THREE-DIMENSIONAL ALEXANDROV SPACES: A SURVEY

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Abstract. We survey several results concerning the geometry and topology of three-dimensional Alexandrov spaces with the aim of providing a panoramic and up-to-date view of the subject. In particular we present the classification of positively and non-negatively curved spaces, the geometrization theorem, a discussion of known results for simply-connected and aspherical spaces, the equivariant and topological classifications of closed three-dimensional Alexandrov spaces with isometric compact Lie group actions, and recent developments on collapsing theory.

Contents

1. Introduction

The natural objects of study in Riemannian geometry are smooth manifolds which carry a smooth Riemannian metric, that is, smooth Riemannian manifolds. Many useful tools have been developed to study these objects, including highly developed theories of geometric and functional analysis on Riemannian spaces. Riemannian manifolds also carry a natural metric space structure allowing not only for local analytic arguments, but also for global results linking geometry and topology.

On account of being metric spaces, compact Riemannian manifolds naturally fall in the context of the Gromov–Hausdorff distance, which makes the collection of (isometry classes of) compact metric spaces into a metric space. It is then natural to ask what the metric closure of the class of compact Riemannian manifolds is under Gromov–Hausdorff convergence. The

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answer to this question was obtained by Cassorla in [16]: this closure consists of all compact inner metric spaces.

On a different note, Riemannian manifolds can be studied via several fundamental invariants, the most important being the curvature tensor. The curvature tensor itself gives rise to different notions of curvature (and, in particular, curvature bounds) such as *sectional*, *Ricci*, and *scalar* curvatures. One could then wonder about the interplay between curvature and Gromov–Hausdorff convergence, or particularly, what the closure of different classes of Riemannian manifolds with given curvature bounds is. This leads in a natural way to the consideration of different curvature notions on non-smooth metric spaces.

Motivated by the preceding considerations, much work has been devoted to extending the definitions of sectional, Ricci, and scalar curvature lower bounds to non-smooth metric spaces. By now, there exists a well established theory of metric spaces with sectional curvature bounded either above or below, known, respectively, as CAT [3, 10] or *Alexandrov* spaces [12, 13]. In the case of Ricci curvature bounded below, starting with the seminal work of Lott, Villani [62], and Sturm [95, 96], the theory of metric spaces with a lower curvature Ricci curvature has evolved into a well developed field that has seen much attention in recent years (see, for example, [36, 37]). Finally, a metric generalization of a uniform lower scalar curvature bound is, at the moment, open and remains the focus of current research (see, for example, [39, 40, 94, 97]).

Alexandrov spaces in particular play an important role in questions involving the global geometry of Riemannian manifolds and arise, for example, as Gromov–Hausdorff limits of convergent sequences of n-dimensional compact Riemannian manifolds with a uniform lower sectional curvature bound. Familiar examples of Alexandrov spaces include Riemannian orbifolds with a uniform lower sectional curvature bound and orbit spaces of isometric compact Lie group actions on complete Riemannian manifolds with sectional curvature bounded below. In addition to their relevance in Riemannian geometry, Alexandrov spaces are objects of interest in their own right and, since complete Riemannian manifolds with a uniform lower sectional curvature bound are Alexandrov spaces, Alexandrov geometry may be seen as a metric generalization of Riemannian geometry. Indeed, many Riemannian theorems, such as the Bonnet–Myers theorem, have corresponding analogues for Alexandrov spaces and much effort has been made in generalizing Riemannian results to Alexandrov spaces. This seemingly simple approach involves significant challenges, as in Alexandrov geometry one must make do without the smooth tools available for Riemannian manifolds, relying instead on purely metric machinery. Thus, in order to recover Riemannian results, one must usually devise new arguments depending only on metric considerations.

As noted above, generic Alexandrov spaces are not manifolds and their topological intricacies increase considerably with their dimension. Indeed, locally, an n-dimensional Alexandrov space is homeomorphic to a cone over an $(n-1)$-dimensional Alexandrov space with curvature bounded below by 1. The latter spaces are far from being classified, even in the Riemannian case. Moreover, even if the space is manifold, the set of metric singularities may be dense. It is natural then to first consider Alexandrov spaces of low dimensions. The topology and geometry of one- and two-dimensional Alexandrov spaces is, essentially, well understood (see for example [12, Corollary 10.10.3]). Therefore, the next step consists in analyzing three-dimensional Alexandrov spaces and in this survey we present an up-to-date and panoramic view of the topology and geometry of such spaces.

Our manuscript is organized as follows. In Section 2 we recall preliminary notions of Alexandrov geometry. In Section 3 we present the classification of positively and non-negatively curved spaces, the geometrization theorem, and a discussion on simply-connected and aspherical three-dimension Alexandrov spaces. We then present Lie group actions and their topological and equivariant classifications for three-dimensional Alexandrov spaces in Section 6. Finally, in Section 7 we give a brief account of results on collapsing Alexandrov spaces in dimension three.

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2. Alexandrov spaces

In this section we recall some basic concepts and general results on Alexandrov geometry. Most of this material can be found in the basic references [12, 13]; see also [83, 91]. We refer the reader to [1, 82] for further results on Alexandrov geometry as well as to Petersen's notes in this same volume.

2.1. **Basic definitions.** Let (X, d) be a metric space. A *curve* (or *path*) in X is, by definition, a continuous function $\gamma\colon I \to X$ defined on an interval $I \subset \mathbb{R}$. We define the *length* of a curve $\gamma\colon [a, b] \to X$ by

$$\mathrm{Length}(\gamma) = \sup_{\mathcal{P}} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions of $[a, b]$, i.e., over all finite collections of points $\mathcal{P} = \{ t_0, \dots, t_n \}$ with $a = t_0 < t_2 < \cdots < t_n = b$. The metric space (X, d) is a *length space* if it is path connected and the distance $d(p, q)$ between points $p, q \in X$ is given by the infimum of lengths of curves joining p and q, i.e., curves $\gamma\colon [a, b] \to X$ with $\gamma(a) = p$ and $\gamma(b) = q$. In this case, d is said to be an *intrinsic* (or *inner*) metric. Note that any connected Riemannian manifold equipped with its Riemannian distance is a length space.

Let $I \subset \mathbb{R}$ be an interval. A curve $\gamma\colon I \to X$ is a *geodesic* if, for each interior value t of I, the restriction of γ to a small interval centered at t is a shortest path. A curve γ between points p and q in a metric space (X, d) is a *minimal geodesic* if $\mathrm{Length}(\gamma) = d(p, q)$. We will assume all geodesics to be minimal unless otherwise stated. It is worth noting that for locally compact and complete length spaces, one can always join every pair of points by a minimal geodesic [12, Theorem 2.5.23]. We will denote a geodesic between p and q by $[pq]$. Note that this terminology differs from the one used in Riemannian geometry, where a geodesic is a curve that locally minimizes distances between any two of its points and may not necessarily realize the distance between its endpoints. This may be easily verified by considering great circles on a round sphere.

Let us now define the *Hausdorff dimension* of a metric space Y. Let $\mathcal{V} = \{V_i\}_{i \in I}$ be a countable covering of Y, let $h \geq 0$ be a real number, and fix $\delta > 0$. Define

$$H^h_\delta(Y) = \inf_{\mathcal{V}} \left\{ \sum_{i \in I} \operatorname{diam}(V_i)^h : \operatorname{diam}(V_i) < \delta \text{ for all } i \in I \right\}, \tag{2.1}$$

where the infimum is taken over all countable coverings of Y by subsets of diameter less than δ. Note that if no such covering exists, then $H^h_\delta(Y) = \infty$. We convene that, if $h = 0$, then every 0^0 term appearing in the sum $\sum_{i \in I} \operatorname{diam}(V_i)^h$ in (2.1) is replaced by 1. We then define the *h-dimensional Hausdorff measure* of Y, denoted by $H^h(Y)$, by letting

$$H^h(Y) = C(h) \lim_{\delta \to 0} H^h_\delta(Y).$$

Here $C(h) > 0$ is a normalization constant chosen so that, if h is a positive integer, then $H^h([0,1]^h) = 1$, where $[0,1]^h$ is the unit cube in Euclidean space $\mathbb{R}^h$. We define the *Hausdorff dimension* of Y by

$$\dim_H(Y) = \inf \left\{ h \geq 0 : H^h(Y) = 0 \right\}.$$

We define the Hausdorff dimension of subsets of Y by considering such subsets as metric spaces equipped with the subspace metric induced by the metric on Y. Note that the Hausdorff dimension of a metric space may not necessarily be an integer. Indeed, many self-similar subspaces of Euclidean spaces, such as the Cantor set in $\mathbb{R}$ or the Sierpinski triangle in $\mathbb{R}^2$, have non-integer Hausdorff dimensions. On the other hand, the Hausdorff dimension of a Riemannian n-manifold equals its topological dimension.

We conclude this subsection by recalling the definitions of the *Hausdorff* and *Gromov–Hausdorff* distances. Let X be a metric space. Given $r > 0$ and a subset $S \subset X$, let $U_r(S) = \{ x \in X : d(x, S) < r \}$, i.e., $U_r(S)$ is an open r-neighborhood of S in X. The *Hausdorff distance* $d_H(A, B)$ between two subsets A, B of X is, by definition,

$$d_H(A, B) = \inf\{ r > 0 : A \subset U_r(B) \text{ and } B \subset U_r(A) \}.$$

Now, let X and Y be compact metric spaces. The *Gromov–Hausdorff* distance $d_{GH}(X, Y)$ between X and Y is given by

$$d_{GH}(X, Y) = \inf \{ d_H(f(X), g(Y)) \},$$

where the infimum is taken over all metric spaces Z and all isometric embeddings $f \colon X \to Z$ and $g \colon Y \to Z$. This distance measures how far X and Y are from being isometric. Indeed, $d_{GH}(X, Y) = 0$ if and only if X is isometric to Y. The space of (isometry classes of) compact metric spaces equipped with the Gromov–Hausdorff distance is itself a metric space.

From now on we will assume that (X, d) is a complete, locally compact length space. This guarantees the existence of geodesics between any two points in (X, d) (see [12, Theorem 2.5.23]). In the Riemannian setting the existence of shortest curves between any pair of points in a complete Riemannian manifold is a consequence of the Hopf–Rinow Theorem (see [22, Ch. 7, Theorem 2.8]). To lighten the notation, we will usually denote the metric space (X, d) simply by X. We will assume all our spaces to be connected.

2.2. **Curvature bounded below.** Rather than considering directly the geometry of a length space X, we will compare distances in X with distances in a given *model space* whose geometry is well understood. Our goal is to define a notion of (sectional) curvature bounded below using only distances in X.

The *model space* M_k^2 with curvature $k \in \mathbb{R}$ is the simply-connected complete Riemannian 2-manifold with constant sectional curvature k. Hence, M_k^2 is isometric to the Euclidean plane $\mathbb{R}^2$ if $k = 0$, to the round sphere of radius $1/\sqrt{k}$ if $k > 0$, or to the hyperbolic plane appropriately rescaled so that its sectional curvature is $k < 0$. Observe that

$$\operatorname{diam}(M_k^2) = \begin{cases} \pi/\sqrt{k} & \text{if } k > 0; \\ \infty & \text{if } k \leq 0. \end{cases}$$

A *geodesic triangle* $\triangle pqr$ in X consists of three different points $p, q, r \in X$ and three geodesics $[pq]$, $[pr]$, $[qr]$ between them. Given $k \in \mathbb{R}$, a *comparison triangle for* $\triangle pqr$ in M_k^2 is a geodesic triangle $\triangle \overline{pqr}$ in M_k^2 with vertices $\overline{p}$, $\overline{q}$, $\overline{r}$ and whose sides have the same length as the corresponding sides of $\triangle pqr$, i.e., $d(p,q) = d(\overline{p}, \overline{q})$, $d(p,r) = d(\overline{p}, \overline{r})$, and $d(q,r) = d(\overline{q}, \overline{r})$.

The angles of $\triangle \overline{pqr}$ at each one of its vertices $\overline{p}$, $\overline{q}$, and $\overline{r}$ are the *comparison angles* and we denote them, respectively, by $\widetilde{\measuredangle} rpq$, $\widetilde{\measuredangle} pqr$, and $\widetilde{\measuredangle} prq$. Given a geodesic triangle $\triangle pqr$ in X, a corresponding comparison triangle exists and is unique (up to an isometry of the model space) if $k \leq 0$, or if $k > 0$ and the perimeter of $\triangle pqr$ is strictly less than $2\pi/\sqrt{k}$.

