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Collapsed 3-Dimensional Alexandrov Spaces: A Brief Survey

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Abstract

We survey two recent developments in the topic of three-dimensional Alexandrov spaces: the topological classification of closed collapsed three-dimensional Alexandrov spaces and the geometrization of sufficiently collapsed closed three-dimensional Alexandrov spaces.

13.1 Introduction

Alexandrov spaces (with curvature bounded below) are metric generalizations of complete Riemannian manifolds with a uniform lower sectional curvature bound. In addition to its intrinsic interest, Alexandrov geometry plays an important role in the proof of finiteness results for certain families of closed (i.e. compact and without boundary) Riemannian manifolds (see, for example, the survey [12]). Indeed, by Gromov's precompactness theorem, the family $\mathcal{M}_k^D(n)$ of closed Riemannian n-manifolds with sectional curvature $\sec \geq k$ and diameter bounded above by $D > 0$ is precompact in the Gromov–Hausdorff topology. Moreover, limits of sequences in $\mathcal{M}_k^D(n)$ are Alexandrov spaces with curvature bounded below by k. More generally, if X is the Gromov–Hausdorff limit of a sequence $\{X_i^n\}_{i=1}^\infty$ of compact n-dimensional Alexandrov spaces with curvature bounded below by k, then X is an Alexandrov space with curvature bounded below by k and (Hausdorff) dimension at most n. Topologically, the case where the limit X is n-dimensional is well understood. Indeed, by Perelman's stability theorem [21], the elements X_i^n of the sequence are homeomorphic to X for i sufficiently large. The complementary phenomenon, in which the dimension of X is strictly less than n, is known as *collapse*. Note that, by Perelman's stability theorem, if sequences in a precompact family of closed Riemannian n-manifolds with a uniform lower sectional curvature bound do not collapse, then the family must consist of finitely many homeomorphism types. This is the case, for example, for the family $\mathcal{M}^D_{k,v}(n)$ of closed Riemannian n-manifolds with $\sec \geq k$, diameter at most $D > 0$ and volume bounded below by $v > 0$.

A simple example of collapse is furnished by rescaling the Riemannian metric of a given flat n-dimensional torus T^n, $n \geq 2$, by $1/k$, $k = 1, 2 \ldots$ In this way, one obtains a sequence $\{T^n_k\}$ of flat n-tori whose diameter decreases as $k \to \infty$. In this case, the sequence of flat tori collapses to a point. By rescaling appropriate factors of T^n we may obtain sequences of flat tori which collapse to flat tori of dimension strictly less than n. Further examples of collapse with a uniform lower sectional curvature bound may be obtained by rescaling the orbits of isometric compact Lie group actions on a given closed Riemannian manifold. In this case, the sequence of metrics converges to the orbit space of the action, which is an Alexandrov space and is, in general, not a manifold.

Motivated by the preceding considerations, one may attempt to understand the topological consequences of collapse. In the Riemannian category, a thorough analysis of collapse of Riemannian 3-manifolds was carried out by Shioya and Yamaguchi in [28, 29]. More recently, Mitsuishi and Yamaguchi obtained topological classification and structure results for collapsed Alexandrov spaces of dimension 3 (see [18]), while the authors of the present survey obtained the geometrization of closed, sufficiently collapsed irreducible 3-dimensional Alexandrov spaces (see [9]), thus extending the Riemannian results to the case of Alexandrov spaces. In this note we give a brief account of these results in Alexandrov geometry in the hope of sparking the interest of the reader.

This chapter is organized as follows. Section 13.2 contains a summary of basic results in Alexandrov geometry. In Section 13.3 we recall the basic results on the topological structure of general Alexandrov spaces of dimension three. Finally, in Section 13.4, we present the topological structure and classification results for closed collapsed 3-dimensional Alexandrov spaces, as well as the geometrization of closed, sufficiently collapsed irreducible Alexandrov spaces of dimension 3.

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13.2 Basic Alexandrov Geometry

In this section we will recall the notation and main aspects of the theory of Alexandrov spaces (of curvature bounded below). Standard references in the subject are [4, 5] (see also the recent manuscript [1]), and we refer the reader to these sources for a detailed account of the theory.

In order to introduce the definition of an Alexandrov space, we first recall some concepts. Alexandrov spaces fall within the class of the so-called *length spaces*. A metric space (X, d) is a *length space* whenever, for every $x, y \in X$,

$$d(x, y) = \inf \{L(\gamma) \mid \gamma(a) = x,\ \gamma(b) = y\}.$$

Here, the infimum is taken over all continuous curves $\gamma : [a, b] \to X$ (for some $a \leq b$) and $L(\gamma)$ stands for the *length* of γ. The length of such a curve is defined as

$$L(\gamma) = \sup \left\{ \sum_{i=1}^{n-1} d\left(\gamma(t_i), \gamma(t_{i+1})\right) \right\},$$

where the supremum is taken over all finite partitions

$$a = t_0 \leq t_1 \leq \ldots \leq t_n = b$$

of $[a, b]$. We require two technical assumptions on any length space X: completeness and local compactness. This ensures the existence of *geodesics* between each pair of points $x, y \in X$, that is, continuous curves $\gamma : [a, b] \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$ and $L(\gamma) = d(x, y)$. A geodesic joining x and y will be denoted by $[xy]$. Note that such a geodesic might not be unique, as can be readily seen by considering geodesics in a round sphere.

