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MATCHING GAMES: THE LEAST CORE AND THE NUCLEOLUS

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A matching game is a cooperative game defined by a graph $G = (N, E)$. The player set is N and the value of a coalition $S \subseteq N$ is defined as the size of a maximum matching in the subgraph induced by S. We show that the nucleolus of such games can be computed efficiently. The result is based on an alternative characterization of the least core, which may be of independent interest. The general case of weighted matching games remains unsolved.

1. Introduction. A *cooperative game* is defined by a set N of *players* and a *characteristic function* $v: 2^N \to \mathbb{R}$, associating a *value* $v(S)$ to every subset (*coalition*) $S \subseteq N$. We assume that $v(\varnothing) = 0$. The value $v(S)$ of a coalition $S \subseteq N$ is interpreted as the total gain the members of S can achieve by cooperating.

The central problem in cooperative game theory is how to allocate the total gain $v^* = v(N)$ among the individual players $i \in N$ in a "fair" way. There are various notions of fairness and corresponding allocation rules (*solution concepts*).

Clearly, a useful solution concept should not only be "fair" in an adequate sense but also efficiently computable. The computational complexity of classical solution concepts has therefore been studied with growing interest during the last years (see, e.g., Deng and Papadimitriou 1994, Granot and Granot 1992, Granot et al. 1996, Faigle et al. 1997, 1998, Faigle et al. 1998a, Deng et al. 1999).

The most prominent and widely accepted solution concept is the *core* of a game:

$$\text{core}(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v^*, x(S) \geq v(S) \text{ for all } S \subset N\}.$$

Here, we use the shorthand notation

$$x(S) := \sum_{i \in S} x_i$$

for $S \subseteq N$. Any $x \in \mathbb{R}^N$ with $x(N) = v^*$ is an *allocation*. So a core allocation $x \in \mathbb{R}^N$ guarantees each coalition $S \subseteq N$ to be satisfied in the sense that it gets at least what it could gain on its own.

If the core is empty (and even in case it is not) one might try to find allocations x in the *least core*, satisfying all coalitions $S \subset N$ as much as possible. To this end we let $\mathcal{F}_0 := \{\varnothing, N\}$ and consider the LP

$$\begin{array}{lll} (P_1) & \max & \epsilon \\ & \text{s.t.} & x(S) \geq v(S) + \epsilon \quad (S \notin \mathcal{F}_0), \\ & & x(N) = v^*, \end{array}$$

with optimum value $\epsilon_1 \in \mathbb{R}$. (Clearly, $\epsilon_1 \geq 0$ if and only if $\text{core}(N, v) \neq \varnothing$.)

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We let $P_1(\epsilon)$ denote the set of all $x \in \mathbb{R}^N$ such that (x, ϵ) satisfies the constraints of (P_1). So core$(N, v)= P_1(0)$. The *least core* is defined as

$$\text{leastcore}(N, v) := P_1(\epsilon_1).$$

The *excess* of a coalition $\varnothing \neq S \neq N$ with respect to an allocation $x \in \mathbb{R}^N$ is defined as

$$e(S, x) := x(S) - v(S).$$

So least core allocations are those that maximize the minimal excess. If the least core is not yet a single point, one might try to find "the best" allocation in the least core by further pursuing the idea of maximizing minimum excess: Given an allocation $x \in \mathbb{R}^N$ define the *excess vector* $\theta(x)$ as the $2^N - 2$ dimensional vector whose components are the excesses $e(S, x)$, $\varnothing \neq S \neq N$, arranged in nondecreasing order. The *nucleolus* (Schmeidler 1969) is then the (unique!) allocation $x^* \in \mathbb{R}^N$ that lexicographically maximizes the excess vector $\theta(x)$.

Although computational aspects will be discussed later, it is immediately clear that computing the nucleolus by explicit lexicographic optimization of the excess vector is intractable: In general, there are exponentially (in $|N|$) many different excess values, whereas an efficient procedure should be polynomial in $|N|$. The standard procedure for computing the nucleolus proceeds by solving up to $|N|$ linear programs (cf. Maschler et al. 1979). To present it we introduce the following notation: For a polyhedron $P \subseteq \mathbb{R}^N$ let

$$\text{Fix}\, P := \{S \subseteq N \mid x(S) = y(S) \text{ for all } x, y \in P\}$$

denote the set of coalitions *fixed* by P.

Now, assume we have determined the least core $P_1(\epsilon_1)$. We then proceed to maximize the minimal excess on those coalitions that are not already fixed, i.e., we solve

$$\begin{array}{lll} (P_2) & \max & \epsilon \\ & \text{s.t.} & x \in P_1(\epsilon_1), \\ & & x(S) \geq v(S) + \epsilon \quad (S \notin \text{Fix}\, P_1(\epsilon_1)), \end{array}$$

and let $\epsilon_2 > \epsilon_1$ be the corresponding optimum value. Extending our previous notation in the obvious way, we let $P_2(\epsilon)$ denote the set of all $x \in \mathbb{R}^N$ satisfying the constraints of (P_2) for $\epsilon \in \mathbb{R}$. Now proceed to solve the problem

$$\begin{array}{lll} (P_3) & \max & \epsilon \\ & \text{s.t.} & x \in P_2(\epsilon_2), \\ & & x(S) \geq v(S) + \epsilon \quad (S \notin \text{Fix}\, P_2(\epsilon_2)), \end{array}$$

and so on, until the problem

$$\begin{array}{lll} (P_r) & \max & \epsilon \\ & \text{s.t.} & x \in P_{r-1}(\epsilon_{r-1}), \\ & & x(S) \geq v(S) + \epsilon \quad (S \notin \text{Fix}\, P_{r-1}(\epsilon_{r-1})), \end{array}$$

defines a unique solution $x^* \in \mathbb{R}^N$, which is equal to the nucleolus of the game (N, v).

In each iteration k at least one subset $S \subseteq N$ is fixed (to $x(S) = v(S) + \epsilon_k$) which was not fixed before. So the dimension of the feasible region decreases by at least 1 in each iteration, implying that $r \leq |N|$. By construction, we proceed until every $S \subseteq N$ is fixed. Hence, finally, in particular, every one-element coalition $\{i\} \subseteq N$ has some fixed value $x(\{i\}) = x_i^*$. So in the end, the feasible set consists of a single point x^*.

Remark 1.1. The set $\operatorname{Fix} P_k(\epsilon_k)$ consists of all coalitions $S \subseteq N$ which were "fixed" to $x(S) = v(S) + \epsilon_j$ in iterations $j = 1, \ldots, k$ *plus* all coalitions $S' \subseteq N$ which are linear combinations of these $S \subseteq N$ (in the sense that $\chi_{S'} = \sum \lambda_S \chi_S$ with $\lambda_S \in \mathbb{R}$ holds for the corresponding incidence vectors). In each iteration we have to identify the set $\operatorname{Fix} P_k(\epsilon_k)$ to determine the (in general exponentially many) inequality constraints, one for each $S \notin \operatorname{Fix} P_k(\epsilon_k)$.

The above "linear programming approach" to the nucleolus is also interesting from a structural point of view, as it implies a nice bound on the size $\langle x^* \rangle$ of the nucleolus (number of bits necessary to represent x^*). Let $\langle v \rangle$ denote the maximum size of the v-values, i.e., $\langle v \rangle := \max\{\langle v(S) \rangle \mid S \subseteq N\}$.

Theorem 1.1. *The nucleolus of* (N, v) *has size bounded polynomially in* $|N|$ *and* $\langle v \rangle$.