Suppose now that X is a complete Riemannian manifold with sectional curvature $\sec \geq k$ for some $k \in \mathbb{R}$. By Toponogov's comparison theorem, if $\triangle pqr$ is a geodesic triangle in X and $\triangle \overline{pqr}$ is a comparison triangle in M_k^2, then

$$d(p,s) \geq d(\overline{p}, \overline{s}), \tag{2.2}$$

where s is a point in the geodesic $[qr]$ in $\triangle pqr$ and $\overline{s}$ is the point in the side $[\overline{qr}]$ of $\triangle \overline{pqr}$ with $d(\bar{s}, \bar{q}) = d(s,q)$. Conversely, if inequality (2.2) holds for any geodesic triangle $\triangle pqr$ and a corresponding comparison triangle $\triangle \overline{pqr}$ in M_k^2, then X has sectional curvature bounded below by k (see [10, Theorem 1A.6]). We now take this metric characterization of sectional curvature bounded below as the definition of a lower (sectional) curvature for length spaces.

Definition 2.1 (Property T_k)**.** Let X be a complete, locally compact length space. A geodesic triangle $\triangle pqr$ in X satisfies *property* T_k *for a given* $k \in \mathbb{R}$ if, for any comparison triangle $\triangle \overline{pqr}$ in M_k^2 and for any point s in the geodesic $[qr]$ in $\triangle pqr$,

$$d(p,s) \geq d(\overline{p}, \overline{s}),$$

where $\bar{s}$ is the point in the side $[\overline{qr}]$ of $\triangle \overline{pqr}$ with $d(\bar{s}, \bar{q}) = d(s,q)$.

Definition 2.2. A complete, locally compact length space X *has curvature bounded below by* $k \in \mathbb{R}$ (denoted by $\operatorname{curv} \geq k$) if every point in X has an open neighborhood U_p where property T_k holds for every geodesic triangle in U_p.

With a metric definition of lower curvature bound now in hand, we define Alexandrov spaces.

Definition 2.3. A complete, locally compact length space of finite (Hausdorff) dimension is an *Alexandrov space with curvature bounded below by* $k \in \mathbb{R}$ if property T_k is satisfied locally.

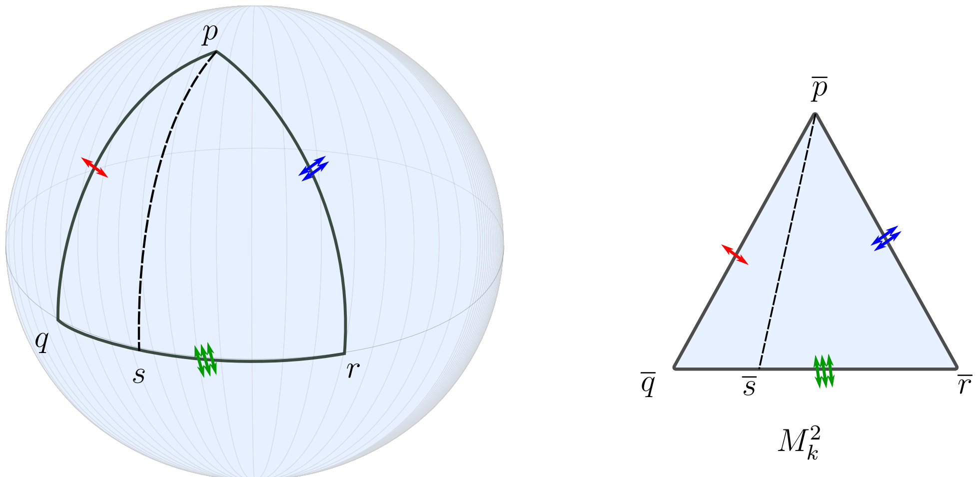


FIGURE 1. The T_k-property. On the left hand side, $\triangle pqr$ is displayed. On the right hand side, a comparison triangle $\triangle\overline{pqr}$ is displayed on M_k^2. In the figure M_k^2 has $k = 0$.

Toponogov's globalization theorem [12, Theorem 10.3.1] implies that property T_k holds globally on any Alexandrov space with curvature bounded below by k, i.e., the property holds for any geodesic triangle in the space. We convene that the line $\mathbb{R}$, the half line $\mathbb{R}_+$, line segments of length greater than $\pi/\sqrt{k}$, and circles of length greater than $2\pi/\sqrt{k}$ are not Alexandrov spaces with curv $\geq k > 0$.

Remark 2.4. Since the definition of lower curvature bound is independent of the Hausdorff dimension of the length space under consideration, it is possible to omit the finiteness of the Hausdorff dimension in Definition 2.3 (see, for example, [83]). Doing so, however, introduces technical difficulties that do not arise in finite dimensions (see, for example, [47]). We refer the reader to [99, 100] for other results on infinite dimensional spaces with curvature bounded below.

The graph in Figure 2, considered as a subset of $\mathbb{R}^2$, equipped with the length metric induced by the usual Euclidean distance, is not an Alexandrov space. Indeed, no neighborhood of a vertex of degree three or more (i.e., where three or more edges meet) has a lower curvature bound. This space, however, has curvature bounded above in the comparison sense (see, for example [12, Ch. 9]). Similarly, a subset of $\mathbb{R}^3$ consisting of a 2-plane attached to an interval by one of its endpoints with the induced length metric is not an Alexandrov space of curvature bounded below.

Alexandrov spaces have several nice topological and geometric properties. For example, they are *non-branching*, i.e., geodesics do not bifurcate. This does not hold, for instance, for the length space in Fig. 2. The Hausdorff dimension of an Alexandrov space is an integer and it equals its topological dimension.

We will denote the class of n-dimensional Alexandrov spaces with curvature bounded below by $k \in \mathbb{R}$ by $\mathrm{Alex}^n(k)$. Observe that $\mathrm{Alex}^n(k)$ contains the set of complete n-dimensional Riemannian manifolds with sec $\geq k$. This inclusion is proper, since not every Alexandrov space is homeomorphic to a topological manifold, as the examples in the following subsection

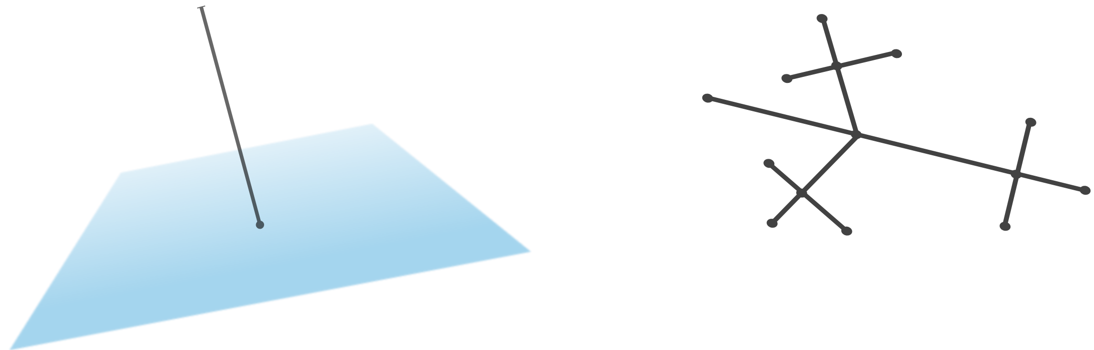


FIGURE 2. These spaces are not Alexandrov spaces.

illustrate. Thus, Alexandrov geometry may be seen as a generalization of Riemannian geometry and, indeed, many Riemannian theorems have corresponding analogues for Alexandrov spaces. One such result is the Bonnet–Myers theorem [12, Theorem 10.4.1], which we will use in subsequent sections.

Theorem 2.5 (Bonnet–Myers)**.** *If X is and Alexandrov space with* $\operatorname{curv} \geq k > 0$*, then* $\operatorname{diam}(X) \leq \pi/\sqrt{k}$*. In particular, X is compact.*

2.3. **Examples and constructions.** We now list some well-known examples and constructions of Alexandrov spaces.

Complete Riemannian manifolds with $\sec \geq k$*.* As stated in the preceding subsection, Toponogov's comparison theorem implies that every complete Riemannian manifold with sectional curvature bounded below by k is an Alexandrov space with $\operatorname{curv} \geq k$.

Convex sets. Any convex subset of an Alexandrov space with $\operatorname{curv} \geq k$ is again an Alexandrov space with $\operatorname{curv} \geq k$.

Convex surfaces. Every *convex surface* in $\mathbb{R}^3$, i.e., the boundary of a convex body in $\mathbb{R}^3$, equipped with the intrinsic metric induced by $\mathbb{R}^3$, is an Alexandrov space of $\operatorname{curv} \geq 0$. More generally, the boundary of any convex body in $\mathbb{R}^n$, $n \geq 3$, is an Alexandrov space of nonnegative curvature. By a result of Buyalo [11, 14], any convex hypersurface N in a complete Riemannian manifold M with sectional curvature bounded below by k is an Alexandrov space of $\operatorname{curv} \geq k$.

Gromov–Hausdorff limits. The limit of a Gromov–Hausdorff convergent sequence of compact Alexandrov spaces with curvature bounded below by k is itself an Alexandrov space of $\operatorname{curv} \geq k$.

Cartesian products. Let (X, d_X) and (Y, d_Y) be Alexandrov spaces with curvature bounded below by k. Motivated by the Pythagorean theorem, we define the *product metric* d on the Cartesian product $X \times Y$ by letting

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. The space $X \times Y$ equipped with the product metric is the *direct metric product of X and Y* and is an Alexandrov space with $\operatorname{curv} \geq k$.

Cones. Let $(X, d) \in \mathrm{Alex}^n(1)$ and let $C(X) = X \times [0, \infty)/X \times \{0\}$ be the cone over X. We define the *Euclidean cone metric* d_C on $C(X)$ by letting

$$d_C((p, s), (q, t)) = \sqrt{s^2 + t^2 - 2st\cos(d(p, q))}$$

for all $(p, s), (q, t) \in C(X)$. Since $\mathrm{curv}(X) \geq 1$, the Bonnet–Myers theorem implies that $\mathrm{diam}(X) \leq \pi$. This ensures that d_C is indeed a metric on $C(X)$ (see, for example, [12, Propsition 3.6.13]). The metric space $(C(X), d_C)$ is the *Euclidean cone* over X and is an Alexandrov space with curv ≥ 0. In particular, the Euclidean cone over the unit round sphere $\mathbb{S}^n(1)$ is isometric to $\mathbb{R}^{n+1}$. Note that $C(X)$ contains an isometric copy of X consisting of all the points at distance one from the vertex of $C(X)$.

Suspensions. Let $(X, d) \in \mathrm{Alex}^n(1)$ and let

$$\mathrm{Susp}(X) = X \times [0, \pi]/\{X \times \{0\}, Y \times \{\pi\}\}$$

be the *suspension* of X. Motivated by the spherical law of cosines, we define the *spherical suspension metric* d_S on $\mathrm{Susp}(X)$ by

$$d_S((p, s), (q, t)) = \cos^{-1}(\cos(s)\cos(t) + \sin(s)\sin(t)\cos(d(p, q)))$$

for all $(p, s), (q, t) \in \mathrm{Susp}(X)$. The space $(\mathrm{Susp}(X), d_S)$ is the *spherical suspension* of X and is an $(n + 1)$-dimensional Alexandrov space with curv ≥ 1. In particular, the spherical suspension of $\mathbb{S}^n(1)$ is isometric to $\mathbb{S}^{n+1}(1)$. Note that $\mathrm{Susp}(X)$ contains an isometric copy of X consisting of the set of points at distance $\pi/2$ from either vertex of the spherical suspension. In analogy with the sphere, we may think of this set as the *equator* of the suspension and of the vertices as the *poles*.

