Boundary Conformal Field Theories on Random Surfaces and The Non-Critical Open String

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Abstract

We analyse boundary conformal field theories on random surfaces using the conformal gauge approach of David, Distler and Kawai. The crucial point is the choice of boundary conditions on the Liouville field. We discuss the Weyl anomaly cancellation for Polyakov's non-critical open bosonic string with Neumann, Dirichlet and free boundary conditions. Dirichlet boundary conditions on the Liouville field imply that the metric is discontinuous as the boundary is approached. We consider the semi-classical limit and argue how it singles out the free boundary conditions for the Liouville field. We define the open string susceptibility, the anomalous gravitational scaling dimensions and a new Yang-Mills Feynman mass critical exponent.

1   Introduction

In 1981 Polyakov [1] showed that when non-critical strings are quantised so as to maintain reparametrisation invariance, the scale of the metric becomes a dynamical degree of freedom even though it decouples classically. Although the associated action is that of a soluble quantum field theory, the Liouville theory, the integration measure is not the usual one encountered in the functional approach to quantum field theory. Consequently it was unclear how to proceed until David, Distler and Kawai [2] showed that the effect of the measure could be accounted for by a simple renormalisation of the action. In this paper we study the effects of boundaries on this approach, extending their results to the cases of open string theory and to the coupling of boundary conformal field theories to 2D quantum gravity.

We start in section 2 with a brief review of the coupling of the minimal models to closed 2D quantum gravity.

In section 3 we consider our solution for the example of Polyakov's non-critical open bosonic string. The key point is the choice of boundary conditions on the Liouville field. Thus, we discuss the Weyl anomaly cancellation for Neumann, Dirichlet and free boundary conditions. We use a linear Coulomb gas perturbative expansion [3, 4] to find the renormalised central charge of the conformally extended Liouville theory that describes the gravitational sector. As expected this will be shown to be the same central charge calculated for the coupling on closed surfaces. Since the metric is to be written as a reference metric multiplied by the exponential of the Liouville field the theory must be independent of a shift in this field together with a compensating Weyl transformation on the reference metric. This leads to the dressing of primary operators that acquire conformal weight $(1, 1)$ on the bulk and conformal weight $(1/2, 1/2)$ on the boundary. Consequently, we show that the Liouville field renormalisation is equal to the one found for closed surfaces both on the bulk and on the boundary of the open surfaces. This only works for Neumann and free boundary conditions on the Liouville field. The Dirichlet boundary conditions freeze the Liouville boundary quantum dynamics so that, it is not possible to cancel all the boundary terms in the Weyl anomaly by a shift in the boundary values of the Liouville field, without leading to a discontinuity in the metric as the boundary is approached. Due to the presence of the boundary we find new renormalised couplings to 2D gravity. Under Weyl invariance at the quantum level we show that they are all determined by the bulk or closed surface couplings as would be expected. We also show how the Coulomb gas screening charge selection rule is a crucial condition for the cancellation of non-local and Weyl anomalous contributions to the correlation functions due to zero modes.

In section 4 we analyse the semi-classical limit which singles out the free boundary conditions on the Liouville field as being the most natural. We define the open string susceptibility, the anomalous gravitational scaling dimensions and a new mass critical exponent. In the context of Yang-Mills theory this mass exponent has an interesting physical interpretation as the critical exponent associated with the Feynman propagator for a test particle which interacts with the gauge fields.

In section 5 we generalise the open string analysis to a natural Feigin-Fuchs representation of $c \leq 1$ minimal conformal field theories on open random surfaces. Finally, we present our conclusions.

2   Minimal Models On Closed Random Surfaces

We now review the aspects of the approach of David, Distler and Kawai [2] to minimal models on closed random surfaces that will be useful when we consider boundaries. The Coulomb gas representation of conformal field theories due to Dotsenko and Fateev [3, 4] has a natural Lagrangian interpretation. We introduce the action

$$S_M[\Phi, \tilde{g}] \;\; = \tfrac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \left[\tfrac{1}{2} \tilde{g}^{ab} \partial_a \Phi \partial_b \Phi + i \left(\beta - 1/\beta \right) \tilde{R} \Phi \right] +$$

$$+\mu^2 \int d^2\xi \sqrt{\tilde{g}} \left(e^{i\beta\Phi} + e^{-i/\beta\Phi} \right) \tag{1}$$

to define the minimal unitary series of conformal field theories on closed surfaces. This is a conformally extended Liouville theory [5] with imaginary coupling, $i\beta$, on a surface with metric $\tilde{g}_{ab}$ and curvature $\tilde{R}$. The central charge of the matter theory is $c_M = 1 - 6(\beta - 1/\beta)^2$ which means the minimal models [6] are at the rational points $\beta^2 = (2 + k')/(2 + k)$. The primary fields are vertex operators given by

$$U(jj') = \int d^2\xi \sqrt{\tilde{g}} \exp\left[-i \left(j\beta - \frac{j'}{\beta} \right) \Phi \right]$$

where $j, j' \geq 0$ are half-integer spins labelling pairs of representations of the Virasoro algebra A_1. To couple this theory to gravity we treat $\tilde{g}_{ab}$ as a dynamical variable and add a cosmological constant term $\mu_0^2 \int d^2\xi \sqrt{\tilde{g}}$ to the action.

In the conformal gauge $\tilde{g}_{ab}$ is decomposed as a reparametrisation of $e^\varphi \hat{g}_{ab}$. Integrating over the matter field and reparametrisations generates a Weyl anomaly which yields a kinetic term for φ if the matter central charge is not balanced by the corresponding reparametrisation ghost charge. For this non-critical theory there results a Liouville field theory for φ

$$S_L[\varphi, \hat{g}] = -\frac{d - 26}{48\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\varphi\hat{\Delta}\varphi + \hat{R}\varphi \right) + \mu_1^2 \int d^2\sqrt{\hat{g}}e^\varphi,$$

where $\hat{\Delta}$ is the covariant Laplacian $-(1/\sqrt{\hat{g}})\partial_a\sqrt{\hat{g}}\hat{g}^{ab}\partial_b$. The functional integral volume element for this theory is induced by the inner product on variations of the Liouville field

$$\|\delta\varphi\|_{\hat{g}}^2 = \int d^2\xi \sqrt{\hat{g}}e^\varphi(\delta\varphi)^2.$$

This theory is deeply non-linear and its complete solution has not yet been found [2, 7, 8, 9]. The reason is the presence of e^φ in the inner product, which means that the volume element is not the usual one that occurs in quantum field theory. According to David, Distler and Kawai this may be replaced by a conventional field theory measure provided the Liouville mode and its couplings to 2D quantum gravity are renormalised:

$$S_L[\phi, \hat{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left[\frac{1}{2}\phi\hat{\Delta}\phi + i \left(\gamma + \frac{1}{\gamma} \right) \hat{R}\phi \right] + \mu_2^2 \int d^2\sqrt{\hat{g}}e^{\alpha\phi}.$$

Since the separation of $\tilde{g}_{ab}$ into the scale e^ϕ and reference metric $\hat{g}_{ab}$ is arbitrary the new theory is required to be invariant under simultaneous shifts in ϕ and compensating scalings of $\hat{g}_{ab}$. Thus a form of Weyl invariance must be preserved at the quantum level. When we integrate ϕ we generate a background Weyl anomaly which we add to the background anomaly coming from the integration of the matter field and the reparametrisation ghosts. The theory is Weyl invariant at the quantum level if this anomaly is absent and the amplitude is independent of the conformal factor of the reference metric.

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The anomaly cancellation sets the total central charge of the system to zero. This gives $\gamma = \pm i\beta$. Also the Liouville field renormalisation parameter α must satisfy $1 - \alpha(\beta + 1/\beta) + \alpha^2 = 0$ if we choose $\gamma = -i\beta$. Then we have two branches $\alpha_+ = \beta$ and $\alpha_- = 1/\beta$. The dressed vertex operators of vanishing conformal weight are

$$U_D(jj') = \int d^2\xi \sqrt{\hat{g}} \exp\left[\left(l\beta - \frac{l'}{\beta}\right)\phi\right] \exp\left[-i\left(j\beta - \frac{j'}{\beta}\right)\Phi\right]$$

where $l = -j$, $l' = j' + 1$ or $l = j + 1$, $l' = -j'$.

It is important to note that Weyl invariance at the quantum level is only possible because we have imposed an independent charge conservation selection rule [2, 3, 4] on the matter and the gravitational sectors. For each sector the Gaussian integrals over Φ and ϕ yield contributions of the form of the exponential of

$$\mathcal{F}^N[g] = \frac{1}{16\pi} \int d^2\xi' d^2\xi'' \sqrt{g(\xi')} J^N(\xi') G(\xi', \xi'') \sqrt{g(\xi'')} J^N(\xi'')$$

where g_{ab} stands for either $\tilde{g}_{ab}$ or $\hat{g}_{ab}$, J^N is the coefficient of the term in the action that is linear in the field and $G(\xi, \xi')$ is the covariant Laplacian's Green's function which satisfies

$$\Delta G(\xi, \xi') = \frac{\delta^2(\xi - \xi')}{\sqrt{g(\xi)}} - \frac{1}{\int d^2\xi'' \sqrt{g(\xi'')}},$$

is symmetric in its arguments and orthogonal to the constant zero-mode

$$\int d^2\xi \sqrt{g(\xi)} G(\xi, \xi') = 0.$$

Due to the presence of the Laplacian's zero-mode we find a non-local Weyl anomaly:

$$\delta_\rho \mathcal{F}^N = -\frac{Q}{8\pi} \int d^2\xi \sqrt{g} J^N \delta_\rho \ln \int d^2\xi \sqrt{g} -$$
$$-\frac{1}{8\pi \int d^2\xi \sqrt{g}} \int d^2\xi \sqrt{g} J^N \int d^2\xi' d^2\xi'' \sqrt{g(\xi')} \rho(\xi') G(\xi', \xi'') \sqrt{g(\xi'')} J^N(\xi''),$$

where Q is either $i(\beta - 1/\beta)$ or $i(\gamma + 1/\gamma)$. When we integrate the zero mode of the fields in each sector the charge selection rule gives $\int d^2\xi \sqrt{g} J^N = 0$ for all non-zero contributions to the amplitude, leading to the cancellation of the non-local anomaly.

Using a simple scaling argument David, Distler and Kawai's approach leads us to the random surfaces critical exponents. We find the susceptibility exponent [2, 11] $\Gamma(\chi_c) = 2 - \chi_c(\beta + 1/\beta)/(2\alpha_\pm)$, where $\chi_c = 2 - 2h$ is the Euler characteristic of the closed Riemann surface given in terms of its genus. It is related to the world-sheet integral of $\tilde{R}$ by the Gauss-Bonnet theorem:

$$\int d^2\xi \sqrt{\tilde{g}} \tilde{R} = 4\pi\chi_c. \tag{2}$$

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The semi-classical limit corresponds to $\beta \to +\infty$ and as expected it selects the solution $\alpha_+ = \beta$. We also get the gravitational scaling dimensions of the matter primary fields [2] $\Delta(jj') = 1 - \beta(jj')_\pm/\alpha_\pm$. Here $\beta(jj')_\pm$ defines the coefficient of the two possible dressings of the primary field $U(jj')$. When this is combined with the bare conformal weight using the equation which defines α it gives the KPZ equation [9]:

$$\Delta - \Delta_0 = -\alpha^2 \Delta(\Delta - 1).$$

These results for the critical exponents of a $c \le 1$ minimal conformal field theory on closed random surfaces agree with the KPZ light-cone analysis on the sphere [9]. Distler, Hlousek and Kawai [12] also used this conformal gauge approach to calculate the Hausdorff dimension of the random surfaces. All the results are in striking agreement with those of the theory of dynamical triangulated random surfaces [13].

Our aim in this paper is to see if this picture still holds when the random surfaces have boundaries and the minimal model becomes a boundary conformal field theory. The main issue is the choice of boundary conditions on the gravitational sector. Since the Liouville theory has a natural generalisation in the presence of boundaries, we expect the coupling of the minimal boundary conformal field theories to 2D quantum gravity to be again described by two conformally extended Liouville theories which are complementary. We start by presenting our solution in the simple case of Polyakov's open bosonic string.

