

RAMIFICATION THEORY FOR HIGHER DIMENSIONAL LOCAL FIELDS

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To 60th birthday of A.N.Parshin.

ABSTRACT. The paper contains a construction of ramification theory for higher dimensional local fields K provided with additional structure given by an increasing sequence of their “subfields of i -dimensional constants”, where $0 \leq i \leq n$ and n is the dimension of K . It is also announced that a local analogue of the Grothendieck Conjecture still holds: all automorphisms of the absolute Galois group of K , which are compatible with ramification filtration and satisfy some natural topological conditions appear as conjugations via some automorphisms of the algebraic closure of K .

0. INTRODUCTION

This paper deals with the formalism of ramification theory of higher dimensional local fields. It comes from I.Zhukov’s approach [Zh], [Ab5] to such a theory in the case of 2-dimensional local fields K , which is based on the introduction of the additional structure on K given by its closed 1-dimensional local subfield K_c of dimension 1 — “the subfield of 1-dimensional constants”. Then the filtration of $\Gamma_K = \text{Gal}(K_{\text{sep}}/K)$ by its ramification subgroups appears in the form of decreasing filtration of Γ_K by normal subgroups $\{\Gamma_K^{(j)}\}_{j \in J(2)}$. Here $J(2) = J_1 \amalg J_2$, where $J_1 = \{j \in \mathbb{Q} \mid j \geq 0\}$, $J_2 = \{j \in \mathbb{Q}^2 \mid j \geq (0, 0)\}$ (with respect to the lexicographical ordering on $\mathbb{Q}^2$), and by definition each element of J_2 is greater than every element of J_1 . For $j \in J_1$, the groups $\Gamma_K^{(j)}$ appear as the preimages of the classical ramification subgroups of $\Gamma_{K_c} = \text{Gal}(K_{c,\text{sep}}/K_c)$ with respect to the natural projection π from Γ_K to Γ_{K_c} . The “2-dimensional part” of ramification filtration of Γ_K appears as a decreasing filtration $\{\Gamma_K^{(j)}\}_{j \in J_2}$ of $\tilde{\Gamma}_K = \text{Ker } \pi$ and its definition can be given in terms of semistable reduction of the arithmetical scheme $\text{Spec } \mathcal{O}_K \rightarrow \text{Spec } \mathcal{O}_{K_c}$ attached to the field extension $K \supset K_c$ (here $\mathcal{O}_K$ and $\mathcal{O}_{K_c}$ are corresponding valuation rings).

The above interpretation of Zhukov’s approach admits a direct generalization to the case of local fields of arbitrary dimension n , which are supposed to be provided with an additional F -structure given by increasing sequence of subfields of i -dimensional constants with $1 \leq i < n$. The techniques developed earlier by the author [Ab1-3] to study the classical ramification filtration can be adjusted to obtain similar results for higher dimensional local fields. In particular, the paper [Ab5] contains an explicit description of the ramification filtration of the maximal p -extension of 2-dimensional local field of characteristic p with Galois group of nilpotence class 2 ($p \geq 3$). Following the strategy from [Ab4] one can use this

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description to prove a local analogue of the Grothendieck Conjecture for higher dimensional local fields. This result is stated in n.6 below. It justifies that the proposed ramification theory is sufficiently nice because it carries practically all information about the original local field. Complete proofs of announced Theorems 1 and 2 are given in the papers [Ab6,7] in the case of local fields of dimension 2. It would be interesting to compare our theory with recent approach to ramification theory from [A-S] as well as with earlier approaches to such a theory by K.Kato, O.Hyodo, etc., which were basically related to the study of arithmetical properties of abelian extensions of higher dimensional local fields. One can find a brief exposition of related results together with necessary references in the book [HLF].¹

1. n -DIMENSIONAL LOCAL FIELDS

By definition L is a local n -dimensional field if either $n = 0$ and L is a finite field, or $n \geq 1$ and L is the quotient field of a complete discrete valuation ring $O_L^{(1)}$ with residue field $\bar{L}$, which is a local field of dimension $n - 1$. With the obvious notation there is the following sequence of epimorphic maps and embeddings of valuation rings and residue fields

$$L := L^{(0)} \supset O_L^{(1)} \longrightarrow \bar{L} := L^{(1)} \supset O_{L^{(1)}}^{(1)} \longrightarrow \dots \longrightarrow \bar{L}^{(n-1)} = L^{(n)}, \quad (1)$$

where $L^{(n)}$ is a finite field. For $0 \leq i \leq n$, denote by $O_L^{(i)}$ the preimage of $L^{(i)}$ in L with respect to the composition of corresponding morphisms from (1). The kernel of the natural projection from $O_L^{(i)}$ to $L^{(i)}$ will be denoted by $m_L^{(i)}$. Notice that $O_L^{(0)} = L$ and $m_L^{(0)} = 0$. The ring $O_L^{(n)}$ will be denoted also by $\mathcal{O}_L$ and will be called the valuation ring of L .

A subfield E of L is closed if it is either finite or it is the fraction field of a closed non-discrete (with respect to the corresponding valuation topology) subring of $O_L^{(1)}$ and the corresponding residue field $\bar{E}$ is a closed subfield of the $(n-1)$ -dimensional local field $\bar{L}$. Then E is provided with a unique induced structure of local field of dimension $\leq n$. On the other hand, if M is a finite extension of L , then M is provided uniquely with a structure of an n -dimensional local field such that L is a closed n -dimensional subfield of M .

In this paper we are going to consider only local fields L , which satisfy one of the following two basic assumptions:

- a) the finite characteristic case, i.e. $\text{char } L^{(0)} = \text{char } L^{(n)}$; in this case the field L is always standard, that is $L \simeq k((t_n)) \dots ((t_1))$, where $k = L^{(n)}$ (for any field F , $F((t))$ is a field of formal Laurent series with coefficients in F);
- b) the mixed characteristic case, i.e. $\text{char } L^{(0)} = 0$ but $\text{char } L^{(1)} = \text{char } L^{(n)} = p > 0$; in this case L is a finite extension of some standard field $K\{\{t_n\}\} \dots \{\{t_2\}\}$, where $[K : \mathbb{Q}_p] < \infty$ (if $F \supset \mathbb{Q}_p$ then $F\{\{t\}\} = F \hat{\otimes}_{\mathbb{Z}_p} \varprojlim \mathbb{Z}/p^M \mathbb{Z}((t))$).

Only in the above two cases the absolute Galois group $\Gamma_L = \text{Gal}(L_{\text{sep}}/L)$ is complicated enough to be provided with interesting ramification filtration.

Let $t_1, \dots, t_n$ be a system of local parameters of L , i.e. for all $1 \leq i \leq n$, $t_i \in m_L^{(i)}$ and $t_i \bmod m_L^{(i-1)}$ is uniformizing element of the complete discrete valuation field $L^{(i-1)} = O_L^{(i-1)} \bmod m_L^{(i-1)}$. Notice that $t_1, \dots, t_n$ is a system of local

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parameters if and only if t_1 is uniformizing element of $O_L^{(1)}$, $t_2, \dots, t_n \in O_L^{(1)*}$ and $t_2 \bmod m_L^{(1)}, \dots, t_n \bmod m_L^{(1)}$ is a system of local parameters in $L^{(1)}$. Clearly, L can be identified with the set of all formal Laurent series

$$l = \sum_{i_1, \dots, i_n} [\alpha_{i_1 \dots i_n}] t_1^{i_1} \dots t_n^{i_n}, \quad (2)$$

where the sum is taken for all multi-indices $(i_1, \dots, i_n)$ such that for some (depending on l) lower boundaries $m, m(i_1), \dots, m(i_1, \dots, i_{n-1})$, one has $i_1 \geq m, i_2 \geq m(i_1), \dots, i_n \geq m(i_1, \dots, i_{n-1})$, and $[\alpha_{i_1 \dots i_n}]$ are Teichmüller representatives of $\alpha_{i_1 \dots i_n} \in L^{(n)}$ (if $\text{char } L = \text{char } L^{(n)}$, then the Teichmüller representative of $\alpha \in L^{(n)}$ is just α itself). This identification has been considered in basic papers on higher dimensional local fields (A.Parshin, K.Kato) via introducing a special topology on L , with respect to which (we shall call it the P -topology) the above series (2) are convergent (the concept of P -topology was analyzed and studied later by I.Fesenko and I.Zhukov). Actually, the P -topology brings into correlation all n discrete valuation topologies of the fields $L = L^{(0)}, \dots, L^{(n-1)}$. Notice that operations of addition and multiplication are sequentially P -continuous in L . If $1 \leq i \leq n$ and the ring $O_L^{(i)} \subset L$ is provided with the induced P -topology, then all natural projections $\text{pr}_i : O_L^{(i)} \rightarrow L^{(i)}$ are continuous. On the other hand, any choice of local parameters $t_1, \dots, t_n$ gives rise to continuous sections $s_i : L^{(i)} \rightarrow O_L^{(i)}$ of projections pr_i and implies a description of elements from L as formal power series (2). It is also known that the P -topology of a finite extension E of L is compatible with that of L with respect to an identification of L -vector spaces $E \simeq L^m$, $m = [E : L]$, given by some choice of L -basis in E . For these and related results we refer again to the book [HLF].

