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TOPOLOGY OF BILLIARD PROBLEMS, II

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Abstract
In this paper we give topological lower bounds on the number of periodic and of closed trajectories in strictly convex smooth billiards $T \subset \mathbf{R}^{m+1}$. Namely, for given n, we estimate the number of n-periodic billiard trajectories in T and also estimate the number of billiard trajectories which start and end at a given point $A \in \partial T$ and make a prescribed number n of reflections at the boundary ∂T of the billiard domain. We use variational reduction, admitting a finite group of symmetries, and apply a topological approach based on equivariant Morse and Lusternik-Schnirelman theories.

1. Introduction

Let $X \subset \mathbf{R}^{m+1}$ be a closed smooth strictly convex hypersurface. We consider the billiard system in the $(m+1)$-dimensional convex body T, bounded by X. Recall that we view the billiard ball as a point that moves in T in a straight line except when it hits $X = \partial T$, where it rebounds, making the angle of incidence equal the angle of reflection.

G. Birkhoff [2] studied periodic billiard trajectories in plane convex billiards. Papers [1] and [8] deal with the problem of estimating the number of periodic trajectories in convex billiards in $\mathbf{R}^{m+1}$, where $m > 1$. In [7] we studied the number of billiard trajectories having fixed distinct end points and making a prescribed number of reflections.

The purpose of this paper, which is a continuation of [7], is twofold. First, we obtain estimates of the number of closed billiard trajectories that start and end at a given point $A \in X$ and make a prescribed number n of reflections at the hypersurface X. This problem may seem to be a special case of the fixed-end billiard problem [7], but, as we show, the presence of symmetry allows us to get much stronger estimates than in [7]. Second, we give a linear in n estimate of the number of n-periodic trajectories.

The following theorem, Theorem 1, gives an estimate of the number of closed billiard trajectories. It deals with $\mathbf{Z}_2$-orbits of billiard trajectories. Any such $\mathbf{Z}_2$-orbit

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is determined by a sequence of reflection points $x_1, x_2, \dots, x_n \in X$ such that $x_i \neq x_{i+1}$ for $i = 1, \dots, n-1$ and $x_1 \neq A$, $x_n \neq A$. The reverse sequence $x_n, x_{n-1}, \dots, x_1$ determines the same $\mathbf{Z}_2$-orbit.

THEOREM 1
Let $X \subset \mathbf{R}^{m+1}$ be a closed smooth strictly convex hypersurface, $A \in X$.

(I) *For any even $n \geq 2$, the number of distinct $\mathbf{Z}_2$-orbits of closed billiard trajectories inside X which start and end at A and make n reflections is at least*

$$\begin{aligned} &n && \text{if } m \geq 3 \text{ is odd,} \\ &\frac{n}{2}+1 && \text{if } m \geq 2 \text{ is even.} \end{aligned} \tag{1.1}$$

(II) *For any even $n \geq 2$, the number of distinct $\mathbf{Z}_2$-orbits of closed billiard trajectories inside X which start and end at A and make n reflections is at least*

$$\begin{aligned} &[\log_2 n] + m - 1 && \text{if } m \geq 3 \text{ is odd,} \\ &[\log_2 n] + m - 2 && \text{if } m \geq 2 \text{ is even and } n \geq 4, \\ &m && \text{if } m \geq 2 \text{ is even and } n = 2. \end{aligned} \tag{1.2}$$

(III) *If $n \geq 2$ is even and if the billiard data (X, A, n) is generic (cf. below), then the number of distinct $\mathbf{Z}_2$-orbits of closed billiard trajectories inside X which start and end at A and make n reflections is at least*

$$\frac{mn}{2}. \tag{1.3}$$

First we explain the *genericity* assumption in statement (III). The billiard data (X, A, n) determines a continuous function

$$X^{\times n} \to \mathbf{R}, \qquad (x_1, \dots, x_n) \mapsto \sum_{i=0}^{n} |x_i - x_{i+1}| \tag{1.4}$$

(the total length), where we understand that $x_0 = A = x_{n+1}$. This function is smooth at all configurations $(x_1, \dots, x_n) \in X^{\times n}$ with $x_i \neq x_{i+1}$ for $i = 0, \dots, n$. The data (X, A, n) is *generic* if any critical configuration $(x_1, \dots, x_n) \in X^{\times n}$ of the total length function (1.4), satisfying the above condition $x_i \neq x_{i+1}$, is Morse (cf. [1], [8]).

Statements (I) and (II) give different lower bounds on the number of closed billiard trajectories. (I) is linear in n; it is better than (II) for large n. On the other hand, (II) may be better than (I) if the dimension $m = \dim X$ of the boundary of billiard domain is large.

Let us compare Theorem 1 with the lower bound on the number of billiard trajectories with fixed distinct end points, obtained in [7]. In Theorem 1 we speak about

$\mathbf{Z}_2$-orbits of billiard trajectories. Each $\mathbf{Z}_2$-orbit contains one or two billiard trajectories. For n even, each $\mathbf{Z}_2$-orbit contains precisely two distinct billiard trajectories. Hence we see that, for n even, statement (I) of Theorem 1 predicts twice the number of closed billiard trajectories, compared to the estimate of [7] for the billiard trajectories with fixed ends. Also, for large m, statements (II) and (III) give much larger lower bounds than the corresponding estimates of [7, Theorem 1].

Statement (III) includes the case $m = 1$ (the plane billiards) and gives the estimate $n/2$. The billiard in the unit circle has precisely $n/2$ orbits of closed billiard trajectories with a given initial point.

It is reasonable to expect that, for any even $n \geq 2$, the number of distinct $\mathbf{Z}_2$-orbits of closed billiard trajectories inside X which start and end at A and make n reflections is at least

$$\begin{aligned} &n+m-1 \quad \text{if } m \geq 3 \text{ is odd,} \\ &\frac{n}{2}+m-1 \quad \text{if } m \geq 2 \text{ is even.} \end{aligned} \tag{1.5}$$

Such an estimate implies both statements (I) and (II) of Theorem 1. The methods of this paper do not prove this assertion, although the gap looks very small.

The proof of Theorem 1 is based on a computation of the cohomology ring of a relevant configuration space of points on the sphere S^m. We apply the technique of the critical point theory, based on the cup-length estimates together with a refinement, suggested by E. Fadell and S. Husseini [6], related to the notion of category weight of cohomology classes.

Next we state the main result concerning n-periodic trajectories.

THEOREM 2

Let $X \subset \mathbf{R}^{m+1}$ be a smooth strictly convex hypersurface. For any odd prime n, there exist at least

$$\begin{aligned} &n \quad \textit{if } m > 1 \textit{ is odd,} \\ &\frac{n+1}{2} \quad \textit{if } m \textit{ is even} \end{aligned} \tag{1.6}$$

distinct D_n-orbits of n-periodic billiard trajectories inside X.

Here D_n denotes the dihedral group of order $2n$, which acts naturally on the billiard trajectories (see [8]).

This theorem complements the results of [8]. In [8] it is shown that, for $m \geq 3$ and n odd, the number of distinct D_n-orbits of n-periodic billiard trajectories inside $X \subset \mathbf{R}^{m+1}$ is not less than $[\log_2(n-1)] + m$ and is at least $(n-1)m$ for generic billiards $X \subset \mathbf{R}^{m+1}$. These results from [8] are similar to statements (II) and (III) of

Theorem 1. Theorem 2 has several advantages compared to [8]. It gives a linear in n estimate that is better for large n than the logarithmic estimate of [8]. Also, it allows the case $m = 2$, which corresponds to convex billiards in 3-dimensional Euclidean space. On the other hand, the result of [8] is better for large m.

The proof of Theorem 2 is based on a computation of the cohomology rings of cyclic configuration spaces of spheres with rational coefficients. The case of $\mathbf{Z}_2$-coefficients was computed in [8].

2. Cohomology of the closed string configuration spaces of spheres

Let

$$G_n = G(S^m; A, A, n) \tag{2.1}$$

denote the closed string configuration space of S^m, that is, the space of all configurations $(x_1, \dots, x_n)$, where $x_i \in S^m$, such that $x_1 \neq A$, $x_n \neq A$, and $x_i \neq x_{i+1}$ for all $i = 1, \dots, n-1$. There is a natural involution

$$T : G_n \to G_n, \qquad T(x_1, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1) \tag{2.2}$$

which is important for the sequel.

THEOREM 3
The cohomology group $H^i(G_n; \mathbf{Z})$ is nonzero only in dimensions

$$i = 0,\ (m-1),\ 2(m-1), \dots, (n-1)(m-1),$$

and for these values i the group $H^i(G_n; \mathbf{Z})$ is free abelian of rank 1. One may choose additive generators

$$\sigma_i \in H^{i(m-1)}(G_n; \mathbf{Z}), \quad i = 0, 1, \dots, n-1,$$

such that, for $m \geq 3$ odd, the multiplication is given by

$$\sigma_i \sigma_j = \begin{cases} \frac{(i+j)!}{i! \cdot j!} \cdot \sigma_{i+j} & \text{if } i + j \leq n-1, \\ 0 & \text{if } i + j > n-1 \end{cases} \tag{2.3}$$

and, for $m \geq 2$ even, it is given by

$$\sigma_i \sigma_j = \begin{cases} \frac{[(i+j)/2]!}{[i/2]! \cdot [j/2]!} \cdot \sigma_{i+j} & \text{if } i + j \leq n-1 \text{ and } i \text{ or } j \text{ is even}, \\ 0 & \text{if either } i + j > n-1 \text{ or both } i \text{ and } j \text{ are odd}. \end{cases} \tag{2.4}$$

Reflection (2.2) acts for $m > 1$ odd by

$$T^*(\sigma_i) = (-1)^i \sigma_i \tag{2.5}$$

and for $m > 1$ even by

$$T^*(\sigma_i) = (-1)^{[i/2]+ni} \sigma_i, \quad i = 0, 1, \dots, n-1. \tag{2.6}$$

Proof
Consider the map

$$G_n = G(S^m; A, A, n) \to S^m - A, \qquad (x_1, \dots, x_n) \mapsto x_n.$$

It is a smooth fibration with fiber $G(S^m; A, B, n-1)$, where $A \neq B$. Since the base $S^m - A$ is contractible, we conclude that the inclusion

$$G(S^m; A, B, n-1) \subset G_n \tag{2.7}$$

is a homotopy equivalence. Hence the integral cohomology ring of G_n coincides with $H^*(G(S^m; A, B, n-1); \mathbf{Z})$, which we calculate below.

Theorem 8 of [7] describes algebra $H^*(G(S^m; A, B, n-1); \mathbf{k})$, where $\mathbf{k}$ is an arbitrary field. From this description it is clear that the dimension of the cohomology does not depend on field $\mathbf{k}$. Therefore we conclude that the integral cohomology $H^i(G(S^m; A, B, n-1); \mathbf{Z})$ has no torsion; it is a free abelian group of rank 1 for $i = r(m-1)$, where $r = 0, 1, \dots, n-1$, and vanishes for all other values of i.

Let $C \in S^m$ be a point distinct from A and B. We obtain an inclusion of configuration spaces $\phi^* : G(S^m - C; A, B, n-1) \to G(S^m; A, B, n-1)$, where we identify $S^m - C$ with $\mathbf{R}^m$. The cohomology algebra $H^*(G(\mathbf{R}^m; A, B, n-1); \mathbf{Z})$ has generators $s_0, \dots, s_{n-1}$, and the full list of relations is described in [7, Proposition 7]. From [7, Remark 9] we know that the induced map ϕ^* on cohomology with an arbitrary field of coefficients $\mathbf{k}$ is injective. This implies that the induced map on integral cohomology

$$\phi^* : H^*\big(G(S^m; A, B, n-1); \mathbf{Z}\big) \to H^*\big(G(\mathbf{R}^m; A, B, n-1); \mathbf{Z}\big) \tag{2.8}$$

is injective and that ϕ^* maps indivisible classes from $H^*(G(S^m; A, B, n-1); \mathbf{Z})$ into indivisible classes in $H^*(G(\mathbf{R}^m; A, B, n-1); \mathbf{Z})$.

