

THE VACUUM FUNCTIONAL AT LARGE DISTANCES

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Abstract

For fields that vary slowly on the scale of the lightest mass the logarithm of the vacuum functional can be expanded as a sum of local functionals, however this does not satisfy the obvious form of the Schrödinger equation. For φ^4 theory we construct the appropriate equation that this expansion does satisfy. This reduces the eigenvalue problem for the Hamiltonian to a set of algebraic equations. We suggest two approaches to their solution. The first is equivalent to the usual semi-classical expansion whilst the other is a new scheme that may also be applied to theories that are classically massless but in which mass is generated quantum mechanically.

For fields that vary slowly on the scale of the lightest mass the logarithm of the vacuum functional of a quantum field theory reduces to a sum of local quantities. The purpose of this letter is to construct the Schrödinger equation that acts directly on this local expansion. This reduces the eigenvalue problem for the Hamiltonian to a set of algebraic equations for which we suggest two methods of solution. One of them is the familiar semi-classical expansion, the other is a new approach that would apply even when mass is generated quantum mechanically in a theory that is classically massless.

To begin with, consider a free massive scalar field theory in $D+1$ dimensions. In the Schrödinger representation the Hamiltonian is $H = -\frac{1}{2}\Delta + \int d^D\mathbf{x} \frac{1}{2}(\nabla\varphi \cdot \nabla\varphi + m^2\varphi^2)$, (see [1] for a review of the Schrödinger representation in field theory). The unregulated Laplacian, Δ , is $\int d\mathbf{x} \delta^2/\delta\varphi^2(\mathbf{x})$, and the vacuum functional $\Psi[\varphi] \equiv \langle\varphi|0\rangle = \exp - \int d^D\mathbf{x} \varphi\Gamma\varphi/2 \equiv \exp W_0$ with $\Gamma = +\sqrt{-\nabla^2 + m^2}$. The corresponding energy eigenvalue, E , is proportional to the functional trace of Γ and so diverges. The Laplacian needs to be regulated so introduce a momentum cut-off, $1/s$, by defining

$$\Delta_s = \int_{p^2 < 1/s} \frac{d^D p}{(2\pi)^D} \frac{\delta^2}{\delta\bar{\varphi}(p)\delta\bar{\varphi}(-p)}, \quad (1)$$

where $\bar{\varphi}(\mathbf{p}) = \int d^D\mathbf{x} \varphi(\mathbf{x}) \exp - i\mathbf{p} \cdot \mathbf{x}$. The vacuum energy density $\mathcal{E} = E/V$ is now well-defined and diverges as the cut-off is removed

$$\mathcal{E} = \frac{1}{2V} \Delta_s W_0 = \frac{1}{2} \int_{p^2 < 1/s} \frac{d^D p}{(2\pi)^D} \sqrt{\mathbf{p}^2 + m^2} \sim \frac{k}{(D+1)s^{(D+1)/2}} \quad \text{as } s \rightarrow 0 \quad (2)$$

Where k is the area of the unit sphere in D dimensions divided by $2(2\pi)^D$.

If φ varies slowly in space on the scale of $1/m$ then $W_0[\varphi]$ simplifies to a local expression obtained by expanding Γ as $m - \nabla^2/(2m) - (\nabla^2)^2/(8m^3) \dots$ to give

$$\int d^D\mathbf{x} \left(\frac{m}{2}\varphi^2 + \frac{1}{4m}(\nabla\varphi)^2 - \frac{1}{16m^3}(\nabla^2\varphi)^2 \dots \right) \equiv W_0^{loc} \quad (3)$$

Similar local expansions may be performed for the wave-functionals given by [2] [3] describing fields coupled to gauge-potentials. A local vacuum-functional for Yang-Mills was first discussed in [4] where it leads heuristically to an area law for the Wilson loop via a kind of dimensional reduction. Quark confinement from dimensional reduction was discussed in [5]. Dimensional reduction resulting from the local nature of the Schrödinger vacuum functional [6] has been generalised by Horiguchi *et al* to the case of the Wheeler-De Witt equation to look for a new phase in quantum gravity beyond the Planck scale [7]. We expect that the vacuum functional of the interacting scalar theory can similarly be expanded as a local quantity for slowly varying φ . The vacuum functional satisfies the Schrödinger equation, the existence of which was shown by Symanzik,[8]. In principle this determines the coefficients of the expansion. However the local expansion does *not* satisfy the obvious form of this equation because expanding in local quantities does not commute

with removing the cut-off, $s \rightarrow 0$, even for the free theory. To see this apply the regulated Laplacian Δ_s directly to the local expansion (3) to obtain the energy density as

$$\frac{1}{2V} \Delta_s W_0^{loc} = \int_{p^2 < 1/s} \frac{d^D p}{(2\pi)^D} \left(\frac{m}{2} - \frac{p^2}{4m} - \frac{(p^2)^2}{16m^3} \dots \right) = \sum_{n=0}^{\infty} \frac{\alpha_n}{(m^2 s)^{n+D/2}} \quad (4)$$