So, it is natural to consider the P -topology as an essential part of the concept of higher dimensional local field. In other words, when working with the category of higher dimensional local fields we shall consider only P -continuous field morphisms. For example, if $t_1, \dots, t_n$ is a system of local parameters in L , then any $\psi \in \text{Aut}_{P\text{-top}}(L)$ is uniquely determined by the images $\psi(t_1), \dots, \psi(t_n)$, which have to form again a system of local parameters in L .

2. F -STRUCTURE

If L is an n -dimensional local field then its F -structure is given by an increasing sequence of its closed subfields $L_{c1} \subset L_{c2} \subset \dots \subset L_{cn} = L$ such that for all $1 \leq i \leq n$,

- L_{ci} is a closed i -dimensional subfield of L ;
- L_{ci} is algebraically closed in L .

The subfields L_{ci} may be treated as subfields of “ i -dimensional constants”. It will be also convenient to introduce the subfield of 0-dimensional constants. If $\text{char } L$ is positive, then the last residue field $L^{(n)}$ can be naturally identified with a unique subfield of L and of all L_{ci} , $1 \leq i \leq n$. So, $L^{(n)}$ may be interpreted as the subfield of “0-dimensional constants” L_{c0} . In the mixed characteristic case L contains $\mathbb{Q}_p$, therefore, L_{c1} is the algebraic closure of $\mathbb{Q}_p$ in L , and we take its maximal unramified over $\mathbb{Q}_p$ subfield as L_{c0} .

If E is a finite extension of L then E is provided with a unique induced F -structure such that for any $1 \leq i \leq n$, E_{ci} is the algebraic closure of L_{ci} in

E . Inversely, any given F -structure on E induces the F -structure of L given by its subfields $L_{c,i} := L \cap E_{c,i}$. In the both cases above we shall call F -structures of E and L compatible. Throughout all this paper all local fields are assumed to be provided with some F -structure. When considering any algebraic extension of n -dimensional local fields we always assume that the corresponding F -structures are compatible. Notice also that for $1 \leq i \leq n$, the subfields $L_{c,i-1}^{(1)} := (L_{c,i} \cap O_L^{(1)}) \bmod m_L^{(1)}$ give the induced F -structure of the first residue field $L^{(1)}$ of L . So, while giving an F -structure on L we provide automatically all residue fields of L with uniquely determined induced F -structures.

Suppose L is a standard field. Then either $L = k((t_n)) \dots ((t_1))$, where k is a finite field, or $L = K\{\{t_n\}\} \dots \{\{t_2\}\}$, where K is a 1-dimensional local field with uniformizing element t_1 . In the both cases $t_1, \dots, t_n$ form a system of local parameters in L . Associate to it the F -structure of L such that for $1 \leq j \leq n$, the subfield $L_{c,j}$ consists of elements l given in terms of corresponding formal series (2) by the condition

$$\alpha_{i_1 \dots i_n} = 0 \text{ if at least one of the indices } i_{j+1}, \dots, i_n \text{ is not zero.}$$

In other words, for $1 \leq j \leq n$, the subfield $L_{c,j}$ consists of elements presented as formal series in variables $t_1, \dots, t_j$. This F -structure of (a standard field) L will be called standard. The following proposition is very well-known application of Epp's result on eliminating wild ramification.

Proposition 1. *Let L be an n -dimensional local field with F -structure. Then there is a finite separable extension E' of $L_{c,n-1}$ such that the induced F -structure on $E := LE'$ is standard.*

Proof. Apply induction on n .

If $n = 1$ then there is nothing to prove.

Let $n > 1$ and let L_{alg} be an algebraic closure of L . By Epp's Theorem [Epp, KZ] there is a finite separable extension M_1 of $L_{c,1}$ in L_{alg} such that if $M = LM_1$, then any uniformizing element t_1 of M_1 appears also as a uniformizing element of M (with respect to its first valuation). Let $M' = M^{(1)}$ be the first residue field of M . Consider its induced F -structure $\{M'_{c,i} \mid 1 \leq i \leq n-1\}$, where $M'_{c,i} = (M_{c,i+1})^{(1)}$ is the first residue field of $M_{c,i+1}$. By induction there is a finite separable extension $\bar{E}'$ of $M'_{c,n-2}$ in $L_{\text{alg}}^{(1)}$ such that the induced F -structure $\{\bar{E}'_{c,i} \mid 1 \leq i \leq n-1\}$ of $\bar{E} := \bar{E}'M'$ is standard, i.e. it is associated to some system of local parameters $\bar{t}_1, \dots, \bar{t}_{n-1}$ of $\bar{E}$.

Let E be one of unramified extensions of M in L_{alg} with the (first) residue field $\bar{E}$. For $1 \leq i \leq n$, denote by $E_{c,i}$ the maximal unramified extension of $M_{c,i}$ in E . It is easy to see that $\{E_{c,i} \mid 1 \leq i \leq n\}$ is F -structure on E , this F -structure is associated to a collection of local parameters $t_1, \dots, t_n$ such that for $i = 2, \dots, n$, $t_i \in O_{E_{c,i}}^{(1)}$ and $t_i \bmod m_{E_{c,i}}^{(1)} = \bar{t}_{i-1}$. The proposition is proved.

Remark. If in the notation of the above proposition M' is a finite extension of E' then the induced F -structure of $M := LM'$ is not generally standard. Nevertheless, we have the following property:

— if $t_1, \dots, t_n$ is a system of local parameters of E , which is associated with its (standard) F -structure, and $u_1, \dots, u_{n-1}$ is a system of local parameters of M' then $u_1, \dots, u_{n-1}, t_n$ is a system of local parameters of M .

3. THE VALUATION v_L

A valuation of rank n of an n -dimensional local field L is a map $v : L \rightarrow \mathbb{Q}^n \cup \{\infty\}$ such that

- $v|_{L^*}$ is a group homomorphism from L^* to $\mathbb{Q}^n$ and $v(0) = \infty$;
- $v(l_1 + l_2) \geq \min\{v(l_1), v(l_2)\}$, where $\mathbb{Q}^n$ is provided with lexicographical ordering and by definition ∞ is greater than any element of $\mathbb{Q}^n$;
- if $1 \leq i \leq n$ and v^i is the i -th coordinate function of the map v , then

$$O_L^{(i)} = \{l \in L \mid (v^1(l), \dots, v^i(l)) \geq \bar{0}_i\},$$

where here and everywhere below $\bar{0}_i := (0, \dots, 0) \in \mathbb{Q}^i$.

As usually, if E is an algebraic extension of L then there is a unique valuation v' of rank n on E such that $v'|_L = v$. Inversely, any valuation v' of E induces the valuation $v = v'|_L$ of L . (In these both situations we often use below the same notation for v and v' .)

Suppose an n -dimensional local field L is provided with some F -structure $\{L_{c,i} \mid 1 \leq i \leq n\}$. A valuation v of L will be called compatible with this F -structure if for all $1 \leq i \leq n$, it holds $v(L_{c,i}^*) \subset \mathbb{Q}^i \oplus \bar{0}_{n-i}$, i.e. for all $l \in L_{c,i}^*$, the last $n-i$ components of $v(l)$ are zeroes. Suppose $[E : L] < \infty$ and the valuation v' on E is the extension of v . Then the compatibility of v with some F -structure on L is equivalent to the compatibility of v' with the corresponding induced F -structure on E .