Proof. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subseteq 2^N$ denote the increasing sequence of fixed sets in $(P_1), \ldots, (P_r)$, i.e., $\mathcal{F}_0 = \{\varnothing, N\}$ and

$$\mathcal{F}_i := \operatorname{Fix} P_i(\epsilon_i) \quad (i = 1, \ldots, r-1).$$

Then, the unique lexicographic optimum of

$$\begin{aligned}
&\text{lex-max } (\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_r, x_1, \ldots, x_{|N|}) \\
&\text{s.t.} \quad x(N) = v^*, \\
&\qquad x(S) \geq v(S) + \tilde{\epsilon}_1 \quad (S \notin \mathcal{F}_0), \\
&\qquad x(S) \geq v(S) + \tilde{\epsilon}_2 \quad (S \notin \mathcal{F}_1), \\
&\qquad \vdots \\
&\qquad x(S) \geq v(S) + \tilde{\epsilon}_r \quad (S \notin \mathcal{F}_{r-1}),
\end{aligned}$$

equals $(\epsilon_1, \ldots, \epsilon_r, x^*)$, where x^* is the nucleolus and $\epsilon_1, \ldots, \epsilon_r$ are the optimum values of $(P_1), \ldots, (P_r)$. Hence $(\epsilon_1, \ldots, \epsilon_r, x^*)$ is a vertex of the feasibility region of the above program. As such its size is polynomial in the dimension $r + |N| = \mathcal{O}(|N|)$ and the maximum size of a constraint (i.e., the *facet complexity*, cf. Grötschel et al. 1993). The latter is bounded by $|N| + \langle v \rangle$. □

As to complexity issues in cooperative game theory, various results have been obtained for particular classes of games and solution concepts. For example, so-called minimum spanning tree games have been studied with respect to core, least core and nucleolus (cf. Megiddo 1978, Granot et al. 1996, Faigle et al. 1997, Faigle et al. 2000). Deng et al. (1999) analyze the core of various combinatorial games with respect to complexity. Granot et al. (1998) study the complexity of the nucleolus in general.

Matchings in graphs are one of the most studied (and best understood) subjects in combinatorial optimization (cf. Lovász and Plummer 1986, Korte and Vygen 2000). From a game theoretical perspective, *matching games* (sometimes called *roommate games*) as introduced below have been studied by many authors. In a classical paper, Shapley and Shubik (1972) introduce the bipartite case (*assignment games*). The problem of computing the nucleolus in the bipartite case is settled by Solymosi and Raghavan (1994). The general (nonbipartite) case is studied in Deng et al. (1999), characterizing when the core is nonempty. The extreme points of the core (in case this is nonempty) are characterized in Eriksson and Karlander (2001). (This paper also touches a "nontransferable utility" version of the roommate problem and provides several references.) Faigle et al. (1998) introduce the *nucleon* as an alternative to the nucleolus, present an efficient algorithm for computing the nucleon, and point out that the problem of computing the nucleolus remains unsolved. Faigle et al. (1998b) prove a general result on the complexity of the so-called *kernel* (a subset of the least core) of a game. As a consequence of this, computing an element in the least core is easy for matching games. The complexity of the nucleolus remains unsolved yet. In the current paper, we solve the "unweighted case" by presenting an efficient algorithm for computing the nucleolus of *cardinality matching games*. Our result is based on a polynomial description of the least core of such games, which might be of independent interest. We would like to remark that this result is generalized in Paulusma (2001), in which the class of cardinality matching games is extended to the class of so-called *node matching games*.

2. Matching games. The classical example of assignment (bipartite matching) games deals with the situation, where n house-owners are to sell their houses to n potential buyers (cf. Shapley and Shubik 1972). Clearly, there is no reason to assume that the group of house-owners is disjoint from the set of potential buyers. We are thus led to consider exchange markets with a set N of agents (players). Each player $i \in N$ can do business with at most one player $j \in N$ (within a given time unit). Players i and j are willing to do business with each other if this results in a common profit $w(i, j) \geq 0$. Obviously, the total profit will be maximal, if all persons in N cooperate and a maximum weight matching can be constructed. The problem of dividing the total profit is an allocation problem.

A *matching game* (N, v) is determined by a graph $G = (N, E)$ with node set N and by a weight function $w \geq 0$ defined on the edge set E. The value $v(S)$ of a coalition $S \subseteq N$ is the value of a maximum weight matching in the subgraph of G induced by S, i.e.,

$$v(S) := \max\{w(M) \mid M \subseteq E(S) \text{ is a matching}\},$$

where $E(S) \subseteq E$ denotes the set of edges joining nodes of S. We also use the standard notation $N(F)$ to denote the set of nodes covered by a subset $F \subseteq E$.

In the following, we restrict ourselves to *cardinality matching games*. These arise when $w \equiv 1$, i.e., the characteristic function is given by

$$v(S) := \max\{|M| \mid M \subseteq E(S) \text{ is a matching}\}.$$

We first consider the matching game (N, v) determined by $G = K_2$, the complete graph on two nodes. The core of this game is equal to

$$\text{core}(N, v) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1 \text{ and } x \geq 0\},$$

and the nucleolus of (N, v) is $x^* = (1/2, 1/2)$.

In the following we will assume that $G \neq K_2$. Below we give some examples of (cardinality) matching games with an empty core. We have also computed the nucleolus of these games.

EXAMPLES. (i) Let $G = (N, E)$ be the graph as shown in Figure 1. N is split into $\{a\} \cup D_1 \cup D_2$. Then (P_1) has a unique optimal solution: the nucleolus x^* given by

$$x_i^* = \begin{cases} \dfrac{4}{7} & \text{if } i = a, \\ \dfrac{3}{7} & \text{if } i \in D_1 \cup D_2, \end{cases}$$

and $\epsilon_1 = -3/7$.

(ii) Let $G = (N, E)$ be the graph as shown in Figure 2. We have $N = D_1 \cup D_2 \cup D_3$. Then $\epsilon_1 = -1$ and $P_1(-1)$ contains all allocations $x \in \mathbb{R}^N$ for which

$$x_i = x_j \quad (i, j \in D_p, p = 1, \ldots, 3),$$
$$x_i + x_j = \frac{1}{2} \quad (i \in D_1, j \in D_2),$$

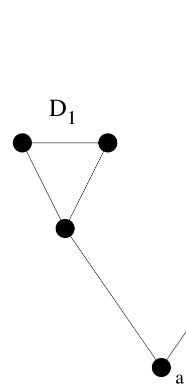

FIGURE 1. A connected graph with empty core.

$$x_i = \frac{1}{2} \quad (i \in D_3),$$
$$x \geq 0.$$

The nucleolus x^* is given by

$$x^* \equiv \frac{1}{4} \quad \text{on } D_1 \cup D_2,$$
$$x^* \equiv \frac{1}{2} \quad \text{on } D_3.$$

In the following, we will need some fundamental results and concepts from matching theory: A (*near-*) *perfect* matching is one that covers all nodes (except one). A graph is *factor-critical* if removing any node results in a perfectly matchable graph.

If $A \subseteq N$, we let as usual $G\backslash A$ denote the graph obtained by removing A. A component of $G\backslash A$ is called *even* or *odd* if it has an even respectively odd number of nodes. We let $\mathscr{C} = \mathscr{C}(A)$ denote the set of even components of $G\backslash A$ and $\mathscr{D} = \mathscr{D}(A)$ the set of odd components of $G\backslash A$. Recall that $A \subseteq N$ is called a *Tutte set* if each maximum matching M of G decomposes as

$$M = M_{\mathscr{C}} \cup M_{A,\mathscr{D}} \cup M_{\mathscr{D}},$$

where $M_{\mathscr{C}}$ is a perfect matching in $\bigcup \mathscr{C}$, the union of all even components. $M_{\mathscr{D}}$ induces a near-perfect matching in all odd components $D \in \mathscr{D}$ and $M_{A,\mathscr{D}}$ is a matching that matches A (completely) into $\bigcup \mathscr{D}$, the union of odd components. Equivalently, A is a Tutte set if

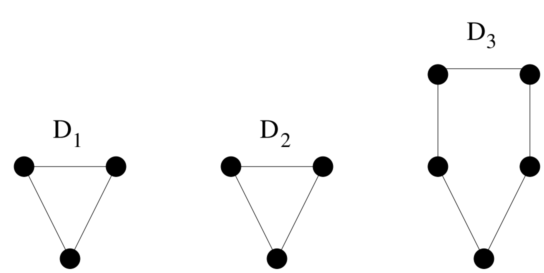

FIGURE 2. A disconnected graph with empty core.

and only if the size v^* of a maximum matching in G equals

$$v^* = \sum_{C\in\mathscr{C}} \frac{|C|}{2} + |A| + \sum_{D\in\mathscr{D}} \frac{(|D|-1)}{2}.$$

Tutte sets can be found efficiently. More precisely, the following is true (see, e.g., Lovász and Plummer 1986).

