On Finding Paths Passing through Specified Vertices

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1 Introduction

We consider undirected finite graphs that have no loops and no multiple edges. A graph is denoted $G = (V_G, E_G)$, where V_G is the set of vertices and E_G is the set of edges.

A graph containment problem is to decide whether one graph can be modified into some other graph by using a number of specified graph operations. For example, by allowing any combination of the four operations edge deletions, edge contractions, vertex deletions and vertex dissolutions the following containment problems are captured: testing on (induced) minors, (induced) topological minors, (induced) subgraphs, induced spanning subgraphs, dissolutions and contractions. Algorithms for detecting specific induced subgraphs such as paths and cycles have been proven to be useful in the design of algorithms that solve more general graph containment problems. We will discuss one open problem in this area and give a potential application afterwards.

We first state some terminology required. Two (not necessarily induced) paths P and Q in a graph G are called *mutually induced* if the following two conditions are both satisfied:

(i) P and Q are vertex-disjoint;

(ii) G has no edges with one end-vertex in P and the other one in Q.

2 Problem Description

We can now formulate the following three decision problems. In each of these problems the input consists of a graph, in which some vertices are specified; we call these vertices *terminals.*

Problem 1
Instance: a graph G with 4 terminals s, t, u, v;
Question: do there exist two mutually induced paths P and Q such that P and Q have end-vertices $\{s, u\}$ and $\{t, v\}$, respectively?

Problem 2
Instance: a graph G with 4 terminals s, t, u, v;
Question: do there exist two mutually induced paths P and Q such that P and Q have end-vertices either $\{s, u\}$ and $\{t, v\}$, or else $\{s, v\}$ and $\{t, u\}$, respectively?

PROBLEM 3
Instance: a graph G with 4 terminals s, t, u, v;
Question: do there exist two mutually induced paths P and Q such that P and Q have end-vertices either $\{s, u\}$ and $\{t, v\}$, or $\{s, v\}$ and $\{t, u\}$, or else $\{s, t\}$ and $\{u, v\}$, respectively?

Fellows [4] showed that PROBLEM 1 is NP-complete. This result also follows from the paper of Bienstock [1] by considering the gadget that he used to prove NP-completeness of the problem 2-IN-A-CYCLE which is to test whether a given graph contains an induced cycle through two given terminals. Lévêque, Lin, Maffray, and Trotignon [6] showed that 2-IN-A-CYCLE is NP-complete even for graphs of maximum degree 3. In both papers, the two terminals are of degree 2. This property can be used for a straightforward reduction that shows that PROBLEM 2 is NP-complete even for graphs of maximum degree 3. We call PROBLEM 3 the 4-ON-2-PATHS problem and pose the following problem.

Problem 2.1. *What is the computational complexity of* 4-ON-2-PATHS*?*

As related work, we mention that Chudnovsky and Seymour [2] showed that the problem 3-IN-A-TREE, which is to test whether a given graph contains an induced tree containing three given terminals, can be solved in polynomial time, whereas the computational complexity of 4-IN-A-TREE is still open. In contract, the problem 3-IN-A-PATH, which is to test whether a given graph contains an induced path passing through three given terminals, is NP-complete even for graphs of maximum degree 3. This follows from the result of Bienstock [1], as observed by Derhy and Picouleau [3].

3 A Possible Application

A graph G contains a graph H as an *induced minor* if H can be obtained from G after a sequence of graph operations that may be vertex deletions and edge contractions. For a fixed graph H (i.e., that is not part of the input), the H-INDUCED MINOR problem is to test whether some given graph contains H as an induced minor.

Let H^* denote the graph obtained from subdividing the centre edge of a double star with two leaves on each side, i.e., H is the graph with vertices $a_1, a_2, b, c, d, e_1, e_2$ and edges a_1b, a_2b, bc, cd, de_1, de_2.

Theorem 3.1 ([5])**.** *For any fixed forest $H \neq H^*$ on at most 7 vertices, the H-*INDUCED MINOR *problem can be solved in polynomial time.*

A possible application of 4-ON-2-PATHS (should it belong to P) would be to use it as a subroutine for solving the missing case in Theorem 3.1.

Problem 3.2. *What is the computational complexity of H^*-*INDUCED MINOR*?*

References

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