3 Open String 2D Quantum Gravity

3.1 Free boundary conditions

For simplicity let us consider Polyakov's open bosonic string partition function Z for the topology of a disc [14, 15, 16]. We take free boundary conditions on the string field X^μ and on the Liouville conformal gauge factor φ. In the case of the reparametrisation ghosts θ^a we consider diffeomorphisms which preserve the parameter domain in $\mathcal{R}^2$ but allow for general reparametrisations along the boundary. This means that the component of θ^a along the outward normal to the boundary must be zero $\tilde{n} \cdot \theta = 0$, but its component along the tangent $\tilde{t} \cdot \theta$ is kept free just like the boundary values of X^μ and φ. More precisely we initially require that X^μ, φ and $\tilde{t} \cdot \theta$ take prescribed values Y^μ, ψ and η on the boundary and then we integrate over these boundary values [16]. The functional $Z[Y, \psi, \eta]$, obtained as an intermediate step, has the physical interpretation of being the tree-level (in the sense of string loops) contribution to the wave-functional of the vacuum for closed string theory in the Schrödinger representation.

The quantum partition function is thus given by

$$Z = \int \mathcal{D}_{\tilde{g}}(Y, \psi, \eta) Z[Y, \psi, \eta]$$

where the wave functional is

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$$Z[Y, \psi, \eta] = \int \mathcal{D}_{\tilde{g}} X \mathcal{D}_{\tilde{g}} \tilde{g} \exp\left\{-S[X, \tilde{g}]\right\}. \tag{3}$$

The action consists of the standard bosonic string matter action of Brink, Di Vecchia and Howe plus renormalisation counterterms:

$$S[X, \tilde{g}] = \frac{1}{16\pi} \int d^2\xi \sqrt{\tilde{g}} \tilde{g}^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \int d^2\xi \sqrt{\tilde{g}} + \lambda_0 \oint d\tilde{s} + \nu_0 \oint d\tilde{s} k_{\tilde{g}}.$$

The cosmological constant terms in the area $\mu_0^2 \int d^2\xi \sqrt{\tilde{g}}$, the invariant length of the boundary $\lambda_0 \oint d\tilde{s}$ and the integral of its geodesic curvature $\nu_0 \oint d\tilde{s} k_{\tilde{g}}$ are the non-trivial pure gravity contributions to the action in two dimensions. The first two are necessary as counterterms due to short distance singularities. Although the geodesic curvature counterterm is not associated with divergencies we will see that it is absolutely necessary for our solution. Here we note that this term can be written as $(\nu_0/2) \int d^2\xi \sqrt{\tilde{g}} \tilde{R}$ if we use the Gauss-Bonnet theorem

$$\int d^2\xi \sqrt{\tilde{g}} \tilde{R} + 2 \oint d\tilde{s} k_{\tilde{g}} = 4\pi \chi_o \tag{4}$$

where χ_o is the Euler characteristic of the open Riemann surface. It is given by $\chi_o = 2 - 2h - b$ where h is the genus of the surface and b the number of smooth boundaries. Note also that in the open string the Gauss-Bonnet theorem cannot fix both the integrals of the scalar curvature $\tilde{R}$ and of the geodesic curvature $k_{\tilde{g}}$ so that we should allow one of these as a pure gravity contribution to the action.

To calculate Z let us first determine the wave functional $Z[Y, \psi, \eta]$. We start by separating X^μ into two parts $X^\mu = X_c^\mu + \bar{X}^\mu$. We define X_c^μ and $\bar{X}^\mu$ in such a way that the string action gets split into two independent pieces, one for X_c^μ which contains all the dependence on the boundary value Y^μ and another for $\bar{X}^\mu$. This is easily done if we fix X_c^μ using Y^μ,

$$\tilde{\Delta} X_c^\mu = 0, \quad X_c^\mu|_B = Y^\mu, \tag{5}$$

and impose on $\bar{X}^\mu$ a homogeneous Dirichlet boundary condition $\bar{X}^\mu|_B = 0$. Here we have used the notation B to say that the fields are evaluated at a point ξ of the boundary B. Eq. (5) is solved in terms of Y^μ using the homogeneous Green's function for the Laplacian with Dirichlet boundary conditions defined for the metric $\tilde{g}_{ab}$. We will separate the boundary value Y^μ into a constant piece, and a piece that is orthogonal with respect to the natural metric on the boundary, i.e. we write $Y^\mu = Y_0^\mu + \bar{Y}^\mu$ where $\oint d\tilde{s} \bar{Y}^\mu = 0$. Then if $\partial_{\tilde{n}}$ is the outward normal derivative on the boundary the solution is

$$X_c^\mu(\xi') = Y_0^\mu - \oint d\tilde{s}(\xi) \partial_{\tilde{n}} \tilde{G}_D(\xi, \xi') \bar{Y}^\mu(\xi) \tag{6}$$

if the point ξ' is not in the boundary and $X_c^\mu|_B = Y^\mu$ if it is. Of course here we have considered

$$\tilde{\Delta} \tilde{G}_D(\xi, \xi') = \frac{\delta^2(\xi - \xi')}{\sqrt{\tilde{g}(\xi)}} \tag{7}$$

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where $\tilde{G}_D(\xi,\xi') = 0$ if either argument lies on the boundary. In this case we can integrate eq. (7) leading to an integral condition on its outward normal derivative

$$\oint d\tilde{s}(\xi)\partial_{\tilde{n}}\tilde{G}_D(\xi,\xi') = -1,$$

which allows the decomposition of X_c^μ given in eq. (6).

The string action can now be cast in the form $S[X,\tilde{g}] = S_c[X_c,\tilde{g}] + S[\bar{X},\tilde{g}]$. The action for $\bar{X}^\mu$ is just the free bosonic action where the kinetic kernel is the covariant Laplacian. To find $S_c[X_c,\tilde{g}]$ as a boundary action we take a total derivative and use eq. (5). We may write the result introducing the boundary kinetic kernel $\tilde{K}_D(\xi,\xi') = -1/(8\pi)\partial_{\tilde{n}}\partial_{\tilde{n}'}\tilde{G}_D(\xi,\xi')$:

$$S_c[X_c,\tilde{g}] = \frac{1}{2}\oint d\tilde{s}(\xi)d\tilde{s}(\xi')Y(\xi)\cdot\tilde{K}_D(\xi,\xi')Y(\xi').$$

In standard fashion [1, 10, 14, 15, 16] the functional integration measure $\mathcal{D}_{\tilde{g}}X$ is characterised by an $\mathcal{L}^2$ norm for variations of X^μ

$$\|\delta X\|_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}\delta X\cdot\delta X, \quad \int \mathcal{D}_{\tilde{g}}\delta X e^{-\|\delta X\|_{\tilde{g}}^2} = 1.$$

When we integrate X^μ keeping Y^μ fixed $\mathcal{D}_{\tilde{g}}X$ is actually $\mathcal{D}_{\tilde{g}}\bar{X}$. For the integration over the metric $\mathcal{D}_{\tilde{g}}\tilde{g}$ we need to consider the similar $\mathcal{L}^2$ norm for $\tilde{g}_{ab}$

$$\|\delta\tilde{g}\|_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}\left(\tilde{g}^{ac}\tilde{g}^{bd} + u\tilde{g}^{ab}\tilde{g}^{cd}\right)\delta\tilde{g}_{ab}\delta\tilde{g}_{cd}$$

where u is a non-negative constant. In the conformal gauge we decompose the integration over $\tilde{g}_{ab}$ into an integration over φ and an integration over θ^a. On the disc an arbitrary infinitesimal variation of $\tilde{g}_{ab}$ is $\delta\tilde{g}_{ab} = \delta\varphi\tilde{g}_{ab} + \tilde{\nabla}_a\delta\theta_b + \tilde{\nabla}_b\delta\theta_a$, where $\tilde{\nabla}_a$ is the covariant derivative in the metric $\tilde{g}_{ab}$. The variations of $\tilde{g}_{ab}$ induced by the reparametrisation ghosts and by Weyl transformations are not orthogonal. They intersect in the conformal Killing vectors $\tilde{P}_{ab}(\delta\theta) = 0$, where $\tilde{P}_{ab}$ acts on vectors to make symmetric, traceless tensor fields $\tilde{P}_{ab}(\delta\theta) = \tilde{\nabla}_a\delta\theta_b + \tilde{\nabla}_b\delta\theta_a - \tilde{g}_{ab}\tilde{\nabla}_e\delta\theta^e$. The adjoint acts on tensor fields to make vectors $\tilde{P}_b^\dagger = -2\tilde{\nabla}_a h_b^a$. Then redefining φ we write

$$\|\delta\tilde{g}\|_{\tilde{g}}^2 = 2(1+2u)\int d^2\xi\sqrt{\tilde{g}}(\delta\varphi)^2 + \int d^2\xi\sqrt{\tilde{g}}\tilde{g}^{ac}\tilde{g}^{bd}\tilde{P}_{ab}(\delta\theta)\tilde{P}_{cd}(\delta\theta).$$

We now split θ^a into a field $\bar{\theta}^a$ vanishing at the boundary and another field ϑ^a such that at the boundary $\vartheta^a = \eta\tilde{t}^a$. Assuming that ϑ^a is fixed by its boundary value in some way we obtain:

$$\|\delta\tilde{g}\|_{\tilde{g}}^2 = 2(1+2u)\int d^2\xi\sqrt{\tilde{g}}(\delta\varphi)^2 + \int d^2\xi\sqrt{\tilde{g}}\delta\bar{\theta}\cdot\tilde{P}^\dagger\tilde{P}(\delta\bar{\theta}).$$

Omiting the renormalisation counterterms we integrate $\bar{X}^\mu$ and $\bar{\theta}$ to find

$$Z[Y,\psi,\eta] = \exp\left\{-S_c[X_c,\tilde{g}]\right\}\int\mathcal{D}_{\tilde{g}}\varphi\left(Det'\tilde{\Delta}\right)^{-d/2}\frac{\sqrt{Det'\tilde{P}^\dagger\tilde{P}}}{Vol(CKV)}$$

where the prime denotes the omission of the zero modes and we have divided by the volume of the space of conformal Killing vectors $Vol(CKV)$. As is well known these infinite determinants generate a Weyl anomaly [1, 10, 14, 15, 16]. If we use the covariant heat kernel to regularise them it is easy to see that the Weyl anomaly only depends on the values of the heat kernels for small proper time cutoff $\sqrt{\varepsilon}$. This means that the Weyl anomaly is a local phenomenon which only reflects the structure of the world-sheet at short distances. Since $\sqrt{\varepsilon}$ can be made infinitesimally small, the bulk and boundary contributions to the anomaly must be independent. Using locality, reparametrisation invariance, dimensional analysis and the commutativity of Weyl transformations we are led to the following expansion in powers of the proper time cutoff $\sqrt{\varepsilon}$:

$$
\delta_\rho \ln \left[\left(Det' \tilde{\Delta} \right)^{-d/2} \frac{\sqrt{Det' \tilde{P}^\dagger \tilde{P}}}{Vol(CKV)} \right] = \frac{d-26}{48\pi} \int d^2\xi \sqrt{\tilde{g}} \tilde{R} \rho + \frac{d-26}{24\pi} \oint d\tilde{s} k_{\tilde{g}} \rho +
$$
$$
+ C_1 \oint d\tilde{s} \partial_{\tilde{n}} \rho + \frac{C_2}{\varepsilon} \int d^2\xi \sqrt{\tilde{g}} \rho + \frac{C_3}{\sqrt{\varepsilon}} \oint d\tilde{s} \rho + O(\sqrt{\varepsilon}),
$$

where the C_i are dimensionless constants which can be determined exactly [14, 16]. Here we will not worry about them because all are absorbed in the renormalisation counterterms.

