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A Remark on the Rank Conjecture

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Abstract. We prove a result about the action of λ-operations on the homology of linear groups. We use this to give a sharper formulation of the rank conjecture as well as some shorter proofs of various known results. We formulate a conjecture about how the sharper formulation of the rank conjecture together with another conjecture could give rise to a different point of view on the isomorphism between $CH^p(F,n) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K_n^{(p)}(F)$ for an infinite field F, and we prove part of this new conjecture.

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1. Introduction

Let A be a commutative ring with identity, and let n be an integer at least equal to 1. Write $K_n(A)_{\mathbb{Q}}$ for $K_n(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If we view $K_n(A)_{\mathbb{Q}}$ as the primitive part of $H_n(GL(A), \mathbb{Q})$, then we get an increasing filtration F_*^{rank} on $K_n(A)_{\mathbb{Q}}$ by setting

$$F_r^{\text{rank}} K_n(A)_{\mathbb{Q}} = \text{Image}(H_n(GL_r(A), \mathbb{Q})) \bigcap K_n(A)_{\mathbb{Q}},$$

for $r \geqslant 1$. Here the image is the image of the natural map $H_n(GL_r(A), \mathbb{Q}) \to H_n(GL(A), \mathbb{Q})$. For $r \leqslant 0$ we put $F_r^{\text{rank}} = 0$. If A is an infinite field F then it is known that the natural map $H_n(GL_n(F), \mathbb{Z}) \to H_n(GL(F), \mathbb{Z})$ is an isomorphism [Su2, Theorem 3.4], hence $F_r^{\text{rank}} K_n(F)_{\mathbb{Q}} = K_n(F)_{\mathbb{Q}}$ if $r \geqslant n$. Note that for a finite field F, $K_n(F)_{\mathbb{Q}} = 0$ for $n \geqslant 1$, so that we shall ignore finite fields from now on.

One has the following conjecture, see, e.g., [Ca, §2.3], which seems due to Suslin (unpublished).

CONJECTURE 1 (Rank Conjecture). Let F be an infinite field. For all $n \geqslant 1$ and all r, we have a direct sum decomposition

$$F_r^{\text{rank}} K_n(F)_{\mathbb{Q}} \bigoplus F_\gamma^{r+1} K_n(F)_{\mathbb{Q}} = K_n(F)_{\mathbb{Q}},$$

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with $F_\gamma^m K_n(F)_\mathbb{Q}$ the mth part of the gamma filtration on $K_n(F)_\mathbb{Q}$, given by $\bigoplus_{j=m}^n K_n^{(j)}(F)$, where $K_n^{(j)}(F)$ is the jth eigenspace for the Adams operations ψ^k on $K_n(F)_\mathbb{Q}$.

We recall from [So, §2] that it is known that $K_n(F)_\mathbb{Q} = \bigoplus_{j=1}^n K_n^{(j)}(F)$ if $n \geqslant 1$ and that $K_n^{(1)}(F) = 0$ if $n \geqslant 2$. We shall, in fact, reprove the first statement in this paper.

There is a related statement, mentioned as a problem by Suslin in [Sa, Problem 4.13], but quoted as a conjecture in [B-Y, Remark 7.7]. The name injectivity conjecture (or injectivity problem) seems appropriate.

CONJECTURE 2. If F is an infinite field, then the natural map

$$H_n(GL_r(F), \mathbb{Q}) \to H_n(GL_{r+1}(F), \mathbb{Q})$$

is an injection for all $r \geqslant 1$.

Little seems to be known about either conjecture. Gerdes proves [Ge, Remark 4.10] that $F_r^{\mathrm{rank}} K_n(F)_\mathbb{Q} \cap F_\gamma^{r+1} K_n(F)_\mathbb{Q} = 0$. Note that the rank conjecture holds for general infinite fields for $r \geqslant n$ by Suslin's stability result, and for $r \leqslant 0$ by the definition of F_r^{rank} and the weights on $K_n(F)_\mathbb{Q}$. We shall show it also holds for $r = 1$ and $r = n - 1$, see Remark 9 below. From this it follows immediately that the rank conjecture holds for $n = 1$, 2, and 3. For Conjecture 2, it is clear it holds for $r = 1$ by the existence of the determinant, and for $r \geqslant n$ by Suslin's stability result. It also holds when $r = 2$ and $n = 3$ by [E-V, Theorem 1.22], so that it holds for $n = 1$, 2, and 3. The reader should compare this with the result by Sah (see [Sa, Theorem 3.0]) that the map $H_3(SL_2(F), \mathbb{Z}) \to H_3(SL_3(F), \mathbb{Z})$ is injective if $F = \mathbb{R}$, an infinite field F satisfying $F^* = (F^*)^6$, or a quaternion algebra over a real closed field, for example, $\mathbb{H}$. (See also Remark 3.19 of loc. cit. for the statement for an arbitrary infinite field F.)

The rank conjecture is also known in full if F is a number field [B-Y]. In this case, it is known that $K_{2m}(F)_\mathbb{Q} = 0$ if $m \geqslant 1$, and $K_{2m-1}(F)_\mathbb{Q} = K_{2m-1}^{(m)}(F)$ [Ra, §1]. So the rank conjecture holds automatically for even $n \geqslant 1$, and Borel and Yang prove the rank conjecture for odd n. More precisely they prove two statements:

(R1) $\quad F_m^{\mathrm{rank}} K_{2m-1}(F)_\mathbb{Q} = K_{2m-1}(F)_\mathbb{Q}$;

(R2) $\quad F_{m-1}^{\mathrm{rank}} K_{2m-1}(F)_\mathbb{Q} = 0$.

We show in Corollary 5 below that for any commutative ring A with identity, we have

$$F_r^{\mathrm{rank}} K_n(A)_\mathbb{Q} \subseteq \bigoplus_{j=1}^r K_n^{(j)}(A). \tag{1}$$

In particular, this immediately implies the result by Gerdes quoted above, and it shows that the bound on weights for the K-theory of infinite fields is a direct consequence of the stabilization isomorphism $H_n(GL_n(F), \mathbb{Z}) \to H_n(GL(F), \mathbb{Z})$.

It also shows that, for number fields, (R2) is a formal consequence of the definition of the rank filtration together with the fact that $K_{2m-1}(F)_{\mathbb{Q}} = K^{(m)}_{2m-1}(F)$, a fact that was used also to see that the rank conjecture is equivalent to (R1) and (R2) above anyway.

Remark 3. It should be pointed out that Borel and Yang prove more about the homology (and cohomology) of $GL_n(F)$ if F is a number field. In particular, they prove Conjecture 2 if F is a number field, see [B-Y, Corollary 7.6 (b)].

1.1. THE RESULTS

We now move on to the main Theorem of this paper. The inclusion (1), which follows readily from the theorem, is perhaps known to some specialists, as it would serve as a motivation for the rank conjecture, but it does not seem to be in the literature. Our method of proving (1), arguing directly in $K_0(\mathbb{Z}[GL_r])$, also has the advantage that we get results for the homology of $GL_r(F)$ with $\mathbb{Z}$-coefficients, rather than for K-theory or homology with $\mathbb{Q}$-coefficients. However, working with $H_n(GL(A), \mathbb{Z})$ rather than with $K_n(A)$ has the disadvantage that we lose some information about torsion. Our proof is simpler in the sense that it shows directly that the bounds on the weights for (infinite) fields are a direct consequence of the stability for homology of $GL_N(F)$. Another advantage is that our results hold for an arbitrary commutative ring A with identity.