Proposition 2. *If v and v_1 are valuations of rank n on L , which are compatible with its F -structure then there is $\bar{d} \in \mathbb{Q}^n$ such that for any $l \in L$, $v(l) = \bar{d}v_1(l)$ — the component-wise product of vectors $\bar{d}$ and $v_1(l)$.*

Proof. By Prop.1 and the uniqueness property of extension of valuations the statement can be reduced to the case of a field L provided with a standard F -structure. Let $t_1, \dots, t_n$ be a system of local parameters associated with such F -structure and let for $1 \leq i \leq n$, $v(t_i) = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbb{Q}^n$ and $v_1(t_i) = (\alpha'_{i1}, \dots, \alpha'_{in}) \in \mathbb{Q}^n$. By the definition of valuation of rank n we have $\alpha_{ij} = \alpha'_{ij} = 0$ for all $i > j$. In addition, F -compatibility of v and v_1 implies $\alpha_{ij} = \alpha'_{ij} = 0$ for all $i < j$. So, we can take $\bar{d} = (\alpha_{11}/\alpha'_{11}, \dots, \alpha_{nn}/\alpha'_{nn})$. The proposition is proved.

If $[E : L] < \infty$, introduce the vector ramification index $\bar{e}_{E/L} = (e_1, \dots, e_n) \in \mathbb{Z}^n$ by setting for $1 \leq i \leq n$,

$$e_i = [E_{c,i} : L_{c,i}E_{c,i-1}] = [E_{c,i} : L_{c,i}][E_{c,i-1} : L_{c,i-1}]^{-1}.$$

Notice that if $L \subset E \subset E_1$ is a tower of finite extensions (with compatible F -structures) then $\bar{e}_{E_1/L} = \bar{e}_{E_1/E}\bar{e}_{E/L}$.

Proposition 3. *Any n -dimensional local field L with an F -structure can be provided with a unique valuation v_L of rank n such that*

- a) *if L has a standard F -structure and $t_1, \dots, t_n$ is a corresponding system of local parameters, then for all $1 \leq i \leq n$, $v_L(t_i) = (\delta_{i1}, \dots, \delta_{in})$, where δ is the Kronecker symbol;*
- b) *if $[E : L] < \infty$, where E has standard F -structure, then $v_L = \bar{e}_{E/L}^{-1}v_E$.*

Proof. a) Clearly, the values $v_L(t_i)$, $1 \leq i \leq n$, determine v_L uniquely and it is easy to see that for any other corresponding system of local parameters $u_1, \dots, u_n$, it holds $v_L(u_i) = v_L(t_i)$, $i = 1, \dots, n$.

b) It will be sufficient to verify that if E and L are provided with standard F -structures, then

$$v_E = \bar{e}_{E/L} v_L. \quad (3)$$

Suppose $t_1, \dots, t_n$ is a corresponding system of local parameters in L and $u_1, \dots, u_n$ is a corresponding system of local parameters in E . Then relation (3) easily follows from the fact that for any $1 \leq j \leq n$, $u_1, \dots, u_{j-1}, t_j$ is a system of local parameters of $L_{c,j} E_{c,j-1}$ and $u_1, \dots, u_j$ is a system of local parameters of $E_{c,j}$. The proposition is proved.

Notice that the above valuation v_L is automatically compatible with given F -structure on L and for any finite extension E of L , it holds $v_E = \bar{e}_{E/L} v_L$. Besides, for any $1 \leq i \leq n$, v_L induces the valuation $v_{L_{c,i}}$ when being restricted to $L_{c,i}$.

On the other hand, if $L^{(i)}$ is the i -th residue field of L , where $1 \leq i \leq n$, then v_L generally does not induce the valuation $v_{L^{(i)}}$ on $L^{(i)}$. But this will be true if e.g. a given F -structure of L is standard.

4. SUBGROUPS $\tilde{\Gamma}_{E/L}$ AND ITS RAMIFICATION SUBGROUPS

Let L_0 be a local field of dimension n with F -structure. Choose an algebraic closure $\bar{L}_0$ of L_0 and suppose everywhere below that any algebraic extension L of L_0 is chosen inside $\bar{L}_0$ and is provided with the induced F -structure $\{L_{c,i} \mid 0 \leq i \leq n\}$.

For any finite normal extension E of L , set $\Gamma_{E/L} = \text{Gal}(E/L_{(i)})$, where $L_{(i)}$ is the maximal purely non-separable extension of L in E . Notice that $\Gamma_{E/L}$ is identified also with the Galois group of the maximal separable extension $E_{(s)}$ of L in E , cf. [Jac], n.8.7. With the above agreement use the induced F -structure on E to introduce the group $\tilde{\Gamma}_{E/L} := \Gamma_{E/LE_{c,n-1}}$. Clearly, there is a natural exact sequence

$$1 \longrightarrow \tilde{\Gamma}_{E/L} \longrightarrow \Gamma_{E/L} \longrightarrow \Gamma_{E_{c,n-1}/L_{c,n-1}} \longrightarrow 1.$$

Let $J_n = \{j \in \mathbb{Q}^n \mid j \geq \bar{0}_n\}$, where $\mathbb{Q}^n$ is provided with the lexicographical ordering. Consider a finite extension M' of $E_{c,n-1}$ in $\bar{L}_0$ such that the induced F -structure $\{\tilde{E}_{c,i} \mid 1 \leq i \leq n\}$ of $\tilde{E} := M'E$ is standard, cf. Prop. 1. Then any system of local parameters $t_1, \dots, t_{n-1}$ of $\tilde{E}_{c,n-1} = M'$ can be extended to a system of local parameters $t_1, \dots, t_{n-1}, \theta$ of $\tilde{E} = EM'$. Let $\tilde{L} = LM'$. Then the extension of 1-dimensional complete discrete valuation fields $\tilde{E}^{(n-1)}/\tilde{L}^{(n-1)}$ is totally ramified and $\theta \bmod m_{\tilde{E}}^{(n-1)}$ is uniformizing element of $\tilde{E}^{(n-1)}$. This implies that $t_1, \dots, t_{n-1}, N_{\tilde{E}/\tilde{L}}\theta$ is a system of local parameters of $\tilde{L}$ and we obtain very important property of monogeneity $\mathcal{O}_{\tilde{E}} = \mathcal{O}_{\tilde{L}}[\theta]$.

Remark. Notice that if M'_1 is any finite extension of M_1 and $\tilde{E}_1 = \tilde{E}M'_1$ and $\tilde{L}_1 = \tilde{L}M'_1$ then we still have the monogeneity property $\mathcal{O}_{\tilde{E}_1} = \mathcal{O}_{\tilde{L}_1}[\theta]$. This follows easily from Remark in n.2.

Let v_E be the valuation of rank n on E from Prop.3. Use the natural identification $\tilde{\Gamma}_{E/L} = \Gamma_{\tilde{E}/\tilde{L}}$ to set for any $g \in \tilde{\Gamma}_{E/L}$,

$$\tilde{v}_{E/L}(g) = v_E(g\theta - \theta) - v_E(\theta).$$

Then $\tilde{i}_{E/L}(g) \in J_n \cup \{\infty\}$ does not depend on the above special choices of the auxiliary field M' and the generator θ (but it definitely depends on the corresponding F -structure on L). For any $j \in J_n$, set

$$\tilde{\Gamma}_{E/L,j} = \{g \in \tilde{\Gamma}_{E/L} \mid \tilde{i}_{E/L}(g) \geq j\}.$$

This is a decreasing filtration of $\tilde{\Gamma}_{E/L}$ by its normal subgroups, which is parametrized by elements of J_n . Define the auxiliary Herbrand function $\tilde{\varphi}_{E/L} : J_n \rightarrow J_n$ by the relation

$$\tilde{\varphi}_{E/L}(j) = \bar{e}_{E/L}^{-1} \int_{\bar{0}_n}^j |\tilde{\Gamma}_{E/L,j}| \, dj.$$

The value of this integral coincides with that of the corresponding integral sum taken for the partition $\bar{0}_n \leq j_1 < \dots < j_s \leq j$ where all breaking points $j_1, \dots, j_s$ are the indices of jumps of the ramification filtration $\{\tilde{\Gamma}_{E/L,j}\}$ between $\bar{0}_n$ and j . This implies for any $j \in J_n$, that

$$\tilde{\varphi}_{E/L}(j) = \bar{e}_{E/L}^{-1} \sum_{g \in \tilde{\Gamma}_{E/L}} \min\{\tilde{i}_{E/L}(g), j\}.$$

Suppose a subfield F of E is normal over L . With the above notation we have a tower of normal extensions $\tilde{E} \supset \tilde{F} := FM' \supset \tilde{L}$. Consider the natural projection $\pi : \tilde{\Gamma}_{E/L} \rightarrow \tilde{\Gamma}_{F/L} := \Gamma_{\tilde{F}/\tilde{L}}$. Then $\text{Ker } \pi = \Gamma_{\tilde{E}/\tilde{F}} = \tilde{\Gamma}_{E/F}$.