THEOREM 2.1 (GALLAI-EDMONDS DECOMPOSITION). *Given $G=(N,E)$, one can efficiently construct a unique Tutte set $A\subseteq N$ such that*

(i) *all odd components $D\in\mathscr{D}$ are factor-critical, and*

(ii) *for each $D\in\mathscr{D}$ there is some maximum matching that does not completely cover D.*

In the following, we assume that $A\subseteq N$ is the (fixed) Tutte set satisfying conditions (i) and (ii) in Theorem 2.1. We will sometimes identify subsets of N with the corresponding induced subgraphs. For example, if $i\in N$ is a node we do not hesitate to write $i\in D$ to indicate that i is a node of the component $D\in\mathscr{D}$. If $x\in\mathbb{R}^N$ is an allocation, we consequently write

$$x(D) = \sum_{i\in D} x(i).$$

Finally, we also extend our general shorthand notation in the following way, if no misunderstanding is possible: If $e=(i,j)\in E$, we write $x(e)=x(\{i,j\})$. More generally, if $M\subseteq E$ is a matching, we let $x(M):=x(N(M))$.

After these preliminaries, let us study the core and the least core of a matching game (N,v) defined by a graph G. We start with the following simple observation (cf. also Deng et al. 1999):

THEOREM 2.2. *The matching game defined by $G=(N,E)$ has nonempty core $(\epsilon_1\geq 0)$ if and only if $|D|=1$ for all $D\in\mathscr{D}$.*

PROOF. "⇐": Suppose $|D|=1$ for all $D\in\mathscr{D}$. Then $x\in\mathbb{R}^N$ defined by

$$\begin{aligned} x &\equiv \tfrac{1}{2} \quad \text{on } \textstyle\bigcup\mathscr{C},\\ x &\equiv 1 \quad \text{on } A,\\ x &\equiv 0 \quad \text{on } \textstyle\bigcup\mathscr{D}, \end{aligned}$$

is easily seen to be in the core.

"⇒": Suppose $D\in\mathscr{D}$ with $|D|\geq 3$. Let $e=(i,j)\in E(D)$. From Theorem 2.1 we conclude that $G\backslash i$ and $G\backslash j$ have matchings of size v^*. So if $x\in\mathbb{R}^N$ were in the core, then

$$x(N\backslash i)\geq v(N\backslash i)=v^* \quad\text{and}\quad x(N\backslash j)\geq v(N\backslash j)=v^*.$$

Furthermore, $x(e)=x(\{i,j\})\geq 1$. Together, these imply $x(N)>v^*$, a contradiction. Hence, the core must be empty. □

Because the Gallai-Edmonds decomposition can be computed efficiently, we can easily check whether the core is empty or not. In the latter case, the least core and the nucleolus are straightforward to compute. This is essentially due to the fact that all ϵ_i are nonnegative:

THEOREM 2.3. *In case of nonempty core $(\epsilon_1\geq 0)$ the least core equals the set of allocations $x\in\mathbb{R}^N$ solving*

$$\begin{aligned} (P_1^+)\quad &\max\quad \epsilon\\ &\text{s.t.}\quad x(e)\geq 1+\epsilon \quad \textit{for all } e\in E,\\ &\qquad\ \ x_i\geq \epsilon \quad\qquad \textit{for all } i\in N,\\ &\qquad\ \ x(N)=v^*. \end{aligned}$$

PROOF. The proof is straightforward, using the fact that the above constraints (for $\epsilon \geq 0$) imply $x(S) \geq v(S) + \epsilon$ for all $S \subset N$. □

REMARK 2.1. Note that the optimum value $\epsilon_1^+ = \epsilon_1$ of the LP problem in Theorem 2.3 is always at most 0. (Recall that we assume that $G \neq K_2$, in which case $\epsilon_1 = 1/2$.) If the core is nonempty, then $\epsilon_1^+ = \epsilon_1 = 0$, and the least core coincides with the core.

Continuing in a similar way, the nucleolus can also be computed easily. We first identify

$$E_1 := \{e \in E \mid e \in \operatorname{Fix} P_1^+(\epsilon_1^+)\} \quad \text{and} \quad N_1 := \{i \in N \mid i \in \operatorname{Fix} P_1^+(\epsilon_1^+)\}$$

and then solve

$$\begin{array}{lll} (P_2^+) & \max & \epsilon \\ & \text{s.t.} & x(e) = 1 + \epsilon_1^+ \quad (e \in E_1), \\ & & x_i = \epsilon_1^+ \quad (i \in N_1), \\ & & x(e) \geq 1 + \epsilon \quad (e \in E \backslash E_1), \\ & & x_i \geq \epsilon \quad (i \in N \backslash N_1), \\ & & x(N) = v^*, \end{array}$$

with optimum value $\epsilon_2^+ = \epsilon_2$ and so on, until we obtain a linear program (P_r^+) that defines a unique solution $\tilde{x} \in \mathbb{R}^N$. Using the fact that the constraints $x \geq 0$ and $x(e) \geq 1$ for all $e \in E$ imply $x(S) \geq v(S)$ for all $S \subset N$, it is clear that $\tilde{x}$ is equal to the nucleolus of (N, v).

REMARK 2.2. Note that also for general weighted matching games with nonempty core, a similar characterization of the (least) core and nucleolus exists.

The above approach fails in the case $\epsilon_1 < 0$. In this case, at least intuitively, large coalitions $S \subset N$ get fixed in the first place rather than small ones (single nodes and edges) as above. The case $\epsilon_1 < 0$ (empty core) is treated in §3.

3. When the core is empty. In the following we assume that the core is empty. Equivalently, $\epsilon_1 < 0$ and $|D| > 1$ for some odd component $D \in \mathcal{D}$. We first state the following simple fact (which in the nonempty core case follows trivially from Theorem 2.3):

LEMMA 3.1. $\operatorname{leastcore}(N, v) \subseteq \mathbb{R}_+^N$.

PROOF. Assume to the contrary that (x, ϵ_1) is an optimal solution of

$$\begin{array}{lll} (P_1) & \max & \epsilon \\ & \text{s.t.} & x(S) \geq v(S) + \epsilon \quad (S \notin \mathcal{F}_0), \\ & & x(N) = v^*, \end{array}$$

and $x_i < 0$ for some $i \in N$.

CLAIM. If $S \subset N$ satisfies $x(S) \geq v(S) + \epsilon_1$ with equality, then $i \in S$.

PROOF. Assume to the contrary that $i \notin S$.

Case 1. $S \subset S \cup i \subset N$. Then $x(S \cup i) < x(S) = v(S) + \epsilon_1 \leq v(S \cup i) + \epsilon_1$ contradicts the feasibility of x.

Case 2. $S \subset S \cup i = N$. Then $x(N) = x(S) + x_i = v(S) + \epsilon_1 + x_i < v(S) \leq v^*$ again contradicts the feasibility of x.

Hence, the claim is true. But then we may slightly increase x on $\{i\}$ and decrease x on $N \backslash i$ uniformly by the same total amount, thereby obtaining a better solution. This proves the lemma. □

REMARK 3.1. In §1 we have actually defined the *prenucleolus* of a game (cf. Schmeidler 1969). However, by Lemma 3.1 the prenucleolus and nucleolus of a matching game coincide.

Due to Lemma 3.1 problem (P_1) defining the least core can equivalently be stated as follows. Let $\mathscr{M}$ denote the set of matchings $M \subseteq E$. Then (recall our notation $x(M) = x(N(M))$ from §2):

$$\begin{aligned} (P_1) \quad & \max \quad \epsilon \\ & \text{s.t.} \quad x(M) \geq |M| + \epsilon \quad (M \in \mathscr{M}), \\ & \qquad\; x(N) = v^*, \\ & \qquad\; x \geq 0. \end{aligned}$$

Proposition 3.1. *Checking whether a given $x \in \mathbb{R}^N$ is an element of $P_1(\epsilon)$ can be done in polynomial time.*

Proof. It suffices to show that for given $x \in \mathbb{R}^N$ and $\epsilon \in \mathbb{R}$ we can sufficiently check whether

$$x(M) \geq |M| + \epsilon \quad (M \in \mathscr{M})$$

holds. This can be done by solving a minimum weight matching problem on $G = (N, E)$ with respect to the edge weights

$$w_{ij} := x_i + x_j - 1 \quad ((i, j) \in E),$$

(see, e.g., Lovász and Plummer 1986). □

As a consequence of Proposition 3.1 we can solve (P_1) efficiently (see Grötschel et al. 1993). Here we aim for more, namely, a concise description of $P_1(\epsilon_1)$.