THEOREM 4. *Let A be a commutative ring with identity. Let n and k be positive integers. Then the image of $H_n(GL_r(A), \mathbb{Z})$ in $H_n(GL(A), \mathbb{Z})$ is annihilated by the operator*

$$\Theta = \sum_{j=0}^{r} (-1)^j \sum_{\substack{I \subset \{1,\ldots,r\} \\ |I|=j}} (-1)^{k^{r-j}} k^I k^{r-j} \lambda^{k^{r-j}},$$

where $k^I = \prod_{i \in I} k^i$.

Proof. In order to prove the theorem, we recall how the operations involved are defined. We refer to [So, §1] or [Kr] for the general theory described here.

Let $BGL(A)$ be the classifying space of $GL(A)$, and similarly for $GL_N(A)$. Let $BGL(A)^+$ be the H-space obtained by applying the $+$ construction to $BGL(A)$, and let $BGL_N(A)^+$ be the H-space obtained similarly from $BGL_N(A)$ for $N \geqslant 3$. There are natural maps $BGL_N(A) \to BGL_{N+1}(A)$, which induce maps $BGL_N(A)^+ \to BGL_{N+1}(A)^+$, natural up to homotopy, for $N \geqslant 3$.

A representation of GL_N gives rise to a map $GL_N \to GL$, and using the H-space structure on $BGL(A)^+$, one can define a map

$$K_0(\mathbb{Z}[GL_N]) \to [BGL_N(A), BGL(A)^+] = [BGL_N(A)^+, BGL(A)^+],$$

where $[\cdot,\cdot]$ are weak homotopy classes of pointed maps, and the last equality follows from the universality of the $+$ construction. The homotopy commutative

H-space structure on $BGL(A)^+$ gives $[BGL_N(A), BGL(A)^+]$ the structure of an Abelian group, and the map above is a homomorphism of Abelian groups. Those classes of maps are compatible with the inclusion of $GL_N \to GL_{N+1}$, and one gets a homomorphism of Abelian groups

$$\begin{aligned}\varprojlim_N K_0(\mathbb{Z}[GL_N]) &\to \varprojlim_N [BGL_N(A), BGL(A)^+] \\ &= \varprojlim_N [BGL_N(A)^+, BGL(A)^+]. \end{aligned} \tag{2}$$

This gives operations on

$$[BGL(A)^+, BGL(A)^+] = \varprojlim_N [BGL_N(A)^+, BGL(A)^+],$$

hence classes in operations on $H_*(GL(A), \mathbb{Z}) = H_*(BGL(A), \mathbb{Z}) = H_*(BGL(A)^+, \mathbb{Z})$ as well as on $K_s(A) = \pi_s(BGL(A)^+)$ for $s \geqslant 1$.

The element $\mathrm{Id}_N - N$, with Id_N the standard N-dimensional representation of GL_N, and N the trivial N-dimensional representation, is part of an element in $\varprojlim_N K_0(\mathbb{Z}[GL_N])$. $K_0(\mathbb{Z}[GL_N])$ has λ-operations, defined via $\lambda^m(V) = \bigwedge^m V$ if V is a representation, and by demanding that the map $\lambda_t : K_0(\mathbb{Z}[GL_N]) \to K_0(\mathbb{Z}[GL_N])[[t]]^*$ defined by $\lambda_t(x) = 1 + \lambda^1(x) + \lambda^2(x) + \cdots$ is a homomorphism of Abelian groups. Those λ-operations are compatible with the natural maps $K_0(\mathbb{Z}[GL_{N+1}]) \to K_0(\mathbb{Z}[GL_N])$ corresponding to the standard inclusion $GL_N \subset GL_{N+1}$. Then the maps λ^m as elements in $\varprojlim_N [BGL_N(A), BGL(A)^+]$ are defined by taking $\{\lambda^m(\mathrm{Id}_N - N)\}_N$ in $\varprojlim_N [BGL_N(A), BGL(A)^+]$, which factors through the natural map to $\varprojlim_N [BGL_N(A)^+, BGL(A)^+]$. Similarly, one defines γ-operations and Adams operations ψ^k.

For our purposes, it therefore suffices to show that the rth component for Θ, $\Theta_r : BGL_r(A) \to BGL(A)^+$, is homotopic to zero. Note that we do not use $BGL_r(A)^+$, so that we do not have to restrict ourself to $r \geqslant 3$. Because the map in (2) is a homomorphism of Abelian groups, we only have to show that

$$\sum_{j=0}^{r} (-1)^j \sum_{\substack{I \subset \{1,\ldots,r\} \\ |I| = j}} (-1)^{k^{r-j}} k^I k^{r-j} \lambda^{k^{r-j}} (\mathrm{Id}_r - r) = 0 \tag{3}$$

in $K_0(\mathbb{Z}[GL_r])$. Because the map from $K_0(\mathbb{Z}[GL_r])$ to $K_0(\mathbb{Z}[GL_r])[[t]]^*$ given by mapping x to $\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots$ is a homomorphism of Abelian groups, we have

$$\begin{aligned}\lambda_t(\mathrm{Id}_r - r) &= 1 + \lambda^1(\mathrm{Id}_r - r)t + \lambda^2(\mathrm{Id}_r - r)t^2 + \cdots \\ &= \lambda_t(\mathrm{Id}_r)\lambda_t(1)^{-r} = \frac{1 + a_1 t + \cdots + a_r t^r}{(1+t)^r}\end{aligned}$$

because $\lambda^m(V) = \bigwedge^m(V)$ if V is a representation of GL_r, which is zero if $m > \dim(V)$. Hence,

$$\sum_{j=0}^{r}\binom{r}{j}\lambda^{m-j}(\mathrm{Id}_r - r) = 0 \tag{4}$$

for $m \geqslant r+1$. We can use the identity in (4) to replace the highest m such that λ^m occurs in (3) with lower ones, until the highest such m equals at most r. Noting that λ^0 never occurs anyway in any of these expressions, we see that the coefficients of λ^j for $j=1$ through r we get this way are the same as the coefficients of X^j in the remainder of division of the polynomial

$$f_r(X) = \sum_{j=0}^{r}(-1)^j \sum_{\substack{I\subset\{1,\ldots,r\}\\ |I|=j}} (-1)^{k^{r-j}} k^I k^{r-j} X^{k^{r-j}}$$

by $X(X+1)^r$ in $\mathbb{Z}[X]$. So in order to prove (3), it suffices to prove that $f_r(X)$ is divisible by $X(X+1)^r$ in $\mathbb{Z}[X]$, or equivalently, $\mathbb{Q}[X]$. Clearly $f(0)=0$, so we only have to verify that $f_r(-1) = f_r^{(1)}(-1) = \cdots = f_r^{(r-1)}(-1) = 0$, which, using multiplication by some nonzero rationals, is the same as saying that

$$\sum_{j=0}^{r}(-1)^j \sum_{\substack{I\subset\{1,\ldots,r\}\\ |I|=j}} (-1)^{k^{r-j}} k^I k^{r-j} \binom{k^{r-j}}{a} X^{k^{r-j}}$$

yields zero for $X=-1$ and $a=0,\ldots,r-1$. Because the $\mathbb{Q}$-vector space spanned by $\binom{z}{a}$ for $a=0,\ldots,r-1$ is the same as the $\mathbb{Q}$-vector space spanned by $1, z, \ldots, z^{r-1}$ it suffices to prove that

$$\sum_{j=0}^{r}(-1)^j \sum_{\substack{I\subset\{1,\ldots,r\}\\ |I|=j}} (-1)^{k^{r-j}} k^I k^{(a+1)(r-j)} X^{k^{r-j}}$$

yields zero for $X=-1$ and $a=0,\ldots,r-1$. But putting $X=-1$ yields

$$\sum_{j=0}^{r}(-1)^j \sum_{\substack{I\subset\{1,\ldots,r\}\\ |I|=j}} k^I (k^{(a+1)})^{r-j}$$

which is also the value for $Y = k^{a+1}$ of the polynomial

$$(Y-k)(Y-k^2)\ldots(Y-k^r),$$

clearly giving zero for $a=0,\ldots,r-1$, thus proving the theorem. In fact, this also shows that $f_r^{(r)}(-1) \neq 0$ so that $X=-1$ is a root of $f_r(X)$ of order exactly r. $\square$