Clearly, for any $\delta \in \tilde{\Gamma}_{E/F}$, it holds $\tilde{i}_{E/L}(\delta) = \tilde{i}_{E/F}(\delta)$ and, therefore, one has for all j , $\tilde{\Gamma}_{E/F,j} = \tilde{\Gamma}_{E/F} \cap \tilde{\Gamma}_{E/L,j}$.

Notice that the extension of penultimate residue fields $\tilde{E}^{(n-1)}/\tilde{F}^{(n-1)}$ is totally ramified, so $t_1, \dots, t_{n-1}, N_{\tilde{E}/\tilde{F}}\theta$ is a system of local parameters of $\tilde{F}$, we still have the monogeneity property $\mathcal{O}_{\tilde{F}} = \mathcal{O}_{\tilde{L}}[N_{\tilde{E}/\tilde{F}}\theta]$, and we can introduce for all $j \in J_n$, the subgroups $\tilde{\Gamma}_{F/L,j}$.

Proposition 4. *For any $j \in J_n$, $\pi(\tilde{\Gamma}_{E/L,j}) = \tilde{\Gamma}_{F/L, \tilde{\varphi}_{E/F}(j)}$.*

Proof. We follow arguments from the proof of corresponding 1-dimensional property from [AN], Ch.1.

Clearly, we have $\omega \in \pi(\tilde{\Gamma}_{E/L,j}) \Leftrightarrow j \leq d(\omega) := \max\{\tilde{i}_{E/L}(\gamma) \mid \pi(\gamma) = \omega\}$ and $\omega \in \tilde{\Gamma}_{F/L, \tilde{\varphi}_{E/F}(j)} \Leftrightarrow \tilde{\varphi}_{E/F}(j) \leq \tilde{i}_{F/L}(\omega)$. So, it is sufficient to prove that

$$\tilde{\varphi}_{E/F}(d(\omega)) = \tilde{i}_{F/L}(\omega).$$

Suppose $\gamma_0 \in \tilde{\Gamma}_{E/L}$ is such that $\pi(\gamma_0) = \omega$ and $\tilde{i}_{E/L}(\gamma_0) = d(\omega)$. For any $\delta \in \tilde{\Gamma}_{E/F}$, we have

$$\tilde{i}_{E/L}(\gamma_0\delta) = \min\{\tilde{i}_{E/L}(\delta), d(\omega)\}. \quad (4)$$

Indeed,

$$\begin{aligned} \tilde{i}_{E/L}(\gamma_0\delta) &= v_E((\gamma_0\delta)\theta - \theta) - v_E(\theta) \geq \\ \min\{v_E(\gamma_0(\delta\theta - \theta)), v_E(\gamma_0\theta - \theta)\} - v_E(\theta) &= \min\{\tilde{i}_{E/L}(\delta), \tilde{i}_{E/L}(\gamma_0)\}, \end{aligned}$$

and this inequality becomes the equality if $\tilde{i}_{E/L}(\delta) < \tilde{i}_{E/L}(\gamma_0)$. On the other hand, if $\tilde{i}_{E/L}(\delta) \geq \tilde{i}_{E/L}(\gamma_0)$, then

$$d(\omega) \geq \tilde{i}_{E/L}(\gamma_0\delta) \geq \min\{\tilde{i}_{E/L}(\delta), \tilde{i}_{E/L}(\gamma_0)\} = \tilde{i}_{E/L}(\gamma_0) = d(\omega)$$

and the equality (4) still holds. So,

$$\tilde{\varphi}_{E/F}(d(\omega)) = \bar{e}_{E/F}^{-1} \sum_{\delta \in \tilde{\Gamma}_{E/F}} \min\{\tilde{i}_{E/F}(\delta), d(\omega)\} = \bar{e}_{E/F}^{-1} \sum_{\substack{\gamma \in \tilde{\Gamma}_{E/F} \\ \pi(\gamma) = \omega}} \tilde{i}_{E/L}(\gamma)$$

(notice that $\tilde{i}_{E/F}(\delta) = \tilde{i}_{E/L}(\delta)$) and our proposition will be implied by the following lemma.

Lemma. *For any $\omega \in \tilde{\Gamma}_{E/L}$, it holds*

$$\bar{e}_{E/F} \tilde{i}_{F/L}(\omega) = \sum_{\substack{\gamma \in \tilde{\Gamma}_{E/L} \\ \pi(\gamma) = \omega}} \tilde{i}_{E/L}(\gamma).$$

Proof. As earlier, we have $\mathcal{O}_{\tilde{E}} = \mathcal{O}_{\tilde{L}}[\theta]$ and $\mathcal{O}_{\tilde{F}} = \mathcal{O}_{\tilde{L}}[\theta']$, where $\theta' = N_{\tilde{E}/\tilde{F}}(\theta)$. Let

$$f(X) = X^m + a_1 X^{m-1} + \cdots + a_m \in \mathcal{O}_{\tilde{F}}[X]$$

be the minimal monic polynomial of θ over $\tilde{F}$. Consider

$$(\omega f)(X) = X^m + \omega(a_1) X^{m-1} + \cdots + \omega(a_m) \in \mathcal{O}_{\tilde{F}}[X].$$

Clearly, $a_m = (-1)^m \theta'$, $\tilde{i}_{F/L}(\omega) + v_F(\theta') = v_F(\omega a_m - a_m) < v_F((\omega a_m - a_m) \theta^{m-n})$ for all $1 \leq n < m$, and therefore,

$$v_E((\omega f)(\theta) - f(\theta)) = \bar{e}_{E/F} v_F((\omega f)(\theta) - f(\theta)) = \bar{e}_{E/F} (\tilde{i}_{F/L}(\omega) + v_F(\theta')).$$

On the other hand, the equality

$$(\omega f)(\theta) - f(\theta) = \prod_{\substack{\gamma \in \tilde{\Gamma}_{E/L} \\ \pi(\gamma) = \omega}} (\theta - \gamma\theta)$$

implies

$$v_E((\omega f)(\theta) - f(\theta)) = \sum_{\substack{\gamma \in \tilde{\Gamma}_{E/L} \\ \pi(\gamma) = \omega}} (\tilde{i}_{E/L}(\gamma) + v_E(\theta))$$

and it remains to notice that $\bar{e}_{E/F} v_F(\theta') = v_E(\theta') = [\tilde{E} : \tilde{F}] v_E(\theta)$.

Proposition 5. *For any $j \in J_n$, it holds*

$$\tilde{\varphi}_{E/L}(j) = \tilde{\varphi}_{F/L}(\tilde{\varphi}_{E/F}(j)). \quad (5)$$

Proof. The both functions are piecewise linear functions taking the same value $\bar{0}_n$ in $\bar{0}_n$. Notice that Prop.4 gives for any $j \in J_n$, the following natural exact sequence of ramification subgroups

$$1 \longrightarrow \tilde{\Gamma}_{E/F,j} \longrightarrow \tilde{\Gamma}_{E/L,j} \longrightarrow \tilde{\Gamma}_{F/L, \tilde{\varphi}_{E/F}(j)} \longrightarrow 1.$$

Therefore, $|\tilde{\Gamma}_{E/L,j}| = |\tilde{\Gamma}_{E/F,j}| |\tilde{\Gamma}_{F/L, \tilde{\varphi}_{E/F}(j)}|$. This relation implies the equality of derivatives of the both sides of (5) in all j except a finite number of edge points coming from jumps of the corresponding ramification filtrations. The proposition is proved.

