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The Complex Bateman Equation

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Abstract. The general solution to the complex Bateman equation is constructed. It is given in implicit form in terms of a functional relationship for the unknown function. The known solution of the usual Bateman equation is recovered as a special case.

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1. Introduction

We define the complex Bateman equation as

$$\det\begin{vmatrix} 0 & \dfrac{\partial\phi}{\partial y_1} & \dfrac{\partial\phi}{\partial y_2} \\ \dfrac{\partial\phi}{\partial x_1} & \dfrac{\partial^2\phi}{\partial x_1\partial y_1} & \dfrac{\partial^2\phi}{\partial x_1\partial y_2} \\ \dfrac{\partial\phi}{\partial x_2} & \dfrac{\partial^2\phi}{\partial x_2\partial y_1} & \dfrac{\partial^2\phi}{\partial x_2\partial y_2} \end{vmatrix} = 0. \tag{1}$$

The real form of the Bateman equation,

$$\left(\frac{\partial\psi}{\partial x}\right)^2\frac{\partial^2\psi}{\partial y^2}+\left(\frac{\partial\psi}{\partial y}\right)^2\frac{\partial^2\psi}{\partial x^2}-2\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} = 0, \tag{2}$$

arises when the unknown function depends upon only two arguments, $x = x_1 + y_1,\ y = x_2 + y_2$

This equation, which is equivalent to the pair of first-order equations

$$\frac{\partial u}{\partial x} = u\frac{\partial u}{\partial y}, \qquad \frac{\partial\psi}{\partial x} = u\frac{\partial\psi}{\partial y}, \tag{3}$$

has a general solution given implicitly by choosing two arbitrary functions of one variable $f(\psi)$ and $g(\psi)$ and constraining them to satisfy the linear relation

$$xf(\psi) + yg(\psi) = c = \quad \text{constant.} \tag{4}$$

An inhomogeneous form of the Bateman equation, the so-called two-dimensional Born–Infeld equation, is equivalent to the equation describing minimal surfaces, and has been solved by Bateman himself, [1] and Barbasov and Chernikov [2] Further properties and generalisations of the real Bateman equation can be found in [3, 4]. The general solution to the complex Bateman equation (1), on the other hand, is given, again implicitly, by identifying two arbitrary functions of three variables, $F(\phi, x_1, x_2)$ which depends upon $(\phi,\ x_1,\ x_2)$ and $G(\phi, y_1, y_2)$ depending upon $(\phi,\ y_1,\ y_2)$ and solving the resulting equality

$$F(\phi, x_1, x_2) \equiv G(\phi, y_1, y_2). \tag{5}$$

implicitly for $\phi(x_1, x_2, y_1, y_2)$. This assertion may be readily verified. This solution encompasses the above solution to the real Bateman equation by the choice of the arbitrary functions $F,\ G$ as

$$F = x_1 f(\phi) + y_1 g(\phi), \qquad G = -x_2 f(\phi) - y_2 g(\phi) + c. \tag{7}$$

It is the purpose of this Letter to explain how this result may be deduced, with an eye to further generalisation. At this point we insert a caveat; this analysis is carried out in the spirit of many investigations in mathematical physics, of being a little cavalier about rigorous questions of differentiability of the functions involved. We assume that the functions with which we deal are twice differentiable, though we are well aware that the real Bateman equation admits solutions of shock wave type, where differentiability at one or more points fails.

2. Proof

The complex Bateman equation (1) is the eliminant of three linearly dependent equations which may be written as

$$\begin{aligned}
&\alpha^1 \frac{\partial \phi}{\partial y_1} + \alpha^2 \frac{\partial \phi}{\partial y_2} = 0,\\
&\frac{\partial \phi}{\partial x_1} - \frac{\partial \alpha^1}{\partial x_1}\frac{\partial \phi}{\partial y_1} - \frac{\partial \alpha^2}{\partial x_1}\frac{\partial \phi}{\partial y_2} = 0,\\
&\frac{\partial \phi}{\partial x_2} - \frac{\partial \alpha^1}{\partial x_2}\frac{\partial \phi}{\partial y_1} - \frac{\partial \alpha^2}{\partial x_2}\frac{\partial \phi}{\partial y_2} = 0.
\end{aligned} \tag{7}$$

Here $\alpha^1,\ \alpha^2$ are functions of the variables $(x_1,\ x_2,\ y_1,\ y_2)$. The linear equations whose eliminant gives (1) are obtained from the first equation of (7) together with the second plus the derivative of the first with respect to x_1 and the third plus the x_2 derivative of the first. These equations admit an obvious generalisation to any number of pairs x_i, y_j. Clearly,

$$\frac{\partial\phi}{\partial x_1} - \frac{\partial}{\partial x_1}\big(\log(\alpha^1) - \log(\alpha^2)\big)\frac{\partial\phi}{\partial y_2}\alpha^2 = 0, \tag{8}$$

$$\frac{\partial\phi}{\partial x_2} - \frac{\partial}{\partial x_2}\big(\log(\alpha^1) - \log(\alpha^2)\big)\frac{\partial\phi}{\partial y_2}\alpha^2 = 0. \tag{9}$$