(where $\alpha_n = km^{D+1} \Gamma(\frac{3}{2}) / (\Gamma(3/2 - n) \Gamma(n+1) (D+2n))$). This expression appears to have divergences of increasing order as $s \rightarrow 0$ unlike (2) which correctly gives the behaviour of the vacuum energy as the cut-off is removed. This is because our local expansion W_0^{loc} is only valid for slowly varying φ , that is for $\bar{\varphi}(\mathbf{p})$ with support in $p^2 < m^2$, hence (4) is only valid for $sm^2 > 1$, i.e. for large s . To obtain information about the vacuum energy for small s from this expansion we must re-sum the series. First define the continuation of the vacuum energy to complex s by

$$\mathcal{E}(s) = \frac{1}{s^{D/2}} \int_{p^2 < 1} \frac{d^D p}{(2\pi)^4} \sqrt{m^2 + p^2/s} \quad (5)$$

By choosing the square-roots to have cuts on the negative real axis this is analytic throughout the complex s -plane with the negative real axis removed. For $|s|m^2 > 1$ it has a large s expansion which is just (4). Let C be a key-hole shaped contour running just under the negative real axis up to $s = -s_0$ with $s_0 > 1/m^2$, around the circle of radius s_0 centred on the origin and then back to $s = -\infty$ running just above the negative real axis. The integral

$$I(\lambda) = \frac{1}{2\pi i} \int_C \frac{ds}{s} \mathcal{E}(s) e^{\lambda s} \quad (6)$$

may be evaluated using the large s expansion (4) as

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{\Gamma(n+1+D/2)} \left(\frac{\lambda}{m^2} \right)^{n+D/2} \quad (7)$$

We can also evaluate the integral by collapsing the contour C until it just surrounds the negative real axis. By taking λ to be real, positive and very large the integrand will be exponentially suppressed away from the vicinity of the origin so that the integral is determined by $\mathcal{E}(s)$ for small s . In fact $I(\lambda) \sim \lambda^{(D+1)/2} \sum_{n=0}^{\infty} \tilde{\alpha}_n \lambda^{-n}$ as $\lambda \rightarrow \infty$ with constant $\tilde{\alpha}_i$ up to exponentially suppressed terms, so that $I(\lambda)$ provides a regularisation of the vacuum energy (2). If instead we had been dealing with a function $\mathcal{E}(s)$ that was finite in the vicinity of the origin with poles and cuts only on the negative real axis we would have obtained $\mathcal{E}(0)$ up to exponentially suppressed terms. In an interacting theory we would only be able to compute a finite number of terms of these expansions. Now (7) is an alternating series so truncating it at $n = N$ gives an error less than $|\frac{\alpha_{N+1} \lambda^{N+1+D/2}}{\Gamma(N+2+D/2) m^{2N+2+D}}|$ which for large N behaves as $(e\lambda/m^2 N)^{(N+1+D/2)}/N$. Since we want to take λ large set $\lambda = N\mu^2$, then the truncation error goes to zero with large N provided the N -independent mass-scale μ is smaller than the particle mass m . This shows that we can extract information about the high momentum cut-off theory by working to a finite order, N , with the local expansion of the vacuum functional valid for slowly varying fields.

We now turn to the interacting theory. Symanzik [8] showed that for $D = 3$ the Schrödinger functional is finite when the cut-off is removed provided that in addition to the usual renormalisation procedure there is a further field renormalisation to take account of divergences associated with the boundary. This was taken further in [9]. The appropriate renormalisation constants are computable in perturbation theory, but since the theory is not asymptotically free these are not reliable so instead we will work with the 1+1 dimensional theory which is super-renormalisable. In the usual treatment without boundaries the only divergent Feynman diagrams for φ^4 theory in 1+1 dimensions are those in which both ends of a propagator are contracted at the same point. These may be eliminated by normal ordering the Hamiltonian with respect to a perturbative vacuum to obtain a regular operator that does not depend on a cut-off:

$$\hat{H} = \int dx : \left(\frac{1}{2} (\hat{\pi}(x)^2 + \hat{\varphi}'(x)^2 + M_r^2 \hat{\varphi}(x)^2) + \frac{g}{4!} \hat{\varphi}^4 \right) : \quad (8)$$