Joins. The *join* of two topological spaces X, Y is the space

$$X * Y = (X \times Y \times [0, \pi])/\sim,$$

where $\sim$ is the equivalence relation given by

$$(x, y, 0) \sim (x, y', 0) \text{ for all } x \in X \text{ and } y, y' \in Y,$$
$$(x, y, \pi) \sim (x', y, \pi) \text{ for all } x, x' \in X \text{ and } y \in Y.$$

If (X, d_X) and (Y, d_Y) are Alexandrov spaces with curv ≥ 1, then we may define a *spherical join metric* d_J on $X * Y$ so that $(X * Y, d_J)$ is an Alexandrov space with curvature bounded below by 1 and dimension $\dim X + \dim Y + 1$. We outline the definition of d_J, following [45]. Since X and Y are Alexandrov spaces with curv ≥ 1, the Euclidean cones $C(X)$ and $C(Y)$ are Alexandrov spaces with curv ≥ 0 and hence the product $C(X) \times C(Y)$ is an Alexandrov space with curv ≥ 0. Let o_X and o_Y denote, respectively, the vertices of the cones $C(X)$ and $C(Y)$. The set of points at unit distance from (o_X, o_Y) in $C(X) \times C(Y)$ is an Alexandrov space with curv ≥ 1 which can be naturally identified with the join $X * Y$. The join $X * Y$ contains isometric copies of X and Y in such a way that all points in X are at distance $\pi/2$ from all points in Y. Moreover, if $A \subset X$ and $B \subset Y$ are Alexandrov spaces isometrically embedded in X and Y, then $A * B$ is isometrically embedded in $X * Y$. We may use this observation to calculate distances between points in $X * Y$ as follows. First, note that any point in $X * Y \setminus (X \cup Y)$ has a unique coordinate representation as (x, t, y) with $x \in X$, $y \in Y$, $t \in (0, \pi/2)$. Now, given two points in the join, let A be a geodesic joining the x-coordinates and let B be a geodesic joining the y-coordinates. Note that we may think of

A and B as being isometrically embedded in $S^1(1)$, since their length is at most π. Then the distance between the original points can be computed in $A * B \subset \mathbb{S}^3(1) = \mathbb{S}^1(1) * \mathbb{S}^1(1)$. Note that when Y is the two point set, $X * Y$ is isometric to the spherical suspension of X. The spherical join of two unit round spheres $\mathbb{S}^n(1)$ and $\mathbb{S}^m(1)$ is isometric to $\mathbb{S}^{n+m+1}(1)$.

Quotients. Let X be an Alexandrov space and let G be a group acting by isometries on X with closed orbits. The orbit space X^* has a metric given by

$$d_Q(p^*, q^*) = \inf \{ d(x, y) \, : \, x \in \mathrm{G}(p), \; y \in \mathrm{G}(q) \}$$

for all $p^*, q^* \in X^*$. If $X \in \mathrm{Alex}^n(k)$, then X^* is also an Alexandrov space with curv $\geq k$. This is a consequence of the fact that the orbit projection map $\mathsf{pr}\colon X \to X^*$ is a *submetry*, i.e., the image under pr of a metric ball of radius $r > 0$ and center $p \in X$ is a metric ball in X^* with radius r and center $p^* \in X^*$.

Doubles and glued spaces. Let $X_1, X_2 \in \mathrm{Alex}^n(k)$ with non-empty boundary (for the definition of boundary see section 2.4 below) and let $f\colon \partial X_1 \to \partial X_2$ be an isometry. The space $X = X_1 \cup X_2/(p \sim f(p))$ has a metric with curv $\geq k$ given by

$$d(p_1, p_2) = \inf \{ \, d_1(p_1, q) + d_2(f(q), p_2) \, : \, q \in \partial X_1 \, \} \, .$$

This was first proved for doubles of Alexandrov spaces with boundary by Perelman [77]. The general gluing result was obtained by Petrunin [81]. It follows from these results that the *double disc*, i.e., the gluing of two copies of a disc in $\mathbb{R}^2$ along their isometric boundaries, is an Alexandrov space with non-negative curvature whose underlying topological space is homeomorphic to a 2-sphere.

We may use the preceding constructions to generate Alexandrov spaces that are neither topological manifolds nor orbifolds. Consider, for example, the complex projective plane $\mathbb{C}P^2$. Equipped with its canonical Fubini–Study metric, $\mathbb{C}P^2$ is a Riemannian manifold with sec ≥ 1 and is, therefore, an Alexandrov space with curv ≥ 1. Hence the spherical suspension $\mathrm{Susp}(\mathbb{C}P^2)$ is again an Alexandrov space with curv ≥ 1 and is homeomorphic neither to a topological manifold nor to an orbifold. Further examples of Alexandrov spaces with curvature bounded below include certain warped products [2, 4] and stratified spaces [8].

2.4. **Local structure.** The local structure of an Alexandrov space is determined by the *space of directions*. To define this space, we first define angles between geodesics. The *angle* between two geodesics $[pq]$, $[pr]$ in an Alexandrov space X is defined as

$$\angle qpr = \lim_{q_1, r_1 \to p} \{ \angle \overline{q}_1 \overline{p} \overline{r}_1 : q_1 \in [pq], \; r_1 \in [pr] \}.$$

Geodesics that make an angle zero determine an equivalence class called *tangent direction*. The set of tangent directions at a point $p \in X$ is denoted by Σ_p' and, when equipped with the angle distance $\angle$, the set Σ_p' is a metric space. This space may not be complete, however, as one can see by considering directions at a point in the boundary of a unit disc in the Euclidean plane. The completion of $(\Sigma_p', \angle)$ is called the *space of directions of X at p* and is denoted by Σ_p. It corresponds, in Alexandrov geometry, to the unit tangent sphere in Riemannian geometry. The space of directions Σ_p is an Alexandrov space with curv ≥ 1 and dimension $\dim X - 1$. Moreover, Σ_p is isometric to the unit round sphere $\mathbb{S}^{n-1}(1)$ on a dense set $\mathcal{R}_X$, called the set of *(metrically) regular points*. The set $\mathcal{S}_X = X \setminus \mathcal{R}_X$ is the set of *(metrically) singular points*. The following example shows that the set $\mathcal{S}_X$ may be dense in X.

Example 2.6 (Otsu–Shioya [75, p. 632])**.** Let P be a convex polyhedron in $\mathbb{R}^3$. For any vertex $p \in P$, let $\angle(P, p)$ be the sum of all inner angles at p of faces of P having p as a vertex. Since P (equipped with the length metric induced by $\mathbb{R}^3$) is an Alexandrov space of non-negative curvature, $\angle(P, p) \leq 2\pi$ and the space of directions Σ_p is a circle of length $\angle(P, p)$. Thus, a vertex $p \in P$ is a metrically singular point if and only if $\angle(P, p) < 2\pi$. We now define inductively a Hausdorff-convergent sequence $\{X_k\}_{k=1}^{\infty}$ of convex polyhedra in $\mathbb{R}^3$ with limit X. Since Hausdorff convergence implies Gromov–Hausdorff convergence, X will be an Alexandrov space of non-negative curvature. We define the X_i so that the set of metrically singular points in X is dense. Let X_1 be a regular tetrahedron in $\mathbb{R}^3$ whose barycenter is the origin $o \in \mathbb{R}^3$. Assume that X_k for some $k > 1$ has been defined. Let us define X_{k+1}. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a monotone decreasing sequence of positive numbers converging to zero. Assume that $0 < \varepsilon_i < 1$ for all i and let $\varepsilon = \Pi_{j=1}^{\infty}(1-\varepsilon_j)$; note that, by construction, $\varepsilon > 0$. Take the barycentric subdivision of X_k and push all the new vertices outward slightly along rays emanating from o while keeping the original vertices of X_k to obtain the convex tetrahedron X_{k+1}. We may assume that

$$2\pi - \angle(X_{k+1}, p) \geq (1 - \varepsilon_k)(2\pi - \angle(X_k, p))$$

for any vertex p of X_k. Define $X \subset \mathbb{R}^3$ to be the Hausdorff limit of $\{X_k\}$. Then X is a non-negatively curved Alexandrov space. For any k and any vertex p of X_k, we obtain

$$\begin{aligned}\lim_{i\to\infty}(2\pi - \angle(X_i, p)) &\geq \Pi_{i=1}^{\infty}(1 - \varepsilon_{k+1})(2\pi - \angle(X_k, p)) \\ &\geq \varepsilon(2\pi - \angle(X_k, p)) \\ &> 0.\end{aligned}$$

The length of the space of directions of X at p is $\lim_{i\to\infty} \angle(X_i, p) < 2\pi$. Thus any vertex of X_k for any k is a singular point of X. Since the maximal length of all the edges of X_k tends to zero as $k \to \infty$, the set S_X of singular points is dense in X.

Example 2.7. Let M be a complete Riemannian manifold with sectional curvature bounded below by k. Recall that, if G is a compact Lie group acting effectively and isometrically on M, then the orbit space M^* is an Alexandrov space with curv $\geq k$. In this case the space of directions Σ_{p^*} at $p^* \in M^*$ consists of geodesic tangent directions and is isometric to $\mathbb{S}_p^{\perp}/G_p$, where $\mathbb{S}_p^{\perp}$ is the unit normal sphere to the orbit $G(p)$ at $p \in M$.

Definition 2.8. Let X be an Alexandrov space and fix $p \in X$. The *tangent cone* of X at a p is the Euclidean cone over the space of directions of X at p. We will denote it by T_pX.

By construction T_pX is an Alexandrov space with curv ≥ 0 and $\dim T_pX = \dim X$. Note that the tangent cone of a complete Riemannian manifold at any one of its points is the usual tangent space and is isometric to a Euclidean space.

The local structure of Alexandrov spaces is given by the following theorem (see [77]).

Theorem 2.9 (Conical Neighborhood Theorem (Perelman))**.** *If X is an Alexandrov space, then every $p \in X$ has a neighborhood pointed-homeomorphic to T_pX.*

Remark 2.10. It is conjectured that the homeomorphism in the preceding theorem should be bi-Lipschitz.

The conical neighborhood theorem implies that the local topology of an Alexandrov space X at a point p is determined by the space of directions Σ_p. Thus, since Σ_p is an Alexandrov space with curvature bounded below by 1, it is important to determine the possible homeomorphism types of such Alexandrov spaces. In dimensions 2 and 3 this classification is complete (see the next section). In dimensions $n \geq 4$, however, the classification problem is open in full generality, even in the case of Riemannian manifolds (see [101] and references therein).

Having defined the space of directions at a point, we may now define the boundary of an Alexandrov space X. If $\dim X = 1$, then X is a manifold (possibly with boundary). We define the boundary of an Alexandrov space inductively by letting

$$\partial X = \{p \in X : \partial\Sigma_p \neq \varnothing\}$$

if $\dim X > 1$. The boundary of X is a closed subset of codimension 1. Note that if $\dim X \leq 2$, then X is a topological manifold (possibly with boundary).

An Alexandrov space is an *Alexandrov manifold* if it is homeomorphic to a topological manifold. We will say that an Alexandrov space is *topologically regular* if every space of directions is homeomorphic to a sphere. Clearly, a topologically regular Alexandrov space is an Alexandrov manifold, but the converse is not necessarily true, as the following example shows.

Example 2.11. Recall that the *Poincaré homology sphere*, which we denote by P^3, is diffeomorphic to a quotient of the 3-sphere by a free smooth action of the binary icosahedral group I^*. Thus $P^3 = \mathbb{S}^3/I^*$ is a compact non-simply-connected 3-manifold without boundary and with the same integral homology groups as $\mathbb{S}^3$. We may assume that I^* acts orthogonally on the unit round 3-sphere, which implies that P^3 inherits a Riemannian metric with constant sectional curvature one. In particular, P^3 is an Alexandrov space of curv ≥ 1 and, hence, its the double spherical suspension $(\mathrm{Susp}^2(P^3), d)$ is a 5-dimensional Alexandrov space with curv ≥ 1. By the Double Suspension Theorem of Edwards and Cannon, $\mathrm{Susp}^2(P)$ is homeomorphic to $\mathbb{S}^5$ (see [15, 24, 23]). It follows that $(\mathrm{Susp}^2(P), d)$ is a five-dimensional Alexandrov manifold. On the other hand, $(\mathrm{Susp}^2(P), d)$ is not topologically regular, since the spaces of directions at the poles of the double suspension are homeomorphic to $\mathrm{Susp}(P)$, which is not a manifold.

Definition 2.12. Let $X \in \mathrm{Alex}^n(k)$. A subset $E \subset X$ is *extremal* if, for $p \in X$ and $q \in E$ with $d(p, q) = d(p, E)$, one has $\Sigma_q = B(p', \frac{\pi}{2})$, where $p' \in \Sigma_q$ is the direction of a geodesic from q to p.

A point p in an Alexandrov space X is extremal if and only if $\mathrm{diam}(\Sigma_p) \leq \pi/2$. Observe that ∂X is extremal. By work of Perelman and Petrunin [78], each extremal set $E \subset X$ can be decomposed into a disjoint union of topological manifolds; along with an n-dimensional open set, this determines a stratification of X into topological manifolds.