5. RAMIFICATION FILTRATION OF Γ_L

As earlier, let L be an n -dimensional local field inside $\bar{L}_0$ provided with induced F -structure $\{L_{c_i} \mid 1 \leq i \leq n\}$. Denote by L_{sep} the separable closure of L in $\bar{L}_0$ and set $\Gamma_L = \text{Gal}(L_{\text{sep}}/L)$. Consider the set of indices $J(n) = J_0 \coprod J_1 \cdots \coprod J_n$, where as earlier $J_i = \{j \in \mathbb{Q}^i \mid j \geq \bar{0}_i \in \mathbb{Q}^i\}$ for all $1 \leq i \leq n$ and $J_0 = \{c\}$, where c is just a symbol. The set $J(n)$ is provided with the ordering coming from lexicographical orderings inside each of its component J_s , $1 \leq s \leq n$, and by setting that for $0 \leq s_1 < s \leq n$, every element of J_{s_1} is less than any element of J_s . We are going to define the ramification filtration $\{\Gamma^{(j)}\}_{j \in J(n)}$ of the absolute Galois group Γ_L .

Consider a finite Galois extension E/L with the induced F -structure $\{E_{c_i} \mid 1 \leq i \leq n\}$. Then for all $1 \leq i \leq n$, E_{c_i}/L_{c_i} is a finite Galois extension of i -dimensional local fields provided with induced compatible F -structures. Besides, for all $1 \leq i \leq n$, we have the natural exact sequences

$$1 \longrightarrow \tilde{\Gamma}_{E_{c_i}/L_{c_i}} \longrightarrow \Gamma_{E_{c_i}/L_{c_i}} \xrightarrow{\pi_i} \Gamma_{E_{c,i-1}/L_{c,i-1}} \longrightarrow 1.$$

Let v_E be the valuation of rank n on E from Prop.3. Then $v_E|_{E_{c_i}} = v_{E_{c_i}}$ is also the valuation of rank i on E_{c_i} from Prop.3 whilst $\mathbb{Q}^i$ being identified with $\mathbb{Q}^i \oplus \bar{0}_{n-i} \subset \mathbb{Q}^n$.

Let $j \in J(n)$. If $j = c \in J_0$ we set $\Gamma_{E/L,c} = \Gamma_{E/L}$. Suppose that $j \in J_i$ with $1 \leq i \leq n$. Consider the subgroup $\tilde{\Gamma}_{E_{c_i}/L_{c_i},j}$ of $\Gamma_{E_{c_i}/L_{c_i}}$ from n.4 and denote by $\Gamma_{E/L,j}$ its preimage with respect to the composition of all projections π_s with $s = i+1, \dots, n$. It is easy to see that $\{\Gamma_{E/L,j}\}_{j \in J(n)}$ is a decreasing filtration by normal subgroups of $\Gamma_{E/L}$. This completes the definition of ramification filtration of the group $\Gamma_{E/L}$ in lower numbering.

Define the Herbrand function $\varphi_{E/L} : J(n) \longrightarrow J(n)$ as follows. For $c \in J_0$, set $\varphi_{E/L}(c) = c$. For $1 \leq i \leq n$ and $j \in J_i \subset J(n)$, set $\varphi_{E/L}(j) = \tilde{\varphi}_{E_{c_i}/L_{c_i}}(j)$.

Clearly, $\varphi_{E/L}$ is a bijection of $J(n)$ such that $\varphi_{E/L}(J_i) = J_i$ for all $0 \leq i \leq n$. Prop.4 implies obviously the following property.

Proposition 6. *Let $E \supset F \supset L$ be a tower of finite Galois extensions provided with compatible F -structures and let π be a natural epimorphism from $\Gamma_{E/L}$ to $\Gamma_{F/L}$. Then for any $j \in J(n)$,*

- a) $\pi(\Gamma_{E/L,j}) = \Gamma_{F/L,\varphi_{E/F}(j)}$;
- b) $\varphi_{E/L}(j) = \varphi_{F/L}(\varphi_{E/F}(j))$.

As usually, introduce the upper numbering of the ramification filtration of $\Gamma_{E/L}$ by setting

$$\Gamma_{E/L,j} = \Gamma_{E/L}^{(\varphi_{E/L}(j))}$$

for all $j \in J(n)$. By the above Prop.6 the ramification filtration in upper numbering behaves well in the projective system of all finite Galois extensions E/L with compatible F -structures and we can introduce for all $j \in J(n)$, the ramification subgroups

$$\Gamma_L^{(j)} = \varprojlim_{E \supset L} \Gamma_{E/L}^{(j)}$$

of the absolute Galois group Γ_L .

Notice that if H is an open subgroup of Γ_L and $E = L_{\text{sep}}^H$, then the decreasing sequence of subgroups

$$\Gamma_L \supset \Gamma_L^{(0)} H \supset \Gamma_L^{(0,0)} H \supset \cdots \supset \Gamma_L^{(\bar{0}_n)} H \supset H$$

corresponds to the tower of algebraic extensions

$$L \subset LE_{c_0} \subset LE_{c_1} \subset \cdots \subset LE_{c_{n-1}} \subset E.$$

In particular, if $\bar{e}_{E/L} = (e_1, \dots, e_n)$, then $e_1 = (\Gamma_L^{(0)} H : \Gamma_L^{(0,0)} H)$, $\dots$, $e_n = (\Gamma_L^{(\bar{0}_n)} H : H)$, i.e. the ramification filtration contains all information about the vector $\bar{e}_{E/L}$.

Similarly to the classical 1-dimensional case the composition property from Prop. 6b) allows to extend the definition of Herbrand's function $\varphi_{E/L}$ to the case of all not necessarily normal finite extensions E/L of n -dimensional local fields with induced F -structures. Equivalently, the Herbrand function can be introduced directly (cf. e.g. [De] for 1-dimensional case): it will be sufficient to replace in all the above constructions the group $\tilde{\Gamma}_{E/L}$ by an appropriate subset $\tilde{I}_{E/L}$ of L -isomorphic embeddings of E into $\bar{L}_0$. Then the Herbrand function is a piece-wise linear function on $J(n)$ and its "edge points" correspond to the jumps of the corresponding filtration $\{I_{E/L,j}\}_{j \in J(n)}$. The above definitions and formal computations with Herbrand's functions imply the following proposition.

Proposition 7. *Suppose E is a finite extension of an n -dimensional local field L and let $\psi_{E/L}$ be the inverse Herbrand function. Then*

- a) *for any $j \in J(n)$, $\Gamma_E^{(\psi_{E/L}(j))} = \Gamma_L^{(j)} \cap \Gamma_E$;*
- b) *if $j \in J_i$ with $1 \leq i \leq n$, then*

$$\psi_{E/L}(j) = \bar{e}_{E/L, \leq i} \int_{\bar{0}_i}^j \left(\Gamma_L^{(j)} \Gamma_E^{(\bar{0}_i)} : \Gamma_E^{(\bar{0}_i)} \right)^{-1} dj \quad (6)$$

(if $\bar{e}_{E/L} = (e_1, \dots, e_n)$ then $\bar{e}_{E/L, \leq i} := (e_1, \dots, e_i)$).

Proof. We can assume that $j \in J_n$.

Let E_1 be any finite Galois extension of L containing E . Then for any $j \in J_n$, $\psi_{E_1/E}(\psi_{E/L}(j)) = \psi_{E_1/L}(j)$ and, therefore,

$$\Gamma_{E_1/E}^{(\psi_{E/L}(j))} = \Gamma_{E_1/E, \psi_{E_1/L}(j)} = \Gamma_{E_1/L, \psi_{E_1/L}(j)} \cap \Gamma_{E_1/E} = \Gamma_{E_1/L}^{(j)} \cap \Gamma_{E_1/E}.$$

Taking the projective limit on E_1 we obtain the property a).