One of the concepts playing a central role in Alexandrov geometry is that of the *model spaces*. Given a real number k, the 2-dimensional *model space* M_k^2 is defined to be the complete, simply-connected 2-dimensional Riemannian manifold of constant sectional curvature k. In other words, depending on the sign of k, M_k^2 is isometric to one of the following spaces:

- $\mathbb{S}_k^2$, the sphere of constant curvature $k > 0$;
- $\mathbb{E}^2$, the Euclidean plane of curvature 0; or
- $\mathbb{H}_k^2$, the hyperbolic plane of constant curvature $k < 0$.

Other prominent objects in the theory are *geodesic triangles*. A geodesic triangle $\triangle\, pqr$ in a length space (X, d) is a collection of three points $p, q, r \in X$ and three geodesics $[pq]$, $[qr]$ and $[rp]$. Once a geodesic triangle $\triangle\, pqr$ in X is given, one says that a geodesic triangle $\triangle\, \widetilde{p}\widetilde{q}\widetilde{r}$ in M_k^2 is a *comparison triangle for* $\triangle\, pqr$ if $d(p, q) = |p, q|$, $d(q, r) = |q, r|$ and $d(r, p) = |r, p|$. Here $|\cdot, \cdot|$ stands for the usual length metric on M_k^2.

With these definitions in hand, we may now recall the definition of an Alexandrov space. We will say that a length space (X, d) has *curvature bounded below by* $k \in \mathbb{R}$, denoted by $\operatorname{curv}(X, d) \geq k$ (or simply $\operatorname{curv}(X) \geq k$), if, for every $x \in X$, there exists an open neighborhood $U \subset X$ of x such that for every geodesic triangle $\triangle\, pqr$ and any comparison triangle $\triangle\, \widetilde{p}\widetilde{q}\widetilde{r}$ in M_k^2 the so called T_k*-property* holds: For every $s \in [pq]$ and $\widetilde{s} \in [\widetilde{p}\widetilde{q}]$ such that $d(p, s) = |\widetilde{p}, \widetilde{s}|$, $d(r, s) \geq |\widetilde{r}, \widetilde{s}|$.

Definition 13.2.1 An *Alexandrov space* is a complete and locally compact length space (X, d) such that $\operatorname{curv}(X) \geq k$ for some $k \in \mathbb{R}$.

It is worth noting that there are several equivalent definitions of Alexandrov spaces (see [4, Thm. 4.3.5]). Here we just mention the following *monotonicity of angles condition*: Let $\gamma_1, \gamma_2 : [a, b] \to X$ be geodesics such that $\gamma_1(0) = \gamma_2(0)$. Then X is an Alexandrov space of curv $\geq k$ if and only if the function

$$\theta_k(s, t) := \angle \widetilde{\gamma_1(s)}\widetilde{\gamma_1(0)}\widetilde{\gamma_2(t)} \tag{13.1}$$

is monotone non-increasing in $s, t \in [a, b]$. Here, $\triangle\, \widetilde{\gamma_1(s)}\widetilde{\gamma_1(0)}\widetilde{\gamma_2(t)}$ is a comparison triangle for $\triangle\, \gamma_1(s)\gamma_1(0)\gamma_2(t)$.

One of the most powerful tools available in Alexandrov geometry is the following globalization theorem, essentially asserting that once the T_k-property is satisfied locally, then it holds in the large (see [4, Thm. 10.3.1]).

Theorem 13.2.2 (Globalization theorem) *Let X be an Alexandrov space with* $\operatorname{curv}(X) \geq k$. *Then the T_k-property is satisfied for any geodesic triangle in X.*

The most familiar examples of Alexandrov spaces are smooth, complete Riemannian manifolds of sectional curvature bounded below. This is guaranteed by the Toponogov distance comparison theorem, [23, Thm. 12.2.2]. The same result implies naturally that if $l < k$ and X is an Alexandrov space of curv $\geq k$, then $\operatorname{curv}(X) \geq l$. However, the class of Alexandrov spaces includes non-smooth spaces. For example, the boundary of an open and convex set in a Euclidean space $\mathbb{R}^n$, regarded with the induced metric, is a non-negatively curved space [27]. There are a number of constructions available to produce new Alexandrov spaces from known examples. In this way one can produce Alexandrov spaces which are not homeomorphic to manifolds. Let us mention the most commonly used constructions.

- *Cartesian products.* Let X and Y be Alexandrov spaces with curv $\geq k$ and $k \leq 0$. The Cartesian product $X \times Y$ with the usual product metric is an Alexandrov space of curv $\geq k$. For $k > 0$, the product is a space of curv $\geq k$ only in the case that one of the spaces is a single point.
- *Euclidean cones.* Let (X, d) be a metric space with $\operatorname{diam}(X) \leq \pi$. Recall that the cone over X is the metric space $(K(X), d_K)$ obtained from $X \times [0, \infty)$ by collapsing $X \times \{0\}$ to a point. The metric d_K is given by

$$d_K\left((x_1, t_1), (x_2, t_2))\right) = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos d(x_1, x_2)}.$$

  The cone $K(X)$ is an Alexandrov space of curv ≥ 0 if and only if X is an Alexandrov space of curv ≥ 1.
- *Spherical suspensions.* Let (X, d) be a metric space with $\operatorname{diam}(X) \leq \pi$. The spherical suspension $(\operatorname{Susp}(X), d_S)$ of X is the metric space obtained from $X \times [0, \pi]$ by collapsing $X \times \{0\}$ and $X \times \{\pi\}$ to single points. A metric d_S is then defined by the equation

$$\cos d_S\left((x_1, t_1), (x_2, t_2)\right) = \cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos d(x_1, x_2).$$

  If X is an Alexandrov space of curv ≥ 1, then $\operatorname{Susp}(X)$ is an Alexandrov space of curv ≥ 1.