COROLLARY 5. In $H_n(GL(A),\mathbb{Z})$, *the intersection of the image of the Hurewicz map from* $K_n(A) = \pi_n(BGL(A)^+)$ *to*

$$H_n(BGL(A)^+,\mathbb{Z}) = H_n(BGL(A),\mathbb{Z}) = H_n(GL(A),\mathbb{Z}),$$

with the image of $H_n(BGL_r(A),\mathbb{Z}) = H_n(GL_r(A),\mathbb{Z})$ is annihilated by the operator

$$\Xi = (\psi^k - k\psi^1) \circ (\psi^k - k^2\psi^1) \circ \cdots \circ (\psi^k - k^r\psi^1).$$

Proof. We have a commutative diagram

$$\begin{tikzcd} K_n(A) \arrow[r] \arrow[d, "\Xi"'] & H_n(GL(A),\mathbb{Z}) \arrow[d, "\Xi"'] & H_n(GL_r(A),\mathbb{Z}) \arrow[l] \arrow[dl, "\Xi_r"] \\ K_n(A) \arrow[r] & H_n(GL(A),\mathbb{Z}) & \end{tikzcd}$$

and similarly for Θ. According to [Kr, Proposition 5.1], $\psi^m = (-1)^{m-1} m\lambda^m$ on $K_n(A)$ for $n \geqslant 1$. Expanding Ξ using the linearity of ψ^k and the fact that $\psi^k \circ \psi^l = \psi^{kl}$, this identity shows that Θ and $-\Xi$ coincide on $K_n(A)$, and from the commutativity of the diagram it follows that the action of $-\Xi$ and Θ on the image of $K_n(A)$ in $H_n(GL(A),\mathbb{Z})$ is the same. Hence, the Corollary follows immediately from Theorem 4. ☐

Because $K_n^{(j)}(A)$ is the kernel of $\psi^k - k^j\psi^1 = \psi^k - k^j$ on $K_n(A)_{\mathbb{Q}}$ for any $k \geqslant 2$, and the Hurewicz map induces an injection $K_n(A)_{\mathbb{Q}} \to H_n(GL(A),\mathbb{Q})$, we immediately get the following Corollary.

COROLLARY 6. $F_r^{\mathrm{rank}} K_n(A)_{\mathbb{Q}} \subseteq \bigoplus_{j=1}^r K_n^{(j)}(A)$. *Hence, for an infinite field F, the rank conjecture is equivalent to the equality*

$$F_r^{\mathrm{rank}} K_n(F)_{\mathbb{Q}} = \bigoplus_{j=1}^{r} K_n^{(j)}(F).$$

Remark 7. If F is an infinite field, it is known that

$$H_n(GL_n(F),\mathbb{Z}) = H_n(GL(F),\mathbb{Z}),$$

see [Su2, Theorem 3.4]. With $\mathbb{Q}$-coefficients, there is a somewhat simpler proof that the map $H_n(GL_n(F),\mathbb{Q}) \to H_n(GL(F),\mathbb{Q})$ is surjective, see [Ya, Corollary 3.12]. This gives, therefore, a quick proof of the bounds on weights for infinite fields, if we ignore torsion. The corresponding proof of this fact without using $\mathbb{Q}$-coefficients in [So] (involving Théorème 1 and Corollaire 1) is more complicated, but also gives more information about the torsion. Note that our method does not give the best possible result for other rings than infinite fields, as the known results for stability for the homology of GL are weaker than the corresponding statements for the stability in Volodin K-theory (see [Su1]), which is a crucial ingredient in the proof in [So].

Remark 8. It was pointed out to me by Ph. Elbaz-Vincent and C. Soulé that one can argue directly on the eigenspaces of the Adams operations analogous to [So, 2.10] to prove Corollary 6. Namely, let x be in $F_r^{\mathrm{rank}} K_n(A)_{\mathbb{Q}}$, and write $x = x_1 + x_2 + \cdots + x_m$ for some m, all x_i in $K_n^{(i)}(A)$ (see [Se, Theorems 1 and 2]). Then for $i > r$, $\gamma^i(x_i) = \lambda^i(\mathrm{Id}_r - r + i - 1)(x_i) = 0$ because $\mathrm{Id}_r - r + i - 1$ is a representation of dimension smaller than i, so its i-th wedge power is zero. But it is known that $\gamma^i(x_i) = \omega_i x_i$ with ω_i a universal non-zero constant, so x_i is zero.

Remark 9. When F is an infinite field we can deduce the rank conjecture for $r = n - 1$ very quickly from results of [Su2] using Corollary 6. Namely, Suslin proves that the natural map $H_n(GL_n(F), \mathbb{Z}) \to H_n(GL(F), \mathbb{Z})$ is an isomorphism (Theorem 3.4 of loc. cit.), and that the natural map

$$\rho\colon\ F^* \times \cdots \times F^* = H_1(GL_1(F)) \times \cdots \times H_1(GL_1(F)) \to H_n(GL_n(F))$$

induced from the external product on homology and the inclusion of $GL_1(F) \times \cdots \times GL_1(F)$ into $GL_n(F)$ as the diagonal matrices, gives rise to an isomorphism

$$\phi_n : K_n^M(F) \xrightarrow{\sim} H_n(GL_n(F), \mathbb{Z})/\mathrm{Image}(H_n(GL_{n-1}(F), \mathbb{Z})) \tag{5}$$

by mapping $\{a_1, \ldots, a_n\}$ to $\rho(a_1, \ldots, a_n)$ (Corollary 2.7.2 of loc. cit.). Moreover, the map

$$\begin{aligned} K_n(F) &= \pi_n(BGL(F)^+) \\ &\to H_n(GL(F), \mathbb{Z}) = H_n(GL_n(F), \mathbb{Z}) \\ &\to H_n(GL_n(F), \mathbb{Z})/\mathrm{Image}(H_n(GL_{n-1}(F), \mathbb{Z})) \cong K_n^M(F) \end{aligned}$$

maps $a_1 \cup \cdots \cup a_n$ to $(-1)^{n-1}(n-1)!\{a_1, \ldots, a_n\}$ for a_i in $K_1(F) = F^*$. Because the map $K_n^M(F) \to K_n(F)$ defined by sending $\{a_1, \ldots, a_n\}$ to $a_1 \cup \cdots \cup a_n$ is a ring homomorphism, $K_n^M(F)$ lands in $K_n^{(n)}(F)$. Tensored with $\mathbb{Q}$, we see that this means that the composition

$$\begin{aligned} K_n^{(n)}(F) \subseteq K_n(F)_{\mathbb{Q}} &\to H_n(GL_n(F), \mathbb{Q})/\mathrm{Image}(H_n(GL_{n-1}(F), \mathbb{Q})) \\ &\cong K_n^M(F)_{\mathbb{Q}} \end{aligned}$$