2.5. **A Riemannian digression.** Alexandrov spaces play an important role in Riemannian geometry and arise, for example, in the following context. Recall that, equipped with the Gromov–Hausdorff distance d_{GH} defined in section 2.1, the collection $\mathcal{X}$ of isometry classes of compact metric spaces is itself a metric space. A key fact is that the class of (isometry classes of) compact Riemannian n-manifolds

$$\mathcal{M}_k^D(n) = \{\, (M^n, g) \,:\, \sec(M^n) \geq k, \mathrm{diam}(M^n) \leq D \,\} \subset (\mathcal{X}, d_{\mathrm{GH}})$$

is precompact under the Gromov–Hausdorff topology for all $n \in \mathbb{N}$, $k \in \mathbb{R}$ and $D > 0$ (see [12, Ch. 7]. Moreover, the points in the closure of $\mathcal{M}_k^D(n)$ are Alexandrov spaces:

Theorem 2.13 (Burago, Gromov, and Perelman [13]; Grove and Petersen [44])**.** *The closure* $\overline{\mathcal{M}}_k^D(n)$ *consists of Alexandrov spaces with* $\operatorname{curv} \geq k$*,* $\operatorname{diam} \leq D$*, and* $\dim \leq n$*.*

Alexandrov spaces therefore naturally appear as limit spaces of Riemannian manifolds. It is not known, however, if every compact Alexandrov space is the limit of a sequence of compact Riemannian manifolds with sectional curvature uniformly bounded below. In the case where the sequence is *non-collapsed*, i.e., where the limit space has the same dimension as the elements of the sequence, it follows from Perelman's stability theorem that the limit space must be a topological manifold (see [57, 79]).

Theorem 2.14 (Stability Theorem (Perelman))**.** *Let X, Y be compact n-dimensional Alexandrov spaces of* $\operatorname{curv} \geq k$*. Then there exists an $\varepsilon = \varepsilon(X) > 0$ such that, if $d_{GH}(X, Y) < \varepsilon$, then Y is homeomorphic to X.*

This theorem, in combination with theorem 2.13, implies the following well-known finiteness result in Riemannian geometry (see, for example, [42]).

Theorem 2.15 (Riemannian Homeomorphism Finiteness Theorem)**.** *For each $n \in N$, $k \in \mathbb{R}$, and $D, v > 0$, the class $\mathcal{M}_{k,v}^D(n)$ of compact Riemannian n-manifolds with* $\operatorname{diam} \leq D$*,* $\sec \geq k$*, and volume* $\operatorname{vol} \geq v$*, contains at most finitely many homeomorphism types.*

Note that a similar finiteness result holds for Alexandrov spaces. Although the class of compact Alexandrov spaces with lower curvature bound k is closed in the Gromov–Hausdorff topology, collapse may occur, that is, it is possible for a Gromov–Hausdorff converging sequence of spaces in $\operatorname{Alex}^n(k)$ to have a limit of dimension less than n. We may observe this phenomenon by considering, for example, a flat torus T^n and rescaling its Riemannian metric by $1/i$ with $i = 1, 2, \ldots$. In this way we get a sequence of flat n-dimensional tori T_i^n whose diameter converges to zero as $i \to \infty$. In this case, the sequence $\{T_i^n\}_{i=1}^{\infty}$ collapses to a point. Collapse imposes strong restrictions on the structure of Riemannian manifolds; for more information we refer the reader to [42, 87]. We will discuss the topology of collapsed 3-dimensional Alexandrov spaces in section 7.

3. Three-dimensional Alexandrov spaces

In this section we discuss the basic topology and geometry of closed (i.e., compact and without boundary) three-dimensional Alexandrov spaces (or, for short, closed *Alexandrov 3-spaces*). We will focus our attention on those that are not homeomorphic to 3-manifolds, as this is where new phenomena arise. The symbol "≈" will denote homeomorphism between topological spaces.

3.1. Basic structure.

Let us first consider the topological structure of closed Alexandrov 3-spaces, following [29]. We refer the reader to [51] for basic results in 3-manifold topology.

Recall that a closed Alexandrov space of dimension one must be homeomorphic to a circle. Then, by Perelman's conical neighborhood theorem, a 2-dimensional Alexandrov space must be homeomorphic to a topological manifold, possibly with boundary. The topological classification of closed, positively curved Alexandrov spaces of dimension two follows now from the Bonnet–Myers theorem, which implies that the fundamental group of a closed, positively curved Alexandrov space must be finite. Therefore, any closed two-dimensional Alexandrov space with curv ≥ 1 is homeomorphic to $\mathbb{S}^2$ or to $\mathbb{R}P^2$. It follows that $\mathbb{S}^2$ and $\mathbb{R}P^2$ are the only possible spaces of directions of an Alexandrov 3-space without boundary. Hence, by the conical neighborhood theorem, an Alexandrov 3-space X without boundary is a topological 3-manifold if and only if each one of its points has space of directions homeomorphic to $\mathbb{S}^2$ i.e., if X is topologically regular).

Let X be a closed Alexandrov 3-space and assume that X is not a topological manifold. Hence at least one point in X has space of directions homeomorphic to $\mathbb{R}P^2$. Since X is compact, the conical neighborhood theorem implies that there are finitely many points in X whose space of directions is $\mathbb{R}P^2$. After removing from X sufficiently small open neighborhoods of these topologically singular points we get a compact non-orientable 3-manifold X_o with a finite number of $\mathbb{R}P^2$-boundary components where we glue in cones over $\mathbb{R}P^2$. It is not difficult to see that X_o must have an even number of boundary components (cf. [51, Proof of Theorem 9.5]) or, equivalently, that X must have an even number of topologically singular points. Let $D(X_o)$ be the double of X_o and consider the natural decomposition of $D(X_o)$ as the union of two copies of X_o glued along ∂X_o. From the Mayer–Vietoris sequence for this decomposition of $D(X_o)$ we obtain that

$$\chi(D(X_o)) = 2\chi(X_o) - \chi(\partial X_o). \tag{3.1}$$

Since $D(X_0)$ is a closed 3-manifold, its Euler characteristic is zero. Hence, equation (3.1) implies that $\chi(\partial X_o)$ is even. Since each connected component of ∂X_o is a real projective space and $\chi(\mathbb{R}P^2) = 1$, it follows that X_o has an even number of boundary components. Therefore, X has an even number of topologically singular points.

Example 3.1. The real projective plane $\mathbb{R}P^2$, equipped with its canonical Riemannian metric of constant sectional curvature 1, is a closed 2-dimensional Alexandrov space with curv ≥ 1. Thus, its spherical suspension $\mathrm{Susp}(\mathbb{R}P^2)$ is a closed Alexandrov 3-space with curv ≥ 1 with exactly two topologically singular points, namely, the poles of the suspension. In this case, $X_o \approx \mathbb{R}P^2 \times [0,1]$ and we obtain $\mathrm{Susp}(\mathbb{R}P^2)$ after capping off each boundary component of X_o with a cone over $\mathbb{R}P^2$. We may also obtain the spherical suspension $\mathrm{Susp}(\mathbb{R}P^2)$ as a quotient of the unit round 3-sphere $\mathbb{S}^3(1)$ as follows. Recall first that $\mathbb{S}^3(1)$ is isometric to the spherical suspension of the unit round 2-sphere $\mathbb{S}^2(1)$. Consider now the involution $\iota\colon \mathbb{S}^3(1) \to \mathbb{S}^3(1)$ corresponding to the suspension of the antipodal map on $\mathbb{S}^2(1)$. This involution is an orientation-reversing isometry of $\mathbb{S}^3(1)$ and its metric quotient $\mathbb{S}^3(1)/\iota$ is isometric to the spherical suspension of $\mathbb{R}P^2$. Note that the involution $\iota\colon \mathbb{S}^3(1) \to \mathbb{S}^3(1)$ has exactly two isolated fixed points; these fixed points project down to the poles of $\mathrm{Susp}(\mathbb{R}P^2)$, giving rise to the two topologically singular points of this space.

The preceding example illustrates a general situation. Given a topologically singular Alexandrov 3-space X, there is a closed, orientable 3-manifold Y and an orientation reversing involution $\iota\colon Y \to Y$ with only isolated fixed points such that $X \approx Y/\iota$. It is important to note that $\iota\colon Y \to Y$ is conjugate to a smooth involution on Y. Hence X is homeomorphic to a smooth non-orientable 3-orbifold. The preceding properties imply that X is the base of a two-fold branched cover $\mathsf{pr}\colon Y \to X$ whose total space Y is a closed, orientable 3-manifold and whose branching set is the set of points with space of directions homeomorphic to $\mathbb{R}P^2$. In sum, up to homeomorphism, any closed Alexandrov 3-space is either a 3-manifold or a quotient of a closed orientable 3-manifold by and orientation reversing smooth involution with only isolated fixed points. Note that the latter spaces are homeomorphic to the *singular* 3-*manifolds* (without boundary) introduced by Quinn in [85] (see [54, 55] as well as [64, Open Problem 6], which asks to develop a theory for such spaces).

The construction of the orientable double branched cover $\mathsf{pr}\colon Y \to X$ relies only on the topology of X. The following theorem brings the geometry of X into play, allowing us to lift the metric of X to Y and turning the double branched cover $\mathsf{pr}\colon Y \to X$ into a metric object compatible with the geometry of X, as illustrated in Example 3.1.

Theorem 3.2 ([29])**.** *Let $X \in \mathrm{Alex}^3(k)$ be topologically singular and let Y be the orientable double branched cover of X. Then the following hold:*

1. *The metric in X can be lifted to Y, so that $Y \in \mathrm{Alex}^3(k)$.*
2. *The involution $\iota\colon Y \to Y$ is an isometry.*
3. *The space of directions $\Sigma_{p'}Y \approx \mathbb{S}^2$ at a fixed point p' of the involution $\iota\colon Y \to Y$ is the canonical Alexandrov double cover of $\Sigma_{p'}X \approx \mathbb{R}P^2$.*

A detailed proof of the preceding theorem would take us beyond the introductory treatment of Alexandrov spaces in this survey. Thus we will only discuss the main ideas in the proof; we refer the reader to [20, Section 2.1] for more details (cf. [46, Sections 2 and 5] and [50]). The initial point in the proof of Theorem 3.2 is the observation that the set $X_o \subset X$ of topologically regular points of X is convex in X. Letting $d_{X_0} = d_X|_{X_0}$ be the restriction of the metric on X to X_o, the convexity of X_o implies that the space (X_o, d_{X_o}) is a non-complete length space that is also a k-*domain*, i.e., the comparison property T_k holds for any geodesic triangle in (X_o, d_{X_o}). Since X_o is a non-orientable topological 3-manifold, it has an orientable double cover Y_o and we may lift the metric d_{X_o} to a metric d_{Y_o} on Y_o. By construction, the metric space (Y_o, d_{Y_o}) is a length space locally isometric to (X_o, d_o) (see [12, Chapters 2.2 and 3.4]). Thus (Y_o, d_{Y_o}) has curvature locally bounded below by k and its metric completion is homeomorphic to the two-fold branched cover Y of X. One then shows, using work of Li [60], that (Y, d_Y) also has curvature bounded below by k. This then implies that the involution $\iota\colon Y \to Y$ is an isometry. Along the way one constructs the space of directions for Y at a fixed point of the involution $\iota\colon Y \to Y$, showing part (3) of the Theorem. We refer the reader to [38] for further explicit examples, besides Example 3.1, of spaces arising from orientation-reversing involutions on closed orientable 3-manifolds.

4. Spaces with positive or non-negative curvature

We now turn our attention to the topological classification of Alexandrov 3-spaces with positive or non-negative curvature. In the Riemannian category this classification follows from Hamilton's classification of closed 3-manifolds with positive or non-negative Ricci curvature [48, 49].

Recall that positively curved Alexandrov spaces arise as spaces of directions and determine the local topology of Alexandrov spaces via the conical neighborhood theorem (see section 2.4). Thus the classification of positively curved spaces is of fundamental importance. This is, however, a challenging problem which has only been solved in dimensions 2 and 3, even in the Riemannian case. We refer the reader to [43, 101] for more information on positively curved Riemannian manifolds.

Recall that, by the Bonnet–Myers theorem, closed Alexandrov spaces with positive curvature have finite fundamental group. Thus, as we have previously seen, a closed 2-dimensional

Alexandrov space of positive curvature is homeomorphic to the 2-sphere or to the real projective plane.