These constructions are in fact special cases of *warped products*. The fact that they indeed produce spaces of curvature bounded below can be easily seen from the so called fiber independence theorem, [1, Thm.

10.1.3]. The problem of obtaining necessary and sufficient conditions for a warped product to be an Alexandrov space was solved in [2, 3]. One of the most important features in Alexandrov geometry is the following:

- *Gromov–Hausdorff limits.* Let $\{X_i\}_{i=1}^{\infty}$ be an infinite sequence of compact Alexandrov spaces with $\operatorname{curv}(X_i) \geq k$ for all i. If X_i converges in the Gromov–Hausdorff sense to a metric space X, then X is an Alexandrov space of curv $\geq k$.

It is in the context of Gromov–Hausdorff limits where the phenomenon of *collapse* occurs. We say that a sequence of compact Alexandrov spaces $\{X_i\}_{i=1}^{\infty}$ of (Hausdorff) dimension n which converges in the Gromov–Hausdorff sense to an Alexandrov space X *collapses* if $\dim X < n$.

As the previous examples indicate, the local geometry and topology of an Alexandrov space may be vastly different from that of a manifold. Nevertheless, there are certain similarities with the manifolds case. It is known that the Hausdorff dimension of an Alexandrov space is either a non-negative integer or infinite. In fact, if the Hausdorff dimension is finite, then, as in the manifold setting, the Hausdorff dimension coincides with the topological dimension [5, Cor. 6.5]. For simplicity we focus on finite-dimensional spaces below.

In the smooth category, the local structure of the space is completely determined by the infinitesimal picture, in the sense that, for every point in a smooth manifold there exists a neighborhood diffeomorphic to the tangent space at the said point. A similar relationship is available in Alexandrov geometry via the *space of directions*. In order to recall this concept, we outline some definitions.

Firstly, let us recall that the monotonicity of angles condition implies the well possessedness of angles between two geodesics which share a starting point. Let X be an Alexandrov space of curv $\geq k$, and assume that $\gamma_1, \gamma_2 : [a, b] \to X$ are two geodesics with $p := \gamma_1(0) = \gamma_2(0)$. As the function $\theta_k(s,t)$ of Equation (13.1) is monotone non-increasing and takes values in $[0, \pi]$, one can define the *angle between* γ_1 *and* γ_2 by

$$\angle(\gamma_1, \gamma_2) := \lim_{s,t} \theta_k(s,t).$$

It is worth noting that the angle is, in fact, independent of k. An equivalence relation between geodesics emanating from the same point is then obtained: Two such geodesics are equivalent if they make a null angle. A *geodesic direction at* $p \in X$ is an equivalence class of geodesics having p as a starting point. The collection of all geodesic directions at a point p

has the structure of a (possibly incomplete) metric space when equipped with the angle as metric. The completion of the space of geodesic directions at p is the *space of directions of* X *at* p and is denoted by $\Sigma_p X$ (or simply by Σ_p).

To obtain a tangent space at $p \in X$ there are at least two natural procedures one can use:

(i) Consider the cone over Σ_p.
(ii) Consider a *blow-up of* X *at* p, i.e. the pointed Gromov–Hausdorff limit of balls $B(p, r_i)$ with the restricted metric rescaled by a factor of $1/r_i$, where $r_i \to 0$. Such a limit exists and is independent of the choice of sequence.

These two methods give rise to isometric metric spaces denoted by $T_p X$ (see [4, Thm. 10.9.3]), directly implying the following structural properties of Σ_p, [4, Cor. 10.9.6].

Theorem 13.2.3 *Let* X *be an* n*-dimensional Alexandrov space and* $p \in X$*. Then the following hold:*

(1) Σ_p *is a compact* $(n-1)$*-dimensional Alexandrov space.*
(2) *If* $n \geq 2$*, then* $\mathrm{curv}(\Sigma_p) \geq 1$*.*
(3) *If* $n = 1$*, then* Σ_p *either consists of two points or a single point.*

The local topology of X at a point p is determined by $T_p X$, a fact that is asserted in the following result of Perelman [22].

Theorem 13.2.4 (Conical neighborhood theorem) *Let* X *be an Alexandrov space and* $p \in X$*. Then, any sufficiently small neighborhood of* p *is pointed-homeomorphic to* $T_p X$*.*

Being one of the most powerful tools in Alexandrov geometry, the previous result allows one, among several other applications, to define inductively the *boundary* of an Alexandrov space. One-dimensional spaces are topological manifolds. Hence the boundary of such a space is defined in the usual manner. Assuming that the boundary of $(n-1)$-dimensional spaces has been defined, one says that a point p in an n-dimensional space is in the boundary if Σ_p has non-empty boundary. The boundary ∂X of an Alexandrov space X is a closed subset of Hausdorff codimension 1.