In order to prove b) notice that

$$\left(\Gamma_L^{(j)} \Gamma_E^{(\bar{0}_n)} : \Gamma_E^{(\bar{0}_n)} \right) = \left(\Gamma_L^{(j)} : \Gamma_L^{(j)} \cap \Gamma_{E_1}^{(\bar{0}_n)} \right) \left(\Gamma_L^{(j)} \cap \Gamma_E^{(\bar{0}_n)} : \Gamma_L^{(j)} \cap \Gamma_{E_1}^{(\bar{0}_n)} \right)^{-1},$$

where the first factor equals $|\Gamma_{E_1/L}^{(j)}| = |\Gamma_{E_1/L, \psi_{E_1/L}(j)}|$ and the second factor equals

$$\left(\Gamma_E^{(\psi_{E/L}(j))} : \Gamma_E^{(\psi_{E/L}(j))} \cap \Gamma_{E_1}^{(\bar{0}_n)} \right)^{-1} = |\Gamma_{E_1/E, \psi_{E_1/L}(j)}|^{-1}.$$

So, the derivative of the right-hand side in (6) equals

$$\begin{aligned} & \bar{e}_{E_1/L} |\Gamma_{E_1/L, \psi_{E_1/L}(j)}|^{-1} \bar{e}_{E_1/E}^{-1} |\Gamma_{E_1/E, \psi_{E_1/L}(j)}| = \\ & \varphi'_{E_1/L}(\psi_{E_1/L}(j))^{-1} \varphi'_{E_1/E}(\psi_{E_1/L}(j)) = \psi'_{E_1/L}(j) \psi'_{E_1/E}(\psi_{E/L}(j))^{-1} = \psi'_{E/L}(j). \end{aligned}$$

The proposition is proved.

Notice that the left-continuity property of the ramification filtration implies

Proposition 8. *For any finite extension E/L of n -dimensional local fields, there is a maximal $j(E/L) \in J(n)$, such that $\Gamma_{E/L}^{(j)}$ acts non-trivially on E if and only if $j \leq j(E/L)$.*

One must be a bit careful about the definition of edge points in $\bar{0}_i \in J_i$ for $1 \leq i \leq n$. They should correspond to tamely ramified sub-extensions. Anyway, if E/L is a p -extension, then such sub-extensions doesn't exist, and we have the following important property.

Proposition 9. *If E/L is a p -extension, then all edge points of the Herbrand function correspond to the jumps of the ramification filtration, and the point $(\varphi_{E/L}^{-1}(j(E/L)), j(E/L))$ is the last edge point of $\varphi_{E/L}$.*

In the paper [Ab5] the definition of ramification filtration was given in slightly different terms: when giving the definition of ramification subgroups $\tilde{\Gamma}_{E/L, j}$ from n.3 we used the extension of a chosen from the very beginning valuation v of the basic field L_0 instead of the canonical valuation v_E . Actually, if $v = v_L$ then the both definitions of ramification filtration for the Galois group Γ_L coincide. So, the main result from [Ab5] gives an explicit description of the ramification filtration of the groups $\Gamma_L / \Gamma_L^{p^M} C_3(\Gamma_L)$, where $M \geq 1$, $C_3(\Gamma_L)$ is the subgroup of commutators of order ≥ 3 and L is a 2-dimensional local field of characteristic p provided with a standard F -structure. This result admits a direct generalization to the case of n -dimensional local fields and plays a crucial role in the proof of a local analogue of the Grothendieck Conjecture, cf. n.6 below.

As usually, let L be an n -dimensional local field with the subfield of i -dimensional constants L_{c_i} and the i -th residue field $L^{(i)}$, $0 \leq i \leq n$. Then there are natural group epimorphisms $\pi_i : \Gamma_L \rightarrow \Gamma_{L_{c_i}}$ and $\pi^{(i)} : \Gamma_L \rightarrow \Gamma_{L^{(i)}}$. By the use of the relation between v_L , $v_{L_{c_i}}$ and $v_{L^{(i)}}$ we obtain the following property.

Proposition 10.

- a) *If $j \in J_l \subset J(n)$ then $\pi_i(\Gamma_L^{(j)}) = e$ if $l > i$ and $\pi_i(\Gamma_L^{(j)}) = \Gamma_{L_{c_i}}^{(j)}$ if $l \leq i$;*
- b) *If L is provided with a standard F -structure and $j \in J(i)$ then $\pi^{(i)}(\Gamma_L^{(\xi(j))}) = \Gamma_{L^{(i)}}^{(j)}$, where $\xi : J(i) \rightarrow J(n)$ is such that for $0 \leq s \leq i$ and $j \in J_s$, it holds $\xi(j) = \bar{0}_{n-i} \times j \in J_{s+n-i}$.*

6. A LOCAL ANALOGUE OF THE GROTHENDIECK CONJECTURE

6.1. The category $\text{FPG}(n)$, $n \in \mathbb{N}$. The objects of this category are profinite groups G with decreasing filtration by its normal closed subgroups $\{G^{(j)}\}_{j \in J(n)}$.

Suppose H is an open subgroup of G . Define “the vector ramification index” $\bar{e}_{GH} = (e_1, \dots, e_n) \in \mathbb{Z}^n$, where $e_1 = (G^{(0)}H : G^{(0,0)}H)$, $\dots$, $e_{n-1} = (G^{(\bar{0}_{n-1})}H : G^{(\bar{0}_{n-1})}H)$, $e_n = (G^{(\bar{0}_n)}H : H)$.

Define also “the inverse Herbrand function” $\psi_{GH} : J(n) \rightarrow J(n)$ by setting $\psi_{GH}(c) = c$ and

$$\psi_{GH}(j) = \bar{e}_{GH, \leq i} \int_{\bar{0}_i}^j \left(G^{(j)} H^{(\bar{0}_i)} : H^{(\bar{0}_i)} \right)^{-1} dj,$$

where $j \in J_i$, $1 \leq i \leq n$, $H^{(\bar{0}_i)} := H \cap G^{(\bar{0}_i)}$ and, as earlier, the vector $\bar{e}_{GH, \leq i} \in \mathbb{Z}^i$ consists of the first i coordinates of the vector $\bar{e}_{GH}$.

If G_1, G_2 are objects of the category $\text{FPG}(n)$, then the set of morphisms $\text{Hom}_{\text{FPG}(n)}(G_1, G_2)$ consists of open embeddings $i : G_1 \rightarrow G_2$ such that for any $j \in J(n)$, it holds

$$i \left(G_1^{(\psi_{G_2^{i(G_1)}(j)})} \right) = i(G_1) \cap G_2^{(j)}. \quad (7)$$

Following arguments from [Ab4], n.1.2, one can verify that the composition of any 2 morphisms in $\text{FPG}(n)$ is again a morphism in $\text{FPG}(n)$ (what is actually equivalent to the composition property of the above inverse Herbrand function). Therefore, $\text{FPG}(n)$ is a category.

Define also the category $\text{FPGP}(n)$. Its objects are objects G of the category $\text{FPG}(n)$ provided with additional structure given by some topology on the maximal abelian quotient H^{ab} of every open subgroup H of G . These topologies must be compatible with natural maps $H_1^{\text{ab}} \rightarrow H^{\text{ab}}$, where H_1 is another open subgroup of G such that $H_1 \subset H$. Morphisms in $\text{FPGP}(n)$ are morphisms $\pi : G_1 \rightarrow G_2$ from $\text{FPG}(n)$ such that for any open subgroup H of G_1 the corresponding map $\pi_H^{\text{ab}} : H^{\text{ab}} \rightarrow \pi(H)^{\text{ab}}$ is continuous with respect to the corresponding topologies of these abelian subquotients.