The topological type of Alexandrov 3-spaces with positive curvature is given by the following theorem (see [29, 50]). Our presentation follows [29]. Recall that a *spherical 3-manifold* is a 3-manifold homeomorphic to $\mathbb{S}^3/\Gamma$, where Γ is a finite group acting freely and orthogonally on the 3-sphere. Note that every spherical 3-manifold is homeomorphic to a three-dimensional *spherical space form*, i.e., a closed Riemannian 3-manifold with constant positive sectional curvature. Conversely, every three-dimensional spherical space form is a spherical 3-manifold.

Theorem 4.1 (Alexandrov 3-spaces of positive curvature)**.** *A closed Alexandrov 3-space of positive curvature is homeomorphic to a spherical 3-manifold or to* $\mathrm{Susp}(\mathbb{R}P^2)$.

Proof. Let X be a closed Alexandrov 3-space with positive curvature. We may assume, after re-scaling the metric if necessary, that $\operatorname{curv} X \geq 1$. Suppose first that X is a manifold. Then, by the Bonnet–Myers theorem, X has finite fundamental group. It follows then from Perelman's proof of the Poincaré Conjecture and Thurston's Elliptization Conjecture that X must be homeomorphic to a spherical manifold, including the 3-sphere.

Suppose now that X is not a topological manifold and let X' be the set of points in X whose space of directions is homeomorphic to $\mathbb{R}P^2$. By hypothesis, X' is nonempty and, by the conical neighborhood theorem, each point in X' has a neighborhood homeomorphic to the Euclidean cone $C_0(\mathbb{R}P^2)$. Since X is compact, the set X' is finite.

Let $p_1, \dots, p_k$ be the points in X'. After removing a neighborhood of each p_i homeomorphic to $C_0(\mathbb{R}P^2)$ we obtain a topological 3-manifold X_o whose boundary consists of k copies of $\mathbb{R}P^2$.

Let $\mathsf{pr}\colon Y \to X$ be the two-fold branched cover over X with branching set X'. Let $q_i = \mathsf{pr}^{-1}(p_i)$, $i = 1, \dots, k$, and let $Y' = \{ q_1, \dots, q_k \}$. By theorem 3.2, Y is an Alexandrov space with $\operatorname{curv} \geq 1$. Hence, by theorem 2.5, Y has finite fundamental group. On the other hand, $\pi_1(Y) \simeq \pi_1(Y \setminus Y')$, since Y' is a finite set of points in Y. Since $\mathsf{pr}\colon Y \setminus Y' \to X \setminus X'$ is a regular two-fold cover, $\mathsf{pr}_*(\pi_1(Y \setminus Y'))$ is a subgroup of index 2 in $\pi_1(X \setminus X')$. Hence, $\pi_1(X \setminus X')$ is finite. It follows from Epstein's theorem (see [51, Chapter 9]) and Perelman's proof of the Poincaré Conjecture, that $X \setminus X'$ is homeomorphic to $\mathbb{R}P^2 \times [0,1]$. Thus, $k = 2$ and we conclude that X is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)$, as desired. Observe that Y is homeomorphic to $\mathbb{S}^3$ and, by work of Hirsch, Smale [52] and Livesay [61], the action of $\mathbb{Z}_2$ corresponding to the two-fold branched cover is equivalent to a linear action given by the suspension of the antipodal map on $\mathbb{S}^2$ (cf. Example 3.1). □

Corollary 4.2. *A closed, simply-connected three-dimensional Alexandrov space of positive curvature is homeomorphic to* $\mathbb{S}^3$ *or to* $\mathrm{Susp}(\mathbb{R}P^2)$.

Corollary 4.3. *The space of directions of a 4-dimensional Alexandrov space without boundary is homeomorphic to* $\mathrm{Susp}(\mathbb{R}P^2)$ *or to a spherical 3-manifold.*

Corollary 4.4. *A closed 4-dimensional Alexandrov space of curvature bounded below by 1 and diameter greater than* $\pi/2$ *is homeomorphic to the suspension of a spherical 3-manifold or to* $\mathrm{Susp}^2(\mathbb{R}P^2)$*, the double suspension of* $\mathbb{R}P^2$.

Corollary 4.2 follows from Perelman's proof of the Poincaré Conjecture. Corollary 4.3 follows from the fact that the space of directions at any point of an n-dimensional Alexandrov space is isometric to a compact $(n-1)$-dimensional Alexandrov space with curvature bounded below by 1. Note that corollary 4.3 implies that 4-dimensional Alexandrov spaces without boundary are, locally, orbifolds without boundary. Finally, corollary 4.4 follows from the fact that an n-dimensional Alexandrov space of curvature bounded below by 1 and diameter greater than $\pi/2$ is homeomorphic to the suspension of a compact $(n-1)$-dimensional Alexandrov space of curvature bounded below by 1.

Corollary 4.5. *Let X^n be an n-dimensional Alexandrov manifold. If $n \leq 4$, then X^n is topologically regular.*

Proof. If $n \leq 3$, the conclusion follows from the fact, recalled at the end of Section 2, that every 1- or 2-dimensional Alexandrov space must be homeomorphic to a topological manifold. Suppose now that $n = 4$ and let X^4 be an Alexandrov 4-manifold. By the conical neighborhood theorem, any sufficiently small neighborhood U of a point $p \in X^4$ is homeomorphic to the cone over the space of directions Σ_p at p. Since a cone over a non-simply-connected 3-manifold cannot be homeomorphic to the 4-ball D^4, the only case we need to consider is when $\Sigma_p X$ is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)$. In this case, a simple calculation using the long exact sequence in homology of the pair $(U, U-p)$ implies that some homology group $H_k(U, U-p)$ is not isomorphic to $H_k(D^4, \mathbb{S}^3)$. Thus X cannot be a topological manifold. ∎

Corollary 4.5 is optimal, since, by Example 2.11, Alexandrov n-manifolds, $n \geq 5$, are not necessarily topologically regular.

The ideas in the proof of Theorem 4.1 can also be used to provide a complete description of closed Alexandrov 3-spaces of non-negative curvature. We denote the non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ by $\mathbb{S}^2\tilde{\times}\mathbb{S}^1$, and the suspension of $\mathbb{R}P^2$ by $\mathrm{Susp}(\mathbb{R}P^2)$. Given two Alexandrov 3-spaces X, Y, we denote their connected sum by $X\#Y$, i.e., $X\#Y$ is the space obtained by removing an open 3-ball from X, an open 3-ball from Y, and then identifying the boundaries of the resulting topological spaces.

Theorem 4.6 (Alexandrov 3-spaces of non-negative curvature [29]; cf. [20])**.** *Let X^3 be a closed, non-negatively curved Alexandrov 3-space.*

(1) *If X^3 is a topological manifold, then one of the following holds:*
   - *X^3 is homeomorphic to a spherical space form,*
   - *X^3 is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$, $\mathbb{R}P^3\#\mathbb{R}P^3$ or $\mathbb{S}^2\tilde{\times}\mathbb{S}^1$; or*
   - *X^3 is isometric to a closed, flat three-dimensional space form.*

(2) *If X^3 has a point with space of directions homeomorphic to $\mathbb{R}P^2$, then either:*
   - *X^3 is homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)$, $\mathrm{Susp}(\mathbb{R}P^2)\#\,\mathrm{Susp}(\mathbb{R}P^2)$, $\mathbb{R}P^3\#\,\mathrm{Susp}(\mathbb{R}P^2)$ or*
   - *X^3 is isometric to a quotient of a closed, orientable, flat three-dimensional manifold by an orientation reversing isometric involution with only isolated fixed points.*

Let us briefly discuss the proof of this theorem. Let X be a closed Alexandrov 3-space with $\mathrm{curv} \geq 0$. As in the proof of theorem 4.1, we consider two possibilities, depending on whether or not X is a topological manifold.

Suppose first that X is a topological manifold. We have two possibilities: either the fundamental group $\pi_1(X)$ is finite or not. If $\pi_1(X)$ is finite, then, as in the proof of theorem 4.1, X is homeomorphic to a spherical space form, by Perelman's resolution of Thurston's elliptization conjecture.

Suppose now that $\pi_1(X)$ is infinite. Then, the splitting theorem for non-negatively curved Alexandrov spaces [12, Theorem 10.5.1] implies that $\tilde{X}$, the universal cover of X, is isometric to a product $\mathbb{R} \times \tilde{Y}$, where $\tilde{Y}$ is a simply-connected Alexandrov 2-space with curv ≥ 0.

Kwun and Tollefson [59], and Luft and Sjerve [63], classified the involutions with only isolated fixed points on closed, orientable, flat three-dimensional space forms and their orbit spaces have been classified. These orbit spaces are the spaces in the second item of part (2) of theorem 4.6 above.

4.1. **Spaces with positive or non-negative Ricci curvature.** One can generalize theorems 4.1 and 4.6 to closed Alexandrov 3-spaces with an arbitrary lower curvature bound and with positive or non-negative Ricci curvature in the sense of Lott–Sturm–Villani (see [62, 95, 96]). In this case one obtains the same list of spaces as in the case of positive or non-negative curvature in the triangle comparison sense (see [20]).

Theorem 4.7 (Alexandrov 3-spaces of positive Ricci curvature [20])**.** *Let X^3 be a closed $\mathsf{CD}^*(2,3)$-Alexandrov space.*

1. *If X^3 is a topological manifold, then it is homeomorphic to a spherical space form.*
2. *If X^3 is not a topological manifold, then it is homeomorphic to* $\mathrm{Susp}(\mathbb{R}P^2)$.

Theorem 4.8 (Alexandrov 3-spaces of non-negative Ricci curvature [20])**.** *Let X^3 be a closed $\mathsf{CD}^*(0,3)$-Alexandrov space.*

1. *If X^3 is a topological manifold, then one of the following holds:*
   - *X^3 is homeomorphic to a spherical space form,*
   - *X^3 is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$, $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$; or*
   - *X^3 is isometric to a closed, flat three-dimensional space form.*
2. *If X^3 is not a topological manifold, then*
   - *X^3 is homeomorphic to* $\mathrm{Susp}(\mathbb{R}P^2)$, $\mathrm{Susp}(\mathbb{R}P^2) \# \mathrm{Susp}(\mathbb{R}P^2)$, $\mathbb{R}P^3 \# \mathrm{Susp}(\mathbb{R}P^2)$ *or*
   - *X^3 is isometric to a quotient of a closed, orientable, flat three- dimensional manifold by an orientation reversing isometric involution with only isolated fixed points.*

Observe that, since X is closed, then its Hausdorff measure $\mathcal{H}^3(X)$ is finite. Hence, the equivalence of the CD and CD^* conditions for (essentially non-branching) spaces with finite measures due to Cavalletti–Milman (see [17, Corollary 13.7]) implies that Theorems 4.7 and 4.8 are still valid for $\mathsf{CD}(2,3)$- and $\mathsf{CD}(0,3)$-Alexandrov spaces, respectively.

5. Topological results

5.1. **Geometrization.** In view of Perelman's resolution of Thurston's geometrization conjecture, geometric 3-manifolds can be considered the building blocks of arbitrary 3-dimensional closed manifolds. It is then natural to ask about the corresponding notion for Alexandrov spaces. Recall that the eight Thurston geometries are $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H} \times \mathbb{R}$, Nil, Sol and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ (see [88]). A closed Alexandrov 3-space X^3 is *geometric* if it can be written as a quotient of one of the eight Thurston geometries by some cocompact lattice. We say that X^3 admits a *geometric decomposition* if there exists a collection of spheres, projective planes, tori and Klein bottles that decompose X^3 into geometric pieces. The following result is proved using the existence of the double branched cover, outlined in the preceding section, in combination with Dinkelbach and Leeb's work on equivariant Ricci flow [21].

Theorem 5.1 (Geometrization of Alexandrov 3-spaces [29])**.** *A closed Alexandrov* 3*-space admits a geometric decomposition into geometric Alexandrov* 3*-spaces.*

The proof of this theorem relies in a key way on the Ricci flow for smooth Riemannian metrics on 3-manifolds. It would be interesting to determine whether it is possible to define Ricci flow for general Alexandrov 3-spaces with no regularity assumptions on the metric.