As a first step we introduce a relaxation $(\hat{P}_1)$ of (P_1) below, which is easier to analyze and, as we will see, defines the same optimum value. To motivate this approach, note that, as mentioned earlier, we expect rather large matchings to become tight when solving (P_1). We let $\mathscr{M}^*$ denote the set of maximum matchings in G. Each $M \in \mathscr{M}^*$ matches A completely in $\mathscr{D}$. By condition (ii) of Theorem 2.1, given $D \in \mathscr{D}$, there is some $M \in \mathscr{M}^*$ matching A into $\mathscr{D} \setminus \{D\}$. We say that M *leaves D uncovered*. Let $\mathscr{M}_{\mathscr{D}}$ denote the set of matchings $M \subseteq E(\bigcup \mathscr{D})$ that are completely contained in the union of the odd components.

We will study the following relaxation of (P_1):

$$\begin{aligned} (\hat{P}_1) \quad & \max \quad \epsilon \\ & \text{s.t.} \quad x(M) \geq |M| + \epsilon \quad (M \in \mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}), \\ & \qquad\; x(N) = v^*, \\ & \qquad\; x \geq 0, \end{aligned}$$

with optimum value $\hat{\epsilon}_1$. (As in the proof of Theorem 2.2, it is easy to see that $\hat{\epsilon}_1 < 0$, cf. also below.)

To investigate the structure of optimal solutions of $(\hat{P}_1)$, let us introduce some notation. As before, $\hat{P}_1(\epsilon)$ denotes the set of $x \in \mathbb{R}^N$ such that (x, ϵ) is feasible for $(\hat{P}_1)$. If $x \in \hat{P}_1(\hat{\epsilon}_1)$ is an optimal solution, we say that $M \in \mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$ is *x-tight*, if $x(M) = |M| + \hat{\epsilon}_1$. Given a feasible solution $x \in \hat{P}_1(\epsilon)$ and $D \in \mathscr{D}$, let

$$x_D := \frac{x(D)}{|D|}$$

denote the average value of x on D. Define $\bar{x} \in \mathbb{R}^N$ by *averaging* x on each component $D \in \mathscr{D}$, i.e.,

$$\bar{x}_i := x_D \quad (i \in D, D \in \mathscr{D})$$

and leaving x unchanged on $A \cup \bigcup \mathscr{C}$.

Lemma 3.2. *If $x \in \widehat{P}_1(\epsilon)$, then $\bar{x} \in \widehat{P}_1(\epsilon)$ and $\epsilon < 0$.*

Proof. Let $x \in \widehat{P}_1(\epsilon)$. It suffices to show that averaging x on some component $D \in \mathscr{D}$ preserves feasibility. Thus, let $D \in \mathscr{D}$ and let $\tilde{x} \in \mathbb{R}^N$ be obtained by averaging x on D, i.e., $\tilde{x}_i = x_D$ $(i \in D)$.

Certainly, $\tilde{x}$ satisfies $\tilde{x} \geq 0$ and $\tilde{x}(N) = v^*$. We are left to check $\tilde{x}(M) \geq |M| + \epsilon$ for $M \in \mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$.

Suppose $M \in \mathscr{M}^*$. Then either M covers D or $M \cap D = D \backslash i$ for some $i \in D$. In the first case, $\tilde{x}(M) = x(M)$ and the claim follows. In the second case, we may assume without loss of generality that $i \in D$ maximizes x_i over D, otherwise we replace M inside D by some other near-perfect matching without changing $\tilde{x}(M) - |M|$. (Recall that D is factor-critical.) Then $x_i \geq \tilde{x}_i$ holds and consequently $\tilde{x}(M) \geq x(M)$, so the claim follows as $x \in \widehat{P}_1(\epsilon)$.

Next, consider $M \in \mathscr{M}_{\mathscr{D}}$ and assume M minimizes $\tilde{x}(M) - |M|$ over $\mathscr{M}_{\mathscr{D}}$. If $\tilde{x} \equiv x_D > 1/2$ on D and $M \cap D \neq \varnothing$, then removing an edge in $M \cap D$ results in a matching $\widehat{M}$ with $\tilde{x}(\widehat{M}) - |\widehat{M}| < \tilde{x}(M) - |M|$. Hence, $M \cap D = \varnothing$ and $\tilde{x}(M) = x(M)$ and the claim follows.

If $\tilde{x} \equiv x_D \leq 1/2$ on D, then $M \cap D$ is without loss of generality a near-perfect matching in D and we argue as we did for $M \in \mathscr{M}^*$.

Finally, let us show that $\epsilon < 0$. Let $D \in \mathscr{D}$ be a component with $|D| > 1$. If $\bar{x} \equiv 0$ on D, then $\epsilon \leq -1$. (Indeed, if $e \in E(D)$, then $0 = \bar{x}(e) \geq 1 + \epsilon$.) If $\bar{x} \equiv x_D > 0$ on D, let $M \in \mathscr{M}^*$ be a maximum matching leaving some $i \in D$ unmatched. Then $\bar{x}(M) \geq v^* + \epsilon$ and $\bar{x}(M) < \bar{x}(M \cup i) \leq \bar{x}(N) = v^*$. Hence, $\epsilon < 0$. □

We conclude that $\hat{\epsilon}_1 < 0$. If $x \in \widehat{P}_1(\hat{\epsilon}_1)$ is an optimal allocation, so is $\bar{x}$. Furthermore, some matchings in $\mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$ must be $\bar{x}$-tight. These can in principle be found by minimizing $\bar{x}(M) - |M|$ over $\mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$. Minimizing $\bar{x}(M) - |M|$ over $\mathscr{M}^*$ amounts to solving a minimum weight maximum matching problem. Minimizing $\bar{x}(M) - |M|$ over $\mathscr{M}_{\mathscr{D}}$ is even trivial: We simply choose a near-perfect matching in each component $D \in \mathscr{D}$ with $\bar{x} \equiv x_D < 1/2$ (plus an arbitrary matching in all components on which $\bar{x} \equiv 1/2$). So computing an $\bar{x}$-tight $M \in \mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$ for given $\bar{x} \in \widehat{P}_1(\hat{\epsilon}_1)$ is easy.

We aim at a more structural characterization of $\bar{x}$-tight matchings for given $\bar{x} \in \widehat{P}_1(\hat{\epsilon}_1)$. Let $\mathscr{D}_{\max} = \mathscr{D}_{\max}(\bar{x}) \subseteq \mathscr{D}$ be the set of odd components on which $\bar{x} \equiv x_D$ is maximum (among all $D \in \mathscr{D}$).

Lemma 3.3. *No $\bar{x}$-tight $M \in \mathscr{M}^*$ covers all $D \in \mathscr{D}_{\max}$. If $\bar{x}$-tight matchings in $\mathscr{M}^*$ exist at all, then for each $D \in \mathscr{D}_{\max}$ there is some $\bar{x}$-tight $M \in \mathscr{M}^*$ leaving D uncovered.*

Proof. Suppose $\overline{M} \in \mathscr{M}^*$ is $\bar{x}$-tight and covers $D \in \mathscr{D}_{\max}$. Let $\widetilde{M} \in \mathscr{M}^*$ be any matching not covering D. (Recall Theorem 2.1.) Let $P \subseteq \overline{M} \cup \widetilde{M}$ be the unique maximal alternating path starting in D (in a node uncovered by $\widetilde{M}$) and ending in, say, $\widetilde{D}$ (in a node uncovered by $\overline{M}$). Reversing $\overline{M}$ along P results in a matching $M \in \mathscr{M}^*$ covering $\widetilde{D}$ instead of D. Since $D \in \mathscr{D}_{\max}$, we have $\bar{x}_D \geq \bar{x}_{\widetilde{D}}$, hence $\bar{x}(M) \leq \bar{x}(\overline{M})$. Thus M must be $\bar{x}$-tight again, proving the second claim.