6.2. The category $\text{DVF}_p(n)$. Choose a basic n -dimensional local field $L_0 = \mathbb{F}_p((t_n)) \dots ((t_1))$ with standard F -structure $\{L_{0i} \mid 0 \leq i \leq n\}$ associated to the system of local parameters $t_1, \dots, t_n$. Let $\bar{L}_0$ be an algebraic closure of L_0 . The direct limit of P -topologies of all finite extensions of L_0 gives the P -topological structure on $\bar{L}_0$. Denote by $\mathcal{C}(n)_p$ the completion of $\bar{L}_0$ with respect to its first valuation $v_0^1 = \text{pr}_1(v_{L_0})$. The P -topological structure on $\mathcal{C}(n)_p$ appears as v_0^1 -adic topology associated with P -topology of $\bar{L}_0$. For $0 \leq i \leq n$, denote by $\mathcal{C}(i)_p$ the completion of the algebraic closure of L_{0i} in $\mathcal{C}(n)_p$. Notice that we have the induced P -topological structures on the fields $\bar{\mathbb{F}}_p = \mathcal{C}(0)_p \subset \mathcal{C}(1)_p \subset \dots \subset \mathcal{C}(n)_p$.

Objects of the category $\text{DVF}_p(n)$ are finite extensions K of L_0 in $\mathcal{C}(n)_p$. Any such field K is provided with induced F -structure $\{K_{ci} \mid 0 \leq i \leq n\}$, where $K_{ci} = K \cap \mathcal{C}(i)_p$. Notice that $\mathcal{C}(n)_p^{\Gamma_K} = R(K)$ — the radical closure (=the completion of the maximal purely non-separable extension) of K in $\mathcal{C}(n)_p$. Similarly, for $0 \leq i \leq n$, it holds that $\mathcal{C}(i)_p^{\Gamma_K} = R(K_{ci})$.

Suppose $K, L \in \text{DVF}_p(n)$. Then the corresponding set of morphisms $\text{Hom}_{\text{DVF}}(K, L)$ in the category $\text{DVF}_p(n)$ consists of all P -continuous field morphisms $\varphi : \mathcal{C}(n)_p \rightarrow \mathcal{C}(n)_p$ such that for $1 \leq i \leq n$,

- a) $\varphi(\mathcal{C}(i)_p) = \mathcal{C}(i)_p$;
- b) $\varphi(K_{ci}) \subset L_{ci}R(L_{c,i-1})$ — the closure of the composite of L_{ci} and $R(L_{c,i-1})$ in $\mathcal{C}(n)_p$;
- c) $L_{ci}R(L_{c,i-1})$ is separable over $\varphi(K_{ci}R(K_{c,i-1}))$.

Notice that for all i , $K_{ci}, L_{ci} \in \text{DVF}_p(i)$ and $\varphi|_{K_{ci}} \in \text{Hom}_{\text{DVF}}(K_{ci}, L_{ci})$.

If $\gamma : K \rightarrow L$ is a separable field embedding then it induces a morphism in $\text{DVF}_p(n)$, which we denote by the same symbol γ .

It is easy to see that $\varphi \in \text{Hom}_{\text{DVF}}(K, L)$ is isomorphism if and only if $LR(L_{c,n-1}) = \varphi(KR(K_{c,n-1}))$. This implies that $L_{ci}R(L_{c,i-1}) = \varphi(K_{ci}R(K_{c,i-1}))$, i.e. $\varphi|_{K_{ci}}$ is an isomorphism in $\text{DVF}_p(i)$ and $R(L_{ci}) = \varphi(R(K_{ci}))$ for all $1 \leq i \leq n$.

Proposition 11. *Any $\varphi \in \text{Hom}_{\text{DVF}}(K, L)$ is uniquely decomposed into the composition of a field embedding and an isomorphism.*

Proof. The proof can be obtained by the use of the following lemma, which is a consequence of Krasner's Lemma and results from [Jac], n.8.7.

Lemma. *Suppose K' is a closed subfield of $K \in \text{DVF}_p(n)$. Then for any finite separable extension $\widetilde{M}$ of $KR(K')$ of some degree d , there is a unique separable extension M of K of degree d such that $\widetilde{M} = MR(K')$. In addition, if M' is the algebraic closure of K' in M then $R(M') = M'R(K')$ and, therefore, $\widetilde{M} = MR(M')$.*

6.3. The category $\text{DVF}_{0p}(n)$. Choose a basic n -dimensional local field $L_0 = \mathbb{Q}_p\{\{t_n\}\} \dots \{\{t_2\}\}$ with the standard F -structure $\{L_{0i} \mid 0 \leq i \leq n\}$ associated to the system of local parameters $p = t_1, t_2, \dots, t_n$. Let $\bar{L}_0$ be an algebraic closure of L_0 . Denote by $\mathbb{C}(n)_p$ the completion of $\bar{L}_0$ with respect to its first valuation. For $0 \leq i \leq n$, denote by $\mathbb{C}(i)_p$ the completion of the algebraic closure of L_{0i} in $\mathbb{C}(n)_p$. As earlier, the P -topological structure of finite extensions of L_0 induces P -topological structures on the fields $\widehat{\mathbb{Q}}_{p,\text{ur}} = \mathbb{C}(0)_p \subset \mathbb{C}(1)_p \subset \dots \subset \mathbb{C}(n)_p$.

The objects of the category $\text{DVF}_{0p}(n)$ are finite extensions K of L_0 in $\mathbb{C}(n)_p$. Any such field K is provided with the induced F -structure $\{K_{ci} \mid 0 \leq i \leq n\}$, where $K_{ci} = K \cap \mathbb{C}(i)_p$. Notice that $\mathbb{C}(n)_p^{\Gamma_K} = K$ cf. [Hy] and, similarly, for all $1 \leq i \leq n$, $\mathbb{C}(i)_p^{\Gamma_K} = K_{ci}$.

Suppose $K, L \in \text{DVF}_{0p}(n)$. Then the corresponding set of morphisms $\text{Hom}_{\text{DVF}}(K, L)$ in the category $\text{DVF}_{0p}(n)$ consists of all P -continuous field morphisms $\varphi : \mathbb{C}(n)_p \rightarrow \mathbb{C}(n)_p$ such that for $1 \leq i \leq n$,

- a) $\varphi(\mathbb{C}(i)_p) = \mathbb{C}(i)_p$;
- b) $\varphi(K_{ci}) \subset L_{ci}$.

6.4. The functor RF_p . Let $K \in \text{DVF}_p(n)$. Then K is provided with canonical F -structure and, therefore, $\text{RF}_p(K) := \Gamma_K = \text{Gal}(K_{\text{sep}}/K)$, where K_{sep} is the separable closure of K in $\mathbb{C}(n)_p$, being provided with the corresponding ramification filtration becomes an object of the category $\text{FPG}(n)$.

Let $L \in \text{DVF}_p(n)$ and $\varphi \in \text{Hom}_{\text{DVF}}(K, L)$. By Lemma from n.6.2 the categories of separable extensions of L and of K are equivalent to the categories of separable extensions of $LR(L_{c,n-1})$ and, respectively, of $KR(K_{c,n-1})$. Therefore, the separable field embedding $\varphi : KR(K_{c,n-1}) \rightarrow LR(L_{c,n-1})$ gives rise to the embedding $\hat{\varphi}$ of the first category into the second and we obtain an open embedding of topological groups $\varphi^* : \Gamma_L \rightarrow \Gamma_K$.

Proposition 12. $\varphi^* \in \text{Hom}_{\text{FPG}}(\Gamma_L, \Gamma_K)$.

Proof. If φ comes from a separable field embedding of K into L then our proposition follows from Prop.7. Therefore, by Prop.11 we can assume that φ is isomorphism and we must prove that for any $j \in J(n)$, it holds $\varphi^*(\Gamma_L^{(j)}) = \Gamma_K^{(j)}$.

Suppose that L' is a finite Galois extension of L , then $K' := \hat{\varphi}(L')$ is Galois over K and we obtain induced group isomorphism $\varphi^* : \Gamma_{L'/L} \rightarrow \Gamma_{K'/K}$. We must verify that for any $j \in J(n)$,

$$\varphi^*(\Gamma_{L'/L,j}) = \Gamma_{K'/K,j}. \quad (8)$$

The compatibility of φ with F -structures on K and L implies that for all $0 \leq i \leq n$, φ^* is compatible with natural projections $\Gamma_{L'/L} \rightarrow \Gamma_{L'_{ci}/L_{ci}}$ and $\Gamma_{K'/K} \rightarrow$

$\Gamma_{K'_{c,i}/K_{c,i}}$, and induces group isomorphisms

$$\varphi_{c,i}^* : \Gamma_{L'_{c,i}/L_{c,i}} \longrightarrow \Gamma_{K'_{c,i}/K_{c,i}}. \quad (9)$$

We can assume by induction that $\varphi_{c,i}^*$ are compatible with ramification filtration for all $i < n$ and, therefore, it is sufficient to prove (8) only for all $j \in J_n$. Notice also that (9) implies that $\bar{e}(L'/L) = \bar{e}(K'/K)$.