Here M_r and g are finite, and the normal-ordering is with respect to oscillators with mass M_r . There are no further divergences in the Schrödinger functional due to the boundaries so in this case there is no further field renormalisation. (This is thought to apply also in Yang-Mills theories [10]). Now define an operator \hat{H}_s constructed from momentum cut off fields

$$\hat{H}_s = \int dx \left(\frac{1}{2} (\hat{\pi}_s^2 + \hat{\varphi}_s'^2 + M^2(s) \hat{\varphi}_s^2) + \frac{g}{4!} \hat{\varphi}_s^4 - \mathcal{E}_s \right) \quad (9)$$

where

$$\hat{\varphi}_s(x) = \int dy \mathcal{G}_s(x, y) \hat{\varphi}(y), \quad \hat{\pi}_s(x) = \int dy \mathcal{G}_s(x, y) \hat{\pi}(y) \quad (10)$$

and the kernel

$$\mathcal{G}_s(x, x') = \int_{p^2 < 1/s} \frac{dp}{2\pi} e^{ip(x-x')}, \quad (11)$$

implements a momentum cut-off. Writing this in terms of normal-ordered operators enables us to express \hat{H} as $\lim_{s \downarrow 0} \hat{H}_s$ provided that

$$\mathcal{E}_s = \frac{1}{2} \int_{p^2 < 1/s} \frac{dp}{2\pi} \left(\sqrt{p^2 + M_r^2} + \frac{M^2(s) - M_r^2}{2\sqrt{p^2 + M_r^2}} \right) + \frac{g}{32} \left(\int \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M^2}} \right)^2 \quad (12)$$

and

$$M_r^2 = M^2(s) + \frac{g}{4} \int_{p^2 < 1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M_r^2}}, \quad (13)$$

so that the Schrödinger equation for the vacuum is just $\lim_{s \downarrow 0} \langle \varphi | \hat{H}_s | 0 \rangle = E \langle \varphi | 0 \rangle$ or

$$\begin{aligned} \lim_{s \downarrow 0} \left(-\frac{1}{2} \Delta_s + \int dx \left(\frac{1}{2} (\varphi_s'(x))^2 + M^2(s) \varphi_s(x)^2 \right) + \frac{g}{4!} \varphi_s(x)^4 - \mathcal{E}(s) \right) \Psi \\ \equiv \lim_{s \downarrow 0} \left(-\frac{1}{2} \Delta_s + V_s \right) \Psi \equiv \lim_{s \downarrow 0} H_s \Psi = E \Psi \end{aligned} \quad (14)$$

If we evaluate this expression for a φ of that has no Fourier modes with momentum greater than $1/s$ then we can replace φ_s in this expression by φ itself.

We will now show that the limit of small s may be taken by re-summing the large s behaviour, just as for the vacuum energy of the free theory, enabling us to compute the coefficients an expansion of the logarithm of the vacuum functional in terms of local quantities, and the eigenvalues of the Hamiltonian. We begin by showing that $\Delta_s \Psi[\varphi]$ extends to an analytic function in the complex s -plane with the negative real axis removed. Consider $\langle \varphi | \hat{\pi}(x, 0) \hat{\pi}(x', 0) | 0 \rangle$, written in terms of a Euclidean functional integral. It will be helpful to make the φ dependence of $\langle \varphi |$ explicit. To do this we can use a bra $\langle D |$ which is annihilated by $\hat{\varphi}$. (D stands for Dirichlet). Then

$$\langle \varphi | = \langle D | e^{i \int dx \varphi(x) \hat{\pi}(x)}, \quad (15)$$

so that

$$\frac{\delta}{\delta \varphi(x)} \langle \varphi | = i \langle \varphi | \hat{\pi}(x), \quad (16)$$

and using the canonical commutation relations $\langle \varphi | \hat{\varphi} = \varphi \langle \varphi |$. Now in Minkowski space

$$T(\hat{\pi}(x, t) \hat{\pi}(x', t')) = \frac{\partial^2}{\partial t \partial t'} T(\hat{\varphi}(x, t) \hat{\varphi}(x', t')) - i \delta(x - x') \delta(t - t') \quad (17)$$

so that if we use the standard relationship between functional integrals and time-ordered expectation values we can express $\langle \varphi | e^{-it_0 \hat{H}} | D \rangle$ as the functional integral

$$\int \mathcal{D}\tilde{\varphi} e^{iS[\tilde{\varphi}] + \int dx (i\varphi \dot{\tilde{\varphi}} + \Lambda \varphi^2)} \quad (18)$$

where $|D\rangle$ implies Dirichlet boundary conditions $\tilde{\varphi} = 0$ at times 0 and t_0 . Λ is a regularisation of the $\delta(0)$ that arises from (17). Similarly the time ordered Green's function

$$T \langle \varphi | e^{-it_0 \hat{H}} \hat{\varphi}(x, t) \hat{\varphi}(x', t') | D \rangle = \int \mathcal{D}\tilde{\varphi} e^{iS[\tilde{\varphi}] + \int dx (i\varphi \dot{\tilde{\varphi}} + \Lambda \varphi^2)} \tilde{\varphi}(x, t) \tilde{\varphi}(x', t'), \quad (19)$$