Cross differentiation shows that $((\partial\phi/\partial y_2)\alpha^2)^{-1}$ is a function of $(\phi,\ y_1,\ y_2)$ and, hence, we may write

$$\log\left(\frac{\alpha^1}{\alpha^2}\right) = K(\phi, y_1, y_2), \tag{10}$$

where K is an arbitrary function of $(\phi,\ y_1,\ y_2)$. Further implications of these equations are as follows:

$$\frac{\partial\phi}{\partial y_1} = -\frac{1}{K'\alpha^1}, \quad \frac{\partial\phi}{\partial y_2} = \frac{1}{K'\alpha^2}, \tag{11}$$

where K' denotes the partial derivative of K with respect to ϕ. As a consequence, $\exists$ a function $U(\phi, y_1, y_2)$ such that

$$\frac{\partial\phi}{\partial y_1} = U(\phi, y_1, y_2)\frac{\partial\phi}{\partial y_2}. \tag{12}$$

Similarly, we can introduce a second function $V(\phi, x_1, x_2)$ such that

$$\frac{\partial\phi}{\partial x_1} = V(\phi, x_1, x_2)\frac{\partial\phi}{\partial x_2} \tag{13}$$

The integrability condition for those two equations, obtained by eliminating the mixed (x_1, y_1) derivatives of ϕ is automatically satisfied. If $U(\phi, y_1, y_2)$ is written in the form

$$U = \frac{(\partial/\partial y_1)G(\phi, y_1, y_2)}{(\partial/\partial y_2)G(\phi, y_1, y_2)} \tag{14}$$

for some function $G(\phi, y_1, y_2)$, where ϕ is regarded as a parameter the partial derivatives with respect to y_1 and y_2 act on the last two arguments of G , then this is simply a first order differential equation for G, which is in principle solvable. However the partial derivatives in (14) may be replaced by total derivatives when ϕ is now regarded as a function of $(x_1,\ x_2,\ y_1,\ y_2)$ since

$$U = \frac{(dG/dy_1)}{(dG/dy_2)} = \frac{(\partial/\partial\phi)G(\phi, y_1, y_2)(\partial\phi/\partial y_1) + (\partial/\partial y_1)G(\phi, y_1, y_2)}{(\partial/\partial\phi)G(\phi, y_1, y_2)(\partial\phi/\partial y_2) + (\partial/\partial y_2)G(\phi, y_1, y_2)} = \frac{(\partial\phi/\partial y_1)}{(\partial\phi/\partial y_2)} \tag{15}$$

using (12). This equation implies that G is a function of ϕ, together with the additional variables in the problem, i.e. $x_1,\ x_2$. So we may write

$$G(\phi, y_1, y_2) = F(\phi, x_1, x_2) \tag{16}$$

for some function F, which is the result announced.

By the same token,

$$V = \frac{(d/dx_1)F(\phi, x_1, x_2)}{(d/dx_2)F(\phi, x_1, x_2)}. \tag{17}$$

3. Conclusions

We have shown that what we have called the complex Bateman equation,

$$\begin{aligned}\frac{\partial\phi}{\partial x_1}\frac{\partial\phi}{\partial y_1}\frac{\partial^2\phi}{\partial x_2\partial y_2} + \frac{\partial\phi}{\partial x_2}\frac{\partial\phi}{\partial y_2}\frac{\partial^2\phi}{\partial x_1\partial y_1} - \\ -\frac{\partial\phi}{\partial x_1}\frac{\partial\phi}{\partial y_2}\frac{\partial^2\phi}{\partial x_2\partial y_1} - \frac{\partial\phi}{\partial x_2}\frac{\partial\phi}{\partial y_1}\frac{\partial^2\phi}{\partial x_1\partial y_2} = 0.\end{aligned} \tag{18}$$

may be solved completely in terms of two arbitrary functions $F(\phi, y_1, y_2)$ and $G(\phi, x_1, x_2)$ which are constrained to be equal. The first-order equations (12) and (13) both separately imply the complex Bateman equation. We expect that the extension of these results to higher dimensions will proceed along similar lines to that for the real Bateman equation [5, 6]. We hope to return to the question of the solution of the complex generalisation in arbitrary dimensions in the near future.

Note Added in Proof.

A similar result, without proof, can be found in T. Chaundy: *The Differential Calculus*, Oxford Univ. Press, 1935, p. 328.

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