Integrating the infinitesimal variation leads to the usual Liouville action plus background contributions depending on the reference metric of the conformal gauge $\hat{g}_{ab}$:

$$
Z[Y, \psi, \eta] = \exp \left\{ -S_c[X_c, \tilde{g}] \right\} \int \mathcal{D}_{\tilde{g}} \varphi \left(Det' \hat{\Delta} \right)^{-d/2} \frac{\sqrt{Det' \hat{P}^\dagger \hat{P}}}{Vol(CKV)} \exp \left\{ -S_L[\varphi, \hat{g}] \right\},
$$

where the Liouville action is given by

$$
S_L[\varphi, \hat{g}] = -\frac{d-26}{48\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + \hat{R} \varphi \right) - \frac{d-26}{24\pi} \oint d\hat{s} k_{\hat{g}} \varphi +
$$
$$
+ \mu_1^2 \int d^2 \sqrt{\hat{g}} e^\varphi + \lambda_1 \oint d\hat{s} e^{\varphi/2} + \nu_1 \oint d\hat{s} \partial_{\hat{n}} \varphi.
$$

Here μ_1^2, λ_1 and ν_1 are arbitrary finite constants left over from the renormalisation process.

Next we start the integration of the Liouville mode and determine the renormalisation of the couplings to 2D quantum gravity.

3.1.1 Anomaly cancellation for coupling renormalisation

To integrate the Liouville mode we start by taking the Coulomb gas perturbative approach expanding the area cosmological constant counterterm. In each order of perturbation theory we split φ in two fields φ_c, $\bar{\varphi}$ in exactly the same way we split X^μ previously. As before the Liouville action becomes the sum of two independent pieces, $S_L[\varphi_c, \hat{g}]$, which contains all the dependence on the boundary value ψ, and

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$S_L[\bar{\varphi}, \hat{g}]$. We further split $\psi = \psi_0 + \bar{\psi}$ into a constant ψ_0 and an orthogonal piece $\bar{\psi}$. The field φ_c is now expressed in terms of $\bar{\psi}$ and ψ_0:

$$\varphi_c(\xi') = \psi_0 - \oint d\hat{s}(\xi)\partial_{\hat{n}}\hat{G}_D(\xi, \xi')\bar{\psi}(\xi). \tag{8}$$

Let us take the lowest order in the area cosmological constant perturbative expansion. When we integrate φ we consider a fixed value of ψ. Then $\mathcal{D}_{\tilde{g}}\varphi = \mathcal{D}_{\tilde{g}}\bar{\varphi}$ and the lowest order contribution to the wave functional is given by

$$Z^{00}[Y, \psi, \eta] = \exp\left\{-S_c[X_c, \tilde{g}] - S_c^0[\varphi_c, \hat{g}]\right\} \left(Det'\hat{\Delta}\right)^{-d/2} \frac{\sqrt{Det'\hat{P}^\dagger\hat{P}}}{Vol(CKV)}\bar{Z}^0[Y, \psi, \eta]$$

where

$$\bar{Z}^0[Y, \psi, \eta] = \int \mathcal{D}_{\tilde{g}}\bar{\varphi}\exp\left\{-\bar{S}^0[\bar{\varphi}, \hat{g}]\right\}.$$

Above we have introduced the lowest order Liouville actions for $\bar{\varphi}$

$$\bar{S}^0[\bar{\varphi}, \hat{g}] = -\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\bar{\varphi}\hat{\Delta}\bar{\varphi} + \hat{R}\bar{\varphi}\right) + \nu_1\oint d\hat{s}\partial_{\hat{n}}\bar{\varphi}$$

and for φ_c

$$S_c^0[\varphi_c, \hat{g}] = -\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi_c\partial_b\varphi_c + \hat{R}\varphi_c\right) - $$
$$-\frac{d-26}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi_c + \lambda_1\oint d\hat{s}e^{\varphi_c/2}. \tag{9}$$

The functional integration measure for the integral over $\bar{\varphi}$ is conformally invariant but non-linear in the Liouville field:

$$\|\delta\bar{\varphi}\|_{\hat{g}}^2 = \int d^2\xi\sqrt{\hat{g}}e^\varphi(\delta\bar{\varphi})^2.$$

To proceed we need to use David, Distler and Kawai's renormalisation ansatz [2]. We may consider a canonical measure in the background $\hat{g}_{ab}$,

$$\left\|\delta\bar{\phi}\right\|_{\hat{g}}^2 = \int d^2\xi\sqrt{\hat{g}}(\delta\bar{\phi})^2,$$

provided we renormalise the Liouville field and its couplings to 2D gravity. Observe that this renormalisation involves the whole Liouville field. As pointed out by Symanzik in the presence of the boundary we should expect to take independent bulk and boundary renormalisations [17]. Since the boundary pieces of the Liouville mode are fixed at the moment we do not need to worry about them for the time being. We also note that the canonical measure can only be introduced if a set of background counterterms is included:

$$S_R(\hat{g}) = \mu_3^2\int d^2\xi\sqrt{\hat{g}} + \lambda_3\oint d\hat{s} + \nu_3\oint d\hat{s}k_{\hat{g}}.$$

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When we renormalise the field $\bar{\varphi} \to \alpha\bar{\phi}$ and its couplings to gravity we get the following renormalised lowest order Liouville action:

$$\bar{S}^0[\bar{\phi}, \hat{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\bar{\phi}\hat{\Delta}\bar{\phi} + Q\hat{R}\bar{\phi} \right) + \nu_2 \oint d\hat{s}\partial_{\hat{n}}\bar{\phi}.$$

The renormalised parameters of the theory are determined by requiring invariance under a shift in ϕ and a compensating Weyl transformation of the reference metric. Once ϕ has been integrated out the result is required to be invariant under Weyl transformations of the metric alone. For the moment we integrate $\bar{\phi}$. To do so we need to follow Alvarez [15] and set ν_2 to zero because the standard way to deal with a term that is linear in the field is to shift the integration variable, in this case by a constant, but this would spoil the homogeneous Dirichlet condition on $\bar{\phi}$. Next we change variables as follows $\sqrt{8\pi}\bar{\phi} \to \bar{\phi} + \hat{O}_Q^0$. Here we have set

$$\hat{O}_Q^0(\xi') = \int d^2\xi \sqrt{\hat{g}(\xi)}\hat{J}_Q^0(\xi)\hat{G}_D(\xi, \xi'), \quad \hat{O}_Q^0|_B = 0$$

and introduced the current $\hat{J}_Q^0 = Q\hat{R}$. As a result we get the free field integrand

$$S_F[\bar{\phi}, \hat{g}] = \frac{1}{2} \int d^2\xi \sqrt{\hat{g}}\bar{\phi}\hat{\Delta}\bar{\phi}$$

plus the non-local functional

$$\mathcal{F}_D^0[\hat{g}] = \frac{Q^2}{16\pi} \int d^2\xi d^2\xi' \sqrt{\hat{g}(\xi)}\hat{R}(\xi)\hat{G}_D(\xi, \xi')\sqrt{\hat{g}(\xi')}\hat{R}(\xi'). \tag{10}$$

Because there is no zero mode $\hat{G}_D(\xi, \xi')$ is Weyl invariant for distinct values of its arguments (coincident values require regularisation which introduces dependence on the scale of the metric). Thus, the Weyl anomaly associated with eq. (10) is determined by the scaling of the current

$$\delta_\rho \sqrt{\hat{g}}\hat{R} = \sqrt{\hat{g}}\hat{\Delta}\rho. \tag{11}$$

Integrating by parts we find:

$$\delta_\rho \mathcal{F}_D^0 = \frac{Q^2}{8\pi} \int d^2\xi \sqrt{\hat{g}}\hat{R}\rho + \frac{Q^2}{8\pi} \oint d\hat{s}(\xi) \int d^2\xi' \rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi, \xi')\sqrt{\hat{g}(\xi')}\hat{R}(\xi'). \tag{12}$$

The product of functional determinants resulting from the integration over the matter field, the reparametrisations and $\bar{\phi}$ also varies under a Weyl transformation:

$$\delta_\rho \ln\left[\left(Det'\hat{\Delta}\right)^{-(d+1)/2} \frac{\sqrt{Det'\hat{P}^\dagger\hat{P}}}{Vol(CKV)} \right] = \frac{d-25}{48\pi} \int d^2\xi \sqrt{\hat{g}}\hat{R}\rho + \frac{d-25}{24\pi} \oint d\hat{s}k_{\hat{g}}\rho +$$

$$+ C'_1 \oint d\hat{s}\partial_{\hat{n}}\rho + \frac{C'_2}{\varepsilon} \int d^2\xi \sqrt{\hat{g}}\rho + \frac{C'_3}{\sqrt{\varepsilon}} \oint d\hat{s}\rho + O(\sqrt{\varepsilon}), \tag{13}$$

where the C'_i are dimensionless constants which as before can be determined exactly.

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Ignoring the counterterms for the moment we cancel the bulk local piece of the Weyl anomaly between eqs. (12) and (13) if we set

$$Q = \pm\sqrt{\frac{25 - d}{6}}.$$

Since ρ is an arbitrary infinitesimal Weyl scaling in the bulk and on the boundary of the surface we also need to deal with the non-local term and with the local boundary contribution in the geodesic curvature found respectively in eqs. (12) and (13). To do so we have to consider the integration over the boundary values of the Liouville field.

First we integrate Y^μ and η. The boundary measures for these fields are induced by the natural reparametrisation invariant inner products on variations of the boundary values:

$$\|\delta Y\|_{\tilde{g}}^2 = \oint d\tilde{s}\,\delta Y \cdot \delta Y, \quad \|\delta\eta\|_{\tilde{g}}^2 = \oint d\tilde{s}(\delta\eta)^2.$$

As the formalism is explicitly reparametrisation invariant the integration over η is trivial leading to an overall factor. For the boundary matter field we find:

$$\int \mathcal{D}_{\tilde{g}}Y \exp\left\{-S_c[X_c, \tilde{g}]\right\} = \left(\frac{Det'\tilde{K}_D}{\oint d\tilde{s}}\right)^{-d/2} \int \prod_\mu dY_0^\mu. \tag{14}$$

Above we took into account the zero mode of the boundary kernel $\hat{K}_D$. Its existence can be seen by considering the eigenvalue problem

$$\oint d\hat{s}(\xi)\hat{K}_D(\xi, \xi')\hat{v}_N(\xi) = \hat{\lambda}_N \hat{v}_N(\xi').$$

These eigenfunctions form a complete and orthonormal set of functions on the boundary:

$$\sum_N \hat{v}_N(\xi)\hat{v}_N(\xi') = \hat{\delta}_B(\xi - \xi'), \quad \oint d\hat{s}(\xi)\hat{v}_N(\xi)\hat{v}_M(\xi) = \delta_{NM}.$$

Here the boundary delta function is defined by $\oint d\hat{s}(\xi)\hat{\delta}_B(\xi - \xi')f(\xi) = f(\xi')$. Then the eigenvalues may be expressed as

$$\hat{\lambda}_N = \oint d\hat{s}(\xi)d\hat{s}(\xi')\hat{v}_N(\xi)\hat{K}_D(\xi, \xi')\hat{v}_N(\xi') =$$
$$= -\frac{1}{8\pi} \oint d\hat{s}(\xi)d\hat{s}(\xi')\hat{v}_N(\xi)\partial_{\hat{n}}\partial_{\hat{n}'}\hat{G}_D(\xi, \xi')\hat{v}_N(\xi').$$

Now define $\hat{V}_N$ to be the solution of Laplace's equation with boundary value v_N:

$$\hat{\Delta}\hat{V}_N = 0, \quad \hat{V}_N|_B = \hat{v}_N.$$

This has the solution

$$\hat{V}_N(\xi') = -\oint d\hat{s}(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\hat{v}_N(\xi),$$

enabling us to write the eigenvalues as

$$\hat{\lambda}_N = \frac{1}{8\pi}\oint d\hat{s}(\xi)\hat{V}_N(\xi)\partial_{\hat{n}}\hat{V}_N(\xi) = \frac{1}{8\pi}\int d^2\sqrt{\hat{g}}\hat{g}^{ab}\partial_a\hat{V}_N\partial_b\hat{V}_N.$$

Thus $\hat{\lambda}_N \geq 0$ and it is only zero when $\hat{V}_N$ is constant. Denoting this solution by $N = 0$ and using the normalisation condition we conclude that $\hat{K}_D$ has the zero mode $\hat{v}_0 = (\oint d\hat{s})^{-1/2}$.