The first claim follows by observing that if $\overline{M}$ would cover all $D \in \mathscr{D}_{\max}$, then $\widetilde{D} \notin \mathscr{D}_{\max}$ (as it is uncovered by $\overline{M}$). But then $\bar{x}_D > \bar{x}_{\widetilde{D}}$ and $\bar{x}(M) < \bar{x}(\overline{M})$, hence

$$\bar{x}(M) < \bar{x}(\overline{M}) = |\overline{M}| + \hat{\epsilon}_1 = |M| + \hat{\epsilon}_1,$$

contradicting $\bar{x} \in \widehat{P}_1(\hat{\epsilon}_1)$. □

Let $\mathscr{M}^*_{\mathscr{D}}$ denote the set of all maximum matchings in $\mathscr{M}_{\mathscr{D}}$.

Lemma 3.4. *Let $x \in \widehat{P}_1(\hat{\epsilon}_1)$. Then*

(i) $x = \bar{x}$,

(ii) $x \leq 1/2$ *on* $\bigcup \mathscr{D}$, *and*

(iii) *each* $M \in \mathscr{M}^*_{\mathscr{D}}$ *is x-tight.*

PROOF. Let $x \in \hat{P}_1(\hat{\epsilon}_1)$. We first prove (ii) and (iii) for $\bar{x}$ and then show that $x = \bar{x}$.

(ii) $\bar{x} \le 1/2$ on $\bigcup \mathscr{D}$: Suppose to the contrary that $\bar{x} > 1/2$ on $D \in \mathscr{D}_{\max}$.

We first consider the case $A \cup \bigcup \mathscr{C} = \varnothing$. If $\mathscr{D}_{\max} = \mathscr{D}$, we had $\bar{x} > 1/2$ on $\bigcup \mathscr{D}$ and hence $\bar{x}(N) > v^*$, a contradiction. Hence $\mathscr{D}_{\max} \subset \mathscr{D}$. Then we may decrease $\bar{x}$ slightly and uniformly on $\bigcup \mathscr{D}_{\max}$ and increase $\bar{x}$ on $\bigcup \mathscr{D} \setminus \bigcup \mathscr{D}_{\max}$ resulting in some $\bar{\bar{x}} \in \hat{P}_1(\hat{\epsilon}_1)$ for which no $M \in \mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$ is tight. This contradicts the optimality of $\hat{\epsilon}_1$.

Now suppose $A \cup \bigcup \mathscr{C} \neq \varnothing$. If $\mathscr{D}_{\max} = \mathscr{D}$, we had $\bar{x} > 1/2$ on $\bigcup \mathscr{D}$ and hence no $M \in \mathscr{M}_{\mathscr{D}}$ were $\bar{x}$-tight. We may thus decrease $\bar{x}$ on $\bigcup \mathscr{D}$ and increase $\bar{x}$ on $A \cup \bigcup \mathscr{C}$ by the same (sufficiently small) amount $\delta > 0$ resulting in some $\bar{x}^\delta \in \hat{P}_1(\hat{\epsilon}_1)$ for which no $M \in \mathscr{M}^* \cup \mathscr{M}_{\mathscr{D}}$ is tight. (Recall Lemma 3.3.) This contradicts the optimality of $\hat{\epsilon}_1$.

If $\mathscr{D}_{\max} \subset \mathscr{D}$, we proceed similarly. Chose $\delta > 0$ sufficiently small and let $\bar{x}^\delta$ arise from $\bar{x}$ by

- Decreasing $\bar{x}_i$ by $\delta/|D|$ ($i \in D, D \in \mathscr{D}_{\max}$).
- Increasing $\bar{x}$ on A by δ' in total, where $(|\mathscr{D}_{\max}| - 1)\delta < \delta' < |\mathscr{D}_{\max}|\delta$.
- Increasing $\bar{x}$ uniformly on $\bigcup \mathscr{D} \setminus \bigcup \mathscr{D}_{\max}$ by $|\mathscr{D}_{\max}|\delta - \delta'$ in total.

For sufficiently small $\delta > 0$, the resulting $\bar{x}^\delta$ has $\bar{x}^\delta(M) > \bar{x}(M)$ for each $\bar{x}$-tight $M \in \mathscr{M}_{\mathscr{D}}$ (because none of these meets $\mathscr{D}_{\max}$) and $\bar{x}^\delta(M) > \bar{x}(M)$ for all $\bar{x}$-tight $M \in \mathscr{M}^*$ by Lemma 3.3. Hence, again $\bar{x}^\delta \in \hat{P}_1(\hat{\epsilon}_1)$ has no tight matchings, contradicting the optimality of $\hat{\epsilon}_1$.

(iii) Each $M \in \mathscr{M}^*_{\mathscr{D}}$ is $\bar{x}$-tight: Because $\bar{x} \le 1/2$ on $\bigcup \mathscr{D}$, each $M \in \mathscr{M}^*_{\mathscr{D}}$ minimizes $\bar{x}(M) - |M|$ over $\mathscr{M}_{\mathscr{D}}$. It therefore suffices to show that *some* matching in $\mathscr{M}_{\mathscr{D}}$ is $\bar{x}$-tight. Assume to the contrary that no matching in $\mathscr{M}_{\mathscr{D}}$ is $\bar{x}$-tight. As above, this excludes $\mathscr{M}^* \subseteq \mathscr{M}_{\mathscr{D}}$, so $A \cup \bigcup \mathscr{C}$ must be nonempty.

Case 1. $\bar{x}(\bigcup \mathscr{D}) > 0$. In this case we may slightly (and uniformly) decrease $\bar{x}$ on $\bigcup \mathscr{D}_{\max}$ and increase it by the same total amount on $A \cup \bigcup \mathscr{C}$. By Lemma 3.3 the resulting $\bar{\bar{x}}$ has no tight matchings, a contradiction.

Case 2. $\bar{x} \equiv 0$ on $\bigcup \mathscr{D}$. Then $\bar{x}(M) = v^*$ for $M \in \mathscr{M}^*$. Because $\hat{\epsilon}_1 < 0$, no $M \in \mathscr{M}^*$ were tight either, a contradiction.

(i) $x = \bar{x}$: For each $D \in \mathscr{D}$ we chose a node $i \in \mathscr{D}$ with maximum x-value and a near-perfect matching covering $D \setminus i$. Let $M \in \mathscr{M}^*_{\mathscr{D}}$ be the union of all these near-perfect matchings. By construction we have $x(M) \le \bar{x}(M)$ with equality if and only if $x \equiv \bar{x}$ on $\bigcup \mathscr{D}$. Because M is $\bar{x}$-tight,

$$x(M) < \bar{x}(M) = |M| + \hat{\epsilon}_1$$

would contradict $x \in \hat{P}_1(\hat{\epsilon}_1)$. $\square$

LEMMA 3.5. *Let $x = \bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$. Then there is some x-tight $M \in \mathscr{M}^*$. Moreover, if $D \in \mathscr{D}_{\max}$ or $|D| > 1$, then there is some x-tight $M \in \mathscr{M}^*$ not covering D.*

PROOF. The lemma is trivial in case $A \cup \bigcup \mathscr{C} = \varnothing$. So we suppose $A \cup \bigcup \mathscr{C} \neq \varnothing$ and we first claim that any $x \in \hat{P}_1(\hat{\epsilon}_1)$ has $x(A \cup \bigcup \mathscr{C}) > 0$. Indeed, any $M \in \mathscr{M}^*$ decomposes as

$$M = M_{\mathscr{C}} \cup M_{A,\mathscr{D}} \cup M_{\mathscr{D}}$$

with $M_{\mathscr{C}}$ a perfect matching of $\bigcup \mathscr{C}$, $M_{A,\mathscr{D}}$ matching A into $\mathscr{D}$, and $M_{\mathscr{D}} \in \mathscr{M}^*_{\mathscr{D}}$. Because $x \in \hat{P}_1(\hat{\epsilon}_1)$, we have

$$x(M) \ge |M_{\mathscr{C}}| + |M_{A,\mathscr{D}}| + |M_{\mathscr{D}}| + \hat{\epsilon}_1.$$

Because $M_{\mathscr{D}}$ is x-tight (cf. Lemma 3.4 (iii)), we have $x(M_{\mathscr{D}}) = |M_{\mathscr{D}}| + \hat{\epsilon}_1$, hence $x(M_{\mathscr{C}} \cup M_{A,\mathscr{D}}) \ge |M_{\mathscr{C}}| + |M_{A,\mathscr{D}}|$. Because $x \le 1/2$ on $\bigcup \mathscr{D}$, we conclude that indeed

$$x\big(A \cup \bigcup \mathscr{C}\big) = x(A \cup M_{\mathscr{C}}) \ge \frac{|A|}{2} + |M_{\mathscr{C}}| > 0.$$

Now let us show that some $M \in \mathcal{M}^*$ is $\bar{x}$-tight. Suppose to the contrary that $x(M) > |M| + \hat{\epsilon}_1$ for all $M \in \mathcal{M}^*$. We could then decrease (somehow) $\bar{x}$ on $A \cup \bigcup \mathcal{C}$ and increase $\bar{x}$ uniformly on $\bigcup \mathcal{D}$ by the same total (sufficiently small) amount. The resulting $\bar{\bar{x}}$ were still in $\widehat{P}_1(\hat{\epsilon}_1)$ and would contradict Lemma 3.4 (iii).