must be surjective. As its kernel must be contained in $K_n^{(n)}(F) \cap F_{n-1}^{\mathrm{rank}} K_n(F)_{\mathbb{Q}}$ by definition, Corollary 6 implies that the map is injective, hence must be an isomorphism. Similarly, arguing at the place of $K_n(F)_{\mathbb{Q}}$, we see that we get an isomorphism of $K_n(F)_{\mathbb{Q}}/F_{n-1}^{\mathrm{rank}} K_n(F)_{\mathbb{Q}}$ with $K_n^M(F)_{\mathbb{Q}}$, which restricted to $K_n^{(n)}(F) \subseteq K_n(F)_{\mathbb{Q}}/F_{n-1}^{\mathrm{rank}} K_n(F)_{\mathbb{Q}}$ (this is an inclusion by Corollary 6) gives the previous isomorphism $K_n^{(n)}(F) \cong K_n^M(F)_{\mathbb{Q}}$. This implies that the rank conjecture holds for $r = n - 1$, and the whole argument gives a slightly different proof that $K_n^{(n)}(F) \cong K_n^M(F)_{\mathbb{Q}}$, cf. [So, p. 506]. It also shows that after tensoring with $\mathbb{Q}$, ϕ_n induces an identification

$$\begin{aligned} \phi_n : K_n^M(F)_{\mathbb{Q}} &\cong H_n(GL_n(F), \mathbb{Q})/\mathrm{Image}(H_n(GL_{n-1}(F), \mathbb{Q})) \\ &= K_n^{(n)}(F). \end{aligned} \tag{6}$$

Using Corollary 6, it follows immediately that the rank conjecture holds for $r = 1$ as $K_n^{(1)}(F) = 0$ for $n \geqslant 2$. As pointed out before, this means that the rank conjecture holds for $n = 1$, 2 and 3.

Remark 10. It is unclear to the author how optimistic one should be for the statement of the rank conjecture to hold for other rings than (infinite) fields. As an example consider $H_3(GL_2(\mathbb{Z}[i]), \mathbb{Q})$. $GL_2(\mathbb{Z}[i])$ has a normal subgroup N of finite index that acts freely and discretely on hyperbolic three space, with non-compact quotient M. From the spectral sequence in group homology, we get that $H_3(GL_2(\mathbb{Z}[i]), \mathbb{Q}) \cong H_3(N, \mathbb{Q})$ and from group homology we know that $H_3(N, \mathbb{Q}) \cong H_3(M, \mathbb{Q})$, which is zero because M is non-compact. But it is known that $K_3^{(2)}(\mathbb{Z}[i]) \cong K_3^{(2)}(\mathbb{Q}(i))$ is a $\mathbb{Q}$-vector space of dimension one ([Bo]), so that the rank conjecture does not hold for $\mathbb{Z}[i]$. (This example showed up in a discussion with D. Blasius.)

Remark 11. One can optimistically hope that for an infinite field F, there is an equality $\mathrm{Ker}(\Theta) = \mathrm{Image}(H_n(GL_r(F), \mathbb{Q}))$ for some $k > 1$ used in the definition of Θ. This would imply the rank conjecture as reformulated in Corollary 6, as follows. If α lies in $\bigoplus_{j=1}^r K_n^{(j)}(F)$, then $\Xi(\alpha) = 0$, and hence $\Theta(\alpha) = 0$ as well, by the equality of Θ and $-\Xi$ on $K_n(F)_\mathbb{Q}$ as used in the proof of Corollary 5. So α lies in $\mathrm{Image}(H_n(GL_r(F), \mathbb{Q})) \cap K_n(F)_\mathbb{Q} = F_r^{\mathrm{rank}} K_n(F)_\mathbb{Q}$ by assumption. Note that this argument only uses that Θ restricts to $-\Xi$ on $\bigoplus_{j=1}^r K_n^{(j)}(F)$, and annihilates $\mathrm{Image}(H_n(GL_r(F), \mathbb{Q}))$. Therefore, one could also try finding other such operators Θ in order to write $\mathrm{Image}(H_n(GL_r(F), \mathbb{Q}))$ as the intersection of such $\mathrm{Ker}(\Theta)$'s. Such an equality is purely homology theoretic, and might be more directly accessible than the rank conjecture itself. Of course, the example in Remark 10 shows that such an equality must depend on some properties of the field F.

1.2. ANOTHER CONJECTURE

Let F be an infinite field. Let Δ^n be the subset of $\mathbb{A}_F^{n+1}$ defined by $\sum_{j=0}^n x_j = 1$, with $x_0, \ldots, x_n$ the standard coordinates on $\mathbb{A}_F^{n+1}$. For $j = 0, \ldots, n+1$ one can embed Δ^n into Δ^{n+1} by the map $\partial_j : (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{j-1}, 0, x_j, \ldots, x_n)$. Using iterations of those embeddings, we get embeddings of Δ^k into Δ^n for $k < n$. Each of those embeddings is called a face. We let $Z^p(F, n)$ be the free Abelian group on codimension p-cycles on Δ^n meeting all faces Δ^k in codimension p for $k < n$. By assumption, there is a pullback map (defined by intersection) $\partial_j^* : Z^p(F, n) \to Z^p(F, n-1)$. Let $d = \sum_{j=0}^n (-1)^j \partial_j^* : Z^p(F, n) \to Z^p(F, n-1)$. Then $Z^p(F, *)$ is a homological chain complex, and Bloch's higher Chow group $CH^p(F, n)$ is defined as its homology in the n-th place. It is known that $K_n^{(p)}(F) \cong CH^p(F, n)$ for all $n \geqslant 0$ and $p \geqslant 0$, see [Bl1] and [Bl2], or [Le], or [Bl3] for a concise discussion. One can define a similar complex using only linear cycles, and we denote the corresponding homology groups by $LCH^p(F, n)$. The inclusion of the linear cycles into all cycles gives an obvious map $LCH^p(F,n) \to CH^p(F,n)$. We can also use $\mathbb{Q}$ as coefficients instead of $\mathbb{Z}$, in which case we denote the corresponding groups by $CH^p(F,n)_{\mathbb{Q}}$ and $LCH^p(F,n)_{\mathbb{Q}}$.

We can get a map $H_n(GL_p(F),\mathbb{Z}) \to LCH^p(F,n)$ for $n \geqslant p$ as follows. Fix $P=(a_1,\dots,a_p)$ in F^p with not all a_i equal to zero. Then $H_*(GL_p(F),\mathbb{Z})$ can be computed using chains involving only tuples $(g_0,\dots,g_k)$ in general position with respect to P, that is, so that every $p\times m$ minor of the $p\times\ (k+1)$ matrix $(g_0(P),\dots,g_k(P))$ has maximal rank $\min(p,m)$. If $k+1\geqslant p$, it defines a codimension p linear space in F^{k+1}, which we can intersect with Δ^k. Deleting the j-th group element in a tuple $(g_0,\dots,g_k)$ corresponds to the pullback coming from the embedding of Δ^{k-1} into Δ^k as the part where the j-th coordinate equals zero. Hence, for $n\geqslant p$ we get a map $H_n(GL_p(F),\mathbb{Z})\to LCH^p(F,n)$, and we can do the same using $\mathbb{Q}$-coefficients everywhere. The map $H_n(GL_p(F);\mathbb{Z}) \to LCH^p(F,n)$ is independent of the choice of P because there is a unique map (up to homotopy) of complexes $R^\bullet(GL_p(F))\to C_F^p(\bullet)$ with $R^\bullet$ the standard resolution of $GL_p(F)$ and $C_F^p(n)$ the set of $(n+1)$-tuples in F^p in general position, a $\mathbb{Z}[GL_p]$-resolution of $\mathbb{Z}$, with boundary operation given by deleting each of the points in term and taking the alternating sum. This map must be given by the maps $R^\bullet(GL_p(F))\to R_P^\bullet(GL_p(F))\to C_F^p(\bullet)$ where $R_P^\bullet(GL_p(F))$ is the resolution of $\mathbb{Z}$ by linear combinations of tuples $(g_0,\dots,g_k)$ such that $(g_0(P),\dots,g_k(P))$ is in general position, and the last map is the map given by $(g_0,\dots,g_k)\mapsto(g_0(P),\dots,g_k(P))$.

