.

A second cause of complexity in *dubitatio* is the fact that the rules, unlike those for *positio*, are not deterministic. For example, Nicholas of Paris's rules for *dubitatio* [4, pp. 72–76] include the following:

- Just as in *positio* a *positum* put forward in the form of the *positum*, and everything convertible to it in the time of positing is to be conceded and its opposite and things convertible with it is to be denied and just as in *depositio* a *depositum* put forward in the form of the *depositum*, with its convertibles, must be denied and its opposite with things convertible with it must be conceded; so in *dubitatio* for a *dubitatum* put forward in the form of *dubitatum* and for its convertibles and moreover for the opposite of the *dubitatum* with its convertibles must be answered "prove!" [4, p. 223].

- For everything irrelevant to the *dubitatum* the response must be according to its quality.

- For everything antecedent to the *dubitatum* the response must be "false" or "prove!" and never "true" [4, p. 224].

- For everything consequent to the *dubitatum* it is possible to reply "it is true" or "prove" and never "it is false" [4, p. 224].

Whereas there is always a unique correct response for Respondent in *positio* (in both the *responsio antiqua* and *nova*), here, the rules give Respondent a range of choices. This non-determinacy means that there is a plurality of ways that Respondent may act, and still be disputing according to the rules, a feature which no other version of *obligatio* has. However, this feature of *dubitatio* seems not to have been noticed by later authors who insisted that *dubitatio* could be reduced to *positio*.

Nicholas's *dubitatio* has similar formal properties to *positio*. Provided that the *dubitatum* is neither a contradiction nor a tautology, it can be proved that

the Respondent can *win* the disputation playing by Nicholas's rules for *dubitatio*: that is, there is never any case where he will be forced either to concede or to deny the *dubitatum* [42, Theorem 24].

6.4 *Impositio / Institutio / Appellatio*

The obligation involved in *impositio*, also called *institutio* 'institution' or *appellatio* 'appellation', functions in a relevantly different manner from the obligation in *positio*, *depositio*, or *dubitatio*. Whereas in these latter three, the Respondent's obligation involves how he is to respond to the *obligatum*, *impositio* involves the redefinition (re-imposition) of certain terms or phrases.[10] *Impositio* can take place in conjunction with any of *positio*, *depositio*, and *dubitatio*; that is, once a new imposition is introduced, then the Respondent may also further be obliged to concede, deny, or doubt the initial *obligatum* of the disputation. Sometimes the imposition is simple and straightforward:

> I impose that 'a' signifies precisely that God exists... I impose that this term 'man' may be converted with this word 'donkey', or I impose that this proposition 'God exists' signifies precisely that man is donkey [36, ¶¶2, 21].

In the first example, 'a' is being instituted as the name of a proposition that signifies that God exists; likewise in the third example, the phrase 'God exists' is instituted as the name of a proposition signifying that man is donkey; thus any time that 'God exists' is asserted in a disputation, it must be understood as meaning 'Man is donkey'. In the second example, the institution is not at the level of propositions but at the level of words; it changes the meaning of the term 'man' so that it no longer means 'man' but instead means 'donkey'. Simple impositions like these are relatively easy; the only skill they require beyond the skills needed for *positio* is the skill to remember the new imposition of the term or proposition. However, much more complicated examples can be provided, such as the following (also due to Lavenham):

> I impose that in every false proposition in which 'a' is put down that it signifies only 'man' and that in every true proposition in which 'a' is put down that it signifies only 'donkey', and that in every doubtful proposition in which 'a' is put down that it signifies alternately with this disjunction 'man or non man' [36, ¶24].

Now suppose that the proposition "Man is a" is put forward. The proposition is either true, false, or of unknown truth value (doubtful). Suppose it is true. Then, it means "Man is a donkey", which is impossible; hence, contrary to supposition, the proposition is in fact false. But if it is false, it means "Man is man"—but this is true! Thus, if it cannot be true or false, then it must be doubtful. But if it is doubtful, then it means "Man is man or not man", which is true, and hence not doubtful! No matter which assumption are made about the value of the proposition, the Respondent is lead into contradiction.

[10] For example, the anonymous author of *Obligationes Parisienses* notes that "*Institutio* is divided into certain *institutio* and uncertain or obscure *institutio*, for example if the name 'Marcus' is fixed that it might be a name of Socrates or Plato, but you would not know of which" [8, p. 28] and Lavenham defines *impositio* as an "obligation by means of which a term or proposition is assigned a [new] signification" [36, ¶21].

6.5 *Petitio.*

In *petitio*, the Opponent asks (petitions) the Respondent to respond in a certain way, for example, by conceding or denying the initial proposition. *Petitio* is rarely treated at any length, because, as a number of authors (Nicholas of Paris [4, p. 183], Marsilius of Inghen [22], Peter of Mantua [39], Paul of Venice [28, pp. 38–39]) argue, *petitio* can be reduced to *positio*. Thus, from the disputational point of view, there is little more than cosmetic differences between *positio* and *petitio*.

6.6 *Rei veritas / sit verum.*

The sixth type, *sit verum* or *rei veritas* 'the truth of things', is rarely discussed by the medieval authors, and sometimes not even explicitly defined. As a result, it is difficult to give a precise explanation or characterization of this type. Many discussions of *sit verum* focus on epistemic aspects of the disputation [41, p. 320]. For instance, Paul of Venice gives the following example of *sit verum*: "Let it be true that you know that you are replying" [28, p. 45]. Nicholas of Paris also gives an example of a *rei veritas* that cannot be sustained which is couched in epistemic terms [4, pp. 166, 233]. However, one cannot generalized too broadly from this, as other examples show more in common with counterfactual reasoning than epistemic reasoning [44].

7 Conclusion

In this chapter we have given an overview of medieval logic which we hope is sufficient to show that the Middle Ages were, in the history of logic, not a period of darkness and crudeness, but rather one, particularly during the 13th and 14th centuries, of new insights into the nature of language and inference. These insights, building on Aristotle's *Organon* but going far beyond him, provided a foundation for the formal education of centuries of young men, regardless of whether they intended to continue their studies in philosophy. There is much that we have not been able to address in great detail in this chapter, and still more that we have not touched on at all. (For example, we have almost completely omitted developments of the syllogistic, both assertoric and modal, as well as the interesting and complex question of how to deal with future contingents.) Nevertheless, what we have shown is that the impact of logic and the study of it in the Middle Ages therefore cannot be dismissed out of hand. Modern study of medieval logic is still, to a large extent, in its early stages, and decades to come will continue to prove the importance and sophistication of the medieval logicians.

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