We claim that, for any $r = 0, 1, \dots, n-1$, there exists an indivisible class

$$\sigma_r \in H^{r(m-1)}\big(G(S^m; A, B, n-1); \mathbf{Z}\big)$$

such that

$$\phi^*(\sigma_r) = \begin{cases} \sum\limits_{0 \le i_1 < \cdots < i_r < n} s_{i_1} \cdots s_{i_r} & \text{for } m \text{ odd}, \\ (-1)^{[r/2]+nr} \cdot \sum\limits_{0 \le i_1 < \cdots < i_r < n} (-1)^{i_1 + \cdots + i_r} s_{i_1} \cdots s_{i_r} & \text{for } m \text{ even} \end{cases} \tag{2.9}$$

(cf. [7, (4.3), (4.4)]). Indeed, applying [7, Remark 9] with $\mathbf{k} = \mathbf{Q}$, we see that the image of the generator of the group

$$H^{r(m-1)}\big(G(S^m; A, B, n-1); \mathbf{Z}\big) \simeq \mathbf{Z}$$

under homomorphism ϕ^* equals an integral multiple of the expression on the right-hand side of (2.9). Since the classes on the right-hand side of (2.9) are indivisible, and since we know that ϕ^* maps indivisible classes to indivisible classes, we conclude that there exists a generator σ_r with the required property.

The product formulae (2.3) and (2.4) for classes σ_r follow since they hold for the products $\phi^*(\sigma_i)\phi(\sigma_j) \in H^*(G(\mathbf{R}^m; A, B, n-1); \mathbf{Z})$, as can be easily checked using the arguments of the proof of [7, Theorem 8].

Now we want to find the action of the reflection $T : G_n \to G_n$ on classes σ_i. It is clear that $T^*(\sigma_i) = \pm\sigma_i$, and we need to calculate the sign. Consider the following diagram of natural inclusions,

$$\begin{tikzcd}
G(\mathbf{R}^m; A, B, n-1) \arrow[r] \arrow[d] & G(S^m; A, B, n-1) \arrow[d] \\
G(\mathbf{R}^m; A, A, n) \arrow[r] & G(S^m; A, A, n)
\end{tikzcd}$$

(where $\mathbf{R}^m = S^m - C$ as above), and the induced diagram of cohomology groups,

$$\begin{tikzcd}
H^*\big(G(\mathbf{R}^m; A, A, n); \mathbf{Z}\big) \arrow[d, "\beta"'] & H^*\big(G(S^m; A, A, n); \mathbf{Z}\big) \arrow[l, "\gamma"'] \arrow[d, "\alpha"', "\simeq"] \\
H^*\big(G(\mathbf{R}^m; A, B, n-1); \mathbf{Z}\big) & H^*\big(G(S^m; A, B, n-1); \mathbf{Z}\big) \arrow[l, "\phi^*"']
\end{tikzcd}$$

where α is an isomorphism and ϕ^* is injective. To understand β, note that $G(\mathbf{R}^m; A, A, n)$ is homotopy equivalent to the cyclic configuration space $G(\mathbf{R}^m, n+1)$ (cf. [8]) and so the cohomology $H^*(G(\mathbf{R}^m; A, A, n); \mathbf{Z})$ has $(m-1)$-dimensional generators $s_0, s_1, \ldots, s_n$ which satisfy the relations of [8, Proposition 2.2]. (We shift indices for convenience.) The proof of [7, Proposition 7] shows that $\beta(s_i) = s_i$ for $i = 0, 1, \ldots, n-1$ and $\beta(s_n) = 0$. Hence β is an epimorphism with a kernel equal to the ideal generated by s_n.

The reflection T also acts on $G(\mathbf{R}^m; A, A, n)$ (by formula (2.2)). It is clear that the induced map $T^* : H^*(G(\mathbf{R}^m; A, A, n); \mathbf{Z}) \to H^*(G(\mathbf{R}^m; A, A, n); \mathbf{Z})$ acts on the generators s_i as follows:

$$T^*(s_i) = (-1)^m s_{n-i}, \quad \text{where } i = 0, 1, \ldots, n. \tag{2.10}$$

Now we may calculate $T^*(\sigma_r)$, where $r = 1, \ldots, n-1$. Fix a subsequence $0 < i_1 < \cdots < i_r < n$. (We avoid indices 0 and n.) Suppose first that m is odd. Then $\phi^*(\alpha(\sigma_r))$ contains monomial $s_{i_1}s_{i_2}\cdots s_{i_r}$; therefore $\gamma(\sigma_r)$ contains the same monomial with coefficient 1. Then $T^*(\gamma(\sigma_r))$ contains monomial $s_{n-i_r}s_{n-i_{r-1}}\cdots s_{n-i_1}$

with coefficient $(-1)^{mr} = (-1)^r$. The last monomial appears in $\gamma(\sigma_r)$ with coefficient 1. Since we know that $T^*(\sigma_r) = \pm\sigma_r$, we conclude that $T^*(\sigma_r) = (-1)^r\sigma_r$.

Assume now that m is even. Then $\phi^*(\alpha(\sigma_r))$ contains monomial $s_{i_1}s_{i_2}\cdots s_{i_r}$ with coefficient

$$(-1)^{[r/2]+nr+i_1+\cdots+i_r}.$$

Applying T^* and using (2.10), we see that the monomial $s_{n-i_r}s_{n-i_{r-1}}\cdots s_{n-i_1}$ appears in $T^*(\gamma(\sigma_r))$ with coefficient

$$(-1)^{nr+i_1+\cdots+i_r}$$

and in $\gamma(\sigma_r)$ with coefficient

$$(-1)^{[r/2]+i_1+\cdots+i_r}.$$

This shows that $T^*(\sigma_r) = (-1)^{[r/2]+nr}\sigma_r$. □

3. Calculation of equivariant cohomology

Our purpose in this section is to compute the cohomology of $G_n/\mathbf{Z}_2$, the factor space of the space of closed string configurations $G_n = G(S^m; A, A, n)$ with respect to the $\mathbf{Z}_2$-action given by the reflection $T : G_n \to G_n$. For n even, T acts freely, and $H^*(G_n/\mathbf{Z}_2; \mathbf{Z})$ coincides with the equivariant cohomology of G_n.

The problem is trivial for $m = 1$; therefore everywhere in this section we assume that $m > 1$.

To compute the equivariant cohomology, we apply the Morse theory method. Namely, we consider the simplest billiard in the standard unit sphere $S^m \subset \mathbf{R}^{m+1}$ and the function of negative total length

$$L : G_n = G(S^m; A, A, n) \to \mathbf{R}, \qquad (x_1, \dots, x_n) \mapsto -\sum_{i=0}^{n} |x_i - x_{i+1}|. \tag{3.1}$$

Here we understand that $x_0 = x_{n+1} = A$. The critical points of L are the billiard trajectories in S^m which start and end at A and make n reflections. All such trajectories can easily be described.

Namely, fix a vector $a \in S^m$, $a \perp A$, orthogonal to A and an angle

$$\psi_k = \frac{2\pi k}{n+1}, \quad k = 1, 2, \dots, \left[\frac{n+1}{2}\right]. \tag{3.2}$$

This choice (a, ψ_k) determines the billiard trajectory $(x_1, \dots, x_n)$, where

$$x_j = A\cos(j\psi_k) + a\sin(j\psi_k), \quad j = 1, \dots, n.$$

Note that for n odd the trajectory determined by the pair $(a, \psi_{(n+1)/2})$ does not depend on a; it has the form $(x_1, \dots, x_n)$, where $x_j = A$ for j even and $x_j = -A$ for j odd.

We denote by

$$V_p \subset G_n, \quad p = 0, 1, \dots, \left[\frac{n-1}{2}\right],$$

the variety of trajectories determined by all pairs (a, ψ_k), where

$$k = \left[\frac{n+1}{2}\right] - p \tag{3.3}$$

and $a \perp A$ is an arbitrary point of the sphere $S^{m-1} \subset S^m$ orthogonal to A.

If n is even, then every submanifold V_p is diffeomorphic to sphere S^{m-1}.

If n is odd, then V_0 is a single point and $V_1, \dots, V_{[(n-1)/2]}$ are diffeomorphic to the sphere S^{m-1}.

The following statement is similar to that of I. Babenko in [1, Proposition 3.1].

PROPOSITION 4
Each $V_p \subset G_n$ is a nondegenerate critical submanifold of function L in the sense of R. Bott.

If n is even, then the index of each V_p equals $2p(m-1)$ for $p = 0, 1, \dots, (n-2)/2$.

If n is odd, then the index of V_0 equals zero and, for $p = 1, \dots, (n-1)/2$, the index of V_p equals $(2p-1)(m-1)$.

Proof
Let $e_1, \dots, e_{m+1} \in \mathbf{R}^{m+1}$ be an orthonormal base. We may assume that $A = e_1$. We want to calculate the Hessian of function L at a billiard trajectory $c_k = (x_1, \dots, x_n) \in G_n$, where

$$x_j = \cos(\psi_k)e_1 + \sin(\psi_k)e_2, \quad j = 1, \dots, n,$$

and

$$\psi_k = \frac{2\pi k}{n+1}, \quad k = 1, \dots, \left[\frac{n+1}{2}\right].$$

Let $x_j^\perp$ denote the vector orthogonal to x_j lying in the (e_1, e_2)-plane, that is,

$$x_j^\perp = \cos\left(\frac{\psi_k + \pi}{2}\right)e_1 + \sin\left(\frac{\psi_k + \pi}{2}\right)e_2.$$

Any tangent vector $Y \in T_{c_k}G_n = \bigoplus_j T_{x_j}S^m$ is determined by numbers $\mu_{r,j} \in \mathbf{R}$, where $r = 0, 1, \dots, m-1$ and $j = 1, \dots, n$, such that the component of Y in $T_{x_j}S^m$ equals

$$\mu_{0,j}x_j^\perp + \sum_{r=1}^{m-1} \mu_{r,j}e_{r+2}.$$

A direct calculation of the Hessian $H(L)_{c_k}(Y, Y)$ of L gives the following quadratic form in variables $\mu_{r,j}$:

$$H(L)_{c_k}(Y, Y) = \frac{1}{2}\sin\Big(\frac{\psi_k}{2}\Big)\cdot\sum_{j=0}^{n}(\mu_{0,j} - \mu_{0,j+1})^2 + \Big(2\sin\Big(\frac{\psi_k}{2}\Big)\Big)^{-1}\cdot\Big[\sum_{r=1}^{m-1} Q_{\psi_k}(\mu_{r,1}, \dots, \mu_{r,n})\Big], \tag{3.4}$$

where in the first sum we understand that $\mu_{0,0} = 0 = \mu_{0,n+1}$ and in the second sum the symbol $Q_\psi(y_1, \dots, y_n)$ denotes the following quadratic form:

$$Q_\psi(y_1, \dots, y_n) = -2\cos(\psi)\cdot\sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n-1} y_i y_{i+1}.$$

We see that the Hessian splits as a direct sum of m quadratic forms corresponding to different values $r = 0, 1, \dots, m-1$. The terms involving $\mu_{0,j}$ (the first sum) give a positive definite quadratic form. The remaining $(m-1)$-forms are identical, and their index and nullity equal the index and nullity of Q_{ψ_k}. Hence we conclude that the index and nullity of the Hessian equals $m-1$ times the index and nullity of the form Q_{ψ_k}.

In order to calculate the index of Q_{ψ_k}, we observe that the eigenvalues of the symmetric $(n \times n)$-matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

are given by

$$\lambda_s = 2\cos\Big(\frac{\pi s}{n+1}\Big), \quad s = 1, 2, \dots, n, \tag{3.5}$$

and the eigenvector $(v_{1,s}, \dots, v_{n,s})$ corresponding to λ_s is given by

$$v_{j,s} = \sin\Big(\frac{\pi j s}{n+1}\Big), \quad j = 1, \dots, n. \tag{3.6}$$

This claim can be checked directly.