The determinant in eq. (14) will generate a new boundary term for the Liouville action. This is the gluing anomaly found in [16]. The kernel $\tilde{K}_D$ has a boundary heat kernel which can only be sensitive to short distance effects, and since the boundary has no intrinsic geometry it can only be sensitive to the invariant length of the boundary. As a consequence covariance and dimensional analysis lead to a contribution to the Weyl anomaly which can be absorbed into the cosmological constant counterterm in the invariant world-sheet length of the boundary.

To cancel the remaining terms in the Weyl anomaly we have to integrate ψ. Just as in the case of $\bar{\varphi}$ we have a non-linear inner product on variations of ψ:

$$\|\delta\psi\|_{\tilde{g}}^2 = \oint d\hat{s}e^{\psi/2}(\delta\psi)^2.$$

We will assume, following David, Distler and Kawai, that we can use the inner product that is more usual for a quantum field in the background $\hat{g}_{ab}$,

$$\|\delta\Psi\|_{\hat{g}}^2 = \oint d\hat{s}(\delta\Psi)^2,$$

provided we renormalise $\psi_0 \rightarrow \alpha_0\Psi_0$, and $\bar{\psi} \rightarrow \alpha_B\bar{\Psi}$ as well as their couplings to 2D quantum gravity. Note that this means we need to introduce an independent field renormalisation for $\bar{\varphi}_c$, the component of φ_c orthogonal to the zero mode ψ_0. According to eq. (8), its explicit expression in terms of $\bar{\psi}$ involves a coupling to 2D gravity. Thus we must also consider $\bar{\varphi}_c \rightarrow \bar{\alpha}_B\bar{\phi}_c$. This is to be done in each order of the perturbative expansion in the length cosmological constant. Note that we have allowed for a different renormalisation of ψ_0 and $\bar{\psi}$. This is because we take independent bulk and boundary renormalisations and ψ_0 is related to the zero mode of the Laplacian on closed surfaces that would be generated if we glued together two disc shaped topologies to obtain a sphere, corresponding to the inner product of the closed string vacuum with itself. Thus ψ_0 is really associated with the Liouville field in the bulk and should be renormalised accordingly.

Now when we decompose ψ into ψ_0 and $\bar{\psi}$ eq. (9) can be rewritten as:

$$S_c^0[\bar{\psi},\psi_0,\hat{g}] = -\frac{d-26}{12}\oint d\hat{s}(\xi)d\hat{s}(\xi')\bar{\psi}(\xi)\hat{K}_D(\xi,\xi')\bar{\psi}(\xi') - \frac{d-26}{24\pi}\oint d\hat{s}k_{\hat{g}}\bar{\psi} +$$
$$+\lambda_1\oint d\hat{s}e^{\bar{\psi}+\psi_0} + \frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}(\xi)}\hat{R}(\xi)\oint d\hat{s}(\xi')\partial_{\hat{n}'}\hat{G}_D(\xi,\xi')\bar{\psi}(\xi') -$$
$$-\frac{d-26}{12}\chi_o\psi_0.$$

Introducing the coupling renormalisation parameters Q_0, Q_B and $\bar{Q}_B$ we write the renormalised lowest order boundary action

$$S_c^{00}[\bar{\Psi}, \Psi_0, \hat{g}] = \tfrac{1}{2} \oint d\hat{s}(\xi) d\hat{s}(\xi') \bar{\Psi}(\xi) \hat{K}_D(\xi, \xi') \bar{\Psi}(\xi') + \oint d\hat{s} \hat{H}_D^{00} \bar{\Psi} +$$
$$+ \frac{Q_0 \chi_o}{2} \Psi_0, \tag{15}$$

where we have the current

$$\hat{H}_D^{00}(\xi) = -\frac{Q_B}{8\pi} \int d^2\xi' \sqrt{\hat{g}(\xi')} \hat{R}(\xi') \partial_{\hat{n}} \hat{G}_D(\xi, \xi') + \frac{\bar{Q}_B}{8\pi} k_{\hat{g}}(\xi). \tag{16}$$

To integrate this we shift out the linear piece in $\bar{\Psi}$. We introduce the Green's function of $\hat{K}_D$ defined by

$$\oint d\hat{s}(\xi'') \hat{K}_D(\xi, \xi'') \hat{G}_K(\xi'', \xi') = \hat{\delta}_B(\xi - \xi') - \frac{1}{\oint d\hat{s}(\xi''')}. \tag{17}$$

The last term on the right-hand side of eq. (17) is necessary to ensure consistency when the equation is integrated with respect to $\hat{s}(\xi)$, since

$$\oint d\hat{s}(\xi) \hat{K}_D(\xi, \xi') = 0.$$

Its value is fixed by the zero mode of $\hat{K}_D$ we have calculated before. Also $\hat{G}_K(\xi, \xi')$ is symmetric in its arguments and is orthogonal to the constant zero mode

$$\oint d\hat{s}(\xi) \hat{G}_K(\xi, \xi') = 0. \tag{18}$$

Then we can consider the shift $\bar{\Psi} \to \bar{\Psi} + \hat{\mathcal{F}}_K^{00}$ where

$$\hat{\mathcal{F}}_K^{00}(\xi') = \oint d\hat{s}(\xi) \hat{H}_D^{00}(\xi) \hat{G}_K(\xi, \xi')$$

is also orthogonal to the zero mode. Thus the integration leads to

$$\int \mathcal{D}_{\hat{g}}(\bar{\Psi}, \Psi_0) \exp\{-S_c^{00}[\bar{\Psi}, \Psi_0, \hat{g}]\} = e^{\mathcal{F}_B^{00}} \left(\frac{Det' \tilde{K}_D}{\oint d\tilde{s}} \right)^{-d/2} \int d\Psi_0 e^{-Q_0 \chi_o \Psi_0 / 2}$$

where

$$\mathcal{F}_B^{00} = \frac{1}{2} \oint d\hat{s}(\xi) d\hat{s}(\xi') \hat{H}_D^{00}(\xi) \hat{G}_k(\xi, \xi') \hat{H}_D^{00}(\xi'). \tag{19}$$

The determinant only changes the background renormalisation counterterm in the world-sheet length. The important contribution to the Weyl anomaly comes from eq. (19). To calculate it we first need the Weyl transformation associated with eq. (16). Using eq. (11) and the corresponding transformation of the geodesic curvature

$$\delta_\rho d\hat{s} k_{\hat{g}} = \frac{1}{2} d\hat{s} \partial_{\hat{n}} \rho,$$

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we take a total derivative and introduce the boundary kernel $\hat{K}_D$ to find:

$$\delta_\rho[d\hat{s}(\xi')\hat{H}_D^{00}(\xi')] = Q_B \oint d\hat{s}(\xi)\rho(\xi)d\hat{s}(\xi')\hat{K}_D(\xi,\xi') +$$
$$+ \tfrac{1}{16\pi}(\bar{Q}_B - 2Q_B)d\hat{s}(\xi')\partial_{\hat{n}'}\rho(\xi'). \qquad (20)$$

Then eqs. (17) and (16) lead us to

$$\delta_\rho\mathcal{F}_B^{00} = -\frac{Q_B^2}{8\pi}\oint d\hat{s}(\xi)\int d^2\xi'\rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\sqrt{\hat{g}(\xi)}\hat{R}(\xi') + \frac{Q_B^2}{4\pi}\oint d\hat{s}k_{\hat{g}}\rho. \qquad (21)$$

Above we have taken $\bar{Q}_B = 2Q_B$ which is a condition needed to eliminate the contribution associated with the outward normal derivative of ρ:

$$\frac{\bar{Q}_B - 2Q_B}{16\pi}\oint d\hat{s}(\xi)d\hat{s}(\xi')\partial_{\hat{n}}\rho(\xi)\hat{G}_K(\xi,\xi')\hat{H}_D^{00}(\xi').$$

Also we note that the zero mode integration defines a net charge selection rule for the gravitational sector just like in the closed string. This allow us to ignore the non-local contributions to eq. (21) coming from the zero mode of the kernel $\hat{K}_D$. They will be generated by eq. (17) and by the non-local Weyl anomaly associated with $\hat{G}_K(\xi,\xi')$. We find (see Appendix A):

$$\delta_\rho\hat{G}_K(\xi,\xi') = -\frac{1}{2\oint d\hat{s}(\xi''')}\oint d\hat{s}(\xi'')\rho(\xi'')\left[\hat{G}_K(\xi'',\xi) + \hat{G}_K(\xi'',\xi')\right]. \qquad (22)$$

This will also contribute when the two points approach each other. In this case we must also include the contribution coming from the regularisation of $\hat{G}_K$ at coincident points. We use the reparametrisation invariant heat kernel

$$\hat{G}_{K\varepsilon}(\xi,\xi') = \int_\varepsilon^\infty dt \left[\hat{\mathcal{G}}_K(t,\xi,\xi') - \frac{1}{\oint d\hat{s}(\xi'')}\right]$$

where $\hat{\mathcal{G}}_K$ satisfies the generalised heat equation

$$\frac{\partial}{\partial t}\hat{\mathcal{G}}_K(t,\xi,\xi') = \oint d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')\hat{\mathcal{G}}_K(t,\xi'',\xi'), \quad \hat{\mathcal{G}}_K(0,\xi,\xi') = \hat{\delta}_B(\xi-\xi').$$

For coincident arguments the regularisation of the Green's function is controlled by the small-t behaviour of the heat kernel which is computable in a standard perturbation series [16]. This thus leads to:

$$\delta_\rho\hat{G}_{K\varepsilon}(\xi',\xi') = 4\rho(\xi') - \frac{1}{\oint d\hat{s}(\xi''')}\oint d\hat{s}(\xi'')\rho(\xi'')\hat{G}_K(\xi'',\xi'). \qquad (23)$$

All these non-local contributions always decouple one of the variables, so they will generate terms in eq. (21) which will all be proportional to the net charge on the whole surface.

To eliminate remaining terms between eqs. (12), (13) and (21) we need $Q_B = Q$. If we finally tune the background cosmological counterterm contributions to zero we get a Weyl invariant lowest order partition function. This shows that we need to include the counterterm in the geodesic curvature because otherwise the finite contribution coming from the reparametrisation ghosts cannot be eliminated. Of course in this particular lowest order case we have a null contribution to the partition function because the net charge, $\oint d\hat{s}(\xi)\hat{H}_D^{00}(\xi)$, is the topological background gravity charge which for the disc is never zero due to the Gauss-Bonnet theorem. However all the terms we have discussed will persist in the more complicated expressions that satisfy the charge selection rule.