Again making use of (17), rotating to Euclidean space and taking the limit $t_0 \rightarrow \infty$ gives for $0 > t \geq t'$

$$\begin{aligned} \langle \varphi | \hat{\pi}(x, t) \hat{\pi}(x', t') | 0 \rangle = \\ - \int \mathcal{D}\tilde{\varphi} e^{-S_E[\tilde{\varphi}] + \int dx (\varphi \dot{\tilde{\varphi}} + \Lambda \varphi^2)} \left(\dot{\tilde{\varphi}}(x, t) \dot{\tilde{\varphi}}(x', t') - \delta(x - x') \delta(t - t') \right) \end{aligned} \quad (20)$$

where the integration variable $\tilde{\varphi}$ is now defined on the Euclidean half-plane, $t \leq 0$, with the boundary condition $\tilde{\varphi}(x, 0) = 0$. To obtain $\langle \varphi | \hat{\pi}(x, 0) \hat{\pi}(x', 0) | 0 \rangle$ we can either take the limit as $t, t' \uparrow 0$ keeping $t > t'$, in which case the δ -functions will not contribute, or we can take $t = t' = 0$ having first cancelled the δ -functions against a divergence coming from $\dot{\tilde{\varphi}}(x, t) \dot{\tilde{\varphi}}(x', t')$. A similar argument applies to the $\Lambda \varphi^2$ terms. Whichever approach we take affects only the φ -independent and $\int dx \varphi^2$ contributions to the connected diagrams in $\Delta_s \Psi$. For example when we integrate against \mathcal{G}_s to get $\Delta_s \Psi$ the delta-functions give a term proportional to Ψ/\sqrt{s} which contributes only to the $1/\sqrt{s}$ part of the vacuum energy in (14). Apart from this the δ -functions have no effect, and henceforth we ignore

them. Setting $t = t' = 0$ in the remaining term, rotate co-ordinates so that the two points $(x, 0)$ and $(x', 0)$ differ by a Euclidean time $\tau = |x - x'|$ to obtain

$$\int \mathcal{D}\tilde{\varphi}_r e^{-S_{Er}[\tilde{\varphi}] + \int dt \varphi \tilde{\varphi}'_r \tilde{\varphi}'_r(0, \tau) \tilde{\varphi}'_r(0, 0)} \quad (21)$$

where now the integration variable $\tilde{\varphi}_r$ is defined on the half-plane $x > 0$. This can be interpreted as the time-ordered vacuum expectation value of fields that evolve in Euclidean time with a Hamiltonian \tilde{H} defined on the half-line $x > 0$

$$T_E \langle 0_r | e^{\int dt \varphi(t) \tilde{\varphi}'(0, t)} \tilde{\varphi}'(0, \tau) \tilde{\varphi}'(0, 0) | 0_r \rangle \quad (22)$$

The vacuum $|0_r\rangle$ corresponds to the rotated Hamiltonian. Expanding the exponential and making the Euclidean time dependence explicit reduces this to a sum of terms of the form

$$\int_{\tau}^{\infty} dt_n \int_{\tau}^{t_n} dt_{n-1} \dots \int_{\tau}^{t_{p+1}} dt_p \int_0^{\tau} dt_{p-1} \dots \int_0^{t_{q+1}} dt_q \int_{-\infty}^0 dt_{q-1} \dots \int_{-\infty}^{t_2} dt_1 \varphi(t_n) \dots \varphi(t_1) \\ \langle 0_r | \tilde{\varphi}' e^{-\tilde{H}(t_n - t_{n-1})} \tilde{\varphi}' \dots e^{-\tilde{H}(t_p - \tau)} \tilde{\varphi}' e^{-\tilde{H}(\tau - t_{p-1})} \tilde{\varphi}' \dots e^{-\tilde{H}t_q} \tilde{\varphi}' e^{\tilde{H}t_{q-1}} \tilde{\varphi}' \dots e^{-\tilde{H}(t_2 - t_1)} \tilde{\varphi}' | 0_r \rangle \quad (23)$$

The integrals may be computed if we first Fourier decompose $\varphi(t_i)$ as

$$\varphi(t_i) = (2\pi)^{-1} \int dk_i \bar{\varphi}(k) \exp(ik_i t_i). \quad (24)$$