5.2. **Simply-connected spaces.** In the sense of topology, the simplest spaces that one can consider are *contractible* spaces in which, a fortiori, all homotopy groups vanish. However, it is easy to show that, as in the manifold case, no closed n-dimensional Alexandrov space X, $n \geq 2$, is contractible. Indeed, on the one hand, if X is orientable, then the top homology group $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ (see, for example, [65, Theorem 1.8]). Then, the Hurewicz theorem readily implies that $\pi_n(X) \neq 0$. If, on the other hand, we assume that X is non-orientable, then [65, Corollary 5.7] gives that $H_{n-1}(X;\mathbb{Z}) \neq 0$, and another application of the Hurewicz theorem grants again that X cannot be contractible. Thus, if one aims at a deeper understanding of the topology of Alexandrov 3-spaces, a natural step is to consider other simple (topologically speaking) classes of spaces, such as simply-connected spaces.

By Perelman's proof of the Poincaré conjecture, a closed, simply-connected 3-manifold must be homeomorphic to the 3-sphere. By Poincaré duality and the Hurewicz theorem, a closed 3-manifold is simply-connected if and only if it is a homotopy sphere. This is no longer the case for Alexandrov 3-spaces, as one sees by considering $\mathrm{Susp}(\mathbb{R}P^2)$. One can still show, however, that a closed Alexandrov 3-space that is also a homotopy sphere is homeomorphic to $\mathbb{S}^3$, thus obtaining an analogue for Alexandrov 3-spaces of the Generalized Poincaré Conjecture (see [29, Proposition 1.4]). On the other hand, there exist closed, geometric, simply-connected Alexandrov 3-spaces that are not homeomorphic to the 3-sphere if and only if the corresponding Thurston geometry is not one of Nil, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ or Sol. To rule out the Nil, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ and Sol geometries, one proves that there are no orientation-reversing involutions with only isolated fixed points on closed geometric 3-manifolds with one of these geometries.

A complete topological description of closed, simply-connected three-dimensional Alexandrov spaces seems currently beyond reach. A naive conjecture would be that any such space is the connected sum of $\mathbb{S}^3$ and suspensions over projective planes. This is true under the presence of an action of a compact Lie group of positive dimension (see the results in the next section). In general, however, there exist simply-connected Alexandrov 3-spaces that are not homeomorphic to these connected sums (see [29, Remark 4.1]). A more feasible goal could be to classify the topological type of non-manifold simply-connected Alexandrov spaces with a given number of topological singularities. In this spirit, we recall the following question:

Question 5.2 (Topological finiteness [34])**.** Does the class of closed, simply-connected Alexandrov 3-spaces with curv ≥ -1 and diam $\leq D$, for fixed $D > 0$, contain finitely many homeomorphism types?

5.3. **Aspherical spaces and the Borel conjecture.** We have considered simply-connected spaces in the previous section, providing a view of the phenomena that can occur under this assumption in three-dimensional Alexandrov geometry. In a similar way, it is natural to try to understand the topology of other topologically simple classes of spaces such as *aspherical* spaces. Recall that a topological space is *aspherical* if all of its higher homotopy groups vanish. One could, in a certain sense, think of the class of aspherical spaces as complementary to the class of simply-connected spaces.

The theorems we present in this subsection provide a basic picture of the topology of closed and aspherical Alexandrov 3-spaces. One of the main fundamental results in 3-manifold topology is the *Borel conjecture*. Recall that the *Borel conjecture* asserts that if two closed, aspherical n-manifolds are homotopy equivalent, then they must be homeomorphic. The proof of this conjecture for three-dimensional manifolds is a consequence of Perelman's proof of Thurston's geometrization conjecture (see [84, Section 2.6]). As we have pointed out in a previous section, a geometrization theorem is available for closed Alexandrov 3-spaces. However, it is unclear whether this implies the Borel conjecture in this more general setting.

In [71, Section 6], Núñez-Zimbrón showed that if two closed, aspherical Alexandrov 3-spaces on which the circle acts effectively and isometrically are homotopy equivalent, then they are homeomorphic. The proof of this result is based on the decomposition that any such space X admits as a connected sum of a closed 3-manifold with a finite number of copies of the suspension of $\mathbb{R}P^2$ (see Theorem 6.4). The argument is based on the observation that no connected sum Y of copies of $\mathrm{Susp}(\mathbb{R}P^2)$ can be aspherical, so that a connected sum of the form $M\#Y$, where M is a closed 3-manifold, is aspherical only if $M\#Y \approx M$. Therefore, if X is a closed aspherical Alexandrov 3-space on which the circle acts effectively and isometrically, then the Borel conjecture holds since X must be homeomorphic to a closed 3-manifold. It is immediate to see that the same argument implies, via the connected sum decomposition of Theorem 6.6, that if two closed, aspherical Alexandrov 3-spaces which admit isometric local circle actions (see Section 6 below for the definition) are homotopy equivalent, then they are homeomorphic.

A different but related result concerning the validity of the Borel conjecture for Alexandrov 3-spaces was obtained by Bárcenas and the second named author in [6]. In their work they reinforce the condition of asphericity further with a constraint on the Hausdorff measure of the spaces with respect to their diameters, as well as with a topological condition of *irreducibility*, originally defined by Galaz-García, Guijarro and the second named author in [30]. A closed Alexandrov 3-space X is *irreducible* if every embedded 2-sphere in X bounds a 3-ball and, in the case that the set of topologically singular points of X is non-empty, it is further required that every 2-sided $\mathbb{R}P^2$ bounds a cone over a real projective plane $\mathbb{R}P^2$. This definition is fashioned after the definition of irreducibility for 3-manifold, which plays a central role in 3-manifold topology (see, for example, [51]). With this definition in hand we recall the main result of [6].

Theorem 5.3 ([6])**.** *For any $D > 0$, there exists $\varepsilon = \varepsilon(D) > 0$ such that, if X is a closed, irreducible and aspherical Alexandrov 3-space satisfying*

- $\mathrm{curv}(X) \geq -1$,

- $\operatorname{diam} X \leq D$ *and*
- $\mathcal{H}^3(X) \leq \varepsilon$,

then X is homeomorphic to a 3-manifold.

The preceding theorem immediately implies the following corollary which asserts the validity of the Borel conjecture for a certain class of Alexandrov 3-spaces.

Corollary 5.4 ([6])**.** *For any $D > 0$, there exists $\varepsilon = \varepsilon(D) > 0$ such that, if X_1 and X_2 are closed, irreducible and aspherical Alexandrov 3-spaces satisfying that*

- $\operatorname{curv}(X_i) \geq -1$,
- $\operatorname{diam} X_i \leq D$, *and*
- $\mathcal{H}^3(X_i) \leq \varepsilon$,

for $i = 1, 2$, then the Borel conjecture holds for X_1 and X_2, that is, if X_1 is homotopy equivalent to X_2 then X_1 is homeomorphic to X_2.

Corollary 5.4 guarantees that given a homotopy equivalence $f\colon X_1 \to X_2$, there exists *some* homeomorphism $\tilde{f}\colon X_1 \to X_2$. In the manifold case, it is true that the initial map f is itself homotopic to a homeomorphism. As the proof of the Borel conjecture in the special case considered in Corollary 5.4 reduces to the Borel conjecture for 3-manifolds, this stronger statement holds true as well.

We now give a rough outline of the proof of Theorem 5.3. By contradiction, let us assume that the thesis does not hold. Then, there exists a sequence of closed, irreducible, and aspherical Alexandrov 3-spaces $\{X_i\}$ with $\operatorname{curv}(X_i) \geq -1$, with uniformly bounded diameters and with their 3-dimensional Hausdorff measures converging to 0. Then Gromov's compactness theorem (see [12, Theorem 10.7.2]) yields that, possibly after passing to a subsequence, there exists a compact Alexandrov space Y (possibly with boundary) such that X_i converges to Y in the Gromov–Hausdorff sense. The Hausdorff dimension of Y must be strictly smaller than 3 as, by the weak convergence of the Hausdorff measures under Gromov–Hausdorff convergence (see [13, Theorem 10.8]), $\mathcal{H}^3(Y) = 0$. At this point we note that, in the terminology of Section 7, the sequence $\{X_i\}$ is a *collapsing* sequence (to Y). Mitsuishi and Yamaguchi have classified the topologies of such X_i for large enough i. To proceed with the proof of Theorem 5.3, we use this classification and a case by case analysis of asphericity to obtain a contradiction.

In general it is not known whether any two closed, aspherical Alexandrov 3-spaces that are homotopy equivalent must also be homeomorphic. However, the evidence gathered by the results in this section seems to point towards the following conjecture, which if true, would answer this question in the affirmative.

Conjecture 5.1. Every closed and aspherical Alexandrov 3-space is homeomorphic to a 3-manifold.

We conclude this section by pointing out that the geometry and topology of non-compact, topologically singular, Alexandrov 3-spaces remains to be explored.

6. Alexandrov 3-spaces with compact Lie group actions

Spaces with large groups of isomorphisms are of interest in different areas of mathematics. In the context of differential geometry, the study of smooth manifolds with smooth actions of compact Lie groups is a subject with a long history (see, for example, [56, 58]) that has brought about further developments in Riemannian geometry. It is therefore natural to consider Alexandrov spaces with isometric group actions and to generalize the theory of compact transformation groups on manifolds [9] to the case of Alexandrov spaces. As in the smooth case, a reasonable starting point in the study of closed Alexandrov spaces with isometric compact Lie group actions is to consider those that support "large" actions. Much work has been done in the Riemannian setting, where this point of view has led to topological and equivariant classification results for smooth manifolds with Riemannian metrics of positive or non-negative sectional curvature, in the context of the *Grove program* (see [41, 43, 98]). One can therefore strive for corresponding results in the context of Alexandrov geometry. There has already been some work in this direction (see, for example, [28, 33, 32, 50, 71]) and we will focus our attention here on results on Alexandrov 3-spaces. For a more general discussion on group actions on Alexandrov spaces the reader may consult [89]. As in the preceding section, we will concentrate on the case where the Alexandrov 3-space is not a manifold.

6.1. **Setup.** Let X be an Alexandrov space. A bijection $f\colon X \to X$ is an *isometry* if $d(f(p), f(q)) = d(p, q)$ for any pair of points $p, q \in X$. We denote the group of isometries of X by $\mathrm{Isom}(X)$. Fukaya and Yamaguchi showed that $\mathrm{Isom}(X)$ is a Lie group [26]. The corresponding result for Riemannian manifolds was proved by Myers and Steenrod [69]. By a theorem of van Dantzig and van der Waerden [18], if X is compact, then $\mathrm{Isom}(X)$ is also compact. Myers and Steenrod also obtained a sharp upper bound on the dimension of the isometry group of a Riemannian manifold, namely, if $\dim(M) = n$, then $\dim(\mathrm{Isom}(M)) \leq n(n+1)/2$, and obtained a rigidity statement in the equality case (see [69]; cf. [58]), showing that such Riemannian manifolds must be isometric to a sphere, a real projective space, Euclidean space or hyperbolic space. In the context of Alexandrov spaces, Galaz-García and Guijarro obtained the same upper bound on $\dim(\mathrm{Isom}(X))$ and generalized the rigidity result, obtaining the same list of spaces as in the Riemannian case (see [28]).

We will consider actions $G \times X \to X$ of a compact Lie group G on X such that the restriction of the action to sets of the form $\{g\} \times X$ are isometries of X. In this case, one says that the action is *isometric* or that G acts *isometrically* (or *by isometries*) on X.

We will denote the orbit of a point $x \in X$ by $G(x)$, that is,

$$G(x) = \{gx \mid g \in G\}.$$

It is easy to show that $G(x)$ is homeomorphic to G/G_x, where

$$G_x = \{g \in G \mid gx = x\}$$

is the *isotropy subgroup* of x in G. The closed subgroup of G given by $\cap_{x\in X} G_x$ is called the *ineffective kernel* of the action. If the ineffective kernel is trivial, we will say that the action is *effective*.

The homeomorphism $G(x) \approx G/G_x$ for each x shows that there is a correspondence between orbits and isotropy groups in the following sense. Given an isotropy subgroup $H \leq G$, one says that $G(x)$ *is of type* (H) if G_x is conjugate to H. The set of orbit types naturally carries a partial ordering defined as follows. We say that $(H) \leq (K)$ if K is conjugate to a subgroup of H. One of the main tools in the theory of compact transformation groups is the principal orbit theorem, obtained for Alexandrov spaces by Galaz-García and Guijarro in [28], (see [41] for the Riemannian case).

