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STRING GRADIENT WEIGHTED MOVING FINITE ELEMENTS FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Moving finite element methods resolve regions containing steep gradients using a manageable number of moving nodes. One such implementation is Gradient Weighted Moving Finite Elements (GWMFE). When applied to a single PDE with one space variable x, the solution $u(x,t)$, is viewed as an evolving parameterized manifold, $\mathbf{u} = [x(\tau,t), u(\tau,t)]$, where τ is a parameter along the manifold. Miller (1997) derived the "normal motion" of the manifold in $[x,u]$ space and discretised in space by making the manifold piecewise linear. For systems of PDEs, he used a separate manifold for each dependent variable but with shared nodes.

However, Miller also proposed a "second GWMFE formulation for systems of PDEs". In the case of two dependent variables $u(x,t)$ and $v(x,t)$, instead of determining the separate normal motion of two manifolds, $[x(\tau,t), u(\tau,t)]$ and $[x(\tau,t), v(\tau,t)]$, using shared nodes, he suggested examining the normal motion of a single manifold $[x(\tau,t), u(\tau,t), v(\tau,t)]$, a "string" embedded in $[x,u,v]$ space . This method, called String Gradient Weighted Moving Finite Elements (SGWMFE), has not previously been implemented and tested.

In this paper we revisit the SGWMFE method, deriving a general form of the equations for normal motion using a projection matrix and implementing the method for the one dimensional shallow water equations and for Sod's shock tube problem.

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1 Introduction

A Second Gradient Weighted Moving Finite Element formulation for systems of PDE's was originally proposed by Miller (1997). The formulation is a variation of the Gradient Weighted Moving Finite Element (GWMFE) method which was developed in detail in Carlson and Miller (1998a) and Carlson and Miller (1998b) for one and two dimensional problems. The one-dimensional version of the second formulation is called the String Gradient Weighted Moving Finite Element Method (SGWMFE). This method has not previously been implemented or tested.

In this paper we develop the SGWMFE for a system of PDE's in one dimension using a projection matrix in a way which can be easily generalized to arbitrary number of equations and multiple space dimensions, although all the results presented in this paper are for one space dimension. We first describe the general structure of the gradient weighted finite element method, then we develop the necessary details of applying the SGWMFE method to two particular systems of partial differential equations, the shallow water equations and the Euler equations for a shock tube, the example we consider being Sod's Shock Tube problem.

For both systems we use an added diffusion in the PDE's, with a small diffusion coefficient, ϵ. We emphasize that the PDE's without diffusion are incomplete. No classical solution exists; it is diffusion (no matter how small the coefficient ϵ) which prevents infinite gradient, prevents multivalued fold over, transmits flux and dissipates energy in the steep portions and essentially causes the shocks as $\epsilon \to 0$ (see section 2.5 and figures 6.3 and 6.10 of Carlson and Miller (1998a) for example). Moreover, even within the setting of "weak solutions" additional information involving "entropy" or the "zero diffusion limit" must be invoked to avoid non-uniqueness. Thus we compute with very small ϵ (as small as we can handle numerically because of the resulting exceedingly thin shocks) in order to approximate the desired "zero diffusion limit" weak solution.

2 Gradient weighted, moving finite elements

2.1 GWMFE for a single PDE in one dimension

Consider a single time evolving PDE,

$$u_t = L(u), \tag{2.1}$$

where L is a first or second order nonlinear differential operator on the one dimensional spatial interval Ω. This is the equation for the vertical motion u_t of the graph of the solution $u(x,t)$. GWMFE instead treats the solution graph as an evolving one-dimensional parameterized manifold embedded in two dimensions based on the equations for the *normal* (rather than vertical) component of the motion of that manifold.

Consider any parameterization of the evolving solution manifold by

$$\mathbf{u}(\tau,t) = (X(\tau,t), U(\tau,t)), \tag{2.2}$$

where τ is a one dimensional parameter. Here $X(\tau,t)$ is any smoothly increasing function of τ and the corresponding $U(\tau,t)$ is given in terms of the original $u(x,t)$ by

$$U(\tau,t) = u(X(\tau,t),t). \tag{2.3}$$

Thus $\mathbf{u}$ represents the two-dimensional position on the evolving manifold of the parameterized point with fixed parameter τ and

$$\dot{\mathbf{u}}(\tau,t) = (\dot{X}(\tau,t), \dot{U}(\tau,t)), \tag{2.4}$$

represents the two-dimensional motion (or velocity) of that parameterized point, where the dots denote time differentiation with τ fixed. By the chain rule in (2.3), we have

$$\dot{U} = u_x \dot{X} - u_t, \tag{2.5}$$

and thus obtain the familiar formula for u_t,

$$u_t = \dot{U} - u_x \dot{X}. \tag{2.6}$$

To avoid proliferation of notation (as is common when using the chain rule and changes of parameterization) we replace the U and X in (2.1)-(2.6) by simply u and x. Thus we replace the notation in (2.1)-(2.6) respectively by

$$u(x,t), \quad x(\tau,t), \quad u(\tau,t) \equiv u(x(\tau,t),t), \quad \mathbf{u}(\tau,t) = (x(\tau,t), u(\tau,t)),$$

so that

$$\dot{\mathbf{u}}(\tau,t) = (\dot{x}(\tau,t), \dot{u}(\tau,t)), \quad \text{and} \quad u_t = \dot{u} - u_x \dot{x}.$$

Moreover, we will just speak of 'the evolving solution manifold $\mathbf{u}$' whether it is in function form $\mathbf{u}(x,t)$ or parameterized form (2.2).

At each point on the evolving manifold we divide the motion vector $\dot{\mathbf{u}} = (\dot{x}, \dot{u})$ into its tangential part, $[\dot{\mathbf{u}}]_T$ and its normal part $[\dot{\mathbf{u}}]_N$. Since the tangential part of this motion makes no change to the manifold itself, we are interested only in getting the correct normal motion, $[\dot{\mathbf{u}}]_N$. Note that

$$[\mathbf{u}]_N = (\dot{\mathbf{u}} \cdot \mathbf{n})\mathbf{n}, \tag{2.7}$$

where

$$\mathbf{n}(\tau,t) = \frac{1}{\sqrt{1+u_x^2}}(-u_x, 1),$$

is the upward normal to the graph at each point and $P(\tau,t) = \mathbf{n}\mathbf{n}^{\mathrm{T}}$ is the orthogonal projection matrix onto the normal direction.