If we identify $GL_{p-1}(F)$ with the subgroup of $GL_p(F)$ of elements of the form $\left(\begin{smallmatrix} A & 0\\ 0 & 1\end{smallmatrix}\right)$, and we take P such that $a_p\neq 0$, and not all of $a_1,\dots,a_{p-1}$ are zero, then $H_*(GL_{p-1}(F),\mathbb{Z})$ can be computed using tuples $(h_0,\dots,h_k)$ such that $(h_0(P),\dots,h_k(P))$ is in general position. But then the composition of maps $H_n(GL_{p-1}(F),\mathbb{Z})\to H_n(GL_p(F),\mathbb{Z})\to LCH^p(F,n)$ is zero for $n\geqslant p$ as $(h_0(P),\dots,h_k(P))$ gives rise to a linear space contained in $\sum_j x_j=0$, which does not meet Δ^n. This shows that for such P, this construction actually gives rise to a map $H_n(GL_p(F),GL_{p-1}(F);\mathbb{Z})\to LCH^p(F,n)$ for $n\geqslant p$, and similarly with $\mathbb{Q}$-coefficients. Note that, because of dimensions, $LCH^p(F,n)$ and $CH^p(F,n)$ are zero if $p>n$, so we get this map for all p.

If the kernel of the map $H_n(GL_p(F),\mathbb{Q})\to H_n(GL(F),\mathbb{Q})$ is contained in the kernel of the map $H_n(GL_p(F),\mathbb{Q})\to LCH^p(F,n)_{\mathbb{Q}}$ defined above (e.g., if Conjecture 2 holds for F), this induces a map

$$\begin{aligned} F_p^{\mathrm{rank}}K_n(F)_{\mathbb{Q}} &\to \mathrm{Image}(H_n(GL_p(F),\mathbb{Q}))\\ &\to LCH^p(F,n)_{\mathbb{Q}}\to CH^p(F,n)_{\mathbb{Q}} \end{aligned}$$

(the image being the image in $H_n(GL(F),\mathbb{Q})$), and we get maps

$$\begin{aligned} \psi_n^{(p)}\colon \frac{F_p^{\mathrm{rank}}K_n(F)_{\mathbb{Q}}}{F_{p-1}^{\mathrm{rank}}K_n(F)_{\mathbb{Q}}} &\to \frac{\mathrm{Image}(H_n(GL_p(F),\mathbb{Q}))}{\mathrm{Image}(H_n(GL_{p-1}(F),\mathbb{Q}))}\\ &\to LCH^p(F,n)_{\mathbb{Q}}\to CH^p(F,n)_{\mathbb{Q}}. \end{aligned}$$

The last group is isomorphic to $K_n^{(p)}(F)$, and if the rank conjecture holds for F, so is the first.

CONJECTURE 12. Let $n \geqslant 1$. Suppose the rank conjecture holds for the infinite field F, and assume the kernel of the map $H_n(GL_p(F), \mathbb{Q}) \to H_n(GL(F), \mathbb{Q})$ is contained in the kernel of the map $H_n(GL_p(F), \mathbb{Q}) \to LCH^p(F, n)_{\mathbb{Q}}$ (e.g., because Conjecture 2 holds for F). Then

(i) the map $\psi_n^{(p)} : F_p^{\text{rank}} K_n(F)_{\mathbb{Q}} / F_{p-1}^{\text{rank}} K_n(F)_{\mathbb{Q}} \to CH^p(F, n)_{\mathbb{Q}}$ is an isomorphism;

(ii) $\psi_n^{(p)}$ is a non-zero rational multiple of the composition of the natural isomorphism $K_n^{(p)}(F) \cong F_p^{\text{rank}} K_n(F)_{\mathbb{Q}} / F_{p-1}^{\text{rank}} K_n(F)_{\mathbb{Q}}$ implied by the rank conjecture, and Bloch's comparison isomorphism $K_n^{(p)}(F) \cong CH^p(F, n)_{\mathbb{Q}}$.

Remark 13. Conjecture 12 is an attempt to unify several other conjectures in this area. Namely, it implies that the map $LCH^p(F, n)_{\mathbb{Q}} \to CH^p(F, n)_{\mathbb{Q}}$ is surjective, a conjecture of Gerdes (see Conjecture 4.7 of [Ge]), proved by him if $p = n$ or $n - 1$. One can also hope that it would allow a proof of the Beilinson–Soulé conjecture that $CH^p(F, n)_{\mathbb{Q}} = 0$ for $2p \leqslant n$ and $n > 0$ by showing that the image of $LCH^p(F, n)$ is zero in this range, which, incidentally, would be an interesting statement to know by itself. Finally, Goncharov's construction for proving part of Zagier's conjecture for $K_5^{(3)}(F)$ for a number field F also begins with defining a map $H_5(GL_3(F), \mathbb{Z}) \to H_5(C_F^3(\bullet))$, although in that case the elements in $C_F^3(\bullet)$ are interpreted as configurations of points, not as defining a codimension three cycle, see Section 3.1 of [Go1]. The connection between the two constructions is partly explained by Proposition 16 below.

Remark 14. We can get a weaker version of Conjecture 12 by only assuming that the kernel of the map $H_n(GL_p(F), \mathbb{Q}) \to H_n(GL(F), \mathbb{Q})$ is contained in the kernel of the map $H_n(GL_p(F, \mathbb{Q})) \to CH^p(F, n)_{\mathbb{Q}}$. Note that that version would allow the injectivity conjecture to fail, for example, in the range (if any) where the Beilinson–Soulé conjecture is true. However, it no longer implies the conjecture by Gerdes mentioned in Remark 13.

We shall verify part (i) of Conjecture 12 for $r = n - 1$ in Proposition 15 below. Note that in this case the rank conjecture is known by Remark 9, and Conjecture 2 is not needed for the conjecture to make sense. We shall, in fact, prove an integral version, from which part (i) of Conjecture 12 follows by tensoring with $\mathbb{Q}$. Namely, we can identify $K_n^M(F)_{\mathbb{Q}}$ with $K_n^{(n)}(F)$ via the map ϕ_n, see Equation (6) in Remark 9, so that the statement follows from Proposition 15 below after tensoring with $\mathbb{Q}$. Finally, in Proposition 16, we verify part (i) of Conjecture 12 for number fields.

We now turn to the case $r = n - 1$. Nesterenko and Suslin show [N-S, Theorem 4.9] that for $n \geqslant 1$, the map

$$\theta_n : \mathbb{Z}[F^* \times \cdots \times F^*] \to CH^n(F, n)$$

given by

$$\theta_n(a_1, \ldots, a_n) = \left(-\frac{a_1}{1 - \sum_i a_i}, \ldots, -\frac{a_n}{1 - \sum_i a_i}, \frac{1}{1 - \sum_i a_i} \right)$$

if $\sum_i a_i \neq 1$ and 0 otherwise, induces an isomorphism $\theta_n : K_n^M(F) \to CH^n(F, n)$, as it factors through the natural maps $\mathbb{Z}[F^* \times \cdots \times F^*] \to F^* \otimes \cdots \otimes F^* \to K_n^M(F)$.