Therefore the eigenvalues of Q_{ψ_k} are

$$2\Big[\cos\Big(\frac{\pi s}{n+1}\Big) - \cos\Big(\frac{2\pi k}{n+1}\Big)\Big], \quad s = 1, 2, \dots, n, \tag{3.7}$$

and the eigenvectors of Q_{ψ_k} are given by (3.6).

Hence the index of Q_{ψ_k} equals the number of integers s such that $2k < s \le n$, which is $n - 2k$ for $2k \le n$ and zero if $k = (n+1)/2$ and n is odd. Since (according to (3.3)) $k = [(n+1)/2] - p$, we conclude that the index of Q_{ψ_k} equals

$$n - 2k = n - 2\Big(\Big[\frac{n+1}{2}\Big] - p\Big) = \begin{cases} 2p & \text{if } n \text{ is even,} \\ 2p - 1 & \text{if } n \text{ is odd.} \end{cases}$$

The special case $k = (n+1)/2$ for n odd corresponds to $p = 0$; in this case the index and nullity of Q_{ψ_k} equal zero.

From (3.7) we see that the nullity of Q_{ψ_k} equals 1 for any k unless n is odd and $k = (n+1)/2$.

The discussion above proves that on any critical submanifold V_p the dimension of the kernel of the Hessian of L equals the dimension of V_p; hence all submanifolds V_p are nondegenerate in the sense of Bott and their indices are as stated. □

The normal bundle $\nu(V_p)$ splits as a direct sum $\nu_+(V_p) \oplus \nu_-(V_p)$ of the positive and negative normal bundles with respect to the Hessian of L. One may describe the negative normal bundle $\nu_-(V_p)$ as follows.

LEMMA 5

The negative normal bundle $\nu_-(V_p)$ to V_p is

$$\nu_-(V_p) = \begin{cases} \underbrace{\xi \oplus \xi \oplus \cdots \oplus \xi}_{2p \text{ times}} & \text{if } n \text{ is even,} \\ \underbrace{\xi \oplus \xi \oplus \cdots \oplus \xi}_{2p-1 \text{ times}} & \text{if } n \text{ is odd and } p > 0, \end{cases} \tag{3.8}$$

where ξ denotes the tangent bundle of sphere S^{m-1}.

Proof

Let $S^{m-1} \subset S^m$ be the equatorial sphere consisting of unit vectors orthogonal to A. Any point $a \in S^{m-1}$ and angle (3.2) determine a critical submanifold V_p. Fix an eigenvalue λ_s (given by (3.5)) such that expression (3.7) is negative. Consider the subbundle $\nu_s(V_p)$ of the normal bundle $\nu(V_p)$ consisting of eigenvectors of the Hessian with eigenvalue λ_s. We show that $\nu_s(V_p)$ is isomorphic to ξ. This would clearly imply the lemma.

Consider a billiard trajectory $c_k = (x_1, \dots, x_n) \in G_n$ in the plane of vectors a and A, where

$$x_j = \cos(\psi_k)A + \sin(\psi_k)a, \quad j = 1, \dots, n,$$

and

$$\psi_k = \frac{2\pi k}{n+1}, \quad k = \left[\frac{n+1}{2}\right] - p.$$

Denote by ξ_a the $(m-1)$-dimensional subspace orthogonal to a and A. We show that there is an isomorphism between the fiber of $\nu_s(V_p)$ over c_k and ξ_a which depends continuously on a.

Let $v_j \in T_{x_j} S^m$, where $j = 1, 2, \ldots, n$, be a sequence of tangent vectors. Using (3.4) and (3.6), we find that a sequence of vectors $(v_1, \ldots, v_n)$ belongs to the fiber of $\nu_s(V_p)$ over the configuration $c_k = (x_1, \ldots, x_n)$ if and only if

$$v_j \in \xi_a \qquad \text{and} \qquad v_j = \frac{\sin(\pi j s/(n+1))}{\sin(\pi s/(n+1))} \cdot v_1, \quad j = 1, \ldots, n. \tag{3.9}$$

We see that the first vector v_1 uniquely determines a tangent vector $(v_1, \ldots, v_n)$ to a configuration c_k in the eigendirection λ_s. Moreover, v_1 can be an arbitrary vector in ξ_a. □

Since ξ is orientable, we obtain the following.

COROLLARY 6

The negative normal bundle $\nu_-(V_p)$ is orientable.

Note that this corollary is trivial for $m > 2$ since the sphere S^{m-1} is then simply connected.

COROLLARY 7

The function $L : G_n \to \mathbf{R}$ (cf. (3.1)) is a perfect Bott function.

Proof

Note that the critical value $L(V_p)$ equals

$$L(V_p) = -2(n+1)\sin\left(\frac{2\pi k}{n+1}\right), \quad \text{where } k = \left[\frac{n+1}{2}\right] - p.$$

Hence, for $p < p'$, we have $L(V_p) < L(V_{p'})$.

Choose constants $c_0, c_1, \ldots, c_{[(n-1)/2]} \in \mathbf{R}$ such that

$$L(V_p) < c_p < L(V_{p+1}), \quad p = 0, 1, \ldots, \left[\frac{n-3}{2}\right],$$

and

$$L(V_{[(n-1)/2]}) < c_{[(n-1)/2]}.$$

Each $F_p = L^{-1}(-\infty, c_p]) \subset G_n$ is a compact manifold with boundary, and we obtain a filtration

$$F_0 \subset F_1 \subset \cdots \subset F_{[(n-1)/2]}.$$

The inclusion $F_{[(n-1)/2]} \to G_n$ is a homotopy equivalence (as follows easily from [7, Proposition 4]). Using Corollary 6 and the Thom isomorphism, we obtain

$$\begin{aligned} H^j(F_p, F_{p-1}; \mathbf{Z}) &\simeq H^{j-\operatorname{ind}(V_p)}(V_p; \mathbf{Z}) \\ &= \begin{cases} \mathbf{Z} & \text{if } j = \operatorname{ind}(V_p) \text{ or } j = \operatorname{ind}(V_p) + m - 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.10}$$

This also holds true for $p = 0$ if we understand that $F_{-1} = \emptyset$.

Suppose that n is even. Then cohomology group $H^j(F_p, F_{p-1}; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ for $j = 2p(m-1)$ and $j = (2p+1)(m-1)$ and vanishes for all other j. Comparing this with the additive structure of $H^*(G_n; \mathbf{Z})$ given by Theorem 3, we find that

$$H^*(G_n; \mathbf{Z}) \simeq \bigoplus_{p=0}^{[(n-1)/2]} H^*(F_p, F_{p-1}; \mathbf{Z}), \tag{3.11}$$

which means that L is perfect.

Suppose now that n is odd. Then $H^j(F_0, F_{-1}; \mathbf{Z})$ is $\mathbf{Z}$ for $j = 0$ and vanishes for all other values of j. If $p > 0$, then

$$H^j(F_p, F_{p-1}; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{for } j = (2p-1)(m-1) \text{ or } j = 2p(m-1), \\ 0 & \text{otherwise,} \end{cases}$$

and thus the perfectness of (3.11) also holds.

Alternatively, for $m > 2$, the perfectness of (3.11) follows without using Theorem 3 by considering the spectral sequence of filtration F_p,

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}; \mathbf{Z}) \Rightarrow H^{p+q}(G_n; \mathbf{Z}),$$

and observing that for any of its differentials d_r, with $r \geq 1$, either the source or the target vanishes. Therefore $E_1 = E_\infty$. Moreover, every diagonal $p + q = c$ of E_∞ contains at most one nonzero group. If $m = 2$, the differential d_1 has a nonzero source and target, and so the above argument does not work. □

From this point on we assume that n is even.

Then the reflection $T : G_n \to G_n$ acts freely, and our purpose is to calculate the cohomology of the factor space $G_n' = G_n/\mathbf{Z}_2$. Function (3.1) is reflection invariant and so determines a smooth function

$$L' : G_n' \to \mathbf{R}.$$

The critical points of L' form nondegenerate (in the sense of Bott) critical submanifolds

$$V'_0, V'_1, \dots, V'_{n/2-1},$$

where $V'_p = V_p/\mathbf{Z}_2$. The index of V'_p equals $2p(m-1)$ (as follows from Proposition 4). Since each V_p can be identified with S^{m-1} and under this identification the reflection T acts as the usual antipodal map, we see that each V'_p is diffeomorphic to the projective space $\mathbf{RP}^{m-1}$.

COROLLARY 8

The Poincaré polynomial of $G'_n = G(S^m; A, A, n)/\mathbf{Z}_2$ with coefficients in field $\mathbf{Z}_2$ is

$$\frac{t^m - 1}{t - 1} \cdot \frac{t^{n(m-1)} - 1}{t^{2(m-1)} - 1},$$

and the sum of Betti numbers with coefficients in $\mathbf{Z}_2$ is $mn/2$.

Proof

We give here a simple proof that works for $m > 2$. The case $m = 2$ follows from Theorem 11.

Consider the filtration $F_0 \subset F_1 \subset \cdots \subset F_{n/2-1} \subset G_n$, as in the proof of Corollary 7. Let F'_p denote $F_p/\mathbf{Z}_2$. We obtain a filtration $F'_0 \subset F'_1 \subset \cdots \subset F'_{n/2-1} \subset G'_n$ such that the inclusion $F'_{n/2-1} \subset G'_n$ is a homotopy equivalence and

$$H^j(F'_p, F'_{p-1}; \mathbf{Z}_2) \simeq H^{j-2p(m-1)}(\mathbf{RP}^{m-1}; \mathbf{Z}_2), \quad p = 0, 1, \dots, \frac{n}{2} - 1$$

(using the Thom isomorphism). Hence $H^j(F'_p, F'_{p-1}; \mathbf{Z}_2)$ is nonzero (and 1-dimensional) only for $2p(m-1) \le j \le (2p+1)(m-1)$. The spectral sequence of filtration F'_p,

$$E_1^{p,q} = H^{p+q}(F'_p, F'_{p-1}; \mathbf{Z}_2) \Rightarrow H^{p+q}(G'_n; \mathbf{Z}_2),$$

has $E_1^{p,q} \simeq \mathbf{Z}_2$ for $p(2m-3) \le q \le p(2m-3) + (m-1)$ and $E_1^{p,q} = 0$ otherwise. Hence, for any differential d_r, where $r \ge 1$, either the source or the target vanishes. Therefore $E_1 = E_\infty$ and our statement follows. □

We now calculate the Stiefel-Whitney classes of the negative normal bundle $\nu_-(V'_p)$. In particular, we find out for which p this bundle is orientable. This information is needed for computing the integral cohomology of G'_n.

LEMMA 9

The total Stiefel-Whitney class of the negative normal bundle $\nu_-(V'_p)$ equals

$$(1+\alpha)^{p(m-1)} \in H^*(V'_p; \mathbf{Z}_2),$$

where $\alpha \in H^1(V'_p; \mathbf{Z}_2) \simeq \mathbf{Z}_2$ denotes the generator.

Proof

As in the proof of Lemma 5, we obtain that the negative normal bundle $\nu_-(V'_p)$ splits as a direct sum of $2p$ vector bundles η_s of rank $m-1$, one for each negative eigenvalue

$$\lambda_s = 2\left[\cos\left(\frac{\pi s}{n+1}\right) - \cos\left(\frac{2\pi k}{n+1}\right)\right]$$

of the Hessian. Here $k = n/2 - p$.

Let τ denote the tangent bundle of $\mathbf{RP}^{m-1}$. Let $\gamma^\perp$ be a rank $m-1$ vector bundle over $\mathbf{RP}^{m-1}$ such that its fiber over a line $\ell \in \mathbf{RP}^{m-1}$ is the orthogonal complement $\ell^\perp$.