The original PDE, (2.1), was the equation for the vertical motion $(0, u_t)$ of the manifold,

$$\begin{pmatrix} 0 \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ L(u) \end{pmatrix}. \tag{2.8}$$

Note that because $(x, u(x,t))$ is one parameterization of the solution manifold (with $x = \tau$ unchanging in t), we automatically have the same normal motion

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix}_N = \begin{pmatrix} 0 \\ u_t \end{pmatrix}_N, \tag{2.9}$$

for any other parameterization. Thus the equation for the normal motion is

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix}_N = \begin{pmatrix} 0 \\ L(u) \end{pmatrix}_N, \tag{2.10}$$

or

$$(\dot{\mathbf{u}} \cdot \mathbf{n})\mathbf{n} = K(u)\mathbf{n}, \tag{2.11}$$

where $u_t/\sqrt{1+u_x^2} = (\dot{\mathbf{u}} \cdot \mathbf{n})$ and $K(u) = L(u)/\sqrt{1+u_x^2}$. These, (2.10) or (2.11), are a system of two PDE's which must be satisfied by the two unknown functions, $u(\tau, t)$ and $x(\tau, t)$ of our parameterization. Of course this system is degenerate because the tangential motion $[\dot{\mathbf{u}}]_T$ is left completely free by these equations.

The GWMFE method is a discretisation of the normal motion equations, (2.10) or (2.11). Once again, to avoid proliferation of notation, we use the same symbols for the numerical solution as for the exact solution. The GWMFE approximation is allowed to be an evolving piece-wise linear one-dimensional manifold with its two-dimensional nodal positions,

$$\mathbf{u}_j(t) = (x_j(t), u_j(t)),$$

all treated as unknowns and with the natural linear parameter τ, $0 \leq \tau \leq 1$, on each cell. The position $\mathbf{u}(\tau, t)$ and motion $\dot{\mathbf{u}}(\tau, t)$ of the parameterized points are thus given by linear interpolation from nodal positions, $\mathbf{u}_j(t)$ and nodal motion, $\dot{\mathbf{u}}_j(t)$, that is

$$\mathbf{u}(\tau, t) = \sum_j \alpha^j(t)\mathbf{u}_j(t), \quad \dot{\mathbf{u}}_j(t) = \sum_j \alpha^j(t)\dot{\mathbf{u}}_j(t), \tag{2.12}$$

where $\alpha^j(t)$ is the usual j^{th} "hat function" which falls linearly from one at the j^{th} node to zero at adjacent nodes.

Miller (1997) introduced a mechanical interpretation for the derivation of the GWMFE ODEs for the motion $\dot{\mathbf{u}}_j$ of the nodes. He interpreted the normal motion equations, (2.10) or (2.11), as being a balance of distributed viscous drag forces and applied forces per unit arclength on the manifold, all in the normal direction. The GWMFE manifold cannot satisfy that force balance at every point; instead it concentrates those distributed forces onto the nodes by the law of leverage and requires that the concentrated forces in (2.11) balance at the nodes. This yields

$$\int (\dot{\mathbf{u}} \cdot \mathbf{n})\mathbf{n}\alpha^j \, \mathrm{d}s = \int K(u)\mathbf{n}\alpha^j \, \mathrm{d}s, \tag{2.13}$$

at the j^{th} node, where s is arc length along the manifold and where the hat functions α^j can be recognised to be the leverage factors for this concentration onto the nodes.

Hence equation (2.13), a system of ordinary differential equations for the nodal positions $\mathbf{u}_j(t)$, states that the GWMFE manifold chooses its nodal motions $\dot{\mathbf{u}}_j(t)$ at each instant such that the distributed viscous drag forces $(\dot{\mathbf{u}} \cdot \mathbf{n})\mathbf{n}$ and applied forces $K(u)\mathbf{n}$, when concentrated onto the nodes by the law of leverages, exactly balance at each node.

GWMFE can also be considered to be a conventional piecewise linear Galerkin discretisation of the degenerate system (2.10) or (2.11) for the unknown functions $x(\tau, t)$ and $u(\tau, t)$, but with the important distinction that we do not integrate with respect to measure $\mathrm{d}\tau$ in parameter

space as would be conventional but with respect to the arc length measure ds on the evolving piecewise linear manifold.

The original variational interpretation for the GWMFE ordinary differential equations was also described in Miller (1997) and Carlson and Miller (1998b). One sees easily that the equations, (2.13), are the variational equations obtained by requiring that the nodal motions $\{\dot{\mathbf{u}}_j\}$ be chosen at each instant so as to minimise the functional

$$\psi = \int \left|\left| \begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix}_N - \begin{pmatrix} 0 \\ L(u) \end{pmatrix}_N \right|\right|^2 \mathrm{d}s \equiv \int |\dot{\mathbf{u}} \cdot \mathbf{n} - K(u)|^2 \mathrm{d}s, \tag{2.14}$$

where this L_2 integral of the residual in the normal motion of the PDEs, (2.10 or(2.11), is taken over the GWMFE manifold.

The mechanical interpretation, (2.13), seems somewhat more natural than the variational interpretation, (2.14) for PDEs whose operator $L(u)$ involves second order terms such as u_{xx}. In that case the second derivative of our piecewise linear approximation is singular at the nodes and hence the L_2 residual function ψ in (2.14) is infinite. This difficulty was resolved in earlier papers, Miller and Miller (1981), Miller (1997), Carlson and Miller (1998b) by the technique of mollification or smoothing of the piecewise linear approximation near the nodes and the taking the limit in the variational equations for this minimisation as the mollification radius tends to zero. However the mechanical interpretation, starting from the concentrated force balance equations, (2.13), seems more natural here because one is used to handling concentrated delta-function type forces in mechanical situations.

At the j^{th} node where the adjacent unit normals $\mathbf{n}_{j-1}$ and $\mathbf{n}_j$ are non-parallel, the viscous drag forces $(\dot{\mathbf{u}} \cdot \mathbf{n})\mathbf{n}$ on the left in (2.13) resist motion of the node in both these normal directions and hence somewhat resist motion in all the two-dimensional directions. However, when adjacent normals are parallel there is absolutely no resistance to tangential motion of the node. Provided adjacent cells are not parallel and provided the cell arc length remains non-zero, the symmetric mass matrix of the system (2.13) remains positive definite. To remove possible degeneracies of the mass matrix one adds 'internodal viscosity' forces which penalise against relative motion of the nodes and which are sufficiently small that they essentially come into play only in the case of adjacent nodes which are nearly parallel.

2.2 GWMFE for systems of PDEs in one dimension

Consider the example of a system of PDE's,

$$u_t = L_1(u, v), \qquad v_t = L_2(u, v), \tag{2.15}$$

for the two unknown functions $u(x,t)$ and $v(x,t)$ on a one-dimensional spatial interval Ω. Here, as in (2.1), L_1 and L_2 are first or second order nonlinear differential operators.

Our GWMFE approximants, u, v will be piecewise linear functions with moving nodes. For a great variety of reasons (which were discussed at length in Miller (1997), Carlson and Miller (1998a) and Carlson and Miller (1998b), one uses a common grid for the two functions. We therefore have three unknowns (x_j, u_j, v_j) at the j^{th} node.