PROPOSITION 15. *The diagram*

$$\begin{tikzcd}
\dfrac{H_n(GL_n(F), \mathbb{Z})}{\mathrm{Image}(H_n(GL_{n-1}(F), \mathbb{Z}))} \arrow[r, "\psi_n"] & CH^n(F, n) \\
& K_n^M(F) \arrow[ul, "\cong", "\phi_n"'] \arrow[u, "\cong", "(-1)^{\frac{1}{2}n(n-1)}\theta_n"']
\end{tikzcd}$$

commutes, where ψ_n is the map $\psi_n^{(n)}$ defined in Conjecture 12, and ϕ_n is the isomorphism in Equation (5) in Remark 9. In other words, $\psi_n \circ \phi_n(\{a_1, \ldots, a_n\}) = \theta_n(\{a_n, \ldots, a_1\})$.

Proof. We start with the case $n = 1$. In that case, $\alpha \in F^*$ corresponds to the element (α) in $H_1(GL_1(F), \mathbb{Z})$ as inhomogeneous cycle, which corresponds to $(1, \alpha)$ as homogeneous cycle, as the inhomogeneous element $(g_1, \ldots, g_n)$ corresponds to the homogeneous element $(1, g_1, g_1 g_2, \ldots, g_1 \ldots g_n)$. Such $(1, \alpha)$ is in general position with respect to $P = (1)$ for all α. As such α with $\alpha \neq 1$ clearly generate F^* and all maps are homomorphisms of Abelian groups, we can assume $\alpha \neq 1$. $(1, \alpha)$ gives the equation $x_0 + \alpha x_1 = 0$, which we have to intersect with $x_0 + x_1 = 1$, so this gives rise to the point $(1 - (1 - \alpha)^{-1}, (1 - \alpha)^{-1}) = \theta_1(\alpha)$. For higher n, let $F_n^{*,\mathrm{sp}}$ be the subset of $F^* \times \cdots \times F^*$ where all coordinates are distinct and distinct from 1. Because ϕ_n is an isomorphism, and elements $\{\alpha_1, \ldots, \alpha_n\}$ in $K_n^M(F)$ with all α_i distinct and different from 1 generate $K_n^M(F)$, it will be sufficient to prove that the diagram

$$\begin{tikzcd}
F_{n+1}^{*,\mathrm{sp}} \arrow[r, "\cup^n \times \mathrm{Id}"] \arrow[d, equal] & H_n(GL_n) \times H_1(GL_1) \arrow[r, "\psi_n \times \psi_1"] \arrow[d] & CH^n(F, n) \times CH^1(F, 1) \arrow[d] & K_n^M(F) \times K_1^M(F) \arrow[l, "\theta_n \times \theta_1"'] \arrow[d] \\
F_{n+1}^{*,\mathrm{sp}} \arrow[r, "\cup^{n+1}"] & H_{n+1}(GL_{n+1}) \arrow[r, "\psi_{n+1}"] & CH^{n+1}(F, n+1) & K_{n+1}^M(F) \arrow[l, "\theta_{n+1}"']
\end{tikzcd}$$

commutes up to $(-1)^n$ in the middle square, where the vertical arrows are the cup products, as this shows our claim by induction on n. Suslin and Nesterenko show that the right-hand part of the diagram commutes [N-S, Lemma 4.6], so we only have to show that the central part of the diagram commutes up to the sign $(-1)^n$.

The map $CH^n(F, n) \times CH^1(F, 1) \to CH^{n+1}(F, n+1)$ can be described as follows, see the proof of [N-S, Lemma 4.6]. If x lies in $CH^n(F, n)$, and y lies in $CH^1(F, 1)$, then (x, y) gets mapped to $\sum_{i=0}^{n}(-1)^i\phi_i(x \times y)$, with ϕ_i the map $\Delta^n \times \Delta^1 \to \Delta^{n+1}$ given by mapping $((a_0, \dots, a_n), (b_0, b_1))$ to

$$(a_0, \dots, a_{i-1}, b_0 - a_0 - \cdots - a_{i-1}, a_0 + \cdots + a_i - b_0, a_{i+1}, \dots, a_n),$$

provided this does not lie on any of the faces, that is, b_0 does not equal $a_0+\cdots+a_{i-1}$ or $a_0 + \cdots + a_i$. For the cup product in homology, recall that if G and H are groups, then the cup product $H_n(G, \mathbb{Z}) \times H_m(H, \mathbb{Z}) \to H_{n+m}(G \times H, \mathbb{Z})$ is given as follows ([N-S, 3.23.2] or [McL, Chapter VIII, Theorem 8.8]). If $(g_1, \dots, g_n)$ and $(h_1, \dots, h_m)$ are inhomogeneous tuples for G and H, respectively, then their product is given by $\sum_\sigma(-1)^\sigma(k_{\sigma(1)}, \dots, k_{\sigma(n+m)})$, where $k_i = (g_i, e_H)$ for $i = 1, \dots, n$, $k_i = (e_G, h_{i-n})$ for $i = n+1, \dots, n+m$, and the sum is over all (n, m)-shuffles σ, that is, all permutations of $1, \dots, n+m$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+m)$. If $(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ is in $F_{n+1}^{*,\mathrm{sp}}$, then the cup product of $(\alpha_1, \dots, \alpha_n)$ in $H_n(GL_n)$ is represented by induction on n by the homogeneous cycle

$$\sum_{\sigma\in S_n}(-1)^\sigma(\mathrm{diag}(1, \dots, 1), \mathrm{diag}(\alpha_{\sigma(1)}, 1, \dots, 1), \dots, \mathrm{diag}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})),$$

where $\mathrm{diag}(c_1, \dots, c_n)$ is the $n \times n$ diagonal matrix with $c_1, \dots, c_n$ on the diagonal. With P the point $(1, \dots, 1)$, the map to $LCH^n(F, n)$ maps each

$$(\mathrm{diag}(1, \dots, 1), \mathrm{diag}(\alpha_{\sigma(1)}, 1, \dots, 1), \dots, \mathrm{diag}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}))$$

to the system of equations (as matrix) obtained by letting σ in S_n act on the indices in

$$\begin{pmatrix} 1 & \alpha_1 & \dots & \dots & \alpha_1 \\ 1 & 1 & \alpha_2 & \dots & \alpha_2 \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 1 & \alpha_n \end{pmatrix}.$$

One checks readily that any n columns of this matrix are linearly independent over F, so that any $(\mathrm{diag}(1, \dots, 1), \mathrm{diag}(\alpha_{\sigma(1)}, 1, \dots, 1), \dots, \mathrm{diag}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}))$ is in general position with respect to P. Hence, we land in $R_P^\bullet(GL_n(F)) \subset R^\bullet(GL_n(F))$. Because the map $R_P^\bullet(GL_n(F)) \subset R^\bullet(GL_n(F)) \to R_P^\bullet(GL_n(F))$ must be homotopic to the identity, we can work with this matrix directly for computing the image of $(\alpha_1, \dots, \alpha_n)$ in $C_F^n(\bullet)$ and hence $LCH^n(F, n)$ or $CH^n(F, n)$. As we have to intersect the linear subspace defined by this with Δ^n given by $\sum_{i=0}^{n} x_i = 1$, we find that in $LCH^n(F, n)$, hence in $CH^n(F, n)$, the image of $\alpha_1 \cup \cdots \cup \alpha_n$ is given by the element $\sum_{\sigma\in S_n}(-1)^\sigma \mathrm{pt}_n(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$, with $\mathrm{pt}_n(\alpha_1, \dots, \alpha_n)$ given by

$$(1 - (1 - \alpha_1)^{-1}, (1 - \alpha_1)^{-1} - (1 - \alpha_2)^{-1}, \dots, (1 - \alpha_{n-1})^{-1} - \\ -(1 - \alpha_n)^{-1}, (1 - \alpha_n)^{-1}).$$