Once one has a well-posed concept of boundary, one can construct more examples of Alexandrov spaces from pairs of them by gluing along the boundaries:

- *Gluing along the boundary.* Let X_1 and X_2 be Alexandrov spaces of curv $\geq k$ with non-empty boundaries such that ∂X_1 is isometric to ∂X_2 when considered with the induced metrics. Let $f : \partial X_1 \to \partial X_2$ be an isometry. Then the adjunct space $X_1 \cup_f X_2$ is an Alexandrov space of curv $\geq k$ [24, Thm. 2.1]. It is possible to glue along more general subsets known as *extremal subsets* in some circumstances [17].

The conical neighborhood theorem suggests the following terminology. A point p on an n-dimensional Alexandrov space X is said to be *topologically regular* if Σ_p is homeomorphic to a sphere $\mathbb{S}^{n-1}$. Otherwise, p is said to be *topologically singular*. Furthermore, p is *metrically regular* if Σ_p is isometric to the unit round sphere $\mathbb{S}^{n-1}$ and *metrically singular* otherwise. In contrast to Riemannian manifolds, an Alexandrov space can be topologically regular (that is, each of its points is topologically regular) but have metrically singular points, (see, for example, [25, Exam. 97]). However, the subset of topologically singular points of X is dimensionally not very large. The codimension of the subset of topologically singular points which are not boundary points is at least 3. This is a consequence of the fact that Alexandrov spaces have a canonical stratification by topological manifolds (see [4, Thm. 10.10.1], [22, Thm. III structure theorem]). This fact will play an important role in the following section.

13.3 Three-Dimensional Alexandrov Spaces

As previously seen, Alexandrov spaces are generalizations of Riemannian manifolds, and, as such, it is interesting to study their topology. In this section we will focus on the three-dimensional case, with the intention of providing the necessary background for the main results in this survey.

It should be observed that Alexandrov spaces of dimensions one or two are respectively topological curves or surfaces. This was already proven in the original [5] (see also [4]), but it is also a consequence of Perelman's conical neighborhood theorem (see Theorem 13.2.4) and the classification of Alexandrov spaces of positive curvature in dimensions zero and one (that are, respectively, one or two points, and circles of length less than 2π or an interval of length less than or equal to π). Therefore, the first dimension where interesting new phenomena occur is dimension 3.

To understand the difference between genuine 3-dimensional Alexandrov spaces and 3-manifolds, it is helpful to think again of Perelman's conical neighborhood theorem; the spaces of directions that can appear in a 3-dimensional Alexandrov space will be compact Alexandrov surfaces of positive curvature. The list of these is also found in [5], where it was proven that, ignoring those with non-empty boundary, they are topologically 2-spheres or projective planes. The former will give rise to manifold points, while the latter will correspond to singular points. We collect these observations in the following statement. As is customary, we will say that a compact space without boundary is *closed.*

Lemma 13.3.1 *Let X be a closed 3-dimensional Alexandrov space. If X is not homeomorphic to a closed topological 3-manifold, then there is an even number of points $p_1, \ldots, p_k$ in X such that X is homeomorphic to the union of k disjoint cones over $\mathbb{R}P^2$s, and a non-orientable 3-dimensional compact manifold Y with a boundary formed by k connected components equal to $\mathbb{R}P^2$.*

Proof Observe that if a point $p \in X$ has $\mathbb{R}P^2$ as its space of directions, then, by Perelman's conical neighborhood theorem, there is a neighborhood U of p such that any other point in $U \setminus \{p\}$ has $\mathbb{S}^2$ as its space of directions. Thus, singular points in X are isolated and, since X is compact, there can only be a finite number of such p's.

Denote by $p_1, \ldots, p_k$ the singular points in X. For each $1 \leq i \leq k$, choose a conical neighborhood of p_i, U_i, such that the collection $\{U_i\}$ consists of pairwise disjoint sets. Then $Y := X \setminus \cup U_i$ is a compact 3-manifold with boundary equal to a disjoint union of k copies of $\mathbb{R}P^2$. Since each boundary component of Y has a collared neighborhood, Y contains two-sided $\mathbb{R}P^2$'s, and is therefore non-orientable. A simple application of Lefschetz duality shows that k is an even number (see, for instance, [11, Exer. 28.25]). □

We can improve upon the preceding description by considering the orientable double cover of Y, as we illustrate in the following proposition.

Proposition 13.3.2 *Let X be a closed 3-dimensional Alexandrov space with singular points. Then there is an orientable closed 3-dimensional manifold M and an orientation-reversing involution $a : M \to M$ with a finite number of isolated fixed points such that X is homeomorphic to the quotient space M/a.*

Proof Let Y be the compact 3-manifold obtained above by removing disjoint open neighborhoods of the singular points of X. Since Y has some boundary components homeomorphic to the projective plane, Y contains two-sided $\mathbb{R}P^2$'s, and is therefore non-orientable. Denote by $\bar{Y}$ its orientable 2-fold cover, and its covering map by $\bar{a} : \bar{Y} \to \bar{Y}$; since $\bar{Y}$ is orientable but Y is not, $\bar{a}$ is orientation reversing. Observe also that $\bar{Y}$ has the same number of boundary components as Y, but containing only $\mathbb{S}^2$'s. Capping out these boundary components by 3-disks, D^3, we get a closed orientable 3-manifold M. The involution $\bar{a} : \bar{Y} \to \bar{Y}$ can be extended to the whole M by identifying each disk D with the Euclidean 3-ball B, and using the involution mapping each point $x \mapsto -x$ in B. It is clear that the extended involution $a : M \to M$ will have only fixed points corresponding to the center of the disks, and will therefore be isolated. □

The above topological description of 3-dimensional Alexandrov spaces is quite useful, since it allows us to switch from the category of metric spaces to that of 3-manifolds, where a lot of information is available. Note that singular Alexandrov 3-spaces are homeomorphic to non-orientable 3-dimensional orbifolds.