This analysis still leaves the parameter Q_0 undetermined. To find it we make the connection with the closed string partition function. As explained earlier this is obtained by identifying the arguments of two copies of $Z[Y, \psi, \eta]$ and integrating over these boundary values. This corresponds to gluing together two discs along their boundaries to produce a sphere. The closed string partition function is

$$Z_{closed} = \int \mathcal{D}_{\tilde{g}}(Y, \psi, \eta) Z_{open}^1[Y, \psi, \eta] Z_{open}^2[Y, \psi, \eta].$$

When we integrate the string field X^μ and the reparametrisation ghosts in each open string wave functional we find:

$$Z_{closed} = \int \mathcal{D}_{\tilde{g}}(\psi, \varphi_1, \varphi_2) \exp\{-S_L[\varphi_1, \hat{g}_1] - S_L[\varphi_2, \hat{g}_2]\}$$
$$\left(Det'\hat{\Delta}_1\right)^{-d/2} \frac{\sqrt{Det'\hat{P}_1^\dagger \hat{P}_1}}{Vol_1(CKV)} \left(Det'\hat{\Delta}_2\right)^{-d/2} \frac{\sqrt{Det'\hat{P}_2^\dagger \hat{P}_2}}{Vol_2(CKV)}.$$

Here the boundary fields Y^μ, η have already been integrated and absorbed in the length renormalisation counterterm. The next step is to perturb in each area renormalisation counterterm and in the common length cosmological constant. Just like before we split each field φ_i, $i = 1, 2$ in two independent fields φ_{ci}, $\bar{\varphi}_i$. In the present case we only need to consider the lowest order in the perturbative expansion. Then we have the following decomposition

$$Z_{closed} = Z_B^{00} \bar{Z}_{open}^1 \bar{Z}_{open}^2$$
$$\left(Det'\hat{\Delta}_1\right)^{-d/2} \frac{\sqrt{Det'\hat{P}_1^\dagger \hat{P}_1}}{Vol_1(CKV)} \left(Det'\hat{\Delta}_2\right)^{-d/2} \frac{\sqrt{Det'\hat{P}_2^\dagger \hat{P}_2}}{Vol_2(CKV)}$$

where the boundary partition function is

$$Z_B^{00} = \int \mathcal{D}_{\tilde{g}}(\psi, \psi_0) \exp\{-S_B^{00}[\psi, \psi_0, \hat{g}]\}.$$

Above we have used the simple property that the outward normal derivative of one of the open surfaces is just the inward normal derivative of the other at the common boundary, plus the Gauss-Bonnet theorems given in eqs. (2) and (4) to find the boundary action:

$$S_B^{00}(\bar{\psi}, \psi_0, \hat{g}) = -\frac{d-26}{12} \oint d\hat{s} d\hat{s}' \bar{\psi}(\xi) \hat{K}_D(\xi, \xi') \bar{\psi}(\xi') - \frac{d-26}{12} \chi_c \psi_0.$$

Next we renormalise the fields and their couplings to 2D gravity to consider canonical measures in the background $\hat{g}_{ab}$. When we integrate $\bar{\phi}_i$ we get the same Weyl anomaly for each field and again using the property of the normal derivative at the common boundary we can easily see that the boundary contributions cancel up to the usual length renormalisation counterterm, leading to $Q_i = Q$, $i = 1, 2$, where the Q_i define the renormalisation of the coupling of the φ_i to the scalar curvature $\hat{R}_i$. The boundary integration is just equal to the zero mode charge selection rule. If $Q_0 = Q$ that is exactly the selection rule we get for the closed string.

3.1.2 Anomaly cancellation for Liouville field renormalisation

So far we have only been able to determine parameters associated with the renormalisation of the couplings to 2D quantum gravity. To go further and calculate the Liouville field renormalisation we need to consider higher orders in the Coulomb gas perturbative expansion. In the case of the couplings we have seen that the renormalised central charge of the conformally extended Liouville field theory is exactly the same as the corresponding central charge of the same theory on a closed surface. We have also proved that the boundary couplings are fixed by this value of the central charge. We have seen that this is all a consequence of the quantum Weyl invariance of the theory. Now we want to find out if the bulk field renormalisation is equal to the corresponding closed string parameter and if the boundary field renormalisation is actually the same as its bulk counterpart as it should happen when we interpret the Liouville field as an arbitrary Weyl scaling defined everywhere on the surface including its boundary. As Symanzik's work makes clear, this is not something we should take for granted. We will now show that this is also a consequence of the quantum Weyl invariance assumed for the theory.

We start with the case where we have a single Liouville vertex operator on the bulk:

$$\int d^2\xi \sqrt{\hat{g}} e^{\alpha_0 \Psi_0 + \alpha \bar{\phi} + \bar{\alpha}_B \bar{\phi}_c}. \tag{24}$$

In this case we find the following action for $\bar{\phi}$

$$S_L^1\left[\bar{\phi}, \hat{g}\right] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2} \bar{\phi} \hat{\Delta} \bar{\phi} + \hat{J}_Q^1 \bar{\phi}\right)$$

where we need the current

$$\hat{J}_Q^1(\xi) = Q\hat{R}(\xi) - 8\pi\alpha \frac{\delta^2(\xi - \xi')}{\sqrt{\hat{g}(\xi)}}.$$

By shifting $\bar{\phi}$ we generate the functional:

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$$\mathcal{F}_D^1[\hat{g}] = \mathcal{F}_D^0[\hat{g}] - \alpha Q \int d^2\xi \sqrt{\hat{g}(\xi)} \hat{R}(\xi) \hat{G}_D(\xi, \xi') + 4\pi\alpha^2 \hat{G}_D(\xi', \xi'). \qquad (25)$$

On the other hand we also find the following renormalised boundary action

$$S_c^{10}[\bar{\Psi}, \Psi_0, \hat{g}] = \tfrac{1}{2} \oint d\hat{s}(\xi) d\hat{s}(\xi') \bar{\Psi}(\xi) \hat{K}_D(\xi, \xi') \bar{\Psi}(\xi') + \oint d\hat{s} \hat{H}_D^{10} \bar{\Psi} + $$
$$+ \left(\tfrac{Q_0 \chi_o}{2} - \alpha_0 \right) \Psi_0$$

where we have introduced the current

$$\hat{H}_D^{10}(\xi) = \hat{H}_D^{00}(\xi) + \bar{\alpha}_B \partial_{\hat{n}} \hat{G}_D(\xi, \xi').$$

In this case we get

$$\mathcal{F}_B^{10} = \mathcal{F}_B^{00} + \bar{\alpha}_B \oint d\hat{s}(\xi) d\hat{s}(\xi'') \hat{H}_D^{00}(\xi) \hat{G}_K(\xi, \xi'') \partial_{\hat{n}''} \hat{G}_D(\xi'', \xi') + $$
$$+ \tfrac{1}{2} \bar{\alpha}_B^2 \oint d\hat{s}(\xi) d\hat{s}(\xi'') \partial_{\hat{n}} \hat{G}_D(\xi, \xi') \hat{G}_K(\xi, \xi'') \partial_{\hat{n}''} \hat{G}_D(\xi'', \xi'). \qquad (26)$$

To analyse the anomaly cancellation in this order of the perturbative expansion we first recall that although $\hat{G}_D(\xi, \xi')$ is Weyl invariant for distinct values of its arguments, at coincident points it requires regularisation which introduces dependence on the scale of the metric. To calculate the correspondent Weyl transformation we represent $\hat{G}_D(\xi, \xi')$ in terms of the Green's function $\hat{G}(\xi, \xi')$ considered on the whole plane

$$\hat{G}_D(\xi, \xi') = \hat{G}(\xi, \xi') - \hat{H}_D(\xi, \xi'),$$

where $\hat{H}_D(\xi, \xi')$ satisfies the boundary-value problem:

$$\hat{\Delta} \hat{H}_D(\xi, \xi') = 0, \quad \hat{H}_D(\xi, \xi')|_{\xi' \in B} = \hat{G}(\xi, \xi')|_{\xi' \in B}.$$

When $\xi = \xi'$ is on the bulk $\hat{H}_D(\xi, \xi)$ is Weyl invariant. Also on the whole plane there is no zero mode. Thus the Weyl transformation of $\hat{G}_D(\xi, \xi)$ is just given by the corresponding well known local change of $\hat{G}(\xi, \xi)$ [1, 10]:

$$\delta_\rho \hat{G}_{D\varepsilon}(\xi, \xi) = \frac{\rho(\xi)}{4\pi}, \quad \xi \notin B. \qquad (27)$$

Then applying eqs. (11), (27) we conclude that the Weyl anomaly of eq. (25) is given by

$$\delta_\rho \mathcal{F}_D^1 = \delta_\rho \mathcal{F}_D^0 - \alpha Q \oint d\hat{s}(\xi) \rho(\xi) \partial_{\hat{n}} \hat{G}_D(\xi, \xi') + (\alpha^2 - \alpha Q) \rho(\xi').$$

On the other hand, ignoring the non-local zero mode contributions which are all proportional to the net charge on the whole surface given in this order by $\oint d\hat{s} \hat{H}_D^{10}$, we use eq. (20) and $\bar{Q}_B = 2Q_B$ to find the Weyl anomaly of eq. (26):

$$\delta_\rho \mathcal{F}_D^{10} = \delta_\rho \mathcal{F}_D^{00} + \bar{\alpha}_B Q_B \oint d\hat{s}(\xi)\rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi').$$

Thus we can easily see that to ensure Weyl invariance at the quantum level we must further set $Q_B = Q$, $\bar{\alpha}_B = \alpha$ and

$$1 - \alpha Q + \alpha^2 = 0.$$

Here we took into account the contribution to the Weyl anomaly of the $\sqrt{\hat{g}}$ present in eq. (24). Introducing the value of Q we find:

$$\alpha_\pm = \frac{1}{2\sqrt{6}}\left(\sqrt{25-d} \pm \sqrt{1-d}\right).$$

As we noted previously, these renormalised parameters only cancel the local contributions to the Weyl anomaly. As in the lowest order case we have to assume the charge selection rule associated with the zero mode integration to eliminate the non-local pieces. To find the renormalised parameters of the charge selection rule we need to glue the two discs to form a sphere enabling us to use the closed string result. We already know the value of Q_0 but now we also want the value of α_0. The calculation goes exactly as before, all boundary contributions cancel out up to the length renormalisation counterterm and we find a zero mode integral which corresponds to a closed string selection rule with two bulk vertex operator charges α_0 and a background gravity charge $Q_0 = Q$. This implies that $\alpha_0 = \alpha$ as expected.

With this calculation we are able to guarantee Weyl invariance at the quantum level for insertions of arbitrary numbers of gravitational Liouville vertex operators in the bulk. To see what happens when operators are inserted on the boundary let us consider the simplest case of just one such operator,

$$\oint d\hat{s} e^{\alpha_0 \Psi_0/2 + \alpha_B \bar{\Psi}/2}. \tag{28}$$

In this case only the boundary integration over ψ gets changed. The renormalised boundary Liouville action is

$$S_c^{01}[\bar{\Psi}, \bar{\Psi}_0, \hat{g}] = \tfrac{1}{2} \oint d\hat{s}(\xi)d\hat{s}(\xi')\bar{\Psi}(\xi)\hat{K}_D(\xi,\xi')\bar{\Psi}(\xi') + \oint d\hat{s}\hat{H}_D^{01}\bar{\Psi} +$$
$$+ \left(\frac{Q_0\chi_o}{2} - \frac{\alpha_0}{2}\right)\bar{\Psi}_0$$

where we have introduced the current

$$\hat{H}_D^{01}(\xi) = \hat{H}_D^{00}(\xi) - \frac{\alpha_B}{2}\hat{\delta}_B(\xi - \xi').$$

The relevant functional is now:

$$\mathcal{F}_B^{01} = \mathcal{F}_B^{00} - \frac{\alpha_B}{2}\oint d\hat{s}(\xi)\hat{H}_D^{00}(\xi)\hat{G}_K(\xi,\xi') + \frac{\alpha_B^2}{8}\hat{G}_K(\xi',\xi').$$

To find out our last renormalised parameter α_B we need the local Weyl transformation of $\hat{G}_K$ at coincident points given in eq. (23). Thus the local anomaly vanish if all the other parameters keep their previous values and $1/2 - \alpha_B Q/2 + \alpha_B^2/2 = 0$, where the $1/2$ term comes from the Weyl transformation of $d\hat{s}$ in eq. (28). Thus $\alpha_B = \alpha$.

Since as before the non-local contributions cancel due to the charge selection rule this result shows that the full perturbative expansion is Weyl invariant at the quantum level for the values of the renormalised parameters found. Whenever we couple distinct Liouville vertex operators in higher orders there are no additional Weyl anomalous contributions.

3.1.3   Comments

Our results for the non-critical open string show that the gravitational sector can be interpreted as a conformally extended boundary Liouville field theory. In this picture Q defines the central charge of the Liouville theory $c_\phi = 1 + 6Q^2$, which has its value fixed by demanding that it cancels the central charges of the matter and ghost systems $c_M + c_{gh} = d - 26$. Thus the central charge of the theory with boundary is equal to the central charge of the theory without boundary. This is to be expected since anomalies are local effects.

We have interpreted the Liouville field as an arbitrary Weyl scaling all over the open surfaces. Then we found that the value of α is exactly right to define a Liouville vertex operator $\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi}$ of zero conformal weight. On the extended field theory it corresponds to a primary field $: e^{\alpha\phi} :$ of weight $(1,1)$. As expected α has the same value it takes when the surfaces are closed. We also found the right value for α_B in the sense that the boundary vertex operator $\oint d\hat{s} e^{\alpha_B \phi/2}$ has zero conformal weight corresponding to the boundary primary field $: e^{\alpha_B \phi/2} :$ of conformal weight $(1/2, 1/2)$. This means that the renormalisation of the Liouville field is the same all over the surface and is equal to the renormalisation on the closed surface as it should be.