Theorem 6.1 (Principal orbit theorem [28])**.** *Let G be a compact Lie group acting isometrically on an n-dimensional Alexandrov space X. Then there is a unique maximal orbit type and the orbits with maximal orbit type, the* principal orbits *of the action, form an open and dense subset of X.*

Given a subset $A \subset X$ we denote its image under the orbit projection map $\pi\colon X \to X/G$ by A^*. In particular, $X^* = X/G$. It was proved in [13] (cf. [12, Proposition 10.2.4]) that the orbit space X^* equipped with the distance between orbits is an Alexandrov space with the same lower curvature bound as X. This is a consequence of the fact that the projection $\pi\colon X \to X^*$ is a *submetry*, that is, π sends balls of radius $r > 0$ in X to balls of radius $r > 0$ in X^*.

Let $x \in X$. Given $A \subset \Sigma_x X$, we define the *set of normal directions to A* as

$$A^{\perp} = \{v \in \Sigma_x X \mid \angle(v, w) = \operatorname{diam}(\Sigma_x X)/2 \text{ for all } w \in A\}.$$

Let S_x denote the tangent unit space to the orbit G/G_x. If $\dim(G/G_x) > 0$ then the set $S_x^{\perp}$ is a compact, totally geodesic Alexandrov subspace of $\Sigma_x X$ with curvature bounded below by 1. Moreover, $\Sigma_x X$ is isometric to the join $S_x * S_x^{\perp}$ with the standard join metric and either $S_x^{\perp}$ is connected or it contains exactly two points at distance π (see [33]).

The slice theorem is a fundamental result in the theory of compact transformation groups and provides a canonical decomposition of a small invariant tubular neighborhood of each orbit as well as more information on how the orbit types are distributed on the space. Several (a posteriori equivalent) definitions of the notion of slice are available depending on the generality. Let us recall the definition of a *slice* from Bredon [9].

A *slice* at a point $x \in X$ is a subset $S \subset X$ containing x which is closed in $G(S)$ and that satisfies the following properties:

- $G(S)$ is an open neighborhood of $G(x)$,
- $G_x(S) = S$, and
- if $(gS) \cap S \neq \varnothing$, then $g \in G_x$.

The existence of a slice at each point can be shown in high generality. Indeed, Montgomery and Yang [67] showed that if a compact Lie group G acts on a completely regular topological space, then there exists a slice at each point. We now state Harvey and Searle's slice theorem for Alexandrov spaces.

Theorem 6.2 (Slice theorem [50])**.** *Let G be a compact Lie group acting isometrically on an Alexandrov space X. Then for all $x \in X$, there exists $r_0 > 0$ such that for all $r < r_0$ there is an equivariant homeomorphism*

$$\Phi : G \times_{G_x} K(S_x^{\perp}) \to B_r(G(x)).$$

It is worth noting that, as the slice theorem shows, in the context of Alexandrov spaces the slice at each point can be taken as the cone over $S_x^{\perp}$, where S_x is the unit tangent space to the orbit G/G_x. An important and immediate consequence of the slice theorem is that $\Sigma_{x^*} X^*$, the space of directions at each $x^* \in X^*$, is isometric to $S_x^{\perp}/G_x$.

One can measure the size of the isometry group $\operatorname{Isom}(X)$ of a closed Alexandrov space X by means of different invariants. Three natural ones are the *symmetry degree*, given by symdeg$(X) = \dim(\mathrm{Isom}(X))$, the *symmetry rank*, given by symrk$(X) = \mathrm{rank}(\mathrm{Isom}(X))$, and the *cohomogeneity* of the action, defined as the dimension of the orbit space $X/\mathrm{Isom}(X)$. Here we will discuss isometric actions of compact connected Lie groups on closed Alexandrov 3-spaces from the point of view of the cohomogeneity of the action.

Let X be a closed Alexandrov 3-space with an isometric action of a compact connected Lie group G. Thus, the cohomogeneity of the action is 0, 1, 2 or 3. We need not consider the case where the cohomogeneity is three, since this implies that G is the identity.

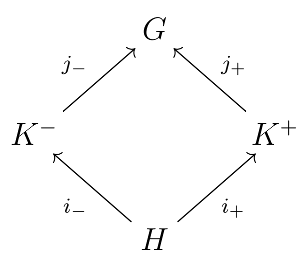
6.2. **Homogeneous spaces.** When the cohomogeneity is 0, X must be a homogeneous space and it follows from work of Berestovskiĭ that X is isometric to a Riemannian manifold (see [7]).

6.3. **Cohomogeneity one spaces.** Topological manifolds with cohomogeneity one actions were first studied by Mostert [68] (see also [35]) and classified in dimension three by Mostert [68] and Neumann [70] (see also [35, 76, 53] for the classification in dimensions at most 7 in the topological and smooth categories). The structure of general closed cohomogeneity one Alexandrov spaces is given by the following result, which generalizes the structure result for closed cohomogeneity-one smooth manifolds:

Theorem 6.3 (Cohomogeneity one Alexandrov spaces [33])**.** *Let X be a closed Alexandrov space with an effective cohomogeneity one isometric action of a compact connected Lie group G with principal isotropy H. Then the following hold:*

- *The orbit space X/G is homeomorphic to a circle or to a closed interval.*

- *If X/G is a circle, then M is equivariantly homeomorphic to a fiber bundle over $\mathbb{S}^1$ with fiber G/H and structure group $N(H)/H$. In particular, X is a manifold.*
- *If $X/G \approx [-1, +1]$, then there is a group diagram (G, H, K^-, K^+) with*



  where $K^\pm$ are the isotropy groups at ± 1 and $K^\pm/H$ are isometric to homogeneous spaces with sec > 0.
- *The space X is the union of two fiber bundles with fiber $C(K^\pm/H)$ and base the singular orbits $G/K^\pm$.*
- *Conversely, any diagram (G, H, K^-, K^+), such that $K^\pm/H$ is a homogeneous space of positive curvature determines an Alexandrov G-space of cohomogeneity* 1.

In Theorem 6.3, if X is a smooth manifold and the action is smooth, then the fibers of the double cone bundle decomposition are disks, i.e., cones over spheres. If X is only assumed to be a topological manifold, then one must consider as fibers, in addition to disks, cones over the Poincaré homology sphere (see [35]).

Let X be a closed cohomogeneity one Alexandrov 3-space. If X is a 3-manifold, then it follows from the work of Mostert and Neumann [68, 70] that X must be one of T^3, $\mathbb{S}^3$,

$\mathbb{L}_{p,q}$, $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$, $\mathbb{K}\mathrm{l} \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$ or $\mathbb{A}$. Here, $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ is the non-trivial $\mathbb{S}^2$ bundle over $\mathbb{S}^1$, $\mathbb{L}_{p,q}$ denotes a lens space, and $\mathbb{K}\mathrm{l}$ is the Klein bottle; the space $\mathbb{A}$ is the manifold $\mathbb{M}\mathrm{b} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times \mathbb{M}\mathrm{b}$, where $\mathbb{M}\mathrm{b}$ is the compact Möbius band and the halves $\mathbb{M}\mathrm{b} \times \mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{M}\mathrm{b}$ intersect canonically in $\mathbb{S}^1 \times \mathbb{S}^1$. If X is not a manifold, then it was proved in [33] that X must be equivariantly homeomorphic to $\mathrm{Susp}(\mathbb{R}P^2)$ with the suspension of the transitive action of $\mathrm{SO}(3)$ on $\mathbb{R}P^2$.

6.4. **Cohomogeneity two spaces.** Let X be a closed Alexandrov 3-space with a cohomogeneity two isometric and effective action of a compact connected Lie group. Since the orbits are one-dimensional, the group acting must be the circle S^1. The topological and equivariant classification in the case where X is a manifold follows from the work of Orlik and Raymond [73, 86], who classified the effective actions of the circle on any closed, connected topological 3-manifold M (see [72]). The orbit space of such an action is a topological 2-manifold, possibly with boundary and each equivariant homeomorphism type is determined by a set of invariants

$$(b; (\varepsilon, g, f, t), \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}).$$

Here, b is the obstruction class (in the sense of obstruction theory, see for example [19, Chapter 7]) for the principal stratum of the action to be a trivial principal S^1-bundle. The symbol ε takes two possible values, corresponding to the orientability of the orbit space. The genus of the orbit space is denoted by g. The number of connected components of the fixed point set is denoted by f, while t is the number of $\mathbb{Z}_2$-isotropy connected components. The pairs $\{(\alpha_i, \beta_i)\}_{i=1}^n$ are the Seifert invariants (see [72] for the definition) associated to the exceptional orbits of the action, if any.

Núñez-Zimbrón carried out the topological and equivariant classification of topologically singular closed Alexandrov 3-spaces with an isometric circle action [71]. Recall that a closed Alexandrov 3-space X that is not a 3-manifold has finitely many topologically singular points, i.e., points whose space of directions is homeomorphic to the real projective plane $\mathbb{R}P^2$. To account for these points, one adds an unordered s-tuple $(r_1, r_2, \dots, r_s)$ of even positive integers to the the set of invariants in the manifold case. The integer s corresponds to the number of boundary components in the orbit space that contain orbits of topologically singular points. The integers r_i correspond to the number of topologically singular points in the i-th boundary component of the orbit space with orbits of topological singularities. If there are no topologically singular points one considers this s-tuple to be empty. The classification is then given by the following theorem. Recall that a homeomorphism $f\colon X \to Y$ between G-spaces X, Y is *weakly equivariant* if there exists an isomorphism $\varphi\colon G \to G$ such that $f(gx) = \varphi(g)f(x)$ for all $g \in G$ and all $x \in X$.

Theorem 6.4 (Spaces with circle actions [71])**.** *Let S^1 act effectively and isometrically on a closed Alexandrov 3-space X. Assume that X has $2r$ topologically singular points, $r \geq 0$. Then the following hold:*

(1) *The set of inequivalent (up to weakly equivariant homeomorphism) effective, isometric circle actions on X is in one-to-one correspondence with the set of unordered tuples*

$$(b; (\varepsilon, g, f, t); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \dots, r_s))$$

where the permissible values for b, ε, g, f, t and $\{(\alpha_i, \beta_i)\}_{i=1}^n$, are the same as in the manifold case and $(r_1, r_2, \dots, r_s)$ is an unordered s-tuple of even positive integers r_i such that $r_1 + \dots + r_s = 2r$.

(2) *X is weakly equivariantly homeomorphic to*

$$M\#\underbrace{\mathrm{Susp}(\mathbb{R}P^2)\#\cdots\#\mathrm{Susp}(\mathbb{R}P^2)}_{r}$$

where M is the closed 3-manifold given by the set of invariants

$$(b;(\varepsilon,g,f+s,t);\{(\alpha_i,\beta_i)\}_{i=1}^n)$$

in the manifold case.

We now outline the main points of the proof of the previous theorem. Observe first that there are different orbit types, which correspond to the possible isotropy groups of the action. These in turn, correspond to the closed subgroups of $\mathbb{S}^1$: the trivial subgroup $\{e\}$, the cyclic subgroups $\mathbb{Z}_k$, $k \geq 2$, and S^1 itself. In particular, since each orbit is homeomorphic to the quotient of S^1 by the corresponding isotropy group, orbits in X are either 0-dimensional or 1-dimensional. This observation and the finiteness of the set of topologically singular points of X imply that topologically singular points are fixed by the action.

We let F be the set of fixed points of the action and let $RF = F \setminus S_X$, the set of topologically regular fixed points. The points whose isotropy is not S^1 are topologically regular, therefore the notion of local orientation makes sense (see for example the remark on orientability in [80, p. 124]). We will say that an orbit with isotropy $\mathbb{Z}_k$ acting without reversing the local orientation is *exceptional*; we will denote the set of points on exceptional orbits by E. An orbit with isotropy $\mathbb{Z}_2$ that acts reversing the local orientation will be called *special exceptional* and the set of points on such orbits will be denoted by SE. The orbits with trivial isotropy will be called *principal*.

An analysis of the structure of X around each orbit via the slice theorem 6.2 yields the structure of the orbit space X^*: It is a compact 2-manifold possibly with boundary in which the interior points correspond to principal orbits except for a finite number (possibly zero) of points which are associated to exceptional orbits. For each of the boundary components one of the following possibilities occurs: The component consists entirely of RF-orbits, SE-orbits or the component can be decomposed as a union of closed non-trivial intervals with SE or RF isotropy in their interiors and with the endpoints corresponding to orbits of topologically singular points. Note that this implies in particular that X must have an even number of topologically singular points, recovering the result mentioned before Example 3.1 above.