We claim that

$$\eta_s \simeq \begin{cases} \tau & \text{if } s \text{ is even,} \\ \gamma^\perp & \text{if } s \text{ is odd.} \end{cases}$$

Indeed, this bundle is obtained from the tangent bundle ξ of S^{m-1} (cf. Lemma 5) by identifying the antipodal points, and under this identification the first vector v_1 should be replaced by the last vector v_n (cf. (3.9)). Formulae (3.9) show that

$$v_n = -\cos(\pi s) \cdot v_1 = (-1)^{s+1} \cdot v_1,$$

and hence the bundle η_s is obtained from ξ by identifying the fibers over points a and $-a$ with a twist $(-1)^{s+1}$. This implies our claim (cf. [12]).

For a given p there is an equal number of negative eigenvalues λ_s of the Hessian on V_p with even and odd s. Therefore the bundle $\nu_-(V'_p)$ is isomorphic to a direct sum of p copies of $\tau \oplus \gamma^\perp$.

The total Stiefel-Whitney class of $\gamma^\perp$ is $(1+\alpha)^{-1}$, and the total Stiefel-Whitney class of τ is $(1+\alpha)^m$ (cf. [12]). Hence the total Stiefel-Whitney class of the negative bundle is

$$\left[(1+\alpha)^{-1} \cdot (1+\alpha)^m\right]^p = (1+\alpha)^{(m-1)p}. \qquad \square$$

COROLLARY 10

If m is odd, then the negative normal bundle $\nu_-(V'_p)$ is orientable for any p. If m is even, then the negative normal bundle $\nu_-(V'_p)$ is orientable for all even p and nonorientable for all odd p.

Proof

By Lemma 9, the first Stiefel-Whitney class of $\nu_-(V'_p)$ is $p(m-1)\alpha$. This implies our statement. $\square$

Recall our permanent assumption that $m > 1$ and n is even.

THEOREM 11
If $m > 1$ is odd, then

$$H^j(G'_n; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{for } j = 2i(m-1), \text{ where } i = 0, 1, \ldots, n/2 - 1, \\ \mathbf{Z}_2 & \text{for } j \text{ even satisfying } 2i(m-1) < j \leq (2i+1)(m-1) \\ & \text{with } i \text{ as above}, \\ 0 & \text{otherwise}. \end{cases}$$

If m is even, then

$$H^j(G'_n; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{for } j = (4r+\epsilon)(m-1), r = 0, 1, \ldots, [(n-2)/4], \epsilon = 0, 1, \\ \mathbf{Z}_2 & \text{for } j = 4r(m-1) + i, \text{ or } j = (4r'+2)(m-1) + i', \text{ where} \\ & i = 2, 4, \ldots, m-2, r \text{ is as above}, i' = 1, 3, \ldots, m-1, \text{ and} \\ & 0 \leq r' \leq (n-4)/4, \\ 0 & \text{otherwise}. \end{cases}$$

Proof
Consider filtration $F'_0 \subset F'_1 \subset \cdots \subset F'_{n/2-1} \subset G'_n$ (cf. the proof of Corollary 8) and the associated spectral sequence

$$E'^{p,q}_1 = H^{p+q}(F'_p, F'_{p-1}; \mathbf{Z}) \Rightarrow H^{p+q}(G'_n; \mathbf{Z}).$$

$F'_p - F'_{p-1}$ contains a single critical submanifold $V'_p \simeq \mathbf{RP}^{m-1}$ with index $2p(m-1)$. The normal bundle to V'_p is orientable if $p(m-1)$ is even and nonorientable if $p(m-1)$ is odd. The Thom isomorphism gives

$$H^j(F'_p, F'_{p-1}; \mathbf{Z}) \simeq \begin{cases} H^{j-2p(m-1)}(\mathbf{RP}^{m-1}; \mathbf{Z}) & \text{if } p(m-1) \text{ is even}, \\ H^{j-2p(m-1)}(\mathbf{RP}^{m-1}; \pm\mathbf{Z}) & \text{if } p(m-1) \text{ is odd}. \end{cases} \tag{3.12}$$

Here $\pm Z$ denotes the nontrivial local system of groups $\mathbf{Z}$ over $\mathbf{RP}^{m-1}$; its monodromy along the generator of $\pi_1(\mathbf{RP}^{m-1})$ is multiplication by -1.

For m even, we have

$$H^j(\mathbf{RP}^{m-1}; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } j = 0 \text{ and } j = m-1, \\ \mathbf{Z}_2 & \text{for } j = 2, 4, \ldots, m-2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^j(\mathbf{RP}^{m-1}; \pm\mathbf{Z}) = \begin{cases} \mathbf{Z}_2 & \text{for } j = 1, 3, \ldots, m-1, \\ 0 & \text{otherwise}. \end{cases}$$

For m odd, we have

$$H^j(\mathbf{RP}^{m-1};\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } j=0, \\ \mathbf{Z}_2 & \text{for } j=2,4,\dots,m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, in the above spectral sequence, ${E'}_1^{p,q}=0$ holds for $p(2m-3)\le q\le p(2m-3)+(m-1)$. This implies that for $m>2$ either the source or the target of any differential d_r vanishes.

Hence, for $m>2$, ${E'}_1={E'}_\infty$ holds and any diagonal $p+q=\text{const}$ contains at most one nonzero group. This proves our statement for $m>2$.

Assume now that $m=2$, and consider the first differential $d_1: {E'}_1^{r-1,r}\to {E'}_1^{r,r}$. We have

$${E'}_1^{r-1,r}\simeq H^{2r-1}(F_{r-1},F_{r-2};\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } r \text{ is odd,} \\ \mathbf{Z}_2 & \text{if } r \text{ is even} \end{cases}$$

and

$${E'}_1^{r,r}\simeq H^{2r}(F_r,F_{r-1};\mathbf{Z}) = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ \mathbf{Z} & \text{if } r \text{ is even.} \end{cases}$$

We see that d_1 vanishes since there are no nonzero homomorphisms ${E'}_1^{r-1,r}\to {E'}_1^{r,r}$ for any r.

The higher differentials d_r, $r\ge 2$, vanish for obvious reasons. Hence the conclusion we made for $m>2$ also holds for $m=2$. □

The following theorem is the main result of this section. It describes the multiplicative structure of $H^*(G_n';\mathbf{Z})$. Recall that we assume that n is even.

THEOREM 12
For $m>1$ odd, $H^*(G_n';\mathbf{Z})$ is the commutative ring given by the sequence of generators

$$\delta_i\in H^{2i(m-1)}(G_n';\mathbf{Z})\simeq\mathbf{Z},\quad i=0,1,2,\dots,$$

and

$$e\in H^2(G_n';\mathbf{Z})\simeq\mathbf{Z}_2,$$

satisfying the following relations:

$$\delta_i\delta_j=\frac{(2i+2j)!}{(2i)!(2j)!}\cdot\delta_{i+j},\qquad \delta_{n/2}=0,\qquad \delta_0=1,$$
$$2e=0,\qquad e^{(m+1)/2}=0.$$

If m is even, then $H^*(G'_n; \mathbf{Z})$ is the graded commutative ring given by the generators

$$\delta_i \in H^{4i(m-1)}(G'_n; \mathbf{Z}) \simeq \mathbf{Z}, \quad i = 0, 1, 2, \dots,$$

and also

$$e \in H^2(G'_n; \mathbf{Z}) \simeq \mathbf{Z}_2, \qquad a \in H^{m-1}(G'_n; \mathbf{Z}) \simeq \mathbf{Z}, \qquad b \in H^{2m-1}(G'_n; \mathbf{Z}) \simeq \mathbf{Z}_2,$$

satisfying the following relations:

$$\begin{aligned} &\delta_i \delta_j = \frac{(2i+2j)!}{(2i)!(2j)!} \cdot \delta_{i+j}, \qquad \delta_{[(n+2)/4]} = 0, \qquad \delta_0 = 1, \\ &2e = 0, \qquad e^{m/2} = 0, \\ &a^2 = 0, \qquad ab = 0, \qquad ae = 0, \\ &2b = 0, \qquad b^2 = 0, \\ &\delta_k b = 0 \quad \textit{(if } n = 4k+2\textit{).} \end{aligned}$$

Remark. For $m = 2$ the generator e disappears since one of the above relations reads $e = 0$. If m is even and $n = 2$, then $b = 0$ since one of the relations gives $\delta_0 b = 0$.

Proof

Consider the universal $\mathbf{Z}_2$-bundle $S^\infty \to \mathbf{RP}^\infty$ and the associated fibration $S^\infty \times_{\mathbf{Z}_2} G_n \to \mathbf{RP}^\infty$, having G_n as the fiber. The total space $S^\infty \times_{\mathbf{Z}_2} G_n$ is homotopy equivalent to G'_n. The Serre spectral sequence of this fibration converges to the cohomology algebra $H^*(G'_n; \mathbf{Z})$. The initial term is

$$E_2^{p,q} = H^p\big(\mathbf{RP}^\infty; \mathscr{H}^q(G_n; \mathbf{Z})\big),$$

where $\mathscr{H}^q(G_n; \mathbf{Z})$, the cohomology of the fiber, is understood as a local system over $\mathbf{RP}^\infty$.

From Theorem 3 we know that $H^q(G_n; \mathbf{Z})$ is either $\mathbf{Z}$ or trivial. There are two types of local systems with fiber $\mathbf{Z}$ over $\mathbf{RP}^\infty$, which we denote $\mathbf{Z}$ and $\pm\mathbf{Z}$. Their structure is determined by the monodromy along any noncontractible loop of $\mathbf{RP}^\infty$, which is 1 in the case of $\mathbf{Z}$ and -1 in the case of $\pm\mathbf{Z}$.

Assume first that $m > 1$ is odd. From formula (2.6) we find that

$$\mathscr{H}^q(G_n; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{for } q = 2i(m-1), \\ \pm\mathbf{Z} & \text{for } q = (2i+1)(m-1), \end{cases}$$

where $i = 0, 1, \ldots, n/2 - 1$. Hence we find that

$$E_2^{p,q} = \begin{cases} \mathbf{Z} & \text{for } p = 0 \text{ and } q = 2i(m-1), \\ \mathbf{Z}_2 & \text{if } p > 0 \text{ is even and } q = 2i(m-1) \\ & \text{or if } p \text{ is odd and } q = (2i+1)(m-1), \\ 0 & \text{otherwise,} \end{cases}$$

where $i = 0, 1, \ldots, n/2 - 1$. As a bigraded algebra, E_2 can be identified with the tensor product

$$E_2^{0,*} \otimes E_2^{*,0} \otimes A,$$

where

$$E_2^{0,*} \simeq H^{2*}(G_n; \mathbf{Z}), \qquad E_2^{*,0} \simeq H^*(\mathbf{RP}^\infty; \mathbf{Z}),$$

and A is an exterior algebra with $A^{0,0} \simeq \mathbf{Z}$ and $A^{1,m-1} \simeq \mathbf{Z}_2$. If $x \in E_2^{1,m-1}$ is the generator, then relation $x^2 = 0$ follows from relation $\sigma_1^2 = 2\sigma_2$ (in the notation of Theorem 3). Here we denote by $H^{2*}(G_n; \mathbf{Z}) \subset H^*(G_n; \mathbf{Z})$ the graded subring

$$H^{2*}(G_n; \mathbf{Z}) = \bigoplus_i H^{2i(m-1)}(G_n; \mathbf{Z}).$$

The structure of the ring $H^{2*}(G_n; \mathbf{Z})$ follows from Theorem 3.

The first nontrivial differential is d_m. Since we know the additive structure of $H^*(G_n'; \mathbf{Z})$ (cf. Theorem 11), we find that the differential $d = d_m : E_2^{1,m-1} \to E_2^{m+1,0}$ must be an isomorphism. On the other hand, $d : E_2^{0,2i(m-1)} \to E_2^{m,(2i-1)(m-1)}$ vanishes (since the range is the zero group). It follows that $d : E_2^{p,j(m-1)} \to E_2^{p+m,(j-1)(m-1)}$ is nonzero if and only if both p and j are odd.