Using the formulas for the cup product

$$CH^n(F,n)\times CH^1(F,1)\to CH^{n+1}(F,n+1)$$

given above, the image of $(\alpha_1\cup\cdots\cup\alpha_n,\alpha_{n+1})$ under the map to $CH^{n+1}(F,n+1)$ is therefore given by

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{\sigma\in S_n}(-1)^\sigma(-1)^i\mathrm{pt}_{n+1}(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(i)},\alpha_{n+1},\alpha_{\sigma(i+1)},\dots,\alpha_{\sigma(n)})\\
&\qquad+\sum_{\sigma\in S_n}(-1)^\sigma(-1)^n\mathrm{pt}_{n+1}(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\dots,\alpha_{\sigma(n)},\alpha_{n+1})\\
&\quad=(-1)^n\sum_{\tau\in S_{n+1}}(-1)^\tau\mathrm{pt}_{n+1}(\alpha_{\tau(1)},\dots,\alpha_{\tau(n+1)}),
\end{aligned}$$

which is $(-1)^n$ times the image of $\alpha_1\cup\cdots\cup\alpha_{n+1}$ in $CH^{n+1}(F,n+1)$. Note that the condition about b_0 in the cup product $CH^n(F,n)\times CH^1(F,1)\to CH^{n+1}(F,n+1)$ translates into α_{n+1} not being equal to either α_i or α_{i+1}, which is satisfied as $(\alpha_1,\dots,\alpha_{n+1})$ is in $F^{*,\mathrm{sp}}_{n+1}$. □

PROPOSITION 16. *Let k be a number field. Then part (i) of Conjecture 12 holds for k.*

Proof. Note that the rank conjecture and Conjecture 2 hold for number fields, so the statement of the Proposition makes sense. Clearly, Conjecture 12 is true if $K_m^{(j)}(k)=0$ as in that case $CH^j(k,n)=0$ as well by Bloch's comparison isomorphism. As k is a number field, it is known that $K_m^{(j)}(k)=0$ for $m\geqslant 1$ unless this is of the form $K_{2n-1}^{(n)}(k)$ for some $n\geqslant 1$. For $n=1$ the statement was already proved in Proposition 15 as $K_1^{(1)}(k)=k^*_{\mathbb{Q}}=K_1^M(k)_{\mathbb{Q}}$, so that we can assume $n\geqslant 2$ from now on. For the proof we shall use that, for $n\geqslant 2$, $K_{2n-1}^{(n)}(k)$ is a finite dimensional $\mathbb{Q}$-vector space and the Borel regulator $\mathrm{reg}_B: K_{2n-1}^{(n)}(k)\to\coprod_\sigma\mathbb{R}_\sigma$ is injective, where σ runs through the embeddings of k into $\mathbb{C}$ ([Bo, or the introduction of Za for a concise statement]). We shall, in fact, show that for every embedding $\sigma: k\to\mathbb{C}$, the triangle in the following diagram commutes, up to multiplication by a non-zero number.

$$\begin{tikzcd}
K_{2n-1}^{(n)}(k) \arrow[r] & H_{2n-1}(GL_n(k),\mathbb{Q}) \arrow[r] \arrow[dr, "\mathrm{reg}_{B,\sigma}"'] & CH^n(k,2n-1)_{\mathbb{Q}} \arrow[d, "\mathrm{reg}_{G,\sigma}"] \\
& & \mathbb{R}
\end{tikzcd}$$

Here the composition of the maps in the top row forms the map $\psi_{2n-1}^{(n)}$, $\mathrm{reg}_{B,\sigma}$ is the σ-component of the Borel regulator, and $\mathrm{reg}_{G,\sigma}$ is described below. Both $K_{2n-1}^{(n)}(k)$ and $CH^n(k,2n-1)$ are $\mathbb{Q}$-vector spaces of the same finite dimension by

Bloch's isomorphism $K^{(n)}_{2n-1}(k) \cong CH^n(k, 2n-1)$, so that it follows that $\psi^{(n)}_{2n-1}$ is an isomorphism for $n \geqslant 2$.

For the description of $\mathrm{reg}_{B,\sigma}$ and $\mathrm{reg}_{G,\sigma}$, we view k as embedded into $\mathbb{C}$ via σ, drop the subscript σ, and describe the corresponding map for $\mathbb{C}$.

Following [Go2, §2.2], given m complex functions $f_1, \ldots, f_m$ on a manifold, $r_m(f_1, \ldots, f_m)$ is defined to be the real-valued $(m-1)$-form

$$\begin{aligned}&\mathrm{Alt}_m \sum_{j \geqslant 0} c_{m,j} \log|f_1| \mathrm{d}\log|f_2| \wedge \cdots \wedge \mathrm{d}\log|f_{2j+1}|\\ &\quad \wedge \mathrm{d}\arg f_{2j+2} \wedge \cdots \wedge \mathrm{d}\arg f_m\end{aligned}$$

with $c_{m,j} = \frac{1}{(2j+1)!(m-2j-1)!}$ and $\mathrm{Alt}_m F(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} (-1)^\sigma F(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. (We change Goncharov's definition with some factors $2\pi i$ that are irrelevant for our purpose.) For $h_1, \ldots, h_{2n}$ $2n$ arbitary hyperplanes in $\mathbb{P}^{n-1}_{\mathbb{C}}$, let h_0 be another hyperplane. Let f_i be a non-zero rational function on $\mathbb{P}^{n-1}_{\mathbb{C}}$ with divisor $h_i - h_0$. Then Goncharov defines

$$\begin{aligned}\mathcal{P}_n(h_1, \ldots, h_{2n}) &= \int_{\mathbb{P}^{n-1}_{\mathbb{C}}} \sum_{j=1}^{2n} (-1)^j r_{2n-1}(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{2n})\\ &= \int_{\mathbb{P}^{n-1}_{\mathbb{C}}} r_{2n-1}\left(\frac{f_1}{f_{2n}}, \ldots, \frac{f_{2n-1}}{f_{2n}}\right),\end{aligned}$$

a convergent integral, which does not depend on the choice of h_0 or of the f_i. Note that this means we can multiply the functions by non-zero constants, and that, due to the definition of r_{2n-1}, $\mathcal{P}_n(h_1, \ldots, h_{2n})$ is alternating in the h_i's. Using a coordinate change on $\mathbb{P}^{n-1}_{\mathbb{C}}$, one als sees that $\mathcal{P}_n(h_1, \ldots, h_{2n}) = \mathcal{P}_n(Mh_1, \ldots, Mh_{2n})$ for any M in $GL_n(\mathbb{C})$.