By Lemma 3.3, this implies that each $D \in \mathcal{D}_{\max}$ is left uncovered by some $\bar{x}$-tight $M \in \mathcal{M}^*$. We are left to prove a corresponding result for $D \in \mathcal{D}$ with $|D| > 1$. Hence, assume $D \in \mathcal{D} \backslash \mathcal{D}_{\max}$ and $|D| > 1$. Then $\bar{x} < 1/2$ on D by Lemma 3.4 (ii), so every $\bar{x}$-tight $M \in \mathcal{M}^*_{\mathcal{D}}$ contains a near-perfect matching of D. Now suppose D is covered by every $\bar{x}$-tight $M \in \mathcal{M}^*$. We may then decrease $\bar{x}$ slightly on $A \cup \mathcal{C}$ and increase $\bar{x}$ uniformly on D by the same (sufficiently small) total amount. The resulting $\bar{\bar{x}}$ would again be in $\widehat{P}_1(\hat{\epsilon}_1)$ and contradict Lemma 3.4 (iii). This finishes the proof. □

Let us call an allocation $x = \bar{x} \in \widehat{P}_1(\hat{\epsilon}_1)$ *flexible* if the conclusion of Lemma 3.5 holds with respect to all $D \in \mathcal{D}$, i.e., if each $D \in \mathcal{D}$ is left uncovered by some $\bar{x}$-tight $M \in \mathcal{M}^*$.

LEMMA 3.6. *Flexible allocations exist.*

PROOF. Let $x = \bar{x} \in \widehat{P}_1(\hat{\epsilon}_1)$. Suppose $\bar{x}$ is not already flexible. Then there exists a component $D = \{i\} \in D$ of size 1 such that every $\bar{x}$-tight $M \in \mathcal{M}^*$ covers i. In particular, this implies that $A \neq \varnothing$. From the proof of Lemma 3.5 it turns out that in that case $x(A \cup \bigcup C) > 0$. We may thus increase $\bar{x}_i$ and decrease $\bar{x}$ on $A \cup \bigcup \mathcal{C}$ by the same total amount δ until $\bar{x}$ becomes "flexible" with respect to $D = \{i\}$. In other words, we choose $\delta > 0$ maximal such that the modification $\bar{x}^{\delta}$ is still in $\widehat{P}(\hat{\epsilon}_1)$. Then $\bar{x}^{\delta}(M) = |M| + \hat{\epsilon}_1$ holds for at least one matching $M \in \mathcal{M}^*$ that does not cover i (and is not $\bar{x}$-tight). Because all matchings in $\mathcal{M}^*$ that were already $\bar{x}$-tight (and contain i) remain tight, the claim follows by induction. □

We are now ready to determine the structure of x-tight matchings in $\mathcal{M}^*$ for flexible $x = \bar{x} \in \widehat{P}_1(\hat{\epsilon}_1)$. Suppose $\hat{x} \in \widehat{P}(\hat{\epsilon}_1)$ is a given flexible allocation. Suppose that $\alpha_0 < \cdots < \alpha_p$ $(p \geq 0)$ are the different values $\hat{x}$ takes on $\bigcup \mathcal{D}$ and let

$$\mathcal{D} = \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_p$$

be the corresponding partition of $\mathcal{D}$. Hence, $\hat{x} \equiv \alpha_i$ on $\bigcup \mathcal{D}_i$ and $\mathcal{D}_p = \mathcal{D}_{\max}$.

PROPOSITION 3.2. *There exists a partition $A = A_0 \cup \ldots \cup A_p$ (with some of the A_i possibly empty) such that $M \in \mathcal{M}^*$ is $\hat{x}$-tight if and only if M matches each A_i into $\mathcal{D}_i$.*

PROOF. If $A = \varnothing$, the claim is true in the sense that nothing is matched into $\mathcal{D}$ and each $M \in \mathcal{M}^*$ is $\hat{x}$-tight. (By Lemma 3.5, some $\hat{x}$-tight $M \in \mathcal{M}^*$ exists and because $A = \varnothing$, all $M \in \mathcal{M}^*$ have the same $\hat{x}$-value.)

In general, recall that $\hat{x}$-tight matchings in $\mathcal{M}^*$ are exactly those that minimize $\hat{x}(M)$ over $\mathcal{M}^*$. For given $\hat{x}$, the value $\hat{x}(M)$ only depends on how many nodes of A are matched into each $\mathcal{D}_i$. (This readily follows from the decomposition $M = M_{\mathcal{C}} \cup M_{A,\mathcal{D}} \cup M_{\mathcal{D}}$.) In other words, $\hat{x}(M)$ only depends on the total $\hat{x}$-weight of nodes in $\bigcup \mathcal{D}$ that are matched with A. The claim, therefore, follows from Lemma 3.7 below. □

LEMMA 3.7. *Consider a bipartite graph $G(A, B)$ with node set $A \cup B$. Suppose $B = B_0 \cup \ldots \cup B_p$ is a partition of B and edges incident with B_i have weight α_i $(\alpha_0 < \cdots < \alpha_p)$. Assume that the set $\mathcal{M}^*$ of matchings that completely match A into B is nonempty and let $\mathcal{M}^*_{\min}$ be the set of $M \in \mathcal{M}^*$ with minimum weight. Suppose finally, that $\mathcal{M}^*_{\min}$ is "flexible" in the sense that each $b \in B$ is left unmatched by some $M \in \mathcal{M}^*_{\min}$. Then there is a partition $A = A_0 \cup \ldots \cup A_p$ of A such that $M \in \mathcal{M}^*_{\min}$ if and only if M matches A_i into B_i $(i = 0, \ldots, p)$.*

PROOF. Let $\mathcal{M}^*_0$ denote the set of maximum matchings in the subgraph G_0 induced by $A \cup B_0$. Clearly, each $M \in \mathcal{M}^*_{\min}$ induces a maximum matching $M_0 \subseteq M$ in $\mathcal{M}^*_0$. (Apply an augmenting path argument.) Hence, we must have

$$(*) \text{ each } b \in B_0 \text{ is left uncovered by some } M_0 \in \mathcal{M}^*_0.$$

Suppose m_0^* is the maximum size of a matching in G_0. As G_0 is bipartite, König's Theorem (see, e.g., Bondy and Murty 1976, p. 74) ensures the existence of a vertex cover $A_0^* \cup B_0^*$ ($A_0^* \subseteq A$, $B_0^* \subseteq B$) of size m_0^*. Each $M \in \mathscr{M}_0^*$ is incident with all nodes in $A_0^* \cup B_0^*$. Hence, by (*) we conclude that $B_0^* = \varnothing$. In other words, each $M \in \mathscr{M}^*_{\min}$ matches A_0^* into B_0. Now let $\mathscr{M}_1^*$ denote the set of maximum matchings in the subgraph G_1 induced by $A \setminus A_0^* \cup B_1$, and proceed in the same way. So the claim follows by induction. □

We are now prepared to present our main result, a simple alternative description of the least core. Consider the LP

$$
\begin{aligned}
(\hat{\hat{P}}_1) \quad & \max \quad \epsilon \\
& \text{s.t.} \quad x = \bar{x}, \\
& \qquad x_i \le \frac{1}{2} \quad (i \in \textstyle\bigcup \mathscr{D}), \\
& \qquad x(e) \ge 1 \quad (e \in E \setminus E(\textstyle\bigcup \mathscr{D})), \\
& \qquad x(N) = v^*, \\
& \qquad x(M) \ge |M| + \epsilon \quad (M \in \mathscr{M}^*_{\mathscr{D}}), \\
& \qquad x \ge 0.
\end{aligned}
$$

Note that $x \equiv \bar{x}$ is just a shorthand for a number of linear equalities of the type $x_i = x_j$. Further note that for $x \equiv \bar{x}$, the value $x(M)$ is independent of the particular choice of $M \in \mathscr{M}_D^*$. Hence, the exponentially many constraints for $M \in \mathscr{M}_D^*$ reduce to one single inequality.