The next natural step is to return to the metric category, and consider whether lifting the metric from X to M produces something of interest. This is contemplated in the following lemma due to Grove and Wilking [13, Sect. 5].

Lemma 13.3.3 *Let X be a closed three-dimensional Alexandrov space with curvature bounded below by k, with $k \geq 0$, and assume that X is not a topological manifold. If M is the orientable double branched cover of X in Lemma 13.3.2, then the following hold:*

(1) *The metric in X can be lifted to M so that M is an Alexandrov space with curvature bounded below by k.*

(2) *The involution $a : M \to M$ is an isometry.*

The proof of this lemma appears in [13], although as it also includes the case of Alexandrov spaces of dimension 4, it is at times brief; for a more detailed version, entirely adapted to dimension three, the reader can refer to [6].

13.3.1 Geometric 3-Alexandrov Spaces

Geometric 3-manifolds can be considered the building blocks of arbitrary 3-dimensional closed manifolds, as Thurston's geometrization conjecture shows. It is then natural to ask about the corresponding notion for Alexandrov spaces. Recall that the eight Thurston geometries are $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H} \times \mathbb{R}$, Nil, Sol and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ (see [26]).

Definition 13.3.4 We say that an Alexandrov space X^3 has a given Thurston geometry (see [26]) if X^3 can be written as a quotient of the corresponding geometry by some cocompact lattice. In that case, we will say that such an Alexandrov 3-space is *geometric*.

The main difference with the manifold case is that we allow for fixed points in the lattice action.

13.3.2 Geometrization of 3-Alexandrov Spaces

Recall that the usual geometrization of closed 3-manifolds requires the manifold to be divided into pieces: first one takes the decomposition into prime manifolds using 2-spheres to subdivide, and later one performs a Jaco–Shalen–Johannson decomposition using 2-tori. In our case, since Alexandrov 3-spaces with singular points contain a non-orientable core, we will require more subdividing surfaces.

Definition 13.3.5 We say that a closed three-dimensional Alexandrov space X admits a geometric decomposition if there exists a collection of spheres, projective planes, tori and Klein bottles that decompose X into geometric pieces.

Theorem 13.3.6 (Geometrization of 3-dimensional Alexandrov spaces) *A closed three-dimensional Alexandrov space admits a geometric decomposition into geometric three-dimensional Alexandrov spaces.*

The proof of the above result can be found in [8]. For an overview of further results on three-dimensional Alexandrov spaces, we refer the reader to [7].

13.4 Collapsed Three-Dimensional Alexandrov Spaces

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of n-dimensional Alexandrov spaces with diameters uniformly bounded above by $D > 0$ and curv $\geq k$ for some $k \in \mathbb{R}$. After passing to a subsequence, Gromov's precompactness theorem implies that there exists an Alexandrov space Y with $\operatorname{diam} Y \leq D$ and $\operatorname{curv} Y \geq k$ such that $X_i \xrightarrow{\text{GH}} Y$. As in the Riemannian case, the sequence X_i is said to *collapse* to Y if $\dim Y < n$. We will also say that an n-dimensional Alexandrov space X *collapses* (or that it is a *collapsing Alexandrov space*) if there exists a sequence of Alexandrov metrics $\{d_i\}_{i=1}^{\infty}$ on X, such that $\{(X, d_i)\}_{i=1}^{\infty}$ is a collapsing sequence. In this section, the last in this survey, we give an overview of the available structure and classification results for collapsed Alexandrov 3-spaces.

13.4.1 General Structure Results

In Riemannian geometry, collapse imposes strong geometric and topological restrictions on the spaces on which it occurs. Indeed, Shioya and Yamaguchi obtained comprehensive structure results for closed, collapsed three-dimensional Riemannian 3-manifolds [27]. In the Alexandrov category, Mitsuishi and Yamaguchi carried out an exhaustive study of collapsed, closed, three-dimensional Alexandrov spaces, and we summarize their results in this section. The main difference between the collapse of three-dimensional Alexandrov spaces and that of three-dimensional Riemannian manifolds resides in the fact that in the Alexandrov case collapse can occur along the fibers of a "generalized" Seifert fibration. Collapsed Alexandrov 3-spaces can be described as unions of certain pieces. Before stating the general structure results, let us give a brief account of those pieces where topological singularities arise.

The space B(pt) Let $D^2 \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{C}$ be equipped with the usual flat product metric. An isometric involution α on $D^2 \times \mathbb{S}^1$ is defined by

$$\alpha((x,y), e^{i\theta}) := ((-x,-y), e^{-i\theta}).$$

The space $\mathrm{B(pt)} := D^2 \times \mathbb{S}^1/\alpha$ is an Alexandrov space of curv ≥ 0 with two topologically singular points corresponding to the image in the quotient of the points $((0,0), e^{i0})$ and $((0,0), e^{i\pi})$, which are fixed by α (see [18, Exam. 1.2]). There is a projection $p : \mathrm{B(pt)} \to K_1(\mathbb{S}^1)$ sending an interval joining the topologically singular points to the vertex o of the cone. To describe it, observe that the quotient of $D^2 \subset \mathbb{R}^2$ by the involution $(x, y) \mapsto (-x, -y)$ is homeomorphic to D^2, and metrically is isometric to $K_1(\mathbb{S}^1)$, where the $\mathbb{S}^1$ taken has length π. The projection $p : \mathrm{B(pt)} \to K_1(\mathbb{S}^1)$ is then obtained by mapping

$$[(x, y), e^{i\theta}] \to [(x, y)].$$

This projection is a fibration on $K_1(\mathbb{S}^1) \setminus \{o\}$.