In order to define reg_B, let $g_1, \ldots, g_{2n}$ be elements in $GL_n(\mathbb{C})$, and let P be an arbitrary point of $\mathbb{P}^{n-1}_{\mathbb{C}}$. Then, up to a non-zero rational number independent of k or σ, reg_B is defined via

$$\begin{aligned}&\mathrm{reg}_B(g_1, \ldots, g_{2n})\\ &\quad = \mathcal{P}_n(\{(x_1, \ldots, x_n) \cdot g_1(P) = 0\}, \ldots, \{(x_1, \ldots, x_n) \cdot g_{2n}(P) = 0\}),\end{aligned}$$

where $x_1, \ldots, x_n$ are coordinates on $\mathbb{P}^{n-1}_{\mathbb{C}}$. (This is [Go, Theorem 4.1(b)] up to some factors $2\pi i$.) Let (AB) be the $n \times 2n$ matrix with column vectors $g_1(P), \ldots, g_{2n}(P)$, with A and B $n \times n$ matrices. Then, using a coordinate change on $\mathbb{P}^{n-1}_{\mathbb{C}}$, we see that we can also compute $\mathcal{P}_n$ using the columns of (IC) with $C = A^{-1}B$. Therefore (with minor abuse of notation), reg_B maps $(g_1, \ldots, g_{2n})$ to $\mathcal{P}_n((x_1, \ldots, x_n)(IC))$.

Goncharov also defines a regulator map on higher Chow groups at the level of complexes, see [Go2, Theorem 6.1]. In the case we are interested in, this is a map

$$\mathrm{reg}_G : CH^n(\mathbb{C}, 2n-1) \to \mathbb{R},$$

described as follows.

Let $Y \subset \Delta_{\mathbb{C}}^{2n-1}$ be an irreducible reduced variety of codimension n meeting all faces in codimension n. View $\Delta_{\mathbb{C}}^m \subset \mathbb{A}_{\mathbb{C}}^{m+1}$ as the subset of $\mathbb{P}_{\mathbb{C}}^m$ consisting of points $[x_0, \dots, x_m]$ with $x_0 + \cdots + x_m \neq 0$ by mapping $(x_0, \dots, x_m)$ to $[x_0, \dots, x_m]$. Then the boundary operations for the codimension one faces of $\Delta_{\mathbb{C}}^m$ correspond to intersection with the hyperplanes $x_i = 0$, $i = 0, \dots, m$. Put

$$\mathrm{reg}_G(Y) = \int_{Y(\mathbb{C})} r_{2n-1}\left(\frac{x_1}{x_{2n}}, \dots, \frac{x_{2n-1}}{x_{2n}}\right)$$

in $\mathbb{R}$, with r_{2n-1} as before.

Now consider the composition of the maps

$$H_{2n-1}(GL_n(k), \mathbb{Z}) \xrightarrow{\phi_P} CH^n(k, 2n-1) \xrightarrow{\mathrm{reg}_G} \mathbb{R},$$

with ϕ_P the map described just before Conjecture 12. Let again A and B be the two $n \times n$ matrices such that (AB) is the $n \times 2n$ matrix whose columns consist of the vectors $g_1(P), \dots, g_{2n}(P)$. With $C = A^{-1}B$, the linear variety this defines via its rows in $\Delta_{\mathbb{C}}^{2n-1}$ inside $\mathbb{P}_{\mathbb{C}}^{2n-1}$ is the same as the one defined by (IC).

Note that we have to intersect the variety defined by the rows of (AB) in $\mathbb{A}_{\mathbb{C}}^{2n}$ with $\Delta_{\mathbb{C}}^{2n-1}$, which gives us the zero element if this intersection is empty. Let us show first that we can always avoid this (modulo boundaries), as we can vary the point P. Namely, we can only get an empty intersection if the system of equations $(IC)(x_1, \dots, x_{2n})^t = (0, \dots, 0)^t$ and $x_1 + \cdots + x_{2n} = 1$ is inconsistent. This is the case if and only if the elements in each of the columns of C sum up to 1. Note that replacing the point P by MP for M in $GL_n(\mathbb{C})$ has the effect of replacing (AB) by $(AMBM)$ and hence C by $M^{-1}CM$. Considering the Jordan canonical form of C, we see that the equations are inconsistent for all M if and only if $C = I$, which is excluded by the general position requirements for $n \geqslant 2$. So, for given C, there is a Zariski open subset of $GL_n(\mathbb{C})$ for which the corresponding system of equations is not inconsistent, so that $\phi_P(g_1, \dots, g_{2n})$ indeed gives a linear variety in $\Delta_{\mathbb{C}}^{2n-1}$. As any homology class will involve only finitely many terms this shows we can choose P such that empty intersections do not occur while finding its image under ϕ_P.

By the previous paragraph, we can assume that $\phi_P(g_1, \dots, g_{2n})$ is the variety Y defined by the rows in (IC) in $\Delta_{\mathbb{C}}^{2n-1} \subset \mathbb{P}_{\mathbb{C}}^{2n-1}$. Parametrize Y by $(x_1, \dots, x_n) = -(y_1, \dots, y_n)C^t$ and $(x_{n+1}, \dots, x_{2n}) = (y_1, \dots, y_n)$. Then $\mathrm{reg}_G \circ \phi_P(g_1, \dots, g_{2n})$ is given by the integral

$$\begin{aligned}
&\int_{\mathbb{P}_{\mathbb{C}}^{n-1}} r_{2n-1}\left(-\left(\frac{y_1}{y_n}, \dots, \frac{y_n}{y_n}\right)C^t, \frac{y_1}{y_n}, \dots, \frac{y_{n-1}}{y_n}\right)\\
&\quad = \int_{\mathbb{P}_{\mathbb{C}}^{n-1}} r_{2n-1}\left(\frac{y_1}{y_n}, \dots, \frac{y_{n-1}}{y_n}, -\left(\frac{y_1}{y_n}, \dots, \frac{y_n}{y_n}\right)C^t\right)\\
&\quad = \mathcal{P}_n(y_1, \dots, y_{n-1}, -(y_1, \dots, y_n)C^t, y_n).
\end{aligned}$$

Writing x_i for y_i we, therefore, find that this equals

$$(-1)^n \mathcal{P}_n((x_1, \ldots, x_n)(I(-C^t)))$$

So we are done if $\mathcal{P}_n((x_1, \ldots, x_n)(I(-C^t))) = \pm\mathcal{P}_n((x_1, \ldots, x_n)(IC))$. As for any D in $GL_n(\mathbb{C})$,

$$\begin{aligned} \mathcal{P}_n((x_1, \ldots, x_n)(ID)) &= \mathcal{P}_n((x_1, \ldots, x_n)D^{-1}(ID)) \\ &= \mathcal{P}_n((x_1, \ldots, x_n)(D^{-1}I)) \\ &= (-1)^n\mathcal{P}_n((x_1, \ldots, x_n)(ID^{-1})), \end{aligned}$$

it suffices to know that $\mathcal{P}_n((x_1, \ldots, x_n)(I(-C^{-1})^t)) = \pm\mathcal{P}_n((x_1, \ldots, x_n)(IC))$. This holds for all $n \geqslant 2$ [Go3]. □

Remark 17. Let k be a number field. As noticed in Remark 13, it follows from Corollary 16 that the map

$$LCH^p(k, n)_{\mathbb{Q}} \to CH^p(k, n)_{\mathbb{Q}}$$

is surjective for $n \geqslant 1$. Gerdes proves this in general (see [Ge, Theorem 4.2]) if k is an arbitrary infinite field, and $p = n$ or $n-1$. He also indicates in [Ge, Remark 4.11] that such a statement would follow from the validity of the rank conjecture for any field k, and in particular for a number field k by the results of Borel and Yang ([B-Y]). Our method is far more direct though.

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