Again, we let $\hat{\hat{P}}_1(\epsilon) := \{x \mid (x, \epsilon) \text{ is feasible for } (\hat{\hat{P}}_1)\}$ and denote the optimum value of $(\hat{\hat{P}}_1)$ by $\hat{\hat{\epsilon}}_1$.

Theorem 3.1. *We have* $\epsilon_1 = \hat{\epsilon}_1 = \hat{\hat{\epsilon}}_1$ *and* $\text{leastcore}(N, v) = P_1(\epsilon_1) = \hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$.

Proof.

- We have $\epsilon_1 \le \hat{\epsilon}_1$ by definition.
- $\hat{\epsilon}_1 \le \hat{\hat{\epsilon}}_1$: Let $\hat{x} \in \hat{P}_1(\hat{\epsilon}_1)$ be flexible with corresponding partitions $\mathscr{D} = \mathscr{D}_0 \cup \ldots \cup \mathscr{D}_p$ and $A = A_0 \cup \ldots \cup A_p$. Define $\hat{\hat{x}} \in \mathbb{R}^N$ by

$$
\begin{aligned}
\hat{\hat{x}} &\equiv \frac{1}{2} \quad \text{on} \textstyle\bigcup \mathscr{C}, \\
\hat{\hat{x}} &\equiv \hat{x} \quad \text{on} \textstyle\bigcup \mathscr{D}, \\
\hat{\hat{x}} &\equiv 1 - \alpha_i \quad \text{on } A_i \quad (0 \le i \le p).
\end{aligned}
$$

We show that $\hat{\hat{x}} \in \hat{\hat{P}}_1(\hat{\epsilon}_1)$ (proving that $\hat{\hat{\epsilon}}_1 \ge \hat{\epsilon}_1$). The only nontrivial constraints to check are $\hat{\hat{x}}(N) = v^*$ and $\hat{\hat{x}}(e) \ge 1$ for $e \in E \setminus E(\bigcup \mathscr{D})$. All other constraints directly follow from Lemma 3.4.

Let $M \in \mathscr{M}^*$ be $\hat{x}$-tight and decompose it as

$$M = M_{\mathscr{C}} \cup M_{A, \mathscr{D}} \cup M_{\mathscr{D}}$$

as usual. Because $M_{\mathscr{D}} \in \mathscr{M}^*_{\mathscr{D}}$ is also $\hat{x}$-tight by Lemma 3.4, we conclude that $\hat{x}(M_{\mathscr{C}} \cup M_{A,\mathscr{D}}) = |M_{\mathscr{C}}| + |M_{A,\mathscr{D}}| = \hat{\hat{x}}(M_{\mathscr{C}} \cup M_{A,\mathscr{D}})$ by definition of $\hat{\hat{x}}$. Hence, $\hat{\hat{x}}(N) = \hat{x}(N) = v^*$.

Second, let us consider $e \in E \setminus E(\bigcup \mathscr{D})$. If $e \in E(A \cup \bigcup \mathscr{C})$ then $\hat{\hat{x}}(e) \ge 1$ by definition of $\hat{\hat{x}}$. (Recall that $\hat{x} = \alpha_i \le 1/2$ on $\bigcup \mathscr{D}_i$.) Thus we are left with edges between A and $\bigcup \mathscr{D}$. Suppose $\hat{\hat{x}}(e) < 1$ for such an edge joining, say, $D \in \mathscr{D}_i$ with $a \in A_j$. Then $\hat{\hat{x}}(e) = \alpha_i + 1 - \alpha_j < 1$, i.e., $\alpha_i < \alpha_j$. Because $\hat{x}$ is flexible, there exists an $\hat{x}$-tight matching $M \in \mathscr{M}^*$ not covering D. Because D is factor-critical (and $\hat{x}$ is constant on D), we may assume that M does not match the endpoint of e in D. Because M is $\hat{x}$-tight, $a \in A_j$ is matched into D_j by some edge $f \in M$ (cf. Proposition 3.2). Then $M' = M \setminus f + e$ has $\hat{x}(M') < \hat{x}(M)$, a contradiction.

• $\hat{\hat{\epsilon}}_1 \le \epsilon_1$: We show that in general $\hat{\hat{P}}_1(\epsilon) \subseteq P_1(\epsilon)$. Suppose $x \in \hat{\hat{P}}_1(\epsilon)$. Then $x(M) \ge |M| + \epsilon$ for all $M \in \mathscr{M}^*_{\mathscr{D}}$. Because $x \le 1/2$ on $\bigcup \mathscr{D}$, this also implies $x(M) \ge |M| + \epsilon$ for all $M \in \mathscr{M}_{\mathscr{D}}$. (Use an augmenting path argument.) Because $x(e) \ge 1$ for all $e \in E \backslash E(\bigcup \mathscr{D})$, we further conclude that $x(M) \ge |M| + \epsilon$ for all $M \in \mathscr{M}$.

• Finally, let us verify that $P_1(\epsilon_1) = \hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$. We have just proved that "$\supseteq$" holds. Conversely, let $x \in P_1(\epsilon_1)$. Then $x \in \hat{P}_1(\hat{\epsilon}_1)$ and, by Lemma 3.4, x satisfies all constraints of $\hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$ except possibly $x(e) \ge 1$ for $e \in E \backslash E(\bigcup \mathscr{D})$. Thus, let $e \in E \backslash E(\bigcup \mathscr{D})$. Pick $M \in \mathscr{M}^*_{\mathscr{D}}$ not covering the endpoint of e in $\bigcup \mathscr{D}$, so that $M \cup e$ is a matching again. Then, because $x \in P_1(\epsilon_1)$, we have $x(M \cup e) \ge |M| + 1 + \epsilon_1$, and because $M \in \mathscr{M}^*_{\mathscr{D}}$ is x-tight, we have $x(M) = |M| + \hat{\epsilon}_1$. Because $\epsilon_1 = \hat{\epsilon}_1$, the claim follows. □

4. The nucleolus. Recall from §1 that the nucleolus is computed by solving the following sequence of LP's:

$$
\begin{aligned}
(P_1) \quad & \max \quad \epsilon \\
& \text{s.t.} \quad x(S) \ge v(S) + \epsilon \quad (S \notin \{\varnothing, N\}), \\
& \qquad\quad x(N) = v^*,
\end{aligned}
$$

with optimum value ϵ_1,

$$
\begin{aligned}
(P_2) \quad & \max \quad \epsilon \\
& \text{s.t.} \quad x \in P_1(\epsilon_1), \\
& \qquad\quad x(S) \ge v(S) + \epsilon \quad (S \notin \operatorname{Fix} P_1(\epsilon_1)),
\end{aligned}
$$

with optimum value ϵ_2, etc. until the nucleolus is finally determined as the unique solution x^*, $\epsilon^* = \epsilon_r$ of

$$
\begin{aligned}
(P_r) \quad & \max \quad \epsilon \\
& \text{s.t.} \quad x \in P_{r-1}(\epsilon_{r-1}), \\
& \qquad\quad x(S) \ge v(S) + \epsilon \quad (S \notin \operatorname{Fix} P_{r-1}(\epsilon_{r-1})).
\end{aligned}
$$

By Theorem 3.1, (P_1) is equivalent to $(\hat{\hat{P}}_1)$ in the sense that they define the same set of optimal solutions. As we will see, similar equivalent formulations can be found for (P_k), $k \ge 2$. Define recursively

$$
\begin{aligned}
(\hat{\hat{P}}_k) \quad & \max \quad \epsilon \\
& \text{s.t.} \quad x \in \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1}), \\
& \qquad\quad x(e) \le 1 + \epsilon_1 - \epsilon \quad (e \in E(\textstyle\bigcup \mathscr{D}),\, e \notin \operatorname{Fix} \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})), \\
& \qquad\quad x(e) \ge 1 - \epsilon_1 + \epsilon \quad (e \in E \backslash E(\textstyle\bigcup \mathscr{D}),\, e \notin \operatorname{Fix} \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})), \\
& \qquad\quad x_i \ge -\epsilon_1 + \epsilon \quad (i \in N,\, i \notin \operatorname{Fix} \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})).
\end{aligned}
$$

As before, let $\hat{\hat{\epsilon}}_k$ denote the optimum value of $(\hat{\hat{P}}_k)$ and define $\hat{\hat{P}}_k(\epsilon)$ in the obvious way.