The usual GWMFE method for systems (which is used in all previous codes, and which Miller in Miller (1997) calls the first GWMFE formulation for systems) involved treating the u and v graphs as two independent evolving two-dimensional manifolds evolving in three dimensions, that is as $(x, u(x,t), 0)$ and $(x, 0, v(x,t)$, as shown in figure 1(a). Under reparameter-

ization with a shared moving $x(\tau,t)$ these two manifolds have parameterized representations $\mathbf{u}(\tau,t)=(x(\tau,t),u(\tau,t),0)$ and $\mathbf{v}(\tau,t)=(x(\tau,t),0,v(\tau,t))$.

The $\mathbf{u}$ manifold has 'vertical' forces $\mathbf{F}_1=(0,f_1,0)$ and the $\mathbf{v}$ manifold has 'vertical' forces $\mathbf{F}_2=(0,0,f_2)$ from the PDEs where $f1=L_1-u_t$ and $f_2=L_2-v_t$. One then forms the normal parts $[\mathbf{F}_1]_N$ and $[\mathbf{F}_2]_N$ of these two forces. These represent normal forces per unit arclength $\mathrm{d}s_1=\sqrt{1+u_x^2}\mathrm{d}x$ and $\mathrm{d}s_2=\sqrt{1+v_x^2}\mathrm{d}x$ upon the two manifolds. An exact manifold solution of the system (2.15) would then satisfy the three-dimensional force balance equations

$$\mathbf{F}=[\mathbf{F}_1]_N+[\mathbf{F}_2]_N=0, \tag{2.16}$$

at every parameterized point (τ,t).

It is these force balance equations (analogous to (2.10) and (2.11)) which GWMFE discretises. Once again, one concentrates these forces onto the nodes, integrating times the leverage factors α^j to yield concentrated forces $\mathbf{F}_{1,j}$ and $\mathbf{F}_{2,j}$ on the j^{th} node of the $\mathbf{u}$ and $\mathbf{v}$ PDEs respectively. Then, since these two j^{th} nodes are rigidly connected at that x_j coordinate, we get the GWMFE force balance equations

$$\mathbf{F}_j=\mathbf{F}_{1,j}+\mathbf{F}_{2,j}=\mathbf{0}, \tag{2.17}$$

at every j^{th} node.

It is easily seen that (2.17) are variational equations for choosing the nodal motions $\{\dot{x}_j,\dot{u}_j,\dot{v}_j\}$ at each instant so as to minimise

$$\psi=\psi_1+\psi_2, \tag{2.18}$$

where ψ_1 and ψ_2 are the L_2 residuals for the u and v normal motion equations,

$$\psi_1=\int\|[\mathbf{F}_1]_N\|^2\,\mathrm{d}s,\qquad \psi_2=\int\|[\mathbf{F}_2]_N\|^2\,\mathrm{d}s. \tag{2.19}$$

2.3 String gradient weighted moving finite elements

This is the second GWMFE formulation for systems, introduced and sketched out in Miller (1997) on one and two space dimensions, but never investigated in detail or implemented numerically. Now we treat the solution graphs for the system (2.15) as a single evolving one dimensional manifold (a 'string') immersed in three dimensions, that is as $(x,u(x,t),v(x,t))$, as shown in figure 1. Here figure 1(a) and 1(b) should make abundantly clear the distinction between the first and second formulations.

Under reparameterization with a moving $x(\tau,t)$, our string becomes an evolving parameterized one dimensional manifold immersed in three dimensions with the representation

$$\mathbf{u}(\tau,t)=(x(\tau,t),u(\tau,t),v(\tau,t)), \tag{2.20}$$

for the evolving reparameterized points.

At each parameterized point on the evolving manifold we divide the motion vector $\dot{\mathbf{u}}=(\dot{x},\dot{u},\dot{v})$ into its tangential, $[\dot{\mathbf{u}}]_T$, and its normal, $[\dot{\mathbf{u}}]_N$, parts. The original system (2.15) was the equation for the 'vertical' motion $(0,u_t,v_t)$ of the manifold

$$\begin{pmatrix}0\\u_t\\v_t\end{pmatrix}=\begin{pmatrix}0\\L_1(u,v)\\L_2(u,v)\end{pmatrix}. \tag{2.21}$$

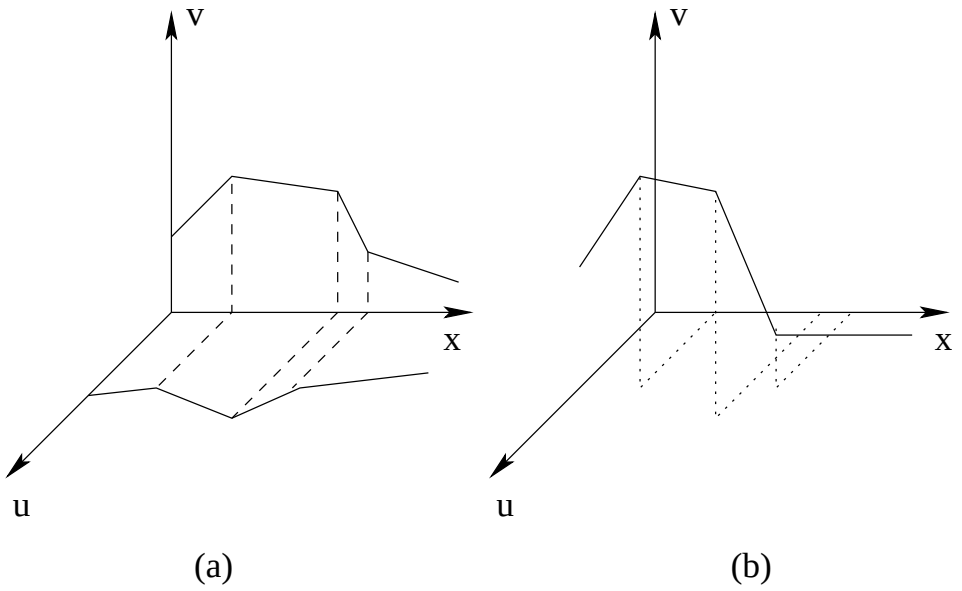

Figure 1: Sketch of piecewise linear representation (a) using GWMFE (b) using SGWMFE

Note that because $(x, u(x,t), v(x,t))$ is one parameterization of the solution manifold we automatically have the same normal motion

$$\begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}_N = \begin{pmatrix} 0 \\ u_t \\ v_t \end{pmatrix}_N , \tag{2.22}$$

for any other parameterization. Thus the equation for the normal motion is

$$\begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}_N = \begin{pmatrix} 0 \\ L_1 \\ L_2 \end{pmatrix}_N , \tag{2.23}$$

which is a system of three PDEs for the three unknown functions $x(\tau,t), u(\tau,t), v(\tau,t)$ (a degenerate system since the tangential component of the motion is left completely free).