Figure 1 shows the nontrivial differential $d = d_m$. The large circles denote group $\mathbf{Z}$, and the small circles denote $\mathbf{Z}_2$.

We conclude that the bigraded algebra E_{m+1} is isomorphic to the tensor product of algebras

$$H^{2*}(G_n; \mathbf{Z}) \otimes H^*(\mathbf{RP}^{m-1}; \mathbf{Z}),$$

where $H^{2i(m-1)}(G_n; \mathbf{Z})$ has bidegree $(0, 2i(m-1))$ and $H^{2j}(\mathbf{RP}^{m-1}; \mathbf{Z})$ has bidegree $(2j, 0)$. It is clear that all further differentials vanish, and hence $E_\infty = E_{m+1}$. Any diagonal $p + q = c$ contains at most one nonzero group, and hence the algebra $H^*(G_n'; \mathbf{Z})$ coincides with E_∞. This proves our statement for $m > 1$ odd.

Assume now that m is even. Recall that we always assume that n is even. From formula (2.6) we find that

$$\mathscr{H}^q(G_n; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{for } q = 4i(m-1) \text{ or } q = (4i+1)(m-1), \\ \pm\mathbf{Z} & \text{for } q = (4i+2)(m-1) \text{ or } q = (4i+3)(m-1), \end{cases}$$

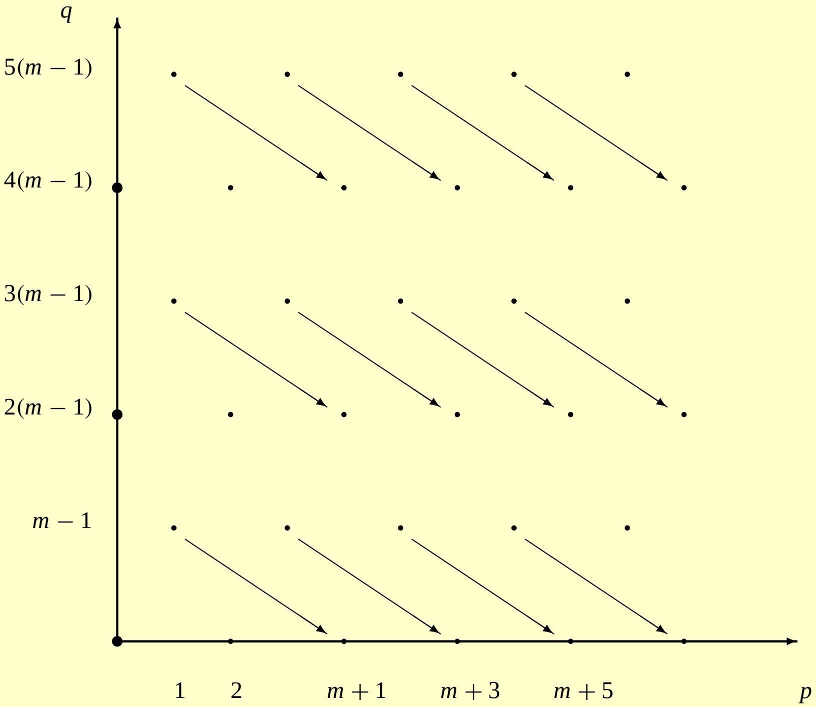

Figure 1. Term E_m of the spectral sequence for m odd

assuming that $q < n(m-1)$. Hence we find that

$$E_2^{p,q} = \begin{cases} \mathbf{Z} & \text{for } p = 0 \text{ and } q = (4i+\epsilon)(m-1), \text{ where } \epsilon = 0, 1, \\ \mathbf{Z}_2 & \text{if } p > 0 \text{ is even and } q = (4i+\epsilon)(m-1) \\ & \text{or if } p \text{ is odd and } q = (4i+2+\epsilon)(m-1), \\ 0 & \text{otherwise.} \end{cases}$$

As a bigraded algebra, E_2 can be identified with the tensor product

$$E_2^{0,*} \otimes E_2^{*,0} \otimes B^{*,*},$$

where

$$E_2^{0,*} \simeq H^{4*}(G_n; \mathbf{Z}) \otimes C^*, \qquad E_2^{*,0} \simeq H^*(\mathbf{RP}^\infty; \mathbf{Z}),$$

C^* is an exterior algebra with $C^0 \simeq \mathbf{Z}$ and $C^{m-1} \simeq \mathbf{Z}$, and $B^{*,*}$ is an exterior bigraded algebra with $B^{0,0} \simeq \mathbf{Z}$ and $B^{1,2(m-1)} \simeq \mathbf{Z}_2$. If $y \in E_2^{1,2(m-1)}$ denotes the generator, then $y^2 = 0$ follows from relation $\sigma_2^2 = 2\sigma_4$ (cf. Theorem 3). We denote

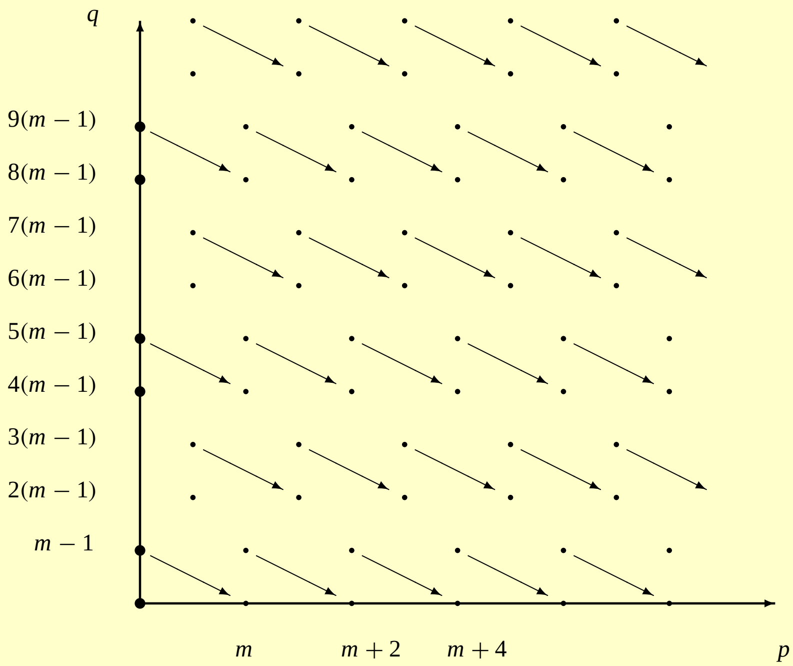

Figure 2. Term E_m of the spectral sequence for m even

by $H^{4*}(G_n; \mathbf{Z}) \subset H^*(G_n; \mathbf{Z})$ the graded subring

$$H^{4*}(G_n; \mathbf{Z}) = \bigoplus_i H^{4i(m-1)}(G_n; \mathbf{Z}).$$

Consider now the first nontrivial differential $d = d_m : E_2^{p,q} \to E_2^{p+m,q-m+1}$. It is clear that it may be nonzero only for q of the form $q = (2i+1)(m-1)$. On the other hand, since we know the additive structure of the limit (cf. Theorem 11), we conclude that $d : E_2^{0,m-1} = \mathbf{Z} \to E_2^{m,0} = \mathbf{Z}_2$ is surjective. Using the multiplicative properties of the spectral sequence, we find that all the differentials shown in Figure 2 are epimorphic. In fact, all differentials in Figure 2, except those that start at the q axis, are isomorphisms (since they act between isomorphic groups). As before, the large circles denote $\mathbf{Z}$ and the small circles denote $\mathbf{Z}_2$.

Hence, moving to the next term, E_{m+1}, classes $\sigma_{4i} = \delta_i$ survive, as do $a = 2\sigma_1$, $e \in E_{m+1}^{2,0} \simeq \mathbf{Z}_2$, and $b \in E_{m+1}^{1,2(m-1)}$ and their products $\delta_i a$, $\delta_i e^j$, and $\delta_i b e^j$ with $j < m/2$. It is clear that all further differentials vanish and that in each diagonal $p + q = c$ there is at most one nonzero group. Therefore we conclude that the ring

$H^*(G'_n; \mathbf{Z})$ is isomorphic to E_{m+1}. Its structure coincides with the description given in Theorem 12. □

4. Equivariant Lusternik-Schnirelman theory via nonsmooth critical point theory

In this section we first recall the basic notions of the critical point theory for nonsmooth functions, suggested recently in [4] and [5]. Then we apply the nonsmooth critical point theory to get a simple independent exposition of a version of the equivariant Lusternik-Schnirelman theory of [11] and [3], which we need for our applications to the billiard problems. One of the advantages of our approach is its applicability to manifolds with boundary.

Let X be a metric space endowed with the metric d. Given a point $p \in X$ and $\delta > 0$, we denote by $B(p, \delta) \subset X$ the ball of radius δ centered at p.

Definition 13

Let $f : X \to \mathbf{R}$ be a continuous function. The *weak slope of* f at a point $p \in X$, denoted $|df|(p)$, is defined as the supremum of all $\sigma \in [0, \infty]$ such that there exist $\delta > 0$ and a continuous deformation $\eta : B(p, \delta) \times [0, \delta] \to X$ with the following properties:

$$d\big(\eta(q, t), q\big) \leq t, \qquad f\big(\eta(q, t)\big) \leq f(q) - \sigma t$$

for all $q \in B(p, \delta)$, $t \in [0, \delta]$.

A point $p \in X$ is said to be a *critical point of function* f if $|df|(p) = 0$.

Example 14

Let X be a smooth Riemannian manifold without boundary, and let $f : X \to \mathbf{R}$ be a smooth function. Then the weak slope $|df|(p)$ coincides with the norm of the differential $||df(p)||$, viewed as a bounded linear functional on the tangent space $T_p(X)$.

Example 15

Let X be a smooth Riemannian manifold with boundary, and let $f : X \to \mathbf{R}$ be a smooth function. A point on the boundary $p \in \partial X$ is a critical point of f if and only if there is no tangent vector $v \in T_pX$ pointing inside X such that the derivative $v(f) < 0$ is negative. The last condition implies that

$$df_p|_{T_p\partial X} = 0, \tag{4.1}$$

that is, that the gradient of f at point $p \in \partial X$ is orthogonal to the boundary ∂X. A point $p \in \partial X$ is a critical point of f if and only if (4.1) holds and the gradient of f at p points inwards. It is clear that the above conditions are independent of the Riemannian metric.

PROPOSITION 16

Let $f : X \to \mathbf{R}$ be a continuous function on a compact metric space X. Then the number of critical points of f (in the sense of Definition 13) is at least $\mathrm{cat}(X)$, *the Lusternik-Schnirelman category of X.*

This follows from the much more general [5, Theorem 3.7].

We apply the nonsmooth critical point theory to the equivariant critical point theory of smooth functions (cf. [11], [3]).

PROPOSITION 17

Let M be a smooth compact G-manifold with boundary, where G is a finite group. Let $f : M \to \mathbf{R}$ be a G-invariant smooth function. Suppose that at points of the boundary $p \in \partial M$ the gradient of f does not vanish and is directed outwards. Then the number of G-orbits of points $p \in M$ with $df_p = 0$ is at least $\mathrm{cat}(M/G)$.

Proof

Let X denote the space of orbits $X = M/G$. Function f determines a continuous function $\tilde{f} : X \to \mathbf{R}$. We want to show that any orbit $x \in X$, representing points $p \in X$ with $df_p \neq 0$, is not a critical point of $\tilde{f} : X \to \mathbf{R}$ in the sense of Definition 13. This implies that the number of critical orbits of f is at least the number of critical points of $\tilde{f}$; the latter can be estimated from below by $\mathrm{cat}(X)$ in Proposition 16.

We assume that M is supplied with a G-invariant Riemannian metric. Let $p \in M$ be a point with $df_p \neq 0$. We want to construct a smooth vector field v in a neighborhood of the orbit of p having the following properties:

(a) $v(f)_p < 0$;
(b) the norm of vector v_p equals 1;
(c) v is G-invariant;
(d) if p belongs the boundary ∂M, the vector v_p points inside M.