Choose a finite extension M of $L'_{c,n-1}$ such that if $\tilde{L}' = L'M$, $\tilde{L} = LM$, $\tilde{K} = \hat{\varphi}(\tilde{L})$ and $\tilde{K}' = \hat{\varphi}(\tilde{L}')$, then:

- a) there is $\tilde{L}'_1$ with standard F -structure such that $L' \subset \tilde{L}'_1 \subset \tilde{L}'$;
- b) there is $\tilde{K}'_1$ with standard F -structure such that $K' \subset \tilde{K}'_1 \subset \tilde{K}'$.

Then there are $\theta_L \in \tilde{L}'_1$ and $\theta_K \in \tilde{K}'_1$ such that $\mathcal{O}_{\tilde{L}'} = \mathcal{O}_{\tilde{L}}[\theta_L]$ and $\mathcal{O}_{\tilde{K}'} = \mathcal{O}_{\tilde{K}}[\theta_K]$, cf. Remark in n.4. Therefore,

$$\mathcal{O}_{\tilde{L}'R(\tilde{L}'_{c,n-1})} = \mathcal{O}_{\tilde{L}R(\tilde{L}_{c,n-1})}[\theta_L] = \mathcal{O}_{\tilde{L}R(\tilde{L}_{c,n-1})}[\varphi(\theta_K)]. \quad (10)$$

Lemma. *If $v_{L'}$ and $v_{K'}$ are valuations of rank n from Prop.3, then $\varphi^*v_{L'} = v_{K'}$, i.e. for any $z \in \mathcal{C}(n)_p$, $v_{K'}(z) = v_{L'}(\varphi(z))$.*

Proof of lemma. Because $\bar{e}(\tilde{L}'/L') = \bar{e}(\tilde{K}'/K')$, it will be sufficient to prove that $\varphi^*v_{\tilde{L}'} = v_{\tilde{K}'}$. By induction we can assume also that $\varphi^*v_{\tilde{L}'}$ and $v_{\tilde{K}'}$ coincide when being restricted to $\tilde{K}'_{c,n-1}$.

Notice, that any system of local parameters of $\tilde{K}'_{c,n-1}$ being completed by θ_K gives a system of local parameters of $\tilde{K}'$. So, we must prove only that $v_{\tilde{L}'}(\varphi(\theta_K)) = v_{\tilde{K}'}(\theta_K)$.

From the definition of valuations $v_{\tilde{L}'}$ and $v_{\tilde{K}'}$, it follows $v_{\tilde{L}'}(\theta_L) = \bar{e}_{\tilde{L}'/\tilde{L}'_1}^{-1} v_{\tilde{L}'_1}(\theta_L) = (0, \dots, 0, 1)$ and, similarly, $v_{\tilde{K}'}(\theta_K) = (0, \dots, 0, 1)$, i.e. $v_{\tilde{K}'}(\theta_K) = v_{\tilde{L}'}(\theta_L)$.

It remains only to note that θ_K and θ_L appear as lifts of uniformizing elements of the $(n-1)$ -th residue fields of the fields $\tilde{L}'R(\tilde{L}'_{c,n-1})$ and $\tilde{K}'R(\tilde{K}'_{c,n-1})$, which are isomorphic under φ . Therefore, $v_{\tilde{L}'}(\varphi(\theta_K)) = v_{\tilde{L}'}(\theta_L)$.

The lemma is proved.

From (10) it follows that we can use $\varphi(\theta_K)$ instead of θ_L to compute ramification invariants of $\Gamma_{L'/L}$. So, for any $\tau \in \Gamma_{L'/L} = \Gamma_{\tilde{L}'/\tilde{L}}$, it holds $i_{L'/L}(\tau) = v_{L'}(\tau(\varphi(\theta_K)) - \varphi(\theta_K)) - v_{L'}(\varphi(\theta_K)) = v_{K'}(\varphi^*(\tau)\theta_K - \theta_K) - v_{K'}(\theta_K) = i_{K'/K}(\varphi^*\tau)$. The proposition is proved.

Now we can set $\text{RF}_p(\varphi) = \varphi^*$ to obtain the functor $\text{RF}_p : \text{DVF}_p(n) \longrightarrow \text{FPG}(n)$. Actually, if $K \in \text{DVF}_p(n)$ then Γ_K can be considered naturally as an object of the category $\text{FPGP}(n)$. Indeed, if $H \subset \Gamma_K$ is an open subgroup then $H = \Gamma_E$ where $[E : K] < \infty$ and H^{ab} is provided with the P -topological structure coming from P -topology on E by Witt-Artin-Schreier duality. Clearly, $\text{RF}_p(\varphi)$ is a morphism of the category $\text{FPGP}(n)$.

The functor RF_p is faithful. This follows from the faithfulness of action of the group of all P -continuous field automorphisms of $KR(K_{c,n-1})$ on the Galois group of the maximal abelian extension of K of exponent p . The proof is based on a suitable version of Artin-Schreier theory. Actually, we have the following local analogue of the Grothendieck Conjecture in characteristic p :

Theorem 1. *The functor $\mathrm{RF}_p : \mathrm{DVF}_p(n) \longrightarrow \mathrm{FPGP}(n)$ is fully faithful.*

The above formalism of ramification theory reduces the above statement to the following result.

Theorem 1'. *Suppose that K is an n -dimensional local field of characteristic p , K' is its subfield of $(n-1)$ -dimensional constants and $\{\Gamma_K^{(j)}\}_{j \in J_n}$ is the “ n -dimensional part of ramification filtration” of Γ_K . Then any continuous group automorphism $\pi : \Gamma_K \rightarrow \Gamma_K$ such that*

a) *for any $j \in J_n$, $\pi(\Gamma_K^{(j)}) = \Gamma_K^{(j)}$;*

b) *for any open subgroup H of Γ_K , $\pi|_{H^{\mathrm{ab}}}$ is P -continuous,*

is induced by a P -continuous field automorphism φ of $KR(K')$ such that $\varphi(R(K')) = R(K')$.

The proof follows the strategy from the proof of the corresponding 1-dimensional property from [Ab4] and will appear in [Ab6] for the case of 2-dimensional local field K . We use the explicit description of ramification filtration of the maximal quotient of the Galois group of the maximal p -extension $\Gamma_K(p)$ of nilpotence class 2. Then we prove that any its group automorphism, which is compatible with ramification filtration and P -continuous on $\Gamma_K(p)^{\mathrm{ab}}$, must satisfy very serious restrictions.

6.5. The functor RF_0 . Let $K \in \mathrm{DVF}_{0p}(n)$. As earlier, K is provided with the canonical F -structure and $\mathrm{RF}_0(K) := \Gamma_K$ is an object of the category $\mathrm{FPGP}(n)$. If $L \in \mathrm{DVF}_{0p}(n)$ and $\varphi \in \mathrm{Hom}_{\mathrm{DVF}}(K, L)$ then $\varphi(K) \subset L$ is a finite extension and the corresponding group embedding $\Gamma_L \subset \Gamma_{\varphi(K)} \xrightarrow{\varphi^*} \Gamma_K$ gives the morphism $\mathrm{RF}_0(\varphi) \in \mathrm{Hom}_{\mathrm{FPGP}}(\Gamma_L, \Gamma_K)$. Again RF_0 is a functor and we have

Theorem 2. *The functor $\mathrm{RF}_0 : \mathrm{DVF}_{0p}(n) \longrightarrow \mathrm{FPGP}(n)$ is fully faithful.*

The proof follows again the strategy from [Ab4]. First of all, we adjust the construction of the field-of-norms functor to the case of higher dimensional local fields. Then we apply it to deduce Theorem 2 from Theorem 1.

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