THEOREM 4.1. *We have $\epsilon_k = \hat{\hat{\epsilon}}_k$ and $P_k(\epsilon_k) = \hat{\hat{P}}_k(\hat{\hat{\epsilon}}_k)$ for $k = 1, \ldots, r$. In particular, the sequence $\hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1) \supset \cdots \supset \hat{\hat{P}}_r(\hat{\hat{\epsilon}}_r) = \{x^*\}$ defines the nucleolus.*

PROOF. For $k = 1$, the claim is equivalent to Theorem 3.1. We proceed by induction on k. Assume that $\epsilon_{k-1} = \hat{\hat{\epsilon}}_{k-1}$ and $P_{k-1}(\epsilon_{k-1}) = \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. The induction step amounts to show the following two things.

(i) $P_k(\epsilon) \subseteq \hat{\hat{P}}_k(\epsilon)$ (implying that $\hat{\hat{\epsilon}}_k \ge \epsilon_k$): Let $x \in P_k(\epsilon)$. Then $x \in P_1(\epsilon_1) = \hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$, so x satisfies $x \ge 0$, $x_i = x_D \le 1/2$ for all $i \in D$, $D \in \mathscr{D}$, and $x(M) = |M| + \epsilon_1$ for all $M \in \mathscr{M}^*_{\mathscr{D}}$.

We first consider $e \in E \backslash E(\bigcup \mathscr{D})$ and show that $x(e) \geq 1 - \epsilon_1 + \epsilon$ unless $e \in \text{Fix}\,\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1}) = \text{Fix}\,P_{k-1}(\epsilon_{k-1})$. Choose $M \in \mathscr{M}_{\mathscr{D}}$ such that $M \cup e$ is a matching. (Existence follows from the fact that each $D \in \mathscr{D}$ is factor-critical.) Because M is fixed by $\hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1) = P_1(\epsilon_1)$, it is fixed by $\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. Hence, $e \in \text{Fix}\,\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$ if and only if $M \cup e \in \text{Fix}\,\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. Because we assume $e \notin \text{Fix}\,\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$, we have $M \cup e \notin \text{Fix}\,P_{k-1}(\epsilon_{k-1})$ and thus $x \in P_k(\epsilon)$ implies $x(M \cup e) \geq |M \cup e| + \epsilon$. Together with $x(M) = |M| + \epsilon_1$ this yields $x(e) \geq 1 - \epsilon_1 + \epsilon$.

In the same way we can show that $x_i \geq -\epsilon_1 + \epsilon$ for a node $i \notin \text{Fix}\,\hat{\hat{P}}(\hat{\hat{\epsilon}}_{k-1})$.

Next, consider $e \in E(\bigcup \mathscr{D})$, say $e \in E(D)$ for $D \in \mathscr{D}$. We show that $x(e) \leq 1 + \epsilon_1 - \epsilon$ unless e is already fixed by $\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1}) = P_{k-1}(\epsilon_{k-1})$. Because $x \equiv x_D$ on $D \in \mathscr{D}$, we conclude that $x(e)$ is independent of the particular choice of $e \in E(D)$. Choose any $M \in M^*_{\mathscr{D}}$ and assume without loss of generality that $e \in M \cap E(D)$ is not fixed by $P_{k-1}(\epsilon_{k-1})$. Because $x(M)$ is fixed (to $|M| + \epsilon_1$), we conclude that $M \backslash e \notin \text{Fix}\,P_{k-1}(\epsilon_{k-1})$. Hence, $x \in P_k(\epsilon)$ implies $x(M \backslash e) \geq |M \backslash e| + \epsilon$. Together with $x(M) = |M| + \epsilon_1$ we get $x(e) \leq 1 + \epsilon_1 - \epsilon$.

(ii) $\hat{\hat{P}}_k(\epsilon) \subseteq P_k(\epsilon)$ (implying that $\epsilon_k \geq \hat{\hat{\epsilon}}_k$): Let $x \in \hat{\hat{P}}_k(\epsilon)$. Again, this implies $x \in \hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$, so $x \geq 0$, $x \equiv x_D \leq 1/2$ on each $D \in \mathscr{D}$ and $x(M_{\mathscr{D}}) = |M_{\mathscr{D}}| + \epsilon_1$ for $M_{\mathscr{D}} \in \mathscr{M}^*_{\mathscr{D}}$. We are to show that $x(S) \geq v(S) + \epsilon$ for $S \subset N$ not yet fixed by $P_{k-1}(\epsilon_{k-1}) = \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. Because $x \geq 0$, we may only consider $S = v(M)$ for $M \in \mathscr{M}$. Furthermore, because $x(e) \geq 1$ on $E \backslash E(\bigcup \mathscr{D})$, we may restrict ourselves to $M \subseteq E(\bigcup \mathscr{D})$. Finally, because $x \equiv x_D$ $(D \in \mathscr{D})$, $x(M)$ only depends on $|M \cap D|$ for each $D \in \mathscr{D}$. So we may without loss of generality assume that $M \subseteq M_{\mathscr{D}}$ for some $M_{\mathscr{D}} \in \mathscr{M}^*_{\mathscr{D}}$. Assume that M is not fixed by $P_{k-1}(\epsilon_{k-1})$. Because $M_{\mathscr{D}}$ is fixed by $P_{k-1}(\epsilon_{k-1})$, we conclude that $M_{\mathscr{D}} \backslash M$ is not fixed by $P_{k-1}(\epsilon_{k-1})$. So at least some $e \in M_{\mathscr{D}} \backslash M$ is not fixed by $P_{k-1}(\epsilon_{k-1})$. Hence, $x \in \hat{\hat{P}}_k(\epsilon)$ implies $x(e) \leq 1 - \epsilon + \epsilon_1$. All other edges $f \in M_{\mathscr{D}} \backslash M$ satisfy $x(f) \leq 1$ (as $x \leq 1/2$ on $\bigcup \mathscr{D}$). Hence, $x(M_{\mathscr{D}}) = |M_{\mathscr{D}}| + \epsilon_1$ implies $x(M) \geq |M| + \epsilon$ as required. □

Clearly, the number of constraints in each linear program $(\hat{\hat{P}}_k)$ is bounded by a polynomial in $|N|$. The size of the parameters $\hat{\hat{\epsilon}}_k$ $(k = 1, \ldots, r)$ is bounded by a polynomial in N and $\langle v \rangle$ (cf. the proof of Theorem 1.1). Then we can conclude that

COROLLARY 4.1. *The nucleolus x^* of a matching game on a graph $G = (N, E)$ with unit edge weights can be computed in polynomial time.*

We end this section by introducing the following class of matching games. Let $G = (N, E)$ model a market situation, in which each person $i \in N$ has a "weight" $w_i \geq 0$ indicating his importance or power. The edges in E correspond with pairs of potential business partners. Now assume that, if i and j do business with each other, their common profit equals $\bar{w}(i, j) = w_i + w_j$. Again, the total profit is maximal if all persons in N cooperate and a maximum weight matching can be constructed. The corresponding matching game is called a *node matching game*.

So a node matching game is determined by a graph $G = (N, E)$ with node weighting w: $N \to \mathbb{R}_+$. The edge weighting $\bar{w}$ is defined by $\bar{w}(i, j) = w_i + w_j$ for all $(i, j) \in E$. Note that a cardinality matching game can been seen as a node matching game by defining a node weighting $w \equiv 1/2$ on N. Also for node matching games the nucleolus can be computed in polynomial time. For the proof we refer to Paulusma (2001).

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