To construct such a vector field v, one first finds a vector $v_p \in T_pM$ for each point p of the orbit so that (a), (b), and (d) are satisfied. It is then possible to extend the vectors v_p to form a smooth vector field v in a neighborhood of the orbit of p with properties (a), (b), and (d). Then (c) can be achieved by averaging.

The flow determined by the vector field v gives a deformation of the ball around the point $\{p\} \in X$, which represents the orbit of p, showing that the slope $|d\tilde{f}|(p)$ is positive. □

5. Proof of Theorem 1

Let $T \subset \mathbf{R}^{m+1}$ be a compact strictly convex domain bounding a smooth hypersurface $X = \partial T$. Given a point $A \in X$ and an integer n, consider the configuration space

$G_n = G(X; A, A, n)$ (cf. (2.1)) and the smooth function

$$L_X : G_n \to \mathbf{R}, \qquad L_X(x_1, \ldots, x_n) = -\sum_{i=0}^{i=n} |x_i - x_{i+1}|$$

(the negative total length), where we understand that $x_0 = A = x_{n+1}$. This function is invariant with respect to the reflection $T : G_n \to G_n$ (cf. (2.2)). Hence L_X determines a continuous function

$$L'_X : G'_n \to \mathbf{R}, \qquad G'_n = G_n/T, \tag{5.1}$$

and the critical points of L'_X (in the sense of Definition 13) are in one-to-one correspondence with the $\mathbf{Z}_2$-orbits of the closed billiard trajectories in X which start and end at A and make n reflections. This follows from [7, Lemma 2] and from the argument in the proof of Proposition 17.

Note that, for n even, T acts freely on G_n, the factor $G'_n = G_n/T$ is a smooth manifold, and the function L'_X is smooth. In this case the nonsmooth critical point theory coincides with the usual one.

We claim the following.

The number of critical points of L'_X is at least the Lusternik-Schnirelman category $\operatorname{cat}(G'_n)$*; moreover, assuming that n is even and the function L'_X is Morse, the number of critical points of (5.1) is at least the sum of the Betti numbers of G'_n with $\mathbf{Z}_2$ coefficients.*

The italicized statement does not follow directly from the traditional Morse-Lusternik-Schnirelman theory since G'_n is not compact. However, as in [8] and [7], one may fix $\epsilon > 0$ and consider the following compact subset:

$$G_\epsilon \subset G_n, \quad G_\epsilon = \left\{(x_1, \ldots, x_n) \in X^{\times n} : \prod_{i=0}^{n} |x_i - x_{i+1}| \geq \epsilon\right\}.$$

If $\epsilon > 0$ is small enough, then

(a) G_ϵ is a compact manifold with boundary;

(b) the inclusion $G_\epsilon \subset G_n$ is a $\mathbf{Z}_2$-equivariant homotopy equivalence;

(c) all critical points of L_X are contained in G_ϵ;

(d) at every point of ∂G_ϵ, the gradient of L_X is directed outwards

(cf. [8, Proposition 4.1], [7, Proposition 4]).

Since $G'_\epsilon = G_\epsilon/T$ is a compact smooth manifold with boundary, we may apply the Morse-Lusternik-Schnirelman theory to it. Condition (d) implies that the critical points of the restriction of L'_X on $\partial G'_\epsilon$ should not be taken into account (cf. Proposition 17). Therefore the number of critical points of $L'_X|_{G'_\epsilon}$ is at least $\operatorname{cat}(G'_\epsilon) = \operatorname{cat}(G'_n)$. If $L'_X|_{G'_\epsilon}$ is Morse, then the number of its critical points is at least the sum of Betti numbers of G'_ϵ, which is the same as the sum of Betti numbers of G'_n.

In the proof of Theorem 1(I), we use the following general simple remark.

For any regular covering map $p : \tilde{X} \to X$ with connected $\tilde{X}$,

$$\mathrm{cat}(X) \geq \mathrm{cat}(\tilde{X}). \tag{5.2}$$

Indeed, if $A \subset X$ is an open subset that is contractible to a point in X, then $\tilde{A} = p^{-1}(A)$ is a disjoint union of open subsets of $\tilde{X}$, such that each is contractible to a point in $\tilde{X}$. Hence any categorical open cover $A_1 \cup A_2 \cup \cdots \cup A_k = X$ produces a categorical open cover $\tilde{A}_1 \cup \tilde{A}_2 \cup \cdots \cup \tilde{A}_k$ of $\tilde{X}$.

The above remark applies to the two-fold cover $G_n \to G'_n$ giving

$$\mathrm{cat}(G'_n) \geq \mathrm{cat}(G_n) \geq \mathrm{cl}(G_n) + 1,$$

where $\mathrm{cl}(G_n)$ is the cohomological cup length of G_n (Froloff-Elsholz theorem).

Now we use Theorem 3 to compute the cup length of G_n. If $m \geq 3$ is odd, then $\sigma_1^{n-1} = (n-1)!\sigma_{n-1} \neq 0 \in H^{(n-1)(m-1)}(G_n; \mathbf{Z})$ and hence $\mathrm{cl}(G_n) = n-1$. Therefore $\mathrm{cat}(G'_n) \geq n$. This proves Theorem 1(I) in the case when $m \geq 3$ is odd.

If m is even, then the longest nontrivial cup product in $H^*(G_n; \mathbf{Z})$ is $\sigma_1\sigma_2^{k-1}$ where $n = 2k$. We conclude that, for m even, the cup length of G_n equals $n/2$ and therefore $\mathrm{cat}(G'_n) \geq n/2 + 1$. Together with the information collected above, this proves Theorem 1(I) for m even.

To prove Theorem 1(II), we use Theorem 12 to estimate the Lusternik-Schnirelman category of G'_n. Note that Theorem 12 requires the assumption that n is even. Suppose first that $m \geq 3$ is odd. Then (in the notation of Theorem 12) we have a nonzero cohomology product

$$\delta_1\delta_2\delta_{2^2}\cdots\delta_{2^s}e^{(m-1)/2}, \tag{5.3}$$

where s is the largest integer with $2^{s+1}-1 \leq n/2-1$, that is, where $s = [\log_2(n)]-2$. Note that the class $e \in H^2(G'_n; \mathbf{Z})$ has order 2; that is, $2e = 0$. Nontriviality of the above product is equivalent to the claim that the product $\delta_1\delta_2\delta_{2^2}\cdots\delta_{2^s}$ is an odd multiple of the class $\delta_{1+2+2^2+\cdots+2^s}$. Indeed, we use the relation

$$\delta_i\delta_j = \begin{pmatrix} 2i+2j \\ 2i \end{pmatrix} \delta_{i+j}$$

and the well-known fact that the binomial coefficient $\binom{2i+2j}{2i}$ is even if and only if i and j, in their binary expansions, have a 1 in the same position.

Now we use the notion of *category weight of a cohomology class*, introduced by Fadell and Husseini [6]. They associate weights to cohomology classes, so that nontriviality of a cup product $v_1 \cup v_2 \cup \cdots \cup v_m$ implies that the Lusternik-Schnirelman category of the space is greater than the sum of the weights of the classes $v_1, v_2, \ldots, v_m$.

Hence, instead of counting each cohomology class v_i as contributing 1 into the total cup-length estimate, in the approach of Fadell and Husseini we count the contribution of v_i according to its category weight.

We claim that the category weight of $e \in H^2(G'_n; \mathbf{Z})$ equals 2. Indeed, e equals the image of a class $e' \in H^1(G'_n; \mathbf{Z}_2)$ under the Bockstein homomorphism $\beta : H^1(G'_n; \mathbf{Z}_2) \to H^2(G'_n; \mathbf{Z})$; that is, $e = \beta(e')$. Hence, by Fadell and Husseini in [6, Theorem (1.2)], the category weight of e is 2. Therefore nontriviality of product (5.3) implies

$$\operatorname{cat}(G'_n) \geq (s+1) + 2 \cdot \frac{m-1}{2} + 1 = [\log_2 n] + m - 1.$$

This proves Theorem 1(II) for $m \geq 3$ odd.

Consider now the case when $m \geq 2$ is even. First we assume that $n \geq 8$ and that $n+2$ is not a power of 2. Then we have a nontrivial cohomological product

$$\delta_1 \delta_2 \delta_{2^2} \cdots \delta_{2^s} b e^{(m-2)/2}, \tag{5.4}$$

where $s = [\log_2[(n+2)/4]] - 1$. As above, nontriviality of (2.3) implies

$$\operatorname{cat}(G'_n) \geq (s+1) + 1 + 2 \cdot \frac{m-2}{2} + 1 = \left[\log_2\left[\frac{n+2}{4}\right]\right] + m \geq [\log_2 n] + m - 2.$$

If $n + 2 = 2^r$ is a power of 2, where $r \geq 4$, then the product

$$\delta_1 \delta_2 \delta_{2^2} \cdots \delta_{2^s} e^{(m-2)/2}$$

(we skip b because of the last relation in Theorem 12) is nonzero, where $s = r - 3$. In this case we obtain

$$\operatorname{cat}(G'_n) \geq (s+1) + 2 \cdot \frac{m-2}{2} + 1 = s + m = [\log_2 n] + m - 2.$$

We are left to consider the cases $n = 2, 4, 6$ with m even. Here we have a nontrivial cup product $be^{(m-2)/2}$, and hence

$$\operatorname{cat}(G'_n) \geq 1 + 2 \cdot \frac{m-2}{2} + 1 = m.$$

This implies the estimate of Theorem 1(II) for the specified values of n and m.

For $m > 1$, Theorem 1(III) follows from Corollary 8. If $m = 1$, then the space $G'_1 = G(S^1; A, A, n)/\mathbf{Z}_2$ consists of $n/2$ connected components and each is contractible; this can be established by arguments similar to those used in [7, §7]. Hence the sum of Betti numbers of G'_1 is $n/2$. □

6. Cohomology of cyclic configuration spaces of spheres

The cyclic configuration space $G(X, n)$ of a space X is defined (cf. [8]) as the set of all configurations $(x_1, \dots, x_n) \in X^{\times n}$ with $x_i \neq x_{i+1}$ for $i = 1, \dots, n-1$ and $x_n \neq x_1$. The dihedral group D_n acts naturally on $G(X, n)$.

In [8] M. Farber and S. Tabachnikov showed that information about the cohomology ring of the factor space $G(S^m, n)/D_n$ leads to estimates of the number of n-periodic orbits of convex billiards in $(m+1)$-dimensional space $\mathbf{R}^{m+1}$. The rings $H^*(G(S^m, n); \mathbf{Z}_2)$ and $H^*(G(S^m, n)/D_n; \mathbf{Z}_2)$ were computed in [8].

In this section we describe the cohomology of the cyclic configuration space $G(S^m, n)$ with other fields of coefficients. It turns out that the answer depends on the parity of m; therefore we state the even- and odd-dimensional cases in the form of two separate theorems.

The results of this section are used in the proof of Theorem 2.

THEOREM 18

Let $m \geq 3$ be odd. The ring $H^*(G(S^m, n); \mathbf{Q})$ is given by generators

$$u \in H^m\big(G(S^m, n); \mathbf{Q}\big), \qquad \sigma_i \in H^{i(m-1)}\big(G(S^m, n); \mathbf{Q}\big), \quad i = 1, \dots, n-2,$$

and relations

$$u^2 = 0, \qquad \sigma_i \sigma_j = \begin{cases} \frac{(i+j)!}{i! \cdot j!} \cdot \sigma_{i+j} & \text{if } i + j \leq n-2, \\ 0 & \text{if } i + j > n-2. \end{cases} \tag{6.1}$$

One may show that the statement of Theorem 18 holds, with $\mathbf{Q}$ replaced by an arbitrary field of coefficients $\mathbf{k}$. However, the short proof we give below works only in the case $\mathbf{k} = \mathbf{Q}$. On the other hand, for the purposes of this paper, it is enough to know the rational cohomology $H^*(G(S^m, n); \mathbf{Q})$. The case of a field $\mathbf{k}$ of positive characteristic may be proven by using [8, Theorem 3] and computing the spectral sequence as in the proof of [8, Theorem 4].