This gives a delta-function conserving the total momentum $\sum k_i$ and for t_i with $i \geq p$ this gives insertions of $(\tilde{H} - i \sum k_i)^{-1}$ whereas for $i < q$ it gives insertions of $(\tilde{H} + i \sum k_i)^{-1}$. The remaining t -integrals may be done by inserting a basis of eigenstates of \tilde{H} between each operator which leads to a sum of products of energy denominators of the form $(E_1 - E_2 - i \sum k_j)^{-1}$ multiplied by exponentials of τ of the form $\exp - (E_1 - i \sum k_j)\tau$, so that the τ dependence may be made explicit as a sum of integrals over the spectrum of \tilde{H}

$$\int dE dk_1 \dots dk_n \bar{\varphi}(k_1) \dots \bar{\varphi}(k_n) \delta(\sum k_i) (\rho_0 + \rho_1 e^{ik\tau} + \rho_2 e^{i(k_1+k_2)\tau} + \dots) e^{-E\tau} \quad (25)$$

Integrating this against the kernel \mathcal{G}_s gives $\Delta_s \Psi$ as a sum of terms like

$$\int_{p^2 < s^{-1}} \frac{dp}{\pi} \int dE dk_1 \dots dk_n \bar{\varphi}(k_1) \dots \bar{\varphi}(k_n) \delta(\sum k_i) \sum \frac{\rho_j(E, k_1, \dots, k_n)}{E - i(p + \sum k_i)} \quad (26)$$

We will make the dependence on s more explicit by setting $p = q/\sqrt{s}$. Also we will scale the function φ with s by evaluating this for φ of the form $\varphi(x) = f(x/\sqrt{s}) \equiv \phi_s(x)$ so that $\bar{\varphi}(k) = \bar{f}(k\sqrt{s})\sqrt{s}$, and we take $\bar{f}(k)$ to vanish outside $|k| < \kappa \ll 1$. This enables us to scale the k_i integrals to obtain

$$\int_{q^2 < 1} \frac{dq}{\pi} \int dE dk_1 \dots dk_n \bar{f}(k_1) \dots \bar{f}(k_n) \sqrt{s} \delta(\sum k_i) \sum \frac{\rho_j(E, k_1, \dots, k_n)}{\sqrt{s}E - i(q + \sum k_i)} \quad (27)$$

The functions ρ_j acquire a dependence on s via the energy denominators that can be written after scaling of k_i as $1/((E_1 - E_2) - i(\sum k)/\sqrt{s})$. Up to this point s has been real and positive but the expressions we obtain for $(\Delta_s \Psi)[\phi_s]$ can now be continued as

analytic functions to the whole complex s -plane excluding the negative real axis, (at least order by order in the expansion in powers of ϕ_s that we used to treat the exponential in (22)). Similarly $((H_s - E)\Psi)[\phi_s]$ extends to an analytic function of s with singularities only on the negative real axis, so its finite value at $s = 0$ can be obtained from an integral over a contour on which $|s|$ is arbitrarily large. Thus the Schrödinger equation becomes

$$\lim_{\lambda \rightarrow \infty} \frac{1}{2\pi i} \int_C \frac{ds}{s} e^{\lambda s} ((H_s - E)\Psi)[\phi_s] = 0 \quad (28)$$

We take C as before. Setting $\Psi = e^W$ gives

$$\lim_{\lambda \rightarrow \infty} \frac{1}{2\pi i} \int_C \frac{ds}{s} e^{\lambda s} \left(-\frac{1}{2} \left(\Delta_s W + \int dx dy \mathcal{G}_s(x, y) \frac{\delta W}{\delta \varphi(x)} \frac{\delta W}{\delta \varphi(y)} \right) - V_s - E \right) [\phi_s] = 0 \quad (29)$$

where we have divided through by $\Psi[\varphi_s]$ which has a finite limit as $s \downarrow 0$ as can be checked in perturbation theory. Since $\kappa \ll 1$ we can replace $\mathcal{G}_s(x, y)$ in (29) by a delta-function. For a general $W[\varphi]$ we would expect to be able to expand in powers of φ to give

$$W[\varphi] = \sum_{n=1}^{\infty} \int \frac{dk_1}{2\pi} \dots \frac{dk_{2n}}{2\pi} \bar{\varphi}(k_1) \dots \bar{\varphi}(k_{2n}) \delta(\sum k_i) \Gamma_n(k_1, \dots, k_{2n}). \quad (30)$$

An expansion in terms of local quantities would result from expanding Γ_n in positive powers of momenta for small momenta. Applying Δ_s to $W[\varphi]$ gives

$$\Delta_s W[\varphi] = \int_{p^2 < \frac{1}{s}} \frac{dp}{2\pi} \sum_{n=1}^{\infty} n(n-1) \int \frac{dk_3}{2\pi} \dots \frac{dk_{2n}}{2\pi} \bar{\varphi}(k_3) \dots \bar{\varphi}(k_{2n}) \delta(\sum k_i) \Gamma_n(p, -p, k_3, \dots, k_{2n}). \quad (31)$$