It is convenient to use here the projection matrix P which projects any given vector, $\mathbf{F}$, into its normal part, $[\mathbf{F}]_N$ at a given point on the 'string', $(x, u(x), v(x))$. The tangential projection is given by

$$[\mathbf{F}]_T = \mathbf{t}\mathbf{t}^{\mathrm{T}}\mathbf{F}, \tag{2.24}$$

where $\mathbf{t} = (1, u_x, v_x)/\sqrt{1+u_x^2+v_x^2}$ is the unit tangent vector to the manifold at this point. Hence

$$[\mathbf{F}]_N = \mathbf{F} - [\mathbf{F}]_T = (\mathbf{I} - \mathbf{t}\mathbf{t}^{\mathrm{T}})\mathbf{F} = P\mathbf{F}, \tag{2.25}$$

where

$$P = \frac{1}{1+u_x^2+v_x^2} \begin{pmatrix} (u_x^2+v_x^2) & -u_x & -v_x \\ -u_x & (1+v_x^2) & -u_x v_x \\ -v_x & -v_x u_x & (1+u_x^2) \end{pmatrix} \tag{2.26}$$

Once again we interpret these equations, (2.23) mechanically, as a balance of viscous drag forces and applied forces per unit arc length of the string, all in the normal direction. We discretise the force balance equations, (2.23), by letting our SGWMFE approximant be an evolving, piecewise linear manifold with its three dimensional nodal positions $\mathbf{u}_j = (x_j(t), u_j(t), v_j(t))$ as unknowns. We proceed as in (2.12) and (2.13), concentrating these distributed normal forces onto the nodes and thus arriving at the balance of three dimensional forces

$$\int \begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}_N \alpha^j \mathrm{d}s = \int \begin{pmatrix} 0 \\ L_1 \\ L_2 \end{pmatrix}_N \alpha^j \mathrm{d}s, \tag{2.27}$$

at each node, much as in (2.13)

This, (2.27), is a system of ODEs for the nodal positions $\mathbf{u}_j(t)$. Once again, one sees easily that these ODEs are the variational equations obtained by requiring that the nodal motions $\{\mathbf{u}_j\}$ be chosen at each instant so as to minimise

$$\psi = \int \left\| \begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}_N - \begin{pmatrix} 0 \\ L_1 \\ L_2 \end{pmatrix}_N \right\|^2 \mathrm{d}s, \tag{2.28}$$

where this is the L_2 integral of the residual of the normal motion PDEs, (2.23).

Thus at a j^{th} node where the adjacent unit tangent vectors are parallel there is no resistance in the left sided forces of (2.27) to tangential motions of the node, but absent this parallelism and absent cells of zero arc length then the symmetric mass matrix of the ODEs (2.27) remains positive definite. Once again, to remove such possible degeneracies of the mass matrix one adds 'internodal viscosity' forces which penalise against relative motion of the nodes and which are sufficiently small that they essentially only come into play in the case of nodes with near parallelism.

3 Implementation

In implementing SGWMFE, as with GWMFE, we repeatedly use the result that after piecewise linear discretisation, spatial derivatives u_x and v_x are constant on each cell. In addition, values of the manifold on a cell are found by linear interpolation between the ends of the cell. Thus, for example, on the first cell, let $\underline{u}_0, \underline{u}_1$ be the position vectors on the manifold at the corresponding left and right end points of the cell. Further we have the corresponding velocity vectors at these same points: $\dot{\underline{u}}_0, \dot{\underline{u}}_1$. Then intermediate values of $\underline{u}$ and $\dot{\underline{u}}$ on the cell are obtained using the hat functions α^0 and α^1 together with the values at either end of the cell. In applying leverage factor α^i to concentrate forces onto the i^{th} node one naturally has integrals which range over two adjacent cells, $cell_i$ and $cell_{i+1}$. In the following we describe the part of the concentrated forces for the i^{th} node from $cell_i$; there will be an analogous part from $cell_{i+1}$ on the i^{th} node.

3.1 Time derivative terms

Using equation (2.27) the time derivatives (the left hand side of our PDE system) in $cell_i$ are concentrated onto the i^{th} node by:

$$\int_{\text{cell}_\text{i}} \begin{pmatrix} 0 \\ u_t \\ v_t \end{pmatrix}_N \alpha^i \mathrm{d}s = \int_{\text{cell}_\text{i}} P \begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix} \alpha^i \mathrm{d}s. \tag{3.1}$$

The resulting 'force' from the cell onto it's i^{th} node is obtained exactly by using Simpson's quadrature as,

$$\int_{\text{cell}_\text{i}} P\dot{\underline{u}}\alpha^i \mathrm{ds} = \mathrm{P}(\frac{1}{3}\dot{\mathbf{u}}_\mathrm{i} + \frac{1}{6}\dot{\mathbf{u}}_{\mathrm{i}-1})\Delta \mathrm{s_i}, \tag{3.2}$$

where

$$\Delta s_i = \|\mathbf{u}_i - \mathbf{u}_{i-1}\|_2 = \sqrt{(x_i - x_{i-1})^2 + (u_i - u_{i-1})^2 + (v_i - v_{i-1})^2}. \tag{3.3}$$

3.2 Flux terms

At this point we restrict the non-linear operators L_1 and L_2 to have a particular form, that of a flux function combined with an artificial diffusivity term, so that in the case of two dependent variables,

$$\begin{aligned} u_t &= -f_x(u,v) + \epsilon u_{xx}, \\ v_t &= -g_x(u,v) + \epsilon v_{xx}, \end{aligned} \tag{3.4}$$

and ϵ determines the magnitude of the artificial diffusion (and note that this is applied equally to *both* equations).

It will be useful to define the following notation. For a scalar function $f(x, u(x), v(x))$ we denote by f_i its value at the i^{th} node and by $[f]_i$, the average over the cell. Thus $f_i = f(x_i, u_i, v_i)$ and $[f]_i = \frac{1}{\Delta x_i}\int_{\text{cell}_\text{i}} f \mathrm{d}x$.

The force contributions from the flux terms in $cell_i$ onto the i^{th} node are obtained using $\mathrm{d}s = \sqrt{1 + u_x^2 + v_x^2}\mathrm{d}x$ so that the contribution from the first of the convective terms can be written

$$\int_{\text{cell}_\text{i}} P \begin{pmatrix} 0 \\ -f_x \\ 0 \end{pmatrix} \alpha^i \mathrm{ds} = P \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \int_{\text{cell}_\text{i}} -f_x \alpha^i \mathrm{ds} \tag{3.5}$$

$$= \begin{pmatrix} -u_x \\ 1 + v_x^2 \\ -v_x u_x \end{pmatrix} \frac{\int_{\text{cell}_\text{i}} -f_x \alpha^i \mathrm{dx}}{\sqrt{1 + u_x^2 + v_x^2}} \tag{3.6}$$

$$= \begin{pmatrix} -u_x \\ 1 + v_x^2 \\ -v_x u_x \end{pmatrix} \frac{[f]_i - f_i}{\sqrt{1 + u_x^2 + v_x^2}} \tag{3.7}$$

Similarly the flux contributions from the $-g_x$ term:

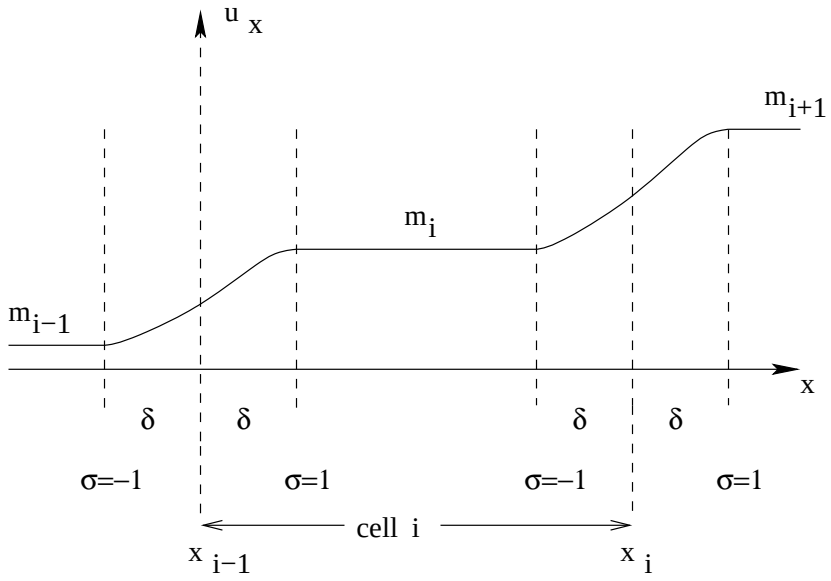

Figure 2: Mollification: the value of u_x on $cell_i$ is m_i, the value of u_x on $cell_{i-1}$ is m_{i-1} and the value of u_x on $cell_{i+1}$ is m_{i+1}. The value of u_x is assumed to vary smoothly in a small neighborhood of each end of the cell of width 2δ and then $\delta \to 0$.

$$\int_{\mathrm{cell_i}} P \begin{pmatrix} 0 \\ 0 \\ -g_x \end{pmatrix} \alpha^i \mathrm{ds} = \begin{pmatrix} -v_x \\ -u_x v_x \\ 1+u_x^2 \end{pmatrix} \frac{[\mathrm{g}]_\mathrm{i} - \mathrm{g_i}}{\sqrt{1+\mathrm{u_x^2}+\mathrm{v_x^2}}} \tag{3.8}$$

3.3 Diffusion terms

Using piecewise linear basis functions means that diffusive terms vanish in the interior of any cell but are undefined at nodes. One way of dealing with this problem is to use formal integration by parts but an equivalent and easily understandable way is to use the mathematical technique of mollification. The first derivative, while being constant over the main body of the cell, is assumed to vary smoothly between cell values in a small neighborhood of width 2δ at each node (see figure 2). Then we take the limit $\delta \to 0$. Thus in any integral involving diffusion terms we only need to take into account the small neighborhoods near each node since u_{xx} and v_{xx} are still identically zero for most part of each cell. The details are presented below for SGWMFE but see also Carlson and Miller (1998a), Miller (1997) or Baines (1994) for further details on how mollification has been carried out for the MFE and GWMFE methods in one dimension.

Denote the values of u_x, v_x on $cell_i$ as m_i, $\tilde{m}_i$ respectively. For the present method we mollify by defining a variable $\sigma(x)$ and then, for instance at the right end of the cell, map a neighborhood of width 2δ of x_i to $-1 \le \sigma(x) \le 1$, map u_x to $m_i \le u_x \le m_{i+1}$, and map v_x to $\tilde{m}_i \le v_x \le \tilde{m}_{i+1}$. Then the normal force on the i^{th} node due to diffusive terms is:

$$\epsilon \int_{-\delta}^{\delta} P \begin{pmatrix} 0 \\ u_{xx} \\ v_{xx} \end{pmatrix} \alpha^i \mathrm{ds} = \epsilon \int_{-\delta}^{\delta} \frac{1}{\sqrt{1+\mathrm{u_x^2}+\mathrm{v_x^2}}} \begin{pmatrix} -u_x u_{xx} - v_x v_{xx} \\ (1+v_x^2)u_{xx} - u_x v_x v_{xx} \\ -v_x u_x u_{xx} + (1+u_x^2) v_{xx} \end{pmatrix} \mathrm{dx} + \mathrm{O}(\delta), \quad (3.9)$$

where the term of order $O(\delta)$ is due to replacing α^i by 1 in the neighborhood of x_i,

The integral is now rewritten using the mappings

$$u_x = m_c + \Delta m_c \sigma(x) \quad (3.10)$$
$$v_x = \tilde{m}_c + \Delta \tilde{m}_c \sigma(x), \quad (3.11)$$

where $m_c = \frac{m_i + m_{i+1}}{2}$, $\tilde{m}_c = \frac{\tilde{m}_i + \tilde{m}_{i+1}}{2}$, $\Delta m_c = \frac{m_{i+1} - m_i}{2}$ and $\Delta \tilde{m}_c = \frac{\tilde{m}_{i+1} - \tilde{m}_i}{2}$.

In the first component of (3.9), let

$$I_A = \int_{-\delta}^{\delta} \frac{u_x u_{xx} + v_x v_{xx}}{\sqrt{1+u_x^2+v_x^2}} \mathrm{dx} \quad (3.12)$$

It is important that we have mollified the jumps in u_x and v_x equally, that is using the same function $\sigma(x)$ for both.

Substituting equations (3.10)-(3.11) into (3.12), so that

$$u_{xx} \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\sigma} u_x \, \mathrm{d}\sigma, \qquad v_{xx} \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\sigma} v_x \, \mathrm{d}\sigma,$$

then under change of variable for the integral, the neighborhood size falls out and we have the following expression for I_A in terms of the parameter σ:

$$I_A = \int_{-1}^{1} \frac{(m_c + \Delta m_c \sigma)\Delta m_c + (\tilde{m}_c + \Delta \tilde{m}_c \sigma)\Delta \tilde{m}_c}{\sqrt{1 + (m_c + \Delta m_c \sigma)^2 + (\tilde{m}_c + \Delta \tilde{m}_c \sigma)^2}} \mathrm{d}\sigma. \quad (3.13)$$

Let

$$a_0 = 1 + m_c^2 + \tilde{m}_c^2, \quad (3.14)$$
$$b_0 = 2 m_0 \Delta m_c + 2 \tilde{m}_c \Delta \tilde{m}_0, \quad (3.15)$$
$$c_0 = {\Delta m_c}^2 + {\Delta \tilde{m}_c}^2, \quad (3.16)$$

then

$$I_A = \frac{b_0}{2} \int_{-1}^{1} \frac{\mathrm{d}\sigma}{\sqrt{a_0 + b_0 \sigma + c_0 \sigma^2}} + c_0 \int_{-1}^{1} \frac{\sigma \mathrm{d}\sigma}{\sqrt{a_0 + b_0 \sigma + c_0 \sigma^2}}. \quad (3.17)$$

If we make further substitutions we may write I_A in a more compact form.