The space B(pt) can also be described as follows (see lines after [18, Exam. 2.60]): take two cones over $\mathbb{R}P^2$, select a disk D_i^2, $i = 0, 1$, on each $\mathbb{R}P^2$-boundary, and glue both cones using some homeomorphism $\varphi : D_0^2 \to D_1^2$. The resulting space does not depend on the gluing homeomorphism φ, and is homeomorphic to B(pt). It is clear that its boundary is obtained by taking two Möbius bands glued by their boundaries, i.e. the boundary of B(pt) is a Klein bottle.

Spaces with 2-dimensional souls We now describe three different closed Alexandrov 3-spaces as quotients of certain involutions:

(i) $\mathrm{B}(S_2) := \mathbb{S}^2 \times [-1, 1]/(\sigma, -\mathrm{id})$, where $\mathbb{S}^2$ is a sphere of non-negative curvature in the Alexandrov sense with an isometric involution σ of $\mathbb{S}^2$ topologically conjugate to the involution on the 2-sphere given by the suspension of the antipodal map on the circle. The resulting space is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2) \setminus \mathrm{int}(D^3)$, where $D^3 \subset \mathrm{Susp}(\mathbb{R}P^2)$ is a closed 3-ball consisting of topologically regular points (see [18, Rem. 2.62]).

(ii) $\mathrm{B}(S_4) := T^2 \times [-1, 1]/(\sigma, -\mathrm{id})$, where T^2 is a flat torus and the involution $\sigma : T^2 \to T^2$ maps (z_1, z_2) to $(\overline{z_1}, \overline{z_2})$ (observe that $\mathbb{T}^2/\sigma$ is homeomorphic to $\mathbb{S}^2$). This space has four topologically singular points, corresponding to the four fixed points of the involution; this can be seen by observing that at each such point, the differential of the involution acts as the antipodal map on the unit tangent sphere. Its oriented branched cover is $\mathbb{T}^2 \times [-1, 1]$.

(iii) $\mathrm{B}(\mathbb{R}P^2) := K^2 \times [-1, 1]/(\sigma, -\mathrm{id})$, where K^2 is a flat Klein bottle and $\sigma : K^2 \to K^2$ is an isometric involution topologically conjugate to the unique involution on K^2 whose quotient is $\mathbb{R}P^2$.

Generalized Seifert fiber spaces A *generalized Seifert fibration* of a topological 3-orbifold M over a topological 2-orbifold B (both possibly with boundaries) is a map $f : M \to B$ whose fibers are homeomorphic to circles or bounded closed intervals. It is required that, for every $x \in B$, there is a neighborhood U_x homeomorphic to a 2-disk such that

(i) if $f^{-1}(x)$ is homeomorphic to a circle, then there is a fiber-preserving homeomorphism of $f^{-1}(U_x)$ to a Seifert fibered solid torus in the usual sense, and
(ii) if $f^{-1}(x)$ is homeomorphic to an interval, then there exists a fiber-preserving homeomorphism of $f^{-1}(U_x)$ to the space B(pt), with respect to the fibration $(\mathrm{B}(\mathrm{pt}), p^{-1}(o)) \to (K_1(\mathbb{S}^1), o)$.

Furthermore, for any compact component C of ∂B there is a collar neighborhood N of C in B such that $f|_{f^{-1}(N)}$ is a usual circle bundle over N. We say that M is a *generalized Seifert fibered space* and we use the notation $M = \mathrm{Seif}(B)$.

Generalized solid tori and Klein bottles A *generalized solid torus* (respectively, *generalized solid Klein bottle*) is a topological 3-orbifold Y with boundary homeomorphic to a torus (respectively, a Klein bottle). It admits a map $Y \to \mathbb{S}^1$ such that the fibers are homeomorphic to either a 2-disk or a Möbius band, and the fiber type can only change at a finite number of *corner points* in $\mathbb{S}^1$. We refer the reader to [18, Def. 1.4] for the precise definitions.

I-bundles over the Klein bottle These are obtained as disk bundles of certain line bundles over the Klein bottle K^2. They are easily described as quotients of $\mathbb{R}^3$ under certain isometric actions. Except for the trivial bundle $K^2 \times I$, the rest are as follows.

(i) $K^2 \tilde{\times} I$: this is the disk bundle in the orientable 3-manifold obtained as the quotient of $\mathbb{R}^3$ under the group generated by

$$(x, y, z) \stackrel{\tilde{\tau}}{\to} (x + 2, y, z), \qquad (x, y, z) \stackrel{\tilde{\sigma}}{\to} (-x, y + 1, -z).$$

Its boundary is given by a 2-torus.

(ii) $K^2 \hat{\times} I$: this is the disk bundle in the non-orientable 3-manifold obtained as the quotient of $\mathbb{R}^3$ under the group generated by

$$(x, y, z) \stackrel{\hat{\tau}}{\to} (x + 1, y, -z), \qquad (x, y, z) \stackrel{\hat{\sigma}}{\to} (-x, y + 1, -z).$$

Its boundary is given by a Klein bottle.