We are now in a position to see that Dirichlet boundary conditions on the Liouville field imply that the metric is discontinuous as the boundary is approached. In this case the calculation stops at S_c^{00} in eq. (15), since we do not integrate over boundary values of the Liouville field, but leave them fixed. The Weyl anomaly of eq. (13) must now be cancelled by the Weyl transform of S_c^{00}, together with a shift in the boundary value of the Liouville field, Ψ. This fixes the latter to be $\delta\Psi = -Q\rho$. Now the full metric is a reparametrisation of $\hat{g}_{ab} e^{\alpha\Psi}$, which should be invariant under this simultaneous Weyl transformation on $\hat{g}_{ab}$ and shift in Ψ, since the separation into reference metric and Liouville field is arbitrary. However it is not because $Q \neq 1/\alpha$, as the correct relation, $1 - \alpha Q + \alpha^2 = 0$, has an extra quantum piece. One way out of this would be to assume that the Liouville field is renormalised differently in the bulk and on the boundary, a phenomenon that occurs in ϕ^4 theory in four dimensions, [17]. However, this implies that the metric is discontinuous as the boundary is approached, and also that the functionals obtained by imposing Dirichlet boundary conditions cannot be sewn together to make closed surface functionals.

As for the closed string we also find the need to restrict the validity of the approach to target space dimensions $d \leq 1$. Only in this way we have real renormalised parameters such that $e^{\alpha\phi}$ and $e^{\alpha_B\phi/2}$ can be interpreted as real Weyl scalings for a real scalar renormalised Liouville field ϕ. From this we can see that our results extend very naturally those found for the closed string by David, Distler and Kawai. Since the analysis is fully local and we can choose the moduli integration measure to be independent of the conformal factor of the metric our results also generalise immediately to higher genus Riemann surfaces with just one boundary. Clearly more general boundary structures can also be considered. Here for simplicity we have just analysed the random surfaces one loop functional defined in euclidean space. Our results hold for an arbitrary number of loops. We also may consider non-smooth boundaries [16].

3.1.4 Tachyon gravitational dressings

Since this formalism is only valid for $d \leq 1$ the string serves as a toy model for the more realistic $c \leq 1$ minimal series of boundary conformal field theories [18]. With this in mind let us see how a bulk tachyon vertex operator gets dressed by the gravitational sector. Taking an n-point function of bulk tachyons with momentum p_j such that the momenta sum to zero we can easily see by following the same path of calculations that the operator $\int d^2\xi_j \sqrt{\tilde{g}_j} e^{ip_j \cdot X(\xi_j)}$ gets dressed to $\int d^2\xi_j \sqrt{\hat{g}_j} e^{\gamma_j \phi} e^{ip_j \cdot X(\xi_j)}$, where quantum Weyl invariance demands

$$\Delta_j^0 - \gamma_j(\gamma_j - Q) = 1, \quad \Delta_j^0 = p_j^2.$$

The above equation shows that the dressed bulk tachyon vertex operator has zero conformal weight. The primary Liouville field : $e^{\gamma_j \phi}$: dresses the tachyon field : $e^{ip_j \cdot X(\xi_j)}$: in such a way that : $e^{\gamma_j \phi} e^{ip_j \cdot X(\xi_j)}$: has conformal weight $(1,1)$. Note that if we solve for γ_j in terms of Q and p_j^2 we get $\gamma_j = (1/2)\left[Q \pm \sqrt{Q^2 + 4(p_j^2 - 1)}\right]$. Just as α should be real for an arbitrary real Weyl scaling so should be γ_j. This implies that $d \leq 1$ and $p_j^2 \geq 0$.

If we now consider a boundary tachyon vertex operator $\oint d\hat{s}_j e^{ip_j \cdot X(\xi_j)/2}$ we may follow the same steps to find the dressed operator $\oint d\hat{s}_j e^{\gamma_j \phi/2 + ip_j \cdot X(\xi_j)/2}$ where γ_j satisfies the same equation as the its bulk counterpart. It means that the dressed field : $e^{ip_j \cdot X(\xi_j)/2} e^{\gamma_j \phi/2}$: considered in the boundary of the surface has conformal weight $(1/2,1/2)$ as a consequence of the quantum Weyl invariance of the theory.

3.2 Neumann boundary conditions

The choice of boundary conditions always depends on the specific physical applications we have in mind. So far we have argued that in a proper coupling to 2D quantum gravity, the boundary conditions on the Liouville field have to be such that it can be interpreted as an arbitrary Weyl scaling on the whole surface and not just on its interior. As we said this rules out Dirichlet boundary conditions but we are free to choose Neumann boundary conditions for the conformal factor. To see what

happens in this case let us for simplicity take also Neumann boundary conditions on the matter field $\partial_{\tilde{n}} X^{\mu} = 0$ and on the reparametrisation ghosts $\tilde{n} \cdot \delta\theta = 0$. Consider first the partition function. We can then follow the same reasoning as in the case of free boundary conditions with much more ease because the Neumann boundary condition simply eliminates the most part of the boundary contributions we had to worry about before. Thus we write the following renormalised Liouville action

$$S_L^N [\phi, \hat{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\phi\hat{\Delta}\phi + Q\hat{R}\phi \right) + \frac{\bar{Q}_B}{8\pi} \oint d\hat{s} k_{\hat{g}}\phi +$$
$$+ \mu_2^2 \int d^2 \sqrt{\hat{g}} e^{\alpha\phi} + \lambda_2 \oint d\hat{s} e^{\alpha_B \phi/2}.$$

Here Q, $\bar{Q}_B$ refer to coupling renormalisation and α, α_B are its field renormalisation counterparts. To ensure quantum Weyl invariance (see Appendix B) we must satisfy the charge conservation selection rule, tune the local reference counterterms to zero, set $\bar{Q}_B = 2Q$, $\alpha_B = \alpha$ plus $Q = \pm\sqrt{(25-d)/6}$ and $1 - \alpha Q + \alpha^2 = 0$.

Starting from a general open string bulk tachyon amplitude it is clear that we may follow the steps of the partition function calculation to find the equation for the gravitational dressing of the bulk tachyon vertex operator. A tachyon vertex operator with momentum p_j gets dressed by the coupling to 2D quantum gravity $\int d^2\xi_j \sqrt{\hat{g}} e^{\gamma_j\phi} e^{ip_j \cdot X(\xi_j)}$, where $\Delta_j^0 - \gamma_j(\gamma_j - Q) = 1$, $\Delta_j^0 = p_j^2$. For the boundary tachyon vertex operator the coupling to gravity leads to the dressed operator $\oint d\hat{s}_j e^{\gamma_j\phi/2 + ip_j \cdot X(\xi_j)/2}$ of zero weight, where γ_j satisfies the same equation as the its bulk counterpart.

Thus so far we conclude that our results are exactly the same for Neumann and for free boundary conditions on the Liouville field.

4 Critical exponents and the saddle point limit

4.1 The open string susceptibility and Yang-Mills Feynman mass exponents

Let us consider again the case of free boundary conditions and start with Polyakov's sum over random surfaces with the topology of a disc. Generalising the closed string case [11] the quantum partition function may now be written as an integral of the partition function for surfaces constrained to have fixed area, A, and perimeter, L, $\Gamma(A, L)$:

$$Z = \int_0^{+\infty} \Gamma(A, L) e^{-\mu_0^2 A - \lambda_0 L} dA dL.$$

After integrating out the matter and reparametrisation ghost fields we renormalise the Liouville field and its couplings to 2D gravity to find the following integral for $\Gamma(A, L)$:

$$\Gamma(A, L) = \int \mathcal{D}_{\hat{g}}(\bar{\Psi}, \bar{\phi}) d\bar{\Psi}_0 (\oint d\hat{s})^{1/2} e^{-S_c^{00}[\bar{\Psi}, \bar{\Psi}_0, \hat{g}] - \bar{S}^0[\bar{\phi}, \hat{g}]} \delta \left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A \right)$$
$$\delta \left(\oint d\tilde{s} e^{\alpha/2\phi} - L \right). \tag{29}$$

Here we have factored out the cosmological constant counterterms left over from the renormalisation process. Note that in this process the initially infinite constants μ_0^2 and λ_0 are changed into the finite constants μ_2^2 and λ_2. As discussed before we have set $\nu_2 = 0$.

To calculate the critical exponents we apply David, Distler and Kawai's scaling argument [2]. Consider the shift of the integration variable ϕ by a constant $\phi \rightarrow \phi + \rho/\alpha$. Since we keep $\hat{g}_{ab}$ fixed our functional integral should scale. Recall that the theory is invariant under arbitrary scalings of the reference metric once we have integrated ϕ. So, it is only invariant under a shift of the integration variable provided this is compensated by a Weyl transformation of the reference metric. Because we consider a translational invariant quantum measure in eq. (29), the scaling behavior is determined by the change in the action $S \equiv S_c^{00}[\bar{\Psi}, \bar{\Psi}_0, \hat{g}] + \bar{S}^0[\bar{\phi}, \hat{g}]$, and in the delta functions which are used to fix the area A and the perimeter L of the surface. Being the shift constant only the zero mode $\bar{\Psi}_0$ is actually changed. Thus the shift in the action is

$$S \rightarrow S + \frac{Q\chi_o}{2\alpha}\rho,$$

and the shifts in the delta functions are

$$\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} - A\right) \rightarrow e^{-\rho}\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} - e^{-\rho}A\right),$$
$$\delta\left(\oint d\hat{s}e^{\alpha\phi/2} - L\right) \rightarrow e^{-\rho/2}\delta\left(\oint d\hat{s}e^{\alpha\phi/2} - e^{-\rho/2}L\right).$$

Then we get the following scaling law:

$$\Gamma(A, L) = e^{-\frac{\rho}{2}\left(\frac{\chi_o Q}{\alpha}+3\right)}\Gamma\left(e^{-\rho}A, e^{-\rho/2}L\right). \tag{30}$$

To be able to introduce critical exponents we have to define the partition function for fixed area A, $\Sigma(A)$, and the partition function for fixed perimeter L, $\Omega(L)$. Factoring out the appropriate counterterms we write:

$$\Sigma(A, \lambda_2) = \int \Gamma(A, L)e^{-\lambda_2 L}dL, \quad \Omega(L, \mu_2^2) = \int \Gamma(A, L)e^{-\mu_2^2 A}dA.$$

Then, from eq. (30), we get:

$$\Sigma(A, \lambda_2) = e^{-\rho\left(\frac{\chi_o Q}{2\alpha}+1\right)}\Sigma\left(e^{-\rho}A, \lambda_2 e^{\rho/2}\right), \tag{31}$$

$$\Omega(L, \mu_2^2) = e^{-\frac{\rho}{2}\left(\frac{\chi_o Q}{\alpha}+1\right)}\Omega\left(e^{-\rho/2}L, \mu_2^2 e^{\rho}\right). \tag{32}$$

The open string susceptibility exponent is defined just like in the closed string. In the case $\lambda_2 = 0$ we can continue to use the scaling argument. As $A \rightarrow +\infty$:

$$\Sigma(A) \sim A^{\sigma(\chi_o)-3}.$$

and

$$\sigma(\chi_o) = 2 - \frac{\chi_o Q}{2\alpha}.$$

The last result is just the expected open string version of the closed string critical exponent. If we take the positive root for Q and the corresponding negative one for α we find that in the semi-classical limit $d \to -\infty$:

$$\sigma(\chi_o) = \frac{d-19}{12}\chi_o + 2.$$

For the open string we can also consider the asymptotic limit $L \to +\infty$ and introduce a mass critical exponent in close analogy with the the asymptotic limit $A \to +\infty$. Here we take $\mu_2^2 = 0$. This case was considered by Durhuus, Olesen and Petersen [19] in connection with the calculation of the Wilson loop quark-antiquark potencial. We define $\omega(\chi_o)$ by

$$\Omega(L) \sim L^{\omega(\chi_o)-3}.$$

Thus we find

$$\omega(\chi_o) = 2 - \frac{\chi_o Q}{\alpha},$$

to which we associate the semi-classical limit

$$\omega(\chi_o) = \frac{d-19}{6}\chi_o + 2.$$

We can interpret of $\omega(\chi_o)$ in the context of Yang-Mills gluon dynamics. To see this first note that the wave functional given in eq. (3) models the Wilson loop ,W, for Yang-Mills theory [15, 19]. Consider the first quantised functional integral representing the propagator of a particle of mass λ_2 moving under the influence of a Yang-Mills field. At coincident points its trace is a gauge invariant expression

$$\int \mathcal{D}Y \, tr \, P \, e^{-\lambda_2 \oint d\tilde{s} - \oint dY \cdot \mathcal{A}} = tr \, G_{\mathcal{A}}(x,x).$$

If this is averaged over the Yang-Mills field we get

$$< tr \, G_{\mathcal{A}}(x,x) >_{\mathcal{A}} = \int \mathcal{D}Y \, e^{-\lambda_2 \oint d\tilde{s}} \, W = \int dL \, e^{-\lambda_2 L} \int \mathcal{D}Y \, \delta(L - \oint d\tilde{s}) \, W$$

but this last functional integral is just what we mean by Ω. Substituting the form that holds for $\mu_2^2 = 0$ we get

$$< tr \, G_{\mathcal{A}}(x,x) >_{\mathcal{A}} \propto \lambda_2{}^{\chi_o Q/\alpha} = \lambda_2{}^{2-\omega(\chi_o)},$$

valid for small λ_2, corresponding to large L. Thus $\omega(\chi_o)$ is the critical exponent associated with the Feynman propagator of a test particle which interacts with the Yang-Mills gauge fields.