For large s the momentum p is small so we can expand $\Gamma_n(p, -p, k_3, \dots, k_{2n})$ in positive powers of p . If we evaluate $\Delta_s W$ for $\varphi = \phi_s$ then we can also expand in positive powers of the other momenta k_i , again obtaining a local expression, since the Fourier transform of ϕ_s has only low momentum modes for large enough s . This coincides with what we would have obtained by applying Δ_s directly to the local expansion of W followed by substituting ϕ_s for φ . This leads to a set of algebraic equations which re-formulate the eigenvalue problem for the Hamiltonian. Concretely, we study

$$\frac{1}{2\pi i} \int_C ds s^{-1} e^{\lambda s} \left(-\frac{1}{2} \left(\Delta_s W_\lambda + \int dx \left(\frac{\delta W_\lambda}{\delta \varphi} \right)^2 \right) - V_s - E_\lambda \right) [\phi_s] = 0 \quad (32)$$

for large λ taking W_λ to be local.

Integration by parts gives linear relations between local expressions so we pick as a linearly independent basis expressions of the form $\int dx \varphi^{v_0} (\varphi')^{v_1} \dots (\varphi^{(n)})^{v_n}$ where v_n , the power of the highest derivative in the expression, must be at least two. So we take as our local expansion of W_λ

$$\begin{aligned} W_\lambda = \int dx & \left(\varphi^2 (b_0 + c_0 \varphi^2) + \varphi'^2 (b_1 + c_1 \varphi^2 + d_1 \varphi'^2) \right. \\ & + \sum_2^{\infty} \varphi^{(n)2} (b_n + c_n \varphi^2 + d_n \varphi'^2 + e_n \varphi \varphi'' + \dots) \\ & \left. + \varphi^4 (f_0 \varphi^2 + f_1 \varphi'^2 + f_2 \varphi''^2 + \dots) + \dots \right) \end{aligned} \quad (33)$$

The coefficients $b_i, c_i, ..$ depend on λ through (32) but since Ψ is finite they are themselves finite as $\lambda \rightarrow \infty$. Substituting this into our Schrödinger equation (32) yields

$$\begin{aligned}
& \int dx \left(\begin{aligned}
& 2\bar{\mathcal{E}}(\lambda) + 2\mathcal{E} + \frac{\sqrt{\lambda}}{\sqrt{\pi}^3} \left(4b_0 + \frac{8b_1\lambda}{9} + \frac{16b_2\lambda^2}{75} + \dots + \frac{2b_n\sqrt{\pi}\lambda^n}{\Gamma(n+3/2)(2n+1)} + \dots \right) \\
& + \varphi^2 \left(-\bar{M}^2(\lambda) + \frac{4b_0^2}{\sqrt{\pi}} + \frac{\sqrt{\lambda}}{\pi} \left(12c_0 + \frac{2c_1\lambda}{3} + \frac{c_2\lambda^2}{5} + \dots \right) \right) \\
& \varphi^4 \left(-\frac{2g}{4!\sqrt{\pi}} + \frac{16b_0c_0}{\sqrt{\pi}} + \frac{\sqrt{\lambda}}{\pi} \left(30f_0 + \frac{2f_1\lambda}{3} + \frac{f_2\lambda^2}{2} + \dots \right) \right) \\
& + \varphi'^2 \left(-\frac{2}{\sqrt{\pi}} + \frac{16b_0b_1}{\sqrt{\pi}} + \frac{\sqrt{\lambda}}{\pi} \left(2c_1 + 2d_1\lambda + \frac{4c_2\lambda}{3} + \frac{d_2\lambda^2}{3} - \frac{e_2\lambda^2}{5} + \dots \right) \right) \\
& + \varphi^2\varphi'^2 \left(\frac{32b_0c_1 + 96b_1c_0}{\sqrt{\pi}} + \frac{12f_1\sqrt{\lambda}}{\pi} + \dots \right) \\
& + \varphi'^2 \left(\frac{32b_0b_2 + 16b_1^2}{3\sqrt{\pi}} + \frac{\sqrt{\lambda}}{\pi} \left(c_2 + \frac{d_1\lambda}{9} - \frac{e_2\lambda}{3} + \dots \right) \right) \\
& + \dots \\
& + (\varphi^{(n)})^2 \left(\frac{\sum 4b_m b_{n-m}}{\Gamma(n+1/2)} + \frac{\sqrt{\lambda}}{\pi} \left(\frac{2c_n}{n!} + \frac{2(d_n - e_n)\lambda}{3(n+1)!} + \dots \right) \right) \\
& + \dots \end{aligned} \right) = 0 \quad (34)
\end{aligned}$$