Letting $\gamma_0 = \frac{b_0}{a_0}$, $\beta_0^2 = \frac{c_0}{a_0}$, we have:

$$I_A = \frac{b_0}{2\sqrt{a_0}} (R_1(\gamma_0, \beta_0) + R_1(-\gamma_0, \beta_0)) + \frac{c_0}{\sqrt{a_0}} (R_2(\gamma_0, \beta_0) - R_2(-\gamma_0, \beta_0)). \quad (3.18)$$

where

$$R_1(\gamma,\beta) = \int_0^1 \frac{\mathrm{d}\sigma}{\sqrt{1+\gamma\sigma+\beta^2\sigma^2}} \tag{3.19}$$

$$= \frac{1}{\beta}\log\left(\frac{2\beta\sqrt{1+\gamma+\beta^2}+2\beta^2+\gamma}{2\beta+\gamma}\right) \tag{3.20}$$

and

$$R_2(\gamma,\beta) = \int_0^1 \frac{\sigma d\sigma}{\sqrt{1+\gamma\sigma+\beta^2\sigma^2}} \tag{3.21}$$

$$= \frac{1}{\beta^2}[\sqrt{1+\gamma+\beta^2}-1] - \frac{\gamma}{2\beta^2}R_1(\gamma,\beta) \tag{3.22}$$

Despite knowing analytic expressions for these integrals, they can be subject to severe roundoff errors and great care has to be taken in their evaluation.

3.4 Regularisation terms

As we have explained, in moving finite element methods it is necessary to prevent the mass matrix from becoming singular. This can be avoided by adding small terms to the matrix which have a negligible effect on the solution but which prevent the mass matrix being truly singular. This is called regularisation and has an associated parameter which scales the added terms. As in the GWMFE approach, we add similar regularisation terms using internodal viscosities as suggested in Miller (1997). These terms remove any degeneracies which occur in the original system of ODE's which may arise when the solution has two consecutive cells with the same slope or when two nodes run into each other or overlap. Each of these cases would result in at least two rows of the mass matrix to be linearly dependent and hence the matrix being singular, see Miller (1997) for a detailed explanation of the possible degeneracies. In order to remove the possibility of this happening, Miller and Miller (1981) show that adding a particular positive definite matrix to the mass matrix removes the degeneracies.

As in GWMFE, see equation (3.11) Miller (1997), we add forces on the i^{th} node whose component in the x, u, v directions are:

$$A^2\sum_j c_{ij}\dot{x}_j, \tag{3.23}$$

$$A^2\sum_j c_{ij}\dot{u}_j, \tag{3.24}$$

$$A^2\sum_j c_{ij}\dot{v}_j. \tag{3.25}$$

The c_{ij} coefficients used are:

$$c_{ij} = \begin{pmatrix} \frac{-1}{\Delta x_i} & for \quad j = i-1 \\ \frac{1}{\Delta x_i}+\frac{1}{\Delta x_{i+1}} & for \quad j = i \\ \frac{-1}{\Delta x_{i+1}} & for \quad j = i+1 \end{pmatrix}, \tag{3.26}$$

In the calculations that follow, we use a regularisation parameter A of relative size suggested in Miller (1997), that is, we use a value that is of the size 2 to 5 times the size of the relative local truncation error tolerance for the non-linear integrator, which in practice has been set to 10^{-3}.

4 Definition of model problems

In order to illustrate implementation of the SGWMFE method in one space dimension we consider two model problems. One is Sod's shock tube problem, see for instance Sod (1978), which involves three dependent variables; while our discussion has focussed on two dependent variables, as explained, the results generalise to more dependent variables in an easy and obvious manner. The second problem is shallow water flow, where there are two dependent variables. We define the two model problems in this section and discuss computed results in the next section.

4.1 Sod's shock tube problem

The shock tube problem we consider was introduced as a model problem by Sod (1978) and takes a fixed length of closed tube in which the gas at one end is at one pressure and the gas at the other end is at a different pressure. At time zero the gas in the two halves is allowed to come into contact. If the left end is at higher pressure, a shock propagates to the right and an expansion wave propagates to the left. The solution is complicated in that a second shock forms in the density profile, called the contact discontinuity. An aim of a computed solution is to capture the various shock strengths and their propagation speed correctly.

The system of equations which we solve are

$$u_t + f_x = \epsilon u_{xx}, \tag{4.1}$$

$$v_t + g_x = \epsilon v_{xx}, \tag{4.2}$$

$$w_t + h_x = \epsilon w_{xx}, \tag{4.3}$$

where $0 \le x \le 1$, time $t \ge 0$ and u, v, w, f, g, h are defined in terms of the primitive variables for density, ρ, velocity c, pressure $p = (\gamma - 1)(e - \frac{1}{2}\rho u^2)$ and total specific energy e by

$$u = \rho, \quad v = \rho c, \quad w = e, \quad p = (\gamma - 1)(w - \frac{1}{2}\frac{v^2}{u}),$$

and

$$f = v, \quad g = p + \frac{v^2}{u}, \quad h = \frac{(w+p)v}{u}.$$

Sod's shock tube problem asks for the "zero diffusion limit" weak solution of (4.1)-(4.3) with the initial conditions

$$(u, v, w) = \left\{ \begin{array}{ll} (1.0, 0.0, 2.5), & \text{if} \quad x \le 0.5, \\ (0.125, 0.0, 0.25), & \text{if} \quad x > 0.5. \end{array} \right\} \tag{4.4}$$

This is a Riemann problem with a well known similarity solution until such time as the disturbances reflect off the boundary. As in Carlson and Miller (1998a) we use $\gamma = 1.4$ and reflection boundary conditions, $u_x = 0$, $v = 0$, $w_x = 0$ imposed at both ends of a tube.

4.2 Shallow water equations model problem

The shallow water equations provide a second model problem to test SGWMFE. For these equations the "dam-break problem" is another Riemann problem with an analytic similarity solution for the "zero diffusion limit" weak problem. We choose instead to study a problem with smooth initial conditions and observe the structure develop. In this problem a 'hump' of stationary water is released at time zero, as the hump subsides under gravity, a wave propagates away and the wave front steepens, forming a shock in the absence of viscosity. In this problem a simulation should capture the shock formation, wave height and front speed.

The shallow water equations (with addition of artificial diffusion) in one dimension for the flow of a fluid with density $\rho = 1$ and a gravitational constant of 1 over a flat bottom can be written as follows:

$$u_t + f_x = \epsilon u_{xx}, \tag{4.5}$$
$$v_t + g_x = \epsilon v_{xx}, \tag{4.6}$$

where $0 \leq x \leq 1$, time $t \geq 0$, u is the height of the fluid from the flat bottom, and v is the fluid momentum in the x direction.

The initial conditions considered for this model problem are given by:

$$u = 0.2 + e^{-x^2}, \tag{4.7}$$
$$v = 0. \tag{4.8}$$
$$\tag{4.9}$$

The boundary conditions are $u_x = 0$ and $v = 0$ at both boundaries($x = 0$ and $x = 1$).