So far we have expanded the cosmological terms so as to linearise the contribution of the exponential terms to the action. We will now discuss a different approach based on the semi-classical expansion.

4.2 The saddle point expansion

When we consider the partition function with free boundary conditions and integrate the matter and ghost fields in the conformal gauge $\tilde{g}_{ab} = e^{\varphi}\hat{g}_{ab}$, the result is

$$\Gamma(A,L) = \int \mathcal{D}_{\tilde{g}}(\psi,\varphi)e^{-S_L[\varphi,\hat{g}]}\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\varphi} - A\right)\delta\left(\oint d\hat{s}e^{\varphi/2} - L\right)$$

where the Liouville action is given by

$$S_L[\varphi,\hat{g}] = \frac{26-d}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \hat{R}\varphi\right) + \frac{26-d}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi.$$

Representing the delta functions by integrals over (imaginary) Lagrange multipliers p,q, gives the Euclidean action

$$S_L[\varphi,\hat{g},p,q] = \frac{26-d}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \hat{R}\varphi\right) + \frac{26-d}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi - $$
$$-p\left(\int d^2\xi\sqrt{\hat{g}}e^{\varphi} - A\right) - q\left(\oint d\hat{s}e^{\varphi/2} - L\right).$$

This action is invariant under Möbius transformations on the upper half-plane, i.e. $SL(2,\mathcal{R})$ invariant. These transformations preserve the conformal gauge, mapping the upper half-plane onto itself $\omega \to \omega' = (a\omega + b)/(c\omega + d)$, $\varphi(\omega,\bar{\omega}) \to \varphi(\omega',\bar{\omega}') + 2\ln|d\omega'/d\omega|$, where $a,b,c,d \in \mathcal{R}$ and $ad - bc = 1$. It will be more convenient to work on the unit disc obtained from the upper half-plane by the complex Möbius transformation $\omega \to z = (i-\omega)/(i+\omega)$, $\varphi(\omega,\bar{\omega}) \to \varphi(z,\bar{z}) + 2\ln|dz/d\omega|$. To get the correspondent invariance on the unit disc we map the $SL(2,\mathcal{R})$ transformation. The result is the conformal mapping of the unit disc onto itself $z \to z' = \exp(i\theta_0)(z + c_0)/(1 + \bar{c}_0 z)$, $\varphi(z,\bar{z}) \to \varphi(z',\bar{z}') + 2\ln|dz/d\omega|$, where $\theta_0 \in \mathcal{R}$ and $|c_0| < 1$.

In the saddle point approximation we expand around the solution of the following classical problem:

$$\tilde{R} = \eta, \quad 2k_{\tilde{g}} = k, \quad \int d^2\xi\sqrt{\tilde{g}} = A, \quad \oint d\tilde{s} = L$$

where $\eta = p\gamma$, $\gamma = 48\pi/(26-d)$ and $2k = q\gamma$. This is a boundary-value problem for the Liouville field of the conformal gauge $\tilde{g}_{ab} = e^{\varphi}\hat{g}_{ab}$. The classical field φ_c must satisfy the Liouville equation

$$\hat{R} + \hat{\Delta}\varphi_c = \eta e^{\varphi_c}, \quad \int d^2\xi\sqrt{\hat{g}}e^{\varphi_c} = A$$

subject to the boundary condition [8]:

$$2k_{\hat{g}} + \partial_{\hat{n}}\varphi_c = ke^{\varphi_c/2}, \quad \oint d\hat{s}e^{\varphi_c/2} = L.$$

Here η and k are not independent. Applying the Gauss-Bonnet theorem gives $\eta A + kL = 4\pi$.

Let us now solve this problem on the unit disc. In the polar coordinates $z = \rho e^{i\theta}$, $\rho \in [0, 1]$, $\theta \in [0, 2\pi]$ we assume that φ_c only depends on ρ to find the following solution:

$$\varphi_c(\rho) = 2 \ln \frac{2A}{L} \left[1 + \left(\frac{4\pi A}{L^2} - 1 \right) \rho^2 \right]^{-1}.$$

Due to the θ independence this is the metric of a spherical cap of length L and area A. Note that $\rho^2 e^{\varphi_c} > 0$ leads to $L^2 < 4\pi A$. We also note that $\eta_c = 8\pi/A[1 - L^2/(4\pi A)]$, $k_c = 2L/A(1 - 2\pi A/L^2)$.

The saddle point tree level approximation is given by the classical functional $e^{-S_L[\varphi_c, \hat{g}, p_c, q_c]}$. Introducing the new coordinate ϱ such that ρ is given by $\rho = \tan(\varrho/2)/\sqrt{4\pi A/L^2 - 1}$, we obtain

$$\Gamma(A, L) = \left(\frac{A}{L} \right)^{(d-26)/6} \left(\frac{L^2}{4\pi A} \right)^{2L^2/A\gamma} e^{-2L^2/A\gamma}.$$

In the semi-classical limit $d \to -\infty$ we get

$$\Gamma(A, L) = e^{d/12\rho} \Gamma \left(e^{-\rho} A, e^{-\rho/2} L \right).$$

If we take the branch α_-, $\chi_o = 1$ and the limit $d \to -\infty$ we reproduce this scaling law from eq. (30) so that in the case of the disc topology both methods match in the asymptotic limit $A \to +\infty$, $L \to +\infty$ such that $A/L^2 \to const$.

If we go to one loop we must consider

$$\Gamma(A, L) = e^{-S_L[\varphi_c, \hat{g}, p_c, q_c]} \int \mathcal{D}_{g_c}(\phi, \chi) \delta \left(\int d^2\xi \sqrt{g_c} \chi \right) \delta \left(\oint ds_c \phi \right) e^{-S_1[\chi, \phi, g_c]}.$$

Here χ is the the quantum flutuation around the classical solution and ϕ is the free value it takes on the boundary. The metric $g_c{}^{ab}$ is given by $e^{\varphi_c} \hat{g}^{ab}$ and the one loop action is

$$S_1[\chi, \phi, g_c] = \frac{1}{2\gamma} \int d^2\xi \sqrt{g_c} g_c{}^{ab} \partial_a \chi \partial_b \chi - \frac{1}{2\gamma} \eta_c \int d^2\xi \sqrt{g_c} \chi^2 - \frac{1}{4\gamma} k_c \oint ds_c \phi^2.$$

Let us separate χ into a fixed background field χ_b and an homogeneous Dirichlet field $\bar{\chi}$. Introducing the operator $\mathcal{O}_c = \Delta_c - \eta_c$ we specify χ_b as the solution to the boundary-value problem $\mathcal{O}_c \chi_b = 0$, $\chi_b|_B = \phi$. Thus:

$$\Gamma(A, L) = e^{-S_L[\varphi_c, \hat{g}, p_c, q_c]} \int \mathcal{D}_{g_c} \phi \, \delta \left(\oint ds_c \chi_b \right) \exp \left(-\frac{1}{2\gamma} \oint ds_c \chi_b \partial_{nc} \chi_b \right)$$
$$\int \mathcal{D}_{g_c} \bar{\chi} \, \delta \left[\int \sqrt{g_c} \left(\bar{\chi} + \chi_b \right) \right] \exp \left(-\frac{1}{2\gamma} \int d^2\xi \sqrt{g_c} \bar{\chi} \mathcal{O}_c^D \bar{\chi} \right).$$

We can use the delta function for the integral along the boundary of ϕ to eliminate the constant zero mode of the covariant Laplacian Δ_c. However unlike the closed string case we still have another delta function which involves the other orthogonal

modes of χ. Unfortunately this means we are left with a functional integral too difficult to be solved here.

All these calculations can be attempted taking homogeneous Neumann boundary conditions on the Liouville field $\partial_{\hat{n}}\varphi = 0$. The results for the critical exponents using the scaling argument are the same. However we run into difficulties in performing the semi-classical expansion because the classical solution φ_c does not satisfy homogeneous Neumann boundary conditions so if the full Liouville field does, then the classical field and the quantum fluctuation are not independent, but rather are related with each other on the boundary $\partial_{\hat{n}}\varphi_c + \partial_{\hat{n}}\chi = 0$. So we conclude that the free boundary conditions are much better suited for the semi-classical expansion.

4.3 The tachyon gravitational scaling dimensions

Let us now calculate the gravitational scaling dimensions of the tachyon vertex operators for free boundary conditions. For the anomalous gravitational scaling dimension of the bulk tachyon vertex operator we consider the expectation value of the 1-point function at fixed area A

$$
< W_j > (A) = \frac{1}{\Gamma(A)} \int \mathcal{D}_{\hat{g}}(\bar{\phi}, \bar{\Psi}) d\bar{\Psi}_0 (\oint d\hat{s})^{1/2} e^{-S_c^{00}[\bar{\Psi}, \bar{\Psi}_0, \hat{g}] - \bar{S}^0[\bar{\phi}, \hat{g}]}
$$
$$
\delta \left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A \right) \int d^2\xi_j \sqrt{\hat{g}} e^{ip_j \cdot X(\xi_j)} e^{\gamma_j \phi}.
$$

By definition the bulk gravitational scaling dimension is as in the closed string $< W_j > (A) \sim A^{1-\Delta_j}$. Applying the scaling argument we find $\Delta_j = 1 - \gamma_j/\alpha$ and this leads to the KPZ equation for the anomalous gravitational dimension in the open string:

$$
\Delta_j - \Delta_j^0 = -\alpha^2 \Delta_j (\Delta_j - 1).
$$

Similarly we define the anomalous gravitational scaling dimension of the boundary tachyon vertex operator by $< W_j^B > (A) \sim A^{1/2 - \Delta_j^B}$. Then the scaling argument gives $\Delta_j^B = \Delta_j/2$.

We can also define critical exponents associated with the expectation values at fixed length L. These should also be interpreted as anomalous gravitational scaling dimensions. In this case the asymptotic limits are $< W_j > (L) \sim L^{1-\Delta_j}$ and $< W_j^B > (L) \sim L^{1/2 - \Delta_j^B}$, where Δ_j and Δ_j^B are given as in the case of fixed area A.

4.4 A connection with matrix models

These results generalise to other models and physical systems. As we observed before the open string is a toy model for the $c \leq 1$ boundary conformal field theories [18] coupled to 2D quantum gravity. In the next section we show that similar results can be written down for this more realistic class of models. Here we finish by considering a comparison with exact results of matrix models at genus zero [20].