where $\varphi = \phi_s$ for $s = 1/\lambda$ Requiring the coefficient of each linearly independent function of φ to vanish gives an infinite set of algebraic equations. These may be solved in the usual semi-classical approach by first ignoring the power series in λ so that to leading order $b_0 = \pi^{1/4}\bar{M}/2$, $c_0 = g/(192b_0)$, $b_1 = 1/(8b_0)$, .. etc. This is the same as ignoring $\Delta_s W$ in (32) and solving the resulting Hamilton-Jacobi equation as a local expansion. This local expansion is possible because the full solution to the Hamilton-Jacobi equation is just the Euclidean action evaluated on-shell and the classical theory is massive. Corrections to the coefficients can now be obtained iteratively by substituting the leading order solutions into $\Delta_s W$.

For theories such as Yang-Mills which are classically massless the classical action does not have a local expansion, so that the semi-classical approach to solving (32) is not viable. However the full theory is believed to be massive, [11], so that our expansions make sense, but we need a different approach to solving the resulting algebraic equations. We now propose a method in which the expansions in λ are truncated at a certain order and the coefficients associated with the higher orders estimated using (27). Consider, for example, the φ -independent contribution to $\Delta_s W$. Using (27)

$$\begin{aligned}
& \frac{\sqrt{\lambda}}{\sqrt{\pi}^3} \left(4b_0 + \frac{8b_1\lambda}{9} + \frac{16b_2\lambda^2}{75} + \dots + \frac{2b_n\sqrt{\pi}\lambda^n}{\Gamma(n+3/2)(2n+1)} + \dots \right) \equiv \sum \lambda^{n+1/2} \xi_n \\
& = \frac{1}{2\pi i} \int_C \frac{ds}{s} e^{\lambda s} \int d\mu(E) \int_{q^2 < 1} dq \frac{1}{\sqrt{s}E - iq} |\langle 0_r | \hat{\varphi}'(0) | E \rangle|^2 \quad (35)
\end{aligned}$$

(In the derivation of this formula we dropped the terms due to the delta-functions in (20) which only contribute a term proportional to $1/\sqrt{s}$ to this expression. Since this has no effect on the terms of higher order in s we ignore it here too). If we expand $(\sqrt{s}E - iq)^{-1}$ for large s in powers of $1/\sqrt{s}$ then for the high order terms the integral over the spectrum is suppressed by a large power of $1/E$ so that the dominant contribution will come from the lowest values of E . These correspond to the lightest particle one-particle states for which

$$d\mu(E)|\langle 0_r|\hat{\varphi}'(0)|E\rangle|^2 = \frac{dp}{E}p^2|\langle 0_r|\hat{\varphi}(0)|E\rangle|^2, \quad p = \sqrt{E^2 - m^2}. \quad (36)$$

For large n

$$\int \frac{dp}{E^{2n+2}}p^2 f(E) = \int dp p^2 e^{-(n+1)\ln(p^2+m^2)} f(E) \approx \int dp p^2 e^{-(n+1)p^2/m^2} m^{-2(n+1)} f(m) \quad (37)$$

This, together with Stirling's formula gives for large n

$$\xi_n \sim (-)^n \frac{\xi}{n^3} \left(\frac{e\lambda}{nm^2} \right)^{n+1/2} \left(1 + O\left(\frac{1}{n}\right) \right), \quad (38)$$

with ξ independent of λ and n and proportional to $m|\langle 0_r|\hat{\varphi}(0)|E=m\rangle|^2$. This shows that at high orders (35) is an alternating series. The error in truncating it at order $\lambda^{N+1/2}$ is proportional to $(e\lambda/Nm^2)^{N+3/2}/N^3$. If we set $\lambda = N\mu^2/e$ then this error goes to zero with increasing N provided the N -independent mass-scale μ satisfies $\mu < m$, and at the same time λ increases as we increase N . Similar, but more complicated calculations show that the coefficients of $\varphi^2, \varphi^4, \varphi'^2$ behave like

$$\sum \frac{(-1)^n}{\sqrt{n^3}} \left(\frac{e\lambda}{nm^2} \right)^{n+1/2}, \quad \sum \frac{(-1)^n}{n} \left(\frac{e\lambda}{nm^2} \right)^{n+1/2}, \quad \sum \frac{(-1)^n}{\sqrt{n}} \left(\frac{e\lambda}{nm^2} \right)^{n+1/2}, \quad (39)$$

so that again if we truncate these series at order $\lambda^{N+1/2}$ the error will go to zero as N is increased. Furthermore, we obtain asymptotic values of the coefficients for large n , e.g.