At this point one must show that each possible orbit space corresponds exactly to a single closed Alexandrov 3-space with an isometric circle action up to equivariant homeomorphism. The main tool is a cross-sectioning theorem asserting the existence of a cross-section to the action in the absence of exceptional orbits, which we recall below. This theorem extends the corresponding result of Orlik and Raymond for circle actions on 3-manifolds. Note, however, that in the manifold case one requires that $F \neq \varnothing$ while for non-manifold Alexandrov 3-spaces this is automatically true as topologically singular points are fixed points.

Proposition 6.5. *If S^1 acts effectively and isometrically on a closed, Alexandrov 3-space X with $E = \varnothing$ and $F \neq \varnothing$, then there exists a cross-section to the action.*

This cross-section given by the preceding proposition can be used to build equivariant homeomorphisms between spaces with isomorphic orbit spaces if no exceptional orbits are present. If $E \neq \varnothing$ one can show that the action is completely determined by the restriction of the cross-section to a tubular neighborhood of E coupled with the information given by the Seifert invariants.

To complete the proof of item (1) in Theorem 6.4 we now must show that each admissible orbit space X^* is indeed the orbit space of a closed Alexandrov 3-space with some isometric circle action. One can easily construct a topological space X with a circle action whose orbit space is X^* by gluing together the "building blocks" obtained via the slice theorem. In other words, each small neighborhood of each orbit type in X^* can be "lifted" uniquely (up to equivariant homeomorphism). A more delicate point is to show that this space X indeed admits an Alexandrov metric. This is achieved by using the branched double cover construction for X, obtaining a topological 3-manifold $\tilde{X}$ which doubly covers X up to a finite number of isolated points. The S^1 action on $\tilde{X}$ is equivalent to a smooth action by the work of Orlik and Raymond on circle actions on 3-manifolds. On $\tilde{X}$, then, one can do an averaging procedure as in [5, Theorem 3.65] to obtain an invariant Riemannian metric which has bounded sectional curvature by compactness and which, in turn, can be projected down to X to obtain an orbifold Riemannian metric on X with sectional curvature from bounded below. We refer the reader to [32] for the details.

The connected sum decomposition of X in item (2) of Theorem 6.4 is obtained by considering orbit spaces which are homeomorphic to 2-disks in which $E = \varnothing$ and with at least two topologically singular points. Proposition 6.5 and the slice theorem 6.2 are then used to prove that these orbit spaces correspond to equivariant connected sums of suspensions of $\mathbb{R}P^2$ with a standard circle action.

6.5. **Spaces with local circle actions.** A wider class of Alexandrov 3-spaces with symmetry results from generalizing the notion of isometric circle action to that of an isometric *local* circle action. An isometric local circle action on a closed Alexandrov 3-space X is a decomposition of X into disjoint, simple, closed curves, which we call *fibers*, each having a tubular neighborhood which admits an effective, isometric circle action (with respect to the restricted metric of X) whose orbits are the curves of the decomposition. We do not exclude the possibility that some of the curves in the decomposition consist of single points. In this case the equivariant and topological classification was obtained by the authors in [32], generalizing the corresponding classifications for closed 3-manifolds obtained by Fintushel [25] and Orlik and Raymond [74].

Several new invariants must be added to those in Theorem 6.4 to account for the fact that tubular neighborhoods of each type of (one-dimensional) orbit may not be orientable, or equivalently, the boundary of such a tubular neighborhood may not be homeomorphic to a 2-torus but, rather, to a Klein bottle. Let us present the classification theorem, followed by the explanation of the invariants appearing in it.

Theorem 6.6 (Spaces with local circle actions [32])**.** *Let X be a closed Alexandrov 3-space with a local isometric S^1-action. If X has $2r \geq 0$ topologically singular points, then the following hold:*

(1) *Isometric local circle actions (up to equivariant equivalence) are in one-to-one correspondence with unordered tuples*

$$\{b; \varepsilon, g, (f, k_1), (t, k_2), (s, k_3); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \dots, r_{s-k_3}); (q_1, q_2, \dots, q_{k_3})\},$$

where the admissible values for b, ε, g, (f, k_1), (t, k_2) and (α_i, β_i) are the same as in the manifold case, and $(r_1, r_2, \dots, r_{s-k_3})$ and $(q_1, q_2, \dots, q_{k_3})$ are unordered

$(s-k_3)$- and k_3-tuples of even non-negative integers r_i, q_j, respectively, such that $r_1+\ldots+r_{s-k_3}+q_1+\ldots q_{k_3}=2r$.

(2) *There is an equivariant equivalence of X with*

$$M\#\underbrace{\mathrm{Susp}(\mathbb{R}P^2)\#\cdots\#\mathrm{Susp}(\mathbb{R}P^2)}_{r\ \text{summands}},$$

where M is the closed 3-manifold determined by the set of invariants

$$\{b;\varepsilon,g,(f+s,k_1+k_3),(t,k_2);\{(\alpha_i,\beta_i)\}_{i=1}^n\}$$

in the manifold case.

Let us now explain the invariants appearing in Theorem 6.6. As in the case of global circle actions, there are several types of fibers, each corresponding to the possible isotropy groups. The fiber types F, E and SE are defined as in the case of global circle actions in the preceding subsection. We denote principal fibers by R and orbits of topologically singular points by SF. Similarly as well, the fiber space X^* is a topological 2-manifold possibly with boundary. When present, the boundary is composed of the images of F-, SE- and topologically singular fibers while the interior of X^* consists of R-fibers and a finite number of E-fibers.

A closed Alexandrov 3-space with an isometric local circle action can be decomposed into the following pieces:

(a) *building blocks* which arise by considering small tubular neighborhoods of connected components of fibers of type F, SF, E and SE;

(b) an S^1-fiber bundle (composed only of R-fibers) with structure group O(2) over a compact 2-manifold with boundary, which corresponds to the complement in X of the union of the building blocks in part (a).

While a similar statement is true in the case of global circle actions, the main differences in the case of local circle actions are, on the one hand, the possible building blocks that appear when examining the neighborhoods of each fiber via the slice theorem and, on the other hand, that in the global case the structure group of the fiber bundle in item (b) can be reduced to SO(2).

A building block is called *simple* if its boundary is homeomorphic to a torus, and *twisted* if its boundary is homeomorphic to a Klein bottle (see [32, Section 3]). A pair (ε,k) where k is a non-negative, even integer and ε takes one of six possible symbolic values can be uniquely associated to the O(2)-bundle of R-fibers, completely characterizing it up to *weak equivalence of bundles* (see [25, Section 1]). We denote the genus of X^* by $g\geq 0$. We let f,t,k_1,k_2 be non-negative integers such that $k_1\leq f$ and $k_2\leq t$, where k_1 is the number of twisted F-blocks and k_2 is the number of twisted SE-blocks. The number $f-k_1$ is the number of simple F-blocks and $t-k_2$ is the number of simple SE-blocks. A non-negative integer n will denote the number of E-fibers and we let $\{(\alpha_i,\beta_i)\}_{i=1}^n$ be the corresponding Seifert invariants. The invariant b is a certain obstruction class defined in similar fashion to that of the global circle actions case (see [74, Section 2] for a precise definition). We let s,k_3 be non-negative integers, where $k_3\leq s$ is the number of twisted SF-blocks. Hence $s-k_3$ is the number of simple SF-blocks, and we let $(r_1,r_2,\ldots,r_{s-k_3})$ and $(q_1,q_2,\ldots,q_{k_3})$ be $(s-k_3)$- and k_3-tuples of non-negative even integers corresponding to the number of topologically singular points in each simple and twisted SF-block, respectively. The numbers k, k_1, k_2, and k_3 satisfy $k_1 + k_2 + k_3 = k$.

It is worth remarking here that all Seifert manifolds admit a local circle action, but not necessarily a global circle action. Prior to the work of Orlik and Raymond [73, 74], a classification of Seifert manifolds in terms of symbolic and numeric invariants was obtained by Seifert in [90]. An analogous classification for generalized Seifert fibered spaces (see Section 7 below) as defined by Mitsuishi and Yamaguchi [66] is still unknown, as is the precise relation such a classification would share with that of the local circle actions on closed Alexandrov 3-spaces of Theorem 6.6.

7. Collapse

We have already mentioned a few instances of collapse in three-dimensional Alexandrov geometry. Let us recall here the definition of collapse. Let $\mathcal{A}^n_{k,D}$ be the class of n-dimensional Alexandrov spaces with lower curvature bound k and diameter bounded above by D. By Gromov's compactness theorem (see [12, Theorem 10.7.2]), $\mathcal{A}^n_{k,D}$ is compact with respect to the topology induced by the Gromov–Hausdorff distance. Let $\{X^n_i\} \subset \mathcal{A}^n_{k,D}$ be a Gromov–Hausdorff converging sequence with limit X. If X is n-dimensional, then, by Perelman's stability theorem, the spaces X^n_i and X are homeomorphic for sufficiently large i (see [57, 79]). If the limit X is lower-dimensional, then the sequence $\{X^n_i\}$ *collapses*. As noted in Section 2, the collapse of Riemannian manifolds in $\mathcal{M}^n_{k,D} \subset \mathcal{A}^n_{k,D}$ is still not well-understood and this question can be explored in the context of the larger class $\mathcal{A}^n_{k,D}$. Observe that closed Alexandrov 3-spaces with an effective, isometric circle action fall within the class $\mathcal{A}^3_{k,D}$.

Shioya and Yamaguchi [92, 93] examined the collapse of Riemannian manifolds in $\mathcal{M}^3_{k,D}$ and determined the possible topology of the elements M_i of the sequence $\{M_i\} \subset \mathcal{M}^3_{k,D}$, for large i, according to the topology of the limit X. More recently, Mitsuishi and Yamaguchi [66] considered the collapse of Alexandrov spaces in $\mathcal{A}^3_{k,D}$. In the case of collapsing sequences $\{X_i\} \subset \mathcal{A}^3_{k,D}$, for sufficiently large i, the collapsing spaces X_i can be written in terms of basic building blocks: so-called *generalized Seifert fiber spaces* (which are similar to Seifert spaces but with some possibly singular fibers), *generalized solid tori*, *generalized Klein bottles* and other, more familiar, spaces, such as interval bundles over the Klein bottle. The structure of collapsed Alexandrov 3-spaces is obtained by a careful analysis of the limit X, which can be zero-, one- or two-dimensional. It follows that X is a point, a closed interval, a circle or an Alexandrov surface with or without boundary. As in the case of Alexandrov 3-spaces with group actions, where the structure of the 3-space can be recovered from that of its orbit space, the topological and metric structure of X determines the structure of X_i for sufficiently large i. For example, when X is one-dimensional, i.e., a closed interval, for large i the spaces X_i are the union of two pieces whose topology can be explicitly determined (see [66, Theorem 1.8] and compare with Theorem 6.3). The general situation for both collapsed Riemannian 3-manifolds and collapsed Alexandrov 3-spaces is rather intricate, though, so we refer the reader to the original articles [66, 92, 93] for precise statements and proofs of these important structure theorems.

Using Mitsuishi and Yamaguchi's classification of collapsing Alexandrov 3-spaces of as well as the classification of local circle actions, Guijarro and the authors obtained a geometrization result for sufficiently collapsed Alexandrov 3-spaces [30]. Roughly speaking, they showed that a closed, irreducible and sufficiently collapsed Alexandrov 3-space X is modeled in one of the eight Thurston geometries (excluding the hyperbolic geometry $\mathbb{H}^3$).

Theorem 7.1 (Geometrization of sufficiently collapsed Alexandrov 3-spaces [30])**.** *For any $D > 0$ there exists $\varepsilon = \epsilon(D) > 0$ such that if X is a closed, irreducible Alexandrov 3-space*

- $\mathrm{curv}(X) \geq -1$*,*
- $\mathrm{diam}(X) \leq D$ *and*
- $\mathcal{H}^3(X) \leq \varepsilon$*,*

then X admits a geometric structure modeled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil *or* Sol.

This result extends part of the work of Shioya and Yamaguchi in [92], formulated in the manifold case, to Alexandrov spaces. The exclusion of hyperbolic geometry in Theorem 7.1 is granted by a result of independent interest proved in [30, Remark B], namely, that a closed collapsing Alexandrov 3-space cannot admit hyperbolic geometry. We refer the reader to [30] and the recent survey [31] for additional details.

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