In this case

$$f = v, \quad g = (\frac{v^2}{u} + \frac{u^2}{2}).$$

5 Results

We now consider results from applying SGWMFE to the two model problems. For each problem the time integrator is a 'simplification' (with the diagonal preconditioning and other special treatment of the mass matrix eliminated) of that described in Carlson and Miller (1998a), but extended to handle the residual and Jacobian of the '2-cell' terms required by the more complicated SGWMFE diffusion expressions.

For the Shallow Water equations we compute with rather a large $\epsilon = 5(10)^{-3}$ because for smaller ϵ we have to use tiny inverse quadratic pressures in the cells to regularize the nodal positioning, much as described in Section 2.7 of Carlson and Miller (1998b) for GWMFE. For Sod's shock tube problem we also compute with a relatively large $\epsilon = (10)^{-3}$, though we can achieve $\epsilon = O((10)^{-4})$ if we use these added pressures. For SGWMFE these added pressure forces in each cell are in the outward tangential direction and of magnitude $C^2/\mathrm{d}s^2$ where $\mathrm{d}s$ is the length of the cell and C is a tiny constant. If we need to make computations for smaller ϵ the coefficient C needs to be chosen much as described in the order of magnitude type argument presented in Section 2.5 of Miller (1997) and in earlier GWMFE papers. For larger ϵ ,such as $5(10)^{-3}$ for the Shallow Water Equations and $(10)^{-3}$ for Sod's shock tube problem,

such internodal pressures were not needed at all. We also note that it is important to have a good initial distribution of nodes to resolve structure which develops near steep gradients. This is discussed to some extent in earlier GWMFE papers and is a key issue currently under investigation.

With a fixed mesh (and refinements of a fixed mesh), comparison between solutions is straight forward at predetermined set of space points. Unfortunately, with moving meshes the situation is not so simple since the solution is computed at different space points and so the only way to compare results is to interpolate, either from one moving mesh onto another or from moving meshes onto fixed, predetermined points. This, of course, means an interpolation error will occur but when we have interpolated solutions, we have used high enough order interpolation for any errors introduced to be of smaller order than errors in the solution. For this study we choose evenly spaced points and use piecewise cubic Hermite interpolation. We found that using interpolation for visualizing the error of exact solution and numerical solution for Sod's shock tube problem to be useful when graphing one solution on top of the other. However, we obtain the same results both when interpolating the moving mesh solution onto a given fixed mesh as we do when comparing values at the moving mesh points.

To examine behavior of the computed solution as ϵ and the number of nodes are varied using the SGWMFE method we start by using Sod's shock tube problem and comparing with the 'zero diffusion limit' solution. We then look at the results for the shallow water equations using ideas suggested from the study of the Sod shock tube problem.

5.1 Shock Tube Problem

The first system of PDEs, Sod's shock tube problem in one dimension, gave results as expected, see figure 3. It is well known that diffusion results in a shock of width of $O(\epsilon)$ and a contact discontinuity of width $O(\sqrt{\epsilon t})$ and both features are reproduced in the computed solution. A solution to the 'zero diffusion limit' using the code from Anderson (1984) has been used to monitor the behavior of the SGWMFE solution as ϵ is varied, see figure 4, where the difference between the SGWMFE solution with artificial diffusion, ϵ, and the 'zero diffusion limit' solution is plotted as ϵ is varied at a fixed time, $t = 0.1$. This shows that the SGWMFE solution approximates the 'zero diffusion limit' solution as ϵ decreases. Note that we would expect the difference to be dominated by the $O(\sqrt{\epsilon t})$ spreading of the finite diffusion solution near the contact discontinuity.

This is a problem where a complicated structure arises immediately out of nothingness (particularly the immediate spike in the momentum). Thus the initial placement of nodes in the initial ramp is critical; there must be nodes in the right places to resolve the immediate structure. We have used an initial placement similar to that shown in figure 7.3 of Carlson and Miller (1998a) for this same test problem.

5.2 Shallow Water Equations

Example results for the second problem, shallow water flow, are shown in figure 5 and are as expected from published solutions using other methods. As the initial hump of water falls under gravity, a jump occurs and is propagated to the right boundary where it is reflected.

One method to examine the convergence of solutions for moving mesh methods has been to overlay the solution using varying number of points and reducing ϵ until the difference on the graphs is smaller than some tolerance. Often "converged" means that there is no difference

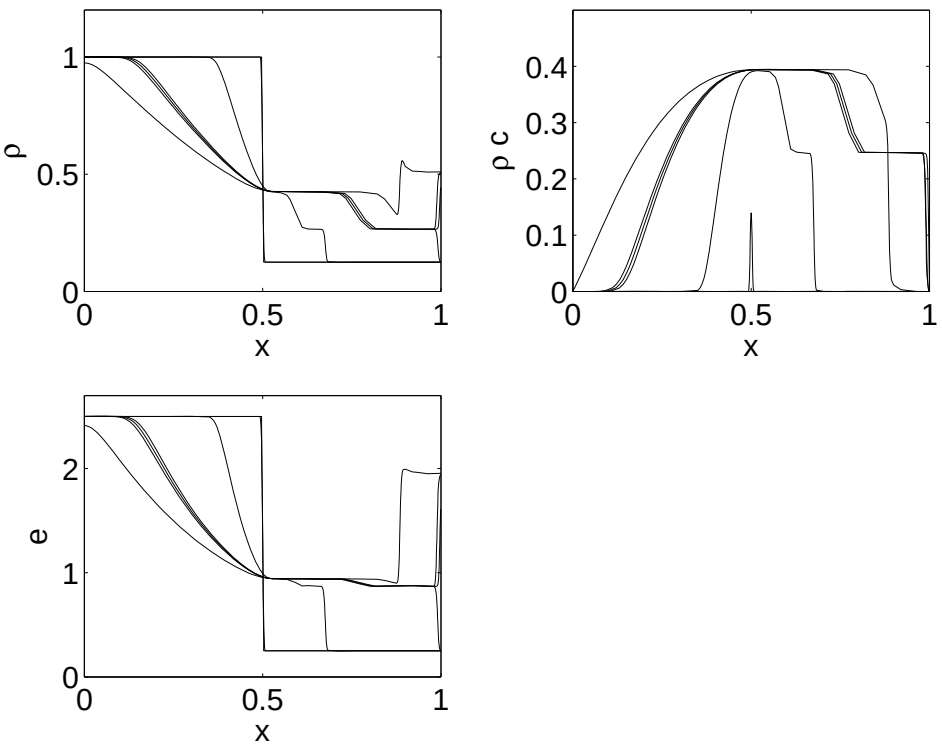

Figure 3: SGWMFE solutions for Sod's shock tube problem using 72 cells. $\epsilon = (10)^{-3}$. Density $= \rho$, velocity $= c$, momentum $= \rho c$, total energy $= e$. Figures show plots of the solutions for times $t = 0.1.28.288.296.4$

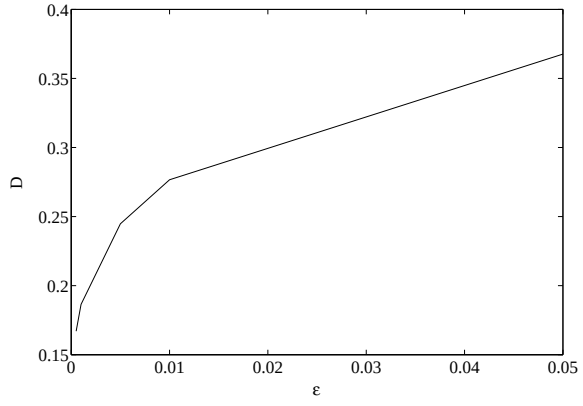

Figure 4: L_1 measure of difference between SGWMFE solution for Sod's shock tube problem at time $t = 0.1$ and the zero diffusion limit problem as artificial diffusion, ϵ, is varied.