The following theorem gives the answer for m even.

THEOREM 19

Let $\mathbf{k}$ be a field of characteristic not equal to 2. For any even $m \geq 2$ and odd $n \geq 3$, the cohomology algebra $H^*(G(S^m, n); \mathbf{k})$ is given by generators

$$w \in H^{2m-1}\big(G(S^m, n); \mathbf{k}\big), \qquad \sigma_{2i} \in H^{2i(m-1)}\big(G(S^m, n); \mathbf{k}\big), \quad i = 1, \dots, \frac{n-3}{2},$$

and relations

$$w^2 = 0, \quad \sigma_{2i} \sigma_{2j} = \begin{cases} \frac{(i+j)!}{i! \cdot j!} \cdot \sigma_{2(i+j)} & \text{if } i + j \leq \frac{n-3}{2}, \\ 0 & \text{if } i + j > \frac{n-3}{2}. \end{cases} \tag{6.2}$$

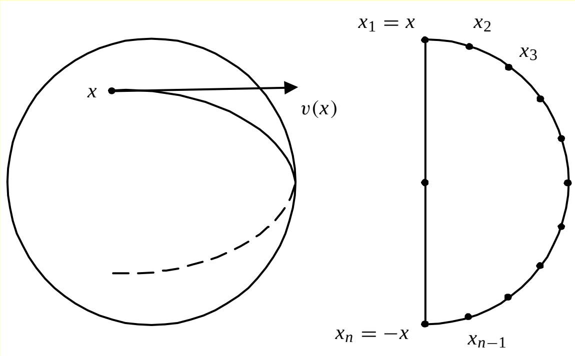

Figure 3. Continuous family of configurations

Proof of Theorem 18

Consider the fibration

$$p : G(S^m, n) \to S^m, \tag{6.3}$$

where the image of a cyclic configuration $(x_1, \ldots, x_n) \in G(S^m, n)$ under projection p is given by $p(x_1, \ldots, x_n) = x_1$. The fiber of p is the configuration space $G(S^m; A, A, n-1)$. Consider the Serre spectral sequence of this fibration. The cohomology of the fiber $G(S^m; A, A, n-1)$ is described by Theorem 3; it has generators $\sigma_1, \ldots, \sigma_{n-2}$, which multiply according to (2.3).

This spectral sequence may have only one nonzero differential d_m. We show that this differential vanishes; that is, $d_m = 0$. This clearly implies our statement.

Since we may write $\sigma_i = (i!)^{-1}(\sigma_1)^i$, it is enough to show that $d_m(\sigma_1) = 0$. Vanishing $d_m(\sigma_1) = 0$ follows from the fact that fibration (6.3) admits a continuous section $s : S^m \to G(S^m, n)$, and thus the transgression is trivial. To construct s, fix a nowhere zero tangent vector field V on the sphere S^m. (Recall that m is odd.) For $x \in S^m$, the tangent vector $V(x)$ determines a half-circle starting at x, tangent to $V(x)$, and ending at the antipodal point $-x$. Then the section s can be defined by

$$s(x) = (x_1, x_2, \ldots, x_n), \quad x \in S^m,$$

where $x_1 = x$, $x_n = -x$ and the points $x_2, \ldots, x_{n-1}$ are situated on the half-circle making equal angles as shown in Figure 3. Analytically, we may write

$$x_j = \cos\left(\frac{(j-1)\pi}{n-1}\right)x + \sin\left(\frac{(j-1)\pi}{n-1}\right)V(x), \quad j = 1, \ldots, n. \qquad \square$$

Proof of Theorem 19

First we assume that $m > 2$; the case $m = 2$ is treated separately later.

We describe the additive structure of $H^*(G(S^m, n); \mathbf{k})$ using the Morse theory approach. Let $S^m \subset \mathbf{R}^{m+1}$ be the unit sphere. Consider the total length function

$$L : G(S^m, n) \to \mathbf{R},$$

where for $(x_1, \dots, x_n) \in G(S^m, n)$ we have

$$L(x_1, x_2, \dots, x_n) = -|x_1 - x_2| - |x_2 - x_3| - \cdots - |x_n - x_1|.$$

The critical points of L are n-periodic billiard trajectories in the unit sphere; hence the critical configurations are regular n-gons lying in 2-dimensional central sections of the sphere. A regular n-gon is determined by two first vectors $x_1, x_2 \in S^m$, which must make an angle of the form

$$\alpha_p = \frac{2\pi}{n} \cdot \Big(\frac{n-1}{2} - p\Big), \quad \text{where } p = 0, 1, \dots, \frac{n-3}{2}.$$

Recall that we assume that n is odd. Fixing $p = 0, 1, \dots, (n-3)/2$, we obtain a variety of critical configurations, which we denote by $V_p \subset G(S^m, n)$. Each V_p has dimension $2m - 1$ and is diffeomorphic to the Stiefel manifold of pairs of mutually orthogonal vectors in $\mathbf{R}^{m+1}$. Since we assume that m is even and that the characteristic of $\mathbf{k}$ is $\neq 2$, we have $H^*(V_p; \mathbf{k}) \simeq H^*(S^{2m-1}; \mathbf{k})$. Note also that V_p is simply connected (since $m > 2$).

Babenko has shown (cf. [1, Proposition 3.1]) that function L is nondegenerate in the sense of Bott and that the index of each critical submanifold V_p equals $2p(m-1)$. Moreover, it is clear that $L(V_p) < L(V_{p'})$ for $p < p'$.

Fix $\varepsilon > 0$ small enough, and consider the submanifold $G_\varepsilon(S^m, n) \subset G(S^m, n)$, where

$$G_\varepsilon(S^m, n) = \Big\{(x_1, \dots, x_n) \in (S^m)^{\times n} : \prod_{i=1}^{n} |x_i - x_{i+1}| \geq \varepsilon\Big\}.$$

If $\varepsilon > 0$ is sufficiently small, then (according to [8, Proposition 4.1]) $G_\varepsilon(S^m, n)$ is a compact manifold with boundary containing all the critical points of L and such that the inclusion $G_\varepsilon(S^m, n) \subset G(S^m, n)$ is a D_n-equivariant homotopy equivalence. Moreover, at every point of the boundary $\partial G_\varepsilon(S^m, n)$, the gradient of L extends outward.

Choose constants $c_0, c_1, \dots, c_{(n-3)/2}$ such that $L(V_p) < c_p < L(V_{p+1})$ for $0 \leq p < (n-3)/2$ and $c_{(n-3)/2} = 0$. Let

$$F_p = L^{-1}\big((-\infty, c_p]\big) \cap G_\varepsilon(S^m, n).$$

We obtain a filtration $F_0 \subset F_1 \subset \cdots \subset F_{(n-3)/2} = G_\varepsilon(S^m, n)$. Since the inclusion $F_{(n-3)/2} \subset G(S^m, n)$ is a homotopy equivalence, we may use the spectral sequence of this filtration to calculate the cohomology of $G(S^m, n)$.

We claim that this filtration is *perfect*, that is, that the Poincaré polynomial of the cyclic configuration space $G(S^m, n)$ equals the sum of the Poincaré polynomials of the pairs (F_p, F_{p-1}). The initial term of the spectral sequence is

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}; \mathbf{k}).$$

Using the Thom isomorphism (recall that V_p is simply connected), we find that $H^j(F_p, F_{p-1}; \mathbf{k})$ is isomorphic to $H^{j-2p(m-1)}(V_p; \mathbf{k})$; hence $H^j(F_p, F_{p-1}; \mathbf{k})$ is 1-dimensional for $j = 2p(m-1)$ and for $j = 2p(m-1) + 2m - 1$ and vanishes for all other values of j. This follows since in $F_{p+1} - F_p$ there is a single nondegenerate critical submanifold V_p which has index $2p(m-1)$ and is diffeomorphic to the Stiefel manifold $V_{m+1,2}$.

The gradient of L at points of the boundary $\partial G_\varepsilon(S^m, n)$ extends outward, and hence, the points of the boundary do not contribute to the usual statements of the Morse-Bott critical point theory.

For a given p there are precisely two values of q such that $E_1^{p,q}$ is nonzero ($q = p(2m-3)$ and $q = p(2m-3) + 2m - 1$). From the geometry of the differentials, we see that all the differentials d_r, $r \geq 1$, must vanish if $m > 2$. This proves that the cohomology $H^j(G(S^m, n); \mathbf{k})$ is 1-dimensional for $j = 2p(m-1)$ and $j = 2p(m-1) + 2m - 1$, where $p = 0, 1, \ldots, (n-3)/2$, and $H^j(G(S^m, n); \mathbf{k})$ vanishes for other values of j.

Having recovered the additive structure of $H^*(G(S^m, n); \mathbf{k})$, we may use Theorem 3 to find its multiplicative structure. The mapping $(x_1, x_2, \ldots, x_n) \mapsto x_1$ is a Serre fibration $G(S^m, n) \to S^m$; its fiber is $G(S^m : A, A, n-1)$. The Serre spectral sequence has only two nonzero columns, and d_m is the only differential that could be nonzero. In the zeroth columns we have classes σ_i in dimensions $i(m-1)$, and in the mth column we have classes $\sigma_i u$ having dimension $i(m-1) + m$ (cf. Theorem 3). Since we already know the additive structure of $H^*(G(S^m, n); \mathbf{k})$, we conclude that the differential

$$d_m : E_2^{0,i(m-1)} \to E_2^{m,(i-1)(m-1)}$$

is an isomorphism for i odd and vanishes for i even. Hence the classes σ_{2i} and $\sigma_{2i+1}u$ survive. Now, we set $w = \sigma_1 u$, and we conclude that $H^*(G(S^m, n); \mathbf{k})$ has the multiplicative structure stated in Theorem 19.

For $m = 2$ the above argument, based on the spectral sequence of the filtration $F_0 \subset F_1 \subset \cdots \subset F_{(n-3)/2} = G_\varepsilon(S^m, n)$, is not sufficient since, in principle, this spectral sequence could have a nonzero differential, as shown in Figure 4. Also, for $m = 2$ the critical submanifolds V_p are not simply connected, and so the Thom isomorphisms for the negative normal bundles of the Hessian may require additional twists by flat line bundles (depending on the orientability of the negative normal bundles of the critical submanifolds).

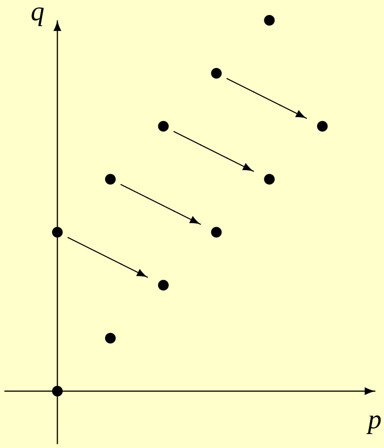

Figure 4. Nonzero terms of the spectral sequence for $m = 2$

However, in the case $m = 2$, a different argument can be applied. Consider the action of SO(3) on $G(S^2, n)$ arising from the standard action of SO(3) on S^2. Fix a point $A \in S^2$, and consider $G(S^2; A, A, n-1)$ as being canonically embedded in $G(S^2, n)$. We obtain the map

$$\mathrm{SO}(3) \times G(S^2; A, A, n-1) \to G(S^2, n), \qquad (R, c) \mapsto Rc, \tag{6.4}$$

given by applying an orthogonal matrix $R \in \mathrm{SO}(3)$ to a configuration of points on the sphere $c \in G(S^2; A, A, n-1)$. It is easy to see that (6.4) is a fibration with fiber S^1. If $c = (A, x_1, \ldots, x_{n-1})$ is a configuration of points on S^2 such that $A \neq x_1$, $x_i \neq x_{i+1}$ for $i = 1, \ldots, n-1$ and $x_{n-1} \neq A$, then the fiber of fibration (6.4) over c consists of the space of all pairs $(R_{-\phi}, R_\phi(c))$, where $R_\phi \in \mathrm{SO}(3)$ denotes the rotation by angle $\phi \in [0, 2\pi]$ about A.