$$b_n \sim \left(-\frac{1}{m^2} \right)^n \frac{\bar{b}}{n^{3/2}}, \quad c_n \sim \left(-\frac{1}{m^2} \right)^n \frac{\bar{c}}{n^{1/2}}, \quad d_n - e_n \sim \left(-\frac{1}{m^2} \right)^n \bar{d}n^{3/2} \quad (40)$$

with constant $\bar{b}, \bar{c}, \bar{d}$. This suggests the following scheme for solving our local expansion of the Schrödinger equation. First truncate the coefficients of each linearly independent local functional of φ at, say, order $\lambda^{N+1/2}$ taking $\lambda = N\mu^2/e$. This gives an infinite set of equations each involving a finite number of unknowns, but with an infinite number of unknowns overall. These can be solved with judicious use of (39) to estimate the discarded coefficients taking care to solve those equations for which the truncation error is smallest. In this approach there is no expansion parameter, but the approximation consists of working with large, but finite, values of N and λ rather than the infinite values that occur in (32). For the purpose of illustration only we take the unrealistically small value $N = 0$. Equating to zero the coefficients of $1, \varphi^2, \varphi'^2, \varphi^4, \varphi^2\varphi'^2, \varphi''^2$ keeping only the terms up to order $\sqrt{\lambda}$ gives six equations in the unknowns $\mathcal{E}, b_0, b_1, b_2, c_0, c_1, f_0, f_1$. Our asymptotic formulae for the coefficients associated with higher orders in λ are only valid for large n

but using them here for small n (after reinstating the factors of $(n+1)/n$ that for large n were replaced by 1) we obtain $c_1 = -3\sqrt{2}c_0/m^2$, $f_1 = -15f_0/m^2$, $b_2 = -2\sqrt{2}b_1/(3\sqrt{3}m^2)$. This gives us six equations in the six unknowns \mathcal{E} , b_0 , b_1 , c_0 , f_0 , m . Thus we obtain

$$c_0 = \frac{\pi}{12\sqrt{\lambda}} \left(\bar{M}^2 - \frac{4b_0^2}{\sqrt{\pi}} \right), \quad b_1 = \frac{1}{8b_0} \left(1 + \frac{\sqrt{\pi}\bar{M}^2 - 4b_0^2}{4\sqrt{2}m^2} \right) \quad (41)$$

where b_0 satisfies

$$gb_0\sqrt{\lambda} - \pi \left(\bar{M}^2 - \frac{4b_0^2}{\sqrt{\pi}} \right) \left(b_0^2(16 - 17\sqrt{2}) + 2m^2 + \frac{\sqrt{2\pi}\bar{M}^2}{4} \right) = 0 \quad (42)$$

Note that when the interaction is switched off, i.e. when $g = 0$, we obtain the free solution from $\bar{M}^2 - 4b_0^2/\sqrt{\pi} = 0$, as we should. Clearly it would be necessary to resort to numerical techniques to make progress with these algebraic equations.

We can extend this approach to the calculation of the particle spectrum. Take the one-particle state corresponding to the lightest particle at rest to have the form of a pre-factor P multiplied by Ψ . For slowly varying fields this pre-factor will be the integral of a local function

$$P = \int dx \left(\varphi + \bar{a}_1\varphi^3 + \bar{a}_2\varphi^5 + \dots + \bar{b}_1\varphi\varphi'^2 + \bar{b}_2\varphi\varphi'^4 + \dots + \bar{c}_1\varphi^3\varphi'^2 + \dots \right) \quad (43)$$

The Schrödinger equation for this state reduces to an equation linear in P

$$\frac{1}{2\pi i} \int_C ds s^{-1} e^{\lambda s} \left(\frac{1}{2}(\Delta_s)P + \int dx \frac{\delta P}{\delta\varphi(x)} \frac{\delta W}{\delta\varphi(x)} + mP \right) [\phi_s] = 0 \quad (44)$$

Again this may be reduced to a finite number of equations for a finite number of unknown coefficients by truncation and using formulae for the asymptotic behaviour of those coefficients derived from the study of $\Delta_s P$ for large s .

To summarise: the logarithm of the vacuum functional reduces to a local functional for fields that vary slowly on the scale of the lightest mass in the theory, but this does not satisfy the obvious form of the Schrödinger equation because expanding in slowly varying fields does not commute with removing a short distance cut-off. Instead we constructed the appropriate form of the equation for the case of φ^4 theory by re-summing the cut-off dependence using the analyticity properties of the Laplacian applied to the vacuum. This Schrödinger equation reduces the eigenvalue problem for the Hamiltonian to an infinite set of algebraic equations. We indicated how these may be solved in the usual semi-classical approximation and suggested a new approximation scheme that would also be viable for theories that are classically massless but acquire mass quantum mechanically such as the O(N) sigma model and Yang-Mills theories.

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