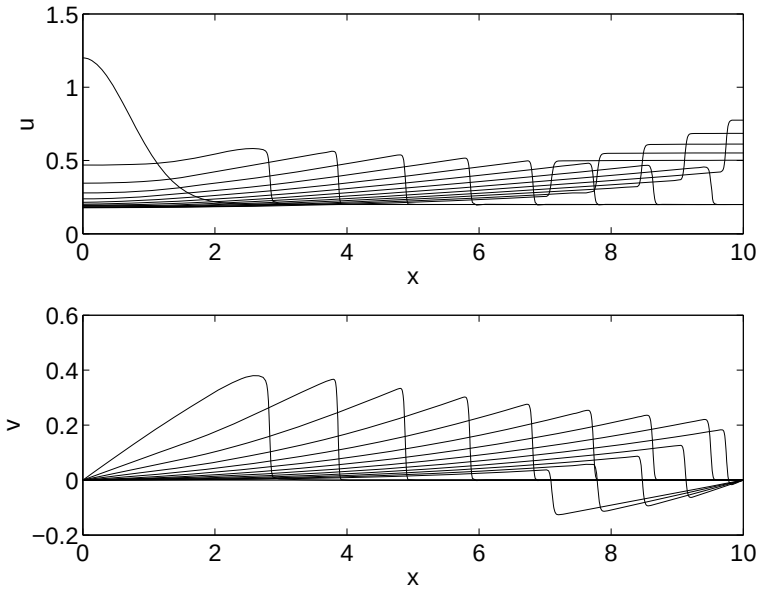

Figure 5: SGWMFE solutions for the Shallow Water Equations. Solution using 80 cells for times $t = 0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$. $\epsilon = 5(10)^{-3}$

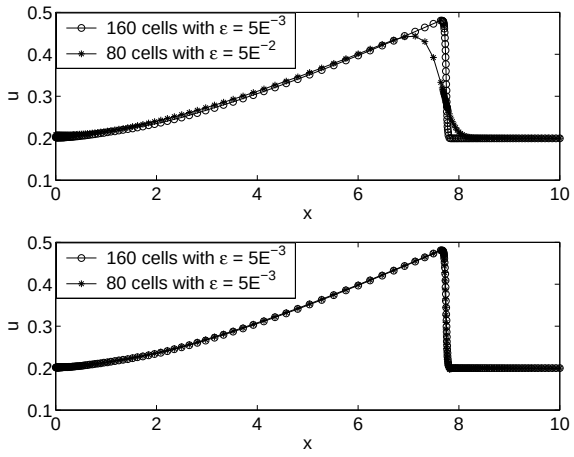

Figure 6: SGWMFE solutions for shallow water equations at $t = 0.7$. Bottom: 160 cells versus 80 cells for the same artificial diffusion, $\epsilon = 5(10)^{-3}$; thus the solution for this fixed ϵ seems quite accurate with 80 or fewer cells. Top: the solutions (both probably quite accurate) with different artificial diffusion, $\epsilon = 5(10)^{-3}$ and $\epsilon = 5(10)^{-2}$. The larger artificial diffusion widens the shock but leaves its position largely unchanged.

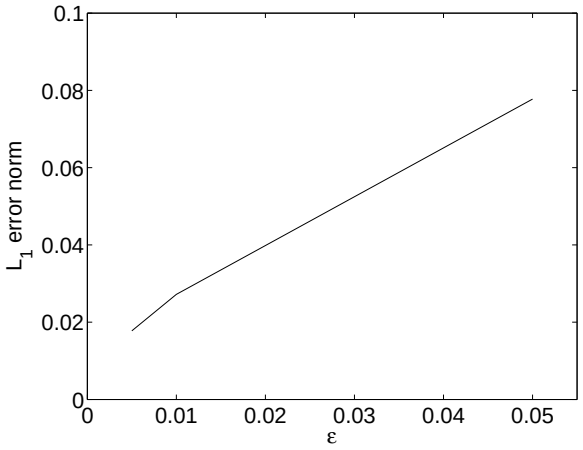

Figure 7: Shallow water equations at $t = 0.7$: L_1 difference between solution using 160 cells and $\epsilon = 5(10)^{-3}$ and solutions using 80 cells and varying ϵ.

in graphs to the naked eye. For example, in the lower part of figure 6 we show the numerical solution using 80 nodes and the solution with 160 nodes for fixed artificial diffusion,$\epsilon = 5\times10^{-3}$. In the upper part of 6 we show the effect of varying the artificial diffusion, with $\epsilon = 5 \times 10^{-2}$ and $\epsilon = 5 \times 10^{-3}$. Both solutions are thought to be accurate solutions.

In the case of the shallow water equations, we show in figure 7 the L_1 norm of the difference between the solution for $\epsilon = 5 \times 10^{-3}$ with 180 nodes and solutions using 80 nodes and varying ϵ, all at fixed time $t = 0.7$, showing that for this case, where the shock is of thickness $O(\epsilon)$, linear decrease in difference as ϵ decreases.

6 Summary

In this paper we have set out the string or second gradient weighted finite element method for one space dimension using a very general formulation in terms of a projection matrix. All of the ideas presented generalise straightforwardly to multiple equations. In a subsequent work, we will examine the generalisation to multiple space dimensions, but that too follows the ideas outlined here. The SGWMFE method is an alternative formulation to the GWMFE method. While it is simpler and more elegant for extension to larger numbers of equations, we cannot say that it has significant computational advantages for the model problems we consider here beyond perhaps being able to achieve solutions for smaller artificial diffusion. We have examined convergence as the number of nodes increases (for fixed artificial diffusion) and then as the artificial diffusion tends to the zero limit. However the initial distribution of nodes is also extremely important for certain problems. Moving finite element methods remain techniques where considerable tuning, particularly of the initial mesh, is required: but for classes of problems described as convection-diffusion like, implementing the SGWMFE method is straightforward and provides considerable fidelity using remarkably few mesh nodes.

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