The identity map in $\mathbb{R}^3$ induces a two-fold Riemannian covering map $\pi : K^2 \tilde{\times} I \to K^2 \hat{\times} I$. At the fundamental group level, π is an injective homomorphism that sends $\tilde{\tau} \to \hat{\tau}^2$ and $\tilde{\sigma} \to \hat{\sigma}$. Furthermore, since the fundamental group of K^2 is the dihedral group, and this group contains a unique subgroup of index 2, it follows that $K^2\tilde{\times}I$ is the unique two-fold cover of $K^2\hat{\times}I$.

With these pieces now in hand, we are ready to state the topological classification and structure theorems for closed, collapsed Alexandrov 3-spaces obtained by Mitsuishi and Yamaguchi in [18]. These results are obtained via a thorough analysis of the local structure of the limit spaces and we refer the reader to [18] for more details. We divide the presentation according to the dimension of the limit space, starting with the case where it is two dimensional. We always assume that our spaces are connected.

13.4.1.1 Collapse to Dimension Two

In this case, the limit space of the collapsing sequence is a compact two-dimensional Alexandrov space, possibly with boundary. Hence, the limit space is topologically a surface.

Theorem 13.4.1 (Collapse to a compact surface without boundary) *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed, three-dimensional Alexandrov spaces with $\mathrm{curv} X_i \geq -1$ and $\mathrm{diam} X_i \leq D$. If X_i GH-converges to a two-dimensional Alexandrov space X^* without boundary, then, for sufficiently large i, X_i is homeomorphic to a generalized Seifert space over X^*.*

In the preceding theorem, singular fibers may occur over *essential singular points* in X^*, i.e. over points whose space of directions has radius at most $\pi/2$.

Theorem 13.4.2 (Collapse to a compact surface with boundary) *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed, three-dimensional Alexandrov spaces with $\mathrm{curv} X_i \geq -1$ and $\mathrm{diam} X_i \leq D$. If X_i GH-converges to a two-dimensional Alexandrov space X^* with non-empty boundary, then, for sufficiently large i, there exist a generalized Seifert fiber space $\mathrm{Seif}_i(X^*)$ over X^* and generalized solid tori or generalized Klein bottles $\pi_{i,k} : Y_{i,k} \to (\partial X^*)_k$ over each component $(\partial X^*)_k$ of ∂X^* such that X_i is homeomorphic to the union of $\mathrm{Seif}_i(X^*)$ and the $Y_{i,k}$, glued along their boundaries, where the fibers of $\mathrm{Seif}_i(X^*)$ over boundary points $x \in (\partial X^*)$ are identified with $\partial\pi_{i,k}^{-1}(x) \approx \mathbb{S}^1$.*

13.4.1.2 Collapse to Dimension One

In this case, the limit space of the collapsing sequence is a compact one-dimensional Alexandrov space, possibly with boundary. Hence, the limit space is topologically a circle or a compact interval.

Theorem 13.4.3 (Collapse to a circle) *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed, three-dimensional Alexandrov spaces with* $\mathrm{curv} X_i \geq -1$ *and* $\mathrm{diam} X_i \leq D$. *If X_i GH-converges to a circle, then, for i sufficiently large, X_i is homeomorphic to the total space of fiber bundle over $\mathbb{S}^1$ with fiber homeomorphic to one of $\mathbb{S}^2$, $\mathbb{R}P^2$, T^2 or the Klein bottle K^2. In particular, X_i is a topological manifold.*

Theorem 13.4.4 (Collapse to a compact interval) *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed, three-dimensional Alexandrov spaces with* $\mathrm{curv} X_i \geq -1$ *and* $\mathrm{diam} X_i \leq D$. *If X_i GH-converges to an interval $I \approx [-1,1]$, then, for i sufficiently large, X_i is homeomorphic to a union $B_i^- \cup B_i^+$ of two spaces $B_i^{\pm}$ glued along their boundary $\partial B_i^- = \partial B_i^+$. The boundary $\partial B_i^{\pm}$ is homeomorphic to one of $\mathbb{S}^2$, $\mathbb{R}P^2$, T^2 or the Klein bottle K^2. The topology of the spaces $B_i^{\pm}$ is determined as follows:*

(1) *If $\partial B_i^{\pm} \approx \mathbb{S}^2$, then $B_i^{\pm}$ is homeomorphic to one of D^3, $\mathbb{R}P^3 - \mathrm{int} D^3$ or $\mathrm{B}(S_2)$ with $S_2 \approx \mathbb{S}^2$.*
(2) *If $\partial B_i^{\pm} \approx \mathbb{R}P^2$, then $B_i^{\pm}$ is homeomorphic to $K_1(P^2)$.*
(3) *If $\partial B_i^{\pm} \approx T^2$, then $B_i^{\pm}$ is homeomorphic to one of $\mathbb{S}^1 \times D^2$, $\mathbb{S}^1 \times \mathrm{Mb}$, $K^2 \tilde{\times} I$ or $\mathrm{B}(S_4)$.*
(4) *If $\partial B_i^{\pm} \approx K^2$, then $B_i^{\pm}$ is homeomorphic to one of $\mathbb{S}^1 \tilde{\times} D^2$, $K^2 \hat{\times} I$, $\mathrm{B}(\mathrm{pt})$, or $\mathrm{B}(S_2)$ with $S_2 \approx \mathbb{R}P^2$.*

13.4.1.3 Collapse to a Point

The last case to consider is collapse to a zero-dimensional space, i.e. to a point.