According to ref. [20] we may deduce from matrix models calculations the following exact expression for $\Gamma(A, L)$ when the surface has the topology of a disc:

$$\Gamma(A, L) = A^x L^y e^{-L^2/A},$$

where $x = -Q/\alpha$ and $y = -3 + Q/\alpha$. This formula is consistent with our scaling laws given in eqs. (30), (31) and (32). Introducing it in the definitions of $\Sigma(A)$ and $\Omega(L)$ we find:

$$\sigma(1) = x + y/2 + 7/2, \quad \omega(1) = 2x + y + 5.$$

When we substitute back the values of x and y we get the same results for $\sigma(1)$ and $\omega(1)$ as we did using the David, Distler and Kawai's scaling argument.

This is an indication that our results should be in agreement with those obtained in models of dynamically triangulated open random surfaces. However it should be emphasised that a full comparison is beyond the scope of the present work.

5 Minimal Models On Open Random Surfaces

The open string analysis can now be easily extended to $c \leq 1$ minimal conformal field theories on open random surfaces if we represent the matter sector by a conformally extended Liouville theory. The curious affinity between the matter and gravitational sector Liouville theories that emerges for closed surfaces generalises to the case with boundaries. We simply take the matter action of eq. (1) with additional boundary terms:

$$S_M[\Phi, \tilde{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \left[\frac{1}{2} \tilde{g}^{ab} \partial_a \Phi \partial_b \Phi + i (\beta - 1/\beta) \tilde{R} \Phi \right] +$$
$$+ \frac{i}{4\pi} (\beta - 1/\beta) \oint d\tilde{s} k_{\tilde{g}} \Phi + \mu^2 \int d^2\xi \sqrt{\tilde{g}} \left(e^{i\beta\Phi} + e^{-i/\beta\Phi} \right) +$$
$$+ \lambda \oint d\tilde{s} \left[e^{i\beta\Phi/2} + e^{-i/(2\beta)\Phi} \right].$$

This is the conformally extended Toda field theory defined on an open surface for the Lie algebra A_1. It has recently been considered as a Coulomb gas description of the $c \leq 1$ minimal conformal matter in the case of Neumann boundary conditions imposed on the matter field [21]. Here we assume without proof that the same is true of when the matter satisfies free boundary conditions. In fact for both free and Neumann boundary conditions we have a full Weyl invariant non-critical theory at the quantum level to all orders in the Coulomb gas perturbation theory.

For definiteness we take here the free boundary conditions on all fields. The central charge of the matter theory is $c_M = 1 - 6(\beta - 1/\beta)^2$. Requiring that the sum of this and the central charges of the gravitational sector Liouville field and the reparametrisation ghosts vanish gives $\gamma = \pm i\beta$, where γ relates to our previous string Q, $Q = i(\gamma + 1/\gamma)$. The Liouville field renormalisation parameter must satisfy the equation $1 - \alpha(\beta + 1/\beta) + \alpha^2 = 0$ which, as before, gives us two branches $\alpha_+ = \beta$

and $\alpha_- = 1/\beta$. All the boundary renormalisation parameters relate to α and γ as happened for the string case. We find dressed vertex operators of vanishing conformal weight on the bulk

$$U^D(jj') = \int d^2\xi \sqrt{\hat{g}} \exp\left[\left(l\beta + \frac{l'}{\beta}\right)\phi\right] \exp\left[-i\left(j\beta - \frac{j'}{\beta}\right)\Phi\right]$$

where $l = -j$, $l' = j' + 1$ or $l = j + 1$, $l' = -j'$. On the boundary we also define dressed primary vertex operators of vanishing conformal weight consistent with the need to consider the Liouville field as an arbitrary Weyl scaling on the whole surface

$$U_B^D(jj') = \oint d\hat{s} \exp\left[\left(l\beta + \frac{l'}{\beta}\right)\frac{\phi}{2}\right] \exp\left[-i\left(j\beta - \frac{j'}{\beta}\right)\frac{\Phi}{2}\right].$$

As occurred for the string, Dirichlet boundary conditions on the Liouville field imply that we have no dynamical quantum degrees of freedom on the boundary, and hence no boundary vertex operators. Although they still allow the cancellation of the Weyl anomaly provided the metric has a discontinuity as the boundary is approached.

The open string formulas for the critical exponents generalise to these models. Thus the susceptibility exponent is $\sigma(\chi_o) = 2 - \chi_o Q/(2\alpha)$, the Feynman mass exponent is $\omega(\chi_o) = 2 - \chi_o Q/\alpha$. The semi-classical limit is obtained for $\beta \to +\infty$ and, just like for closed surfaces, selects the classical branch $\alpha_+ = \beta$. As in the open string the saddle point expansion singles out the free boundary conditions on the Liouville field. Similarly we find the same expressions for the anomalous gravitational scaling dimensions of the primary vertex operators. In the end the gravitational scaling dimension of a boundary operator is half that of a bulk operator, the latter being related to its bare conformal dimension by the KPZ equation.

6   Conclusions

In this paper we have shown how to extend the approach of David, Distler and Kawai to the coupling of boundary conformal field theories to 2D quantum gravity. The organising principal behind their approach is Weyl invariance at the quantum level applied to a perturbative expansion analogous to the Coulomb gas. We used this to determine the renormalised parameters, gravitational dressings and surface critical exponents such as the susceptibility of random surfaces, the anomalous gravitational scaling dimensions of primary vertex operators and the Feynman mass exponent. The crucial problem is the choice of boundary conditions on the Liouville field. We have discussed free, Neumann and Dirichlet boundary conditions on the Liouville field. The first two lead to similar results within this perturbative approach, but Dirichlet conditions imply that the metric is discontinuous as the boundary is approached. We have also considered the semi-classical expansion and advocated the free boundary conditions for the Liouville field, since homogeneous Neumann boundary conditions do not allow a clean split between the classical and quantum pieces of the field, but rather couple them together. As would be expected the

bulk properties are equal for open and closed surfaces. This approach may also be naturally extended to higher genus and more complex boundary structures. Unfortunately as for closed surfaces the results only apply to the weak coupling of $c \leq 1$ boundary conformal field theories to gravity. In the case of the open string this means unrealistic target space dimensions $d \leq 1$. Finally, we found the same close affinity between the matter sector when represented by a Liouville theory and the gravitational sector in this weak Coulomb gas phase as occurs in the case of closed surfaces.

Note added

After submitting this paper we were informed of refs. [22] and [23]. In ref. [22] the open string 2D quantum gravity with Neumann boundary conditions has been analysed. The results agree with ours. In ref. [23] it is conjectured that Neumann and free boundary conditions are equivalent, although as our discussion shows the free boundary conditions are in fact better suited to the semi-classical expansion.

Appendix A

THE NON-LOCAL WEYL CHANGE OF $\hat{G}_K(\xi, \xi')$

Start by taking eq. (17) and multiply it by $d\hat{s}(\xi)$. Since $d\hat{s}(\xi)d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')$ and $d\hat{s}(\xi)\hat{\delta}_B(\xi-\xi')$ are Weyl invariant use $\delta_\rho d\hat{s}(\xi) = (1/2)\rho(\xi)d\hat{s}(\xi)$ to get

$$d\hat{s}(\xi)\oint d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')\delta_\rho\hat{G}_K(\xi'',\xi') = -\frac{\rho(\xi)d\hat{s}(\xi)}{2\oint d\hat{s}(\eta)} + \frac{d\hat{s}(\xi)\oint d\hat{s}(\zeta)\rho(\zeta)}{2[\oint d\hat{s}(\eta)]^2}. \tag{33}$$

Next multiply eq. (33) by $\hat{G}_K(\xi,\xi''')$ and integrate on ξ. Using eqs. (17) and (18) find:

$$\delta_\rho\hat{G}_K(\xi,\xi') = -\frac{\oint d\hat{s}(\xi'')\hat{G}_K(\xi'',\xi)\rho(\xi'')}{2\oint d\hat{s}(\xi''')} + \frac{\oint d\hat{s}(\xi'')\delta_\rho\hat{G}_K(\xi'',\xi')}{\oint d\hat{s}(\xi''')}$$

Finally the Weyl transformation of eq. (18)

$$\oint d\hat{s}(\xi'')\delta_\rho\hat{G}_K(\xi'',\xi') = -\frac{1}{2}\oint d\hat{s}(\xi'')\rho(\xi'')\hat{G}_K(\xi'',\xi')$$

leads to eq. (22).

Appendix B

ANOMALY CANCELLATION FOR NEUMANN 2D QUANTUM GRAVITY

The charge conservation selection rule is $\int d^2\xi\sqrt{\hat{g}}\hat{J}_N^{MM'} = 0$. Here we have written $\hat{J}_N^{MM'} = Q\hat{R} + \bar{Q}_B k_{\hat{g}}\hat{\delta}_B^2 - 8\pi\alpha\sum_{P=1}^M \delta^2(\xi-\xi_P)/\sqrt{\hat{g}(\xi)} - 4\pi\alpha_B\sum_{P=1}^{M'}\hat{\delta}_B^2(\xi-\xi_P)$, where $\int d^2\xi\sqrt{\hat{g}}\hat{\delta}_B^2 = \oint d\hat{s}$. The Neumann Green's function satisfies

$$\hat{\Delta}\hat{G}_N(\xi, \xi') = \frac{\delta^2(\xi - \xi')}{\sqrt{\hat{g}(\xi)}} - \frac{1}{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}}, \quad \partial_{\hat{n}}\hat{G}_N(\xi, \xi') = 0,$$

$$\int d^2\xi \sqrt{\hat{g}(\xi)}\hat{G}_N(\xi, \xi') = 0.$$

We find the non-local functional:

$$\mathcal{F}_N^{MM'} = \frac{1}{16\pi} \int d^2\xi' d^2\xi'' \sqrt{\hat{g}(\xi')}\hat{J}_N^{MM'}(\xi')\hat{G}_N(\xi', \xi'')\sqrt{\hat{g}(\xi'')}\hat{J}_N^{MM'}(\xi'').$$

Consider just one bulk Liouville vertex operator. Then:

$$\delta_\rho \mathcal{F}_N^{10} = \frac{Q}{8\pi} \int d^2\xi \sqrt{\hat{g}}\hat{J}_N^{10}\rho + \alpha^2 \rho(\xi_1) - \frac{Q}{8\pi} \int d^2\xi \sqrt{\hat{g}}\hat{J}_N^{10}\delta_\rho \ln \int d^2\xi \sqrt{\hat{g}} -$$
$$-\frac{1}{8\pi \int d^2\xi \sqrt{\hat{g}(\xi)}} \int d^2\xi \sqrt{\hat{g}}\hat{J}_N^{10} \int d^2\xi' d^2\xi'' \sqrt{\hat{g}(\xi')}\rho(\xi')\hat{G}_N(\xi', \xi'')\sqrt{\hat{g}(\xi'')}\hat{J}_N^{10}(\xi''),$$

where $\bar{Q}_B = 2Q$. This eliminates the terms in $\partial_{\hat{n}}\rho$. The α^2 term comes from the Weyl change of $\hat{G}_N(\xi, \xi)$, $\xi \notin B$. We have:

$$\hat{G}_N(\xi, \xi') = \hat{G}(\xi, \xi') + \hat{H}_N(\xi, \xi')$$

where $\hat{H}_N(\xi, \xi')$ is defined by:

$$\hat{\Delta}\hat{H}_N(\xi, \xi') = -\frac{1}{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}}, \quad \partial_{\hat{n}}\hat{H}_N(\xi, \xi') = -\partial_{\hat{n}}\hat{G}(\xi, \xi').$$

Then the local change is $\delta_\rho \hat{G}_{N\varepsilon}(\xi, \xi) = \rho(\xi)/(4\pi)$. Thus $Q = \pm\sqrt{(25 - d)/6}$ and $1 - \alpha Q + \alpha^2 = 0$.

Now consider just one boundary Liouville vertex operator. When $\xi = \xi'$ is on the boundary, $\hat{H}_N(\xi, \xi)$ is divergent because $\hat{G}(\xi, \xi)$ is singular. In a neighbourhood of order $\sqrt{\varepsilon}$ around ξ the shape of the boundary is flat. Then $\hat{G}_N(\xi, \xi)$ is defined by the method of images so that $\hat{H}_N(\xi, \xi) = \hat{G}(\xi, \xi)$. Hence the local change now is $\delta_\rho \hat{G}_{N\varepsilon}(\xi, \xi) = \rho(\xi)/(2\pi)$. Thus $\alpha_B = \alpha$.

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