The cohomology algebra of the total space of this fibration,

$$H^*\big(\mathrm{SO}(3) \times G(S^2; A, A, n-1); \mathbf{k}\big) \simeq H^*(S^3; \mathbf{k}) \otimes H^*\big(G(S^2; A, A, n-1); \mathbf{k}\big),$$

is given by Theorem 3. It has a generator w with $\deg w = 3$ (coming from a generator of $H^3(\mathrm{SO}(3); \mathbf{k})$) and also classes σ_i, where $i = 0, 1, \ldots, n-2$, with $\deg \sigma_i = i$, which are pullbacks of the generators of $H^*(G(S^2; A, A, n-1); \mathbf{k})$ (cf. Theorem 3). We have the relation $w^2 = 0$, and each product $\sigma_i \sigma_j$ equals a multiple of σ_{i+j}, the coefficient indicated in formula (2.4).

Let us show that *the restriction map from the total space to the fiber*

$$H^1\big(\mathrm{SO}(3) \times G(S^2; A, A, n-1); \mathbf{k}\big) \to H^1(S^1; \mathbf{k})$$

is surjective. Since $H^1(\mathrm{SO}(3); \mathbf{k}) = 0$, our statement is equivalent to the following. Let $c = (A, x_1, \ldots, x_{n-1})$ be a fixed configuration. We obtain an embedding $f : S^1 \to G(S^2; A, A, n-1)$ given by $\phi \mapsto R_\phi(c)$, where $\phi \in [0, 2\pi]$. We claim that the induced map $f^* : H^1(G(S^2; A, A, n-1); \mathbf{k}) \to H^1(S^1; \mathbf{k})$ is surjective. In other words, we want to show that the cohomology class $f^*(\sigma_1) \in H^1(S^1; \mathbf{k})$ is nonzero.

We may assume that the antipode A' of A does not appear in the configuration c. Identify $S^2 - A'$ with $\mathbf{R}^2$ using the stereographic projection with A' as a center; this leads to the following commutative diagram:

$$\begin{tikzcd} S^1 \arrow[r, "g"] \arrow[dr, "f"'] & G(\mathbf{R}^2; 0, 0, n-1) \arrow[d, "h"] \\ & G(S^2; A, A, n-1) \end{tikzcd}$$

where g is given by rotations of a fixed configuration $c' = (0, y_1, y_2, \ldots, y_{n-1})$ of points on the plane, $c' \in G(\mathbf{R}^2; 0, 0, n-1)$, around the origin $0 \in \mathbf{R}^2$. Clearly, the space $G(\mathbf{R}^2; 0, 0, n-1)$ is homotopy equivalent to $G(\mathbf{R}^2, n)$, and thus the cohomology algebra $H^*(G(\mathbf{R}^2; 0, 0, n-1); \mathbf{k})$, as given by [8, Proposition 2.2], has 1-dimensional generators $s_1, \ldots, s_n$ which satisfy the relations $s_i^2 = 0$ for $i = 1, \ldots, n$ and also a relation of degree $n-1$ (cf. [8, formula (4)]). From [7, Remark 9] we obtain

$$h^*(\sigma_1) = \sum_{i=1}^{n} (-1)^{i+1} s_i. \tag{6.5}$$

Let $s \in H^1(S^1; \mathbf{k})$ denote the generator corresponding to the usual anticlockwise orientation of the circle. Then

$$g^*(s_i) = s, \quad i = 1, 2, \ldots, n. \tag{6.6}$$

Indeed, $g^*(s_i) = d_i s$, where d_i is the degree of the following map $S^1 \to S^1$,

$$\phi \mapsto \frac{R_\phi(y_i) - R_\phi(y_{i-1})}{|R_\phi(y_i) - R_\phi(y_{i-1})|} = R_\phi\Big(\frac{y_i - y_{i-1}}{|y_i - y_{i-1}|}\Big), \quad \phi \in [0, 2\pi],$$

and hence it is clear that $d_i = 1$. Here R_ϕ denotes the plane rotation by angle ϕ. Comparing (6.5) and (6.6), we obtain

$$f^*(\sigma_1) = g^* h^*(\sigma_1) = g^*\Big(\sum_{i=1}^{n} (-1)^{i+1} s_i\Big) = s,$$

where we have used the assumption that n is odd. (Note that, for n even, the above arguments give $f^*(\sigma_1) = 0$.)

Let us examine the Serre spectral sequence of fibration (6.4). First we observe that the fundamental group of the base acts trivially on the cohomology of the fiber. This follows since (6.4) is induced from the standard fibration

$$q : \mathrm{SO}(3) \to S^2, \quad \text{where } q(R) = R(A),\ R \in \mathrm{SO}(3),$$

by the map $G(S^2, n) \to S^2$ given by $(x_1, \ldots, x_n) \mapsto x_1$. The Serre spectral sequence of (6.4) has two rows and may have one nontrivial differential. Since we know that the fundamental class of the fiber $s \in H^1(S^1; \mathbf{k})$ survives, that is, applying the differential to it gives zero, it follows that all the differentials in the Serre spectral sequence vanish. We conclude that the cohomology algebra of the base $H^*(G(S^2, n); \mathbf{k})$ is the factor of $H^*(\mathrm{SO}(3) \times G(S^2; A, A, n-1); \mathbf{k})$ with respect to the ideal generated by class σ_1. Since $\sigma_{2i+1} = \sigma_1\sigma_{2i}$, we obtain that $H^*(G(S^2, n); \mathbf{k})$ has generators w with $\deg w = 3$ and σ_{2i} with $\deg \sigma_{2i} = 2i$, where $i = 0, 1, \ldots, (n-3)/2$, which satisfy relations (6.2). □

Relation between cyclic configuration spaces of spheres and spaces of paths

In this section we describe the relations between the cyclic configuration spaces of spheres and some spaces of paths. We do not give the proofs since they are similar (almost identical) to the proof of [7, Theorem 12]. The results mentioned here are not used in the rest of this paper.

Let $\mathscr{L}(S^m)$ denote the space of free loops in S^m, that is, the space of all H^1-smooth maps $S^1 \to S^m$. We refer to [9, Chapter 5] for the basic definitions. We denote by $\mathscr{L}(S^m)_n \subset \mathscr{L}(S^m)$ the subspace of all loops having length less than πn.

For n even, there is a continuous map $\psi : G(S^m, n) \to \mathscr{L}(S^m)_n$ which acts as follows. Given a configuration of points $c = (x_1, x_2, \ldots, x_n) \in G(S^m, n)$, define a sequence $(y_1, \ldots, y_n) \in (S^m)^{\times n}$, where $y_i = (-1)^i x_i$. Then for any $i = 1, \ldots, n$, the points y_i and y_{i+1} are not antipodal and hence we may join them by a shortest geodesic arc of S^m; here y_{n+1} is understood to be y_1. Union of these arcs gives a closed loop $\psi(c)$ having length $< \pi n$. In a manner similar to that of [7, Theorem 12], one may show that ψ is a homotopy equivalence.

For the billiard problem we are mainly interested in the spaces $G(S^m, n)$ with n odd. The corresponding space of paths can be described as follows. Let $\mathscr{L}^*(S^m)$ denote the space of all H^1-smooth paths $\omega : [0, 1] \to S^m$, such that the end points are antipodal, $\omega(0) = -\omega(1)$. Projecting onto $\mathbf{RP}^m$, we obtain closed noncontractible loops in $\mathbf{RP}^m$. In fact, $\mathscr{L}^*(S^m)$ may be identified with a two-fold cover of the connected component of the free loop space $\mathscr{L}(\mathbf{RP}^m)$, consisting of noncontractible free loops in $\mathbf{RP}^m$.

We denote by $\mathscr{L}^*(S^m)_n \subset \mathscr{L}^*(S^m)$ the subspace of paths having length less than πn. In a manner similar to the case of n even and [7, Theorem 12], one may show the following.

For any $n > 2$ odd, there exists a natural homotopy equivalence $\psi : G(S^m, n) \to \mathscr{L}^*(S^m)_n$. In particular, for odd $n \to \infty$, the space $G(S^m, n)$ approximates the space $\mathscr{L}^*(S^m)$.

It is easy to show that, for m odd, the path space $\mathscr{L}^*(S^m)$ is homotopy equivalent to the free loop space $\mathscr{L}(S^m)$. It follows using the remark that, for m odd, the antidiagonal $D : S^m \to S^m \times S^m$, where $D(x) = (x, -x)$, is homotopic to the diagonal $x \mapsto (x, x)$.

For m even, the spaces $\mathscr{L}(S^m)$ and $\mathscr{L}^*(S^m)$ are not homotopy equivalent. For example, $\pi_1(\mathscr{L}(S^2)) \simeq \mathbf{Z}$ and $\pi_1(\mathscr{L}^*(S^2)) \simeq \mathbf{Z}_2$.

7. Proof of Theorem 2

Let $T \subset \mathbf{R}^{m+1}$ be a compact strictly convex domain with smooth boundary $X = \partial T$. Consider the smooth function

$$L_X : G(X, n) \to \mathbf{R}, \qquad L_X(x_1, \dots, x_n) = -\sum_{i=1}^{i=n} |x_i - x_{i+1}|$$

(the negative total length), where we understand the indices cyclically modulo n, that is, where $x_{n+1} = x_1$. The critical points of L_X are in one-to-one correspondence with n-periodic billiard trajectories in X.

Fix $\epsilon > 0$, and consider

$$G_\epsilon \subset G(X, n), \quad G_\epsilon = \left\{(x_1, \dots, x_n) \in X^{\times n} : \prod_{i=1}^{n} |x_i - x_{i+1}| \geq \epsilon\right\}.$$

According to [8, Proposition 4.1], if $\epsilon > 0$ is small enough, then

(a) G_ϵ is a compact manifold with boundary;

(b) the inclusion $G_\epsilon \subset G(X, n)$ is a D_n-equivariant homotopy equivalence;

(c) all critical points of L_X are contained in G_ϵ;

(d) at every point of ∂G_ϵ, the gradient of L_X extends outwards.

Now we apply Proposition 17 with $M = G_\epsilon$, $f = L_X$, and $G = D_n$. We conclude that the number of D_n-orbits of n-periodic billiard trajectories in X is at least

$$\operatorname{cat}(G_\epsilon / D_n) = \operatorname{cat}\big(G(X, n)/D_n\big).$$

Since we assume that n is an odd prime, the action of D_n on $G(X, n)$ is free, and we may use inequality (5), which gives

$$\operatorname{cat}\big(G(X, n)/D_n\big) \geq \operatorname{cat}\big(G(X, n)\big) \geq \operatorname{cl}\big(G(S^m, n)\big) + 1.$$

Theorems 18 and 19 allow us to estimate $\operatorname{cat}(G(S^m, n))$. Assume first that m is odd, $m > 1$. Then (according to Theorem 18) we have a nonzero cup product

$$\sigma_1^{n-2} u,$$

which shows that the cup length of $G(S^m, n)$ for odd $m > 1$ is at least $n-1$. This gives a lower bound n on the number of D_n-orbits of n-periodic billiard trajectories in X for m odd.

If m is even, then (by Theorem 19) we have a nonzero cup product

$$\sigma_2^{(n-3)/2} w \in H^*\big(G(S^m, n); \mathbf{k}\big),$$

where $\mathbf{k}$ is a field of characteristic not equal to 2. This shows that, for m even, the cup length of $G(S^m, n)$ is at least $(n-1)/2$. This gives a lower bound $(n+1)/2$ on the number of D_n-orbits of n-periodic billiard trajectories in X. □

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