Theorem 13.4.5 (Collapse to a point) *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed, three-dimensional Alexandrov spaces with* $\mathrm{curv} X_i \geq -1$ *and* $\mathrm{diam} X_i \leq D$. *If X_i GH-converges to a point, then, for i sufficiently large, X_i is homeomorphic to some space among the following:*

(1) *generalized Seifert fiber spaces as in the conclusion of Theorem 13.4.1 with base an Alexandrov surface with non-negative curvature;*
(2) *spaces in the conclusion of Theorem 13.4.2 with base an Alexandrov surface with non-negative curvature;*
(3) *spaces in the conclusion of Theorems 13.4.3 and 13.4.4;*
(4) *closed Alexandrov three-dimensional spaces with non-negative curvature having finite fundamental group.*

By the work in [8], a manifold in item (4) in Theorem 13.4.5 is homeomorphic to a three-dimensional spherical space form or to one of $\mathrm{Susp}(\mathbb{R}P^2)$, $\mathrm{Susp}(\mathbb{R}P^2)\#\mathrm{Susp}(\mathbb{R}P^2)$ or $\mathbb{R}P^3\#\mathrm{Susp}(\mathbb{R}P^2)$.

13.4.2 Geometrization of Sufficiently Collapsed Three-Dimensional Alexandrov Spaces

We conclude this chapter with a brief discussion of the geometrization of sufficiently collapsed closed Alexandrov 3-spaces. We refer the reader to [9] for further details.

Recall the eight Thurston geometries: $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil and Sol. As stated in Section 13.3, a closed (i.e. compact and without boundary) 3-manifold is *geometric* if it admits a geometric structure modeled on one of these geometries. In this context, Shioya and Yamaguchi [28] obtained a geometrization result for sufficiently collapsed Riemannian 3-manifolds. More precisely, they showed that, for any $D > 0$, there exists a constant $\varepsilon = \varepsilon(D) > 0$ such that if a closed, prime 3-manifold admits a Riemannian metric with diameter at most D and sectional curvature bounded below by -1 with volume $< \varepsilon$, then it admits a geometric structure modeled on one of the seven geometries $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil and Sol (see [28, Cor. 0.9]).

Recall that a non-trivial closed 3-manifold M is *prime* if it cannot be presented as a connected sum of two non-trivial closed 3-manifolds. A closed 3-manifold is *irreducible* if every embedded 2-sphere bounds a 3-ball. It is known that, with the exception of manifolds homeomorphic to $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ (the non-trivial 2-sphere bundle over $\mathbb{S}^1$), a closed 3-manifold is prime if and only if it is irreducible (see [14, Lem. 3.13]). Since $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ are geometric, one can think of the geometrization of sufficiently collapsed prime Riemannian 3-manifolds as a result pertaining to irreducible 3-manifolds. Therefore, in seeking a generalization to Alexandrov spaces, one may focus on the irreducible case. This leads to the following definition of *irreducibility* for this more general class of spaces.

Definition 13.4.6 Let X be a closed Alexandrov 3-space. We say that X is *irreducible* if every embedded 2-sphere in X bounds a 3-ball, and, in the case that the set of topologically singular points of X is non-empty, it is further required that every 2-sided $\mathbb{R}P^2$ bounds a $K(\mathbb{R}P^2)$, a cone over $\mathbb{R}P^2$.

With this definition in hand, we may now state the geometrization of sufficiently collapsed Alexandrov spaces.

Theorem 13.4.7 ([9, Thm. A]) *For any $D > 0$ there exists $\varepsilon = \varepsilon(D) > 0$ such that, if X is a closed, irreducible Alexandrov 3-space with* $\mathrm{curv} \geq -1$, $\mathrm{diam} X \leq D$, *and* $\mathrm{vol} X < \varepsilon$, *then X admits a geometric structure modeled on one of the seven geometries* $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$, Nil *and* Sol.

Here, the *volume* of an Alexandrov 3-space is its 3-dimensional Hausdorff measure, normalized so that the volume of 3-dimensional Riemannian manifolds agrees with the usual Riemannian volume. As in the Riemannian case, one can rule out hyperbolic geometry by combining the fact that the simplicial volume of a collapsing Alexandrov space is zero (see [19, Cor. 1.7]) with the fact that the simplicial volume of a hyperbolic manifold must be bounded below by the Riemannian volume (see [30, Thm. 6.2]).

Theorem 13.4.7 is proven by carefully studying the metric and topological structure of collapsed irreducible Alexandrov 3-spaces and their orientable double branched covers (in the case where the space is not a manifold). Combining Theorems 13.4.1–13.4.5 with the irreducibility hypothesis, one obtains fairly explicit topological descriptions of closed, collapsed, irreducible Alexandrov 3-spaces, exhibiting them as geometric 3-manifolds or their quotients by orientation-reversing involutions with only isolated fixed points. Classification results of Alexandrov spaces with (local) circle actions (see [10, 20]) and classical results on involutions on 3-manifolds (see [15, 16]) also play an important role in the proof. We refer the reader to [9] for precise details, as the proof is based on a case-by-case analysis and is of a rather technical nature.

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