DUKE MATHEMATICAL JOURNAL
Vol. 115, No. 3, © 2002

TOPOLOGY OF BILLIARD PROBLEMS, I

MICHAEL FARBER

Abstract
Let $T \subset \mathbf{R}^{m+1}$ be a strictly convex domain bounded by a smooth hypersurface $X = \partial T$. In this paper we find lower bounds on the number of billiard trajectories in T which have a prescribed initial point $A \in X$, a prescribed final point $B \in X$, and make a prescribed number n of reflections at the boundary X. We apply a topological approach based on the calculation of cohomology rings of certain configuration spaces of S^m.

1. Introduction

In the early 1900s, G. Birkhoff initiated the mathematical theory of convex plane billiards. His main interest was in estimating the number of periodic billiard trajectories. He pioneered the use of topological methods based on variational reduction and using the critical point theory.

Periodic trajectories in convex billiards in Euclidean spaces of dimension greater than 2 were studied in [2] and [10]. The high-dimensional problem also allows an approach based on the critical point theory, and the main difficulty lies in the more complicated topology of the appropriate configuration space. Thus the major effort of [10] was in computing the cohomology algebra of the cyclic configuration space of the sphere.

The purpose of this paper is to strengthen the estimates obtained in [10]. In particular, we obtain estimates, linear in n, of the number of n-periodic billiard trajectories, improving the logarithmic estimates of [10]. On the other hand, in this work we study a larger variety of billiard problems: besides the periodic trajectories, we are also interested in the number of ways the billiard ball can be brought from a given initial point to a given final point after making a prescribed number n of reflections at the boundary of the billiard domain.

Let $X \subset \mathbf{R}^{m+1}$ be a closed smooth strictly convex hypersurface. The $(m+1)$-dimensional convex body T, bounded by X, serves as our billiard table. The billiard

DUKE MATHEMATICAL JOURNAL
Vol. 115, No. 3, © 2002
Received 18 May 2000. Revision received 31 August 2001.
2000 *Mathematics Subject Classification*. Primary 37D50, 55R80.
Author's work partially supported by the United States–Israel Binational Science Foundation and by the Minkowski Center for Geometry.

ball is a point that moves in T in a straight line, except when it hits $X = \partial T$, where it rebounds making the angle of incidence equal the angle of reflection. In other words, if $A, B, C \in X$ are three subsequent reflection points, then the normal to X at point B bisects the angle between the vectors BA and BC (see Figure 1).

We consider the following billiard problems.

PROBLEM A

Given two distinct points $A, B \in X$ and a number n, estimate the number of billiard trajectories inside X which start at point A, end at point B, and make n reflections at the hypersurface X.

PROBLEM B

Estimate the number of billiard trajectories inside X which start and end at a given point $A \in X$ and make a prescribed number n of reflections at the hypersurface X.

Problem B deals with *closed* billiard trajectories. It is clear that any closed billiard trajectory starting and ending at $A \in X$ determines another closed billiard trajectory that is obtained by passing the same route in the reverse order. This explains that there is a natural $\mathbf{Z}_2$-action on the set of closed billiard trajectories, and in Problem B one actually asks about *the number of* $\mathbf{Z}_2$*-orbits* of closed billiard trajectories.

Using this $\mathbf{Z}_2$-symmetry, we give a better estimate for Problem B than the estimate for Problem A given by Theorem 1.

PROBLEM C

Estimate the number of n-periodic billiard trajectories inside the billiard domain T.

In [10] we showed that the number of n-periodic billiard trajectories is at least $[\log_2(n-1)] + m$ and is at least $(n-1)m$ in the generic case. Here $[x]$ denotes the integer part of x, that is, the largest integer not exceeding x.

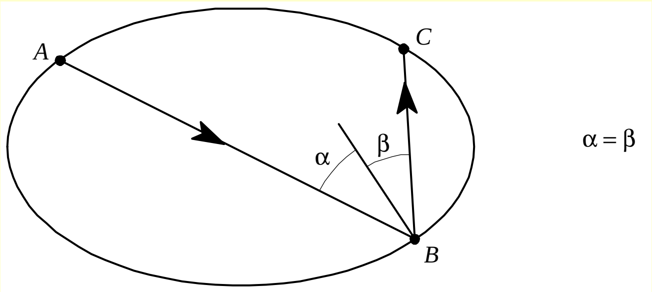

Figure 1

In Part I of this paper we give an answer to Problem A.

THEOREM 1

Let $X \subset \mathbf{R}^{m+1}$ *be a closed smooth strictly convex hypersurface, and let* $A, B \in X$ *be two distinct points. Then for any integer* n*, the number of billiard trajectories inside* X *which start at* A*, end at* B*, and make* n *reflections is at least* $n+1$ *if* m *is odd and* $[(n+1)/2]+1$ *if* m *is even. In a generic situation (cf. Definition 3), for any* m *and* n *the number of billiard trajectories inside* X *which start at* A*, end at* B*, and make* n *reflections is at least* $n+1$.

Consider the following simple example. Let $X = S^m \subset \mathbf{R}^{m+1}$ be the unit sphere. Any billiard trajectory $A = A_0, A_1, \ldots, A_n, A_{n+1} = B$ must lie in a 2-plane passing through the center of the sphere O. If the endpoints A, B are distinct and not antipodal, then there is a unique 2-plane passing through A, B, and O; the circle L, the intersection of this 2-plane with S^m, must contain all the reflection points $A_1, \ldots, A_n$. Fix an orientation on L, and let $\phi \in (0, 2\pi)$ be the angle from A to B. Then the angle between A_i and A_{i+1} must be independent of i and may take the values

$$\alpha_k = \frac{\phi + 2\pi k}{n+1}, \quad \text{where } k = 0, 1, \ldots, n.$$

Hence we see that in this example there exist precisely $n+1$ billiard trajectories starting at A, ending at B, and making n reflections.

This example shows that the statement of Theorem 1 for the generic case and for m odd cannot be improved. It looks reasonable to conjecture that for even m the lower bound on the number of billiard trajectories is also $n+1$.

Problems B and C will be studied in Part II.

2. Billiard ball problem and Lusternik-Schnirelman category of configuration spaces

In this section we use the variational method of Birkhoff to show that the problem of estimating the number of billiard trajectories can be reduced to the topological problem of estimating the Lusternik-Schnirelman category of a space of configurations of n points on the sphere S^m.

Let X be a manifold. Suppose that $A, B \in X$ are two fixed points. The symbol $G(X; A, B, n)$ denotes the subspace of the Cartesian power $X^{\times n} = X \times X \times \cdots \times X$, consisting of the configurations $(x_1, \ldots, x_n) \in X^{\times n}$, such that $x_i \neq x_{i+1}$ for all $i = 1, \ldots, n-1$, and $A \neq x_1$ and $x_n \neq B$. In the case $A \neq B$ we call $G(X; A, B, n)$ *the open string configuration space*. The space $G(X; A, A, n)$ is *the closed string configuration space*.

The configuration space $G(X; A, B, n)$ is closely related to the *cyclic configuration space* $G(X, n)$ introduced in [10], which consists of all n-tuples $(x_1, \ldots, x_n)$ of points of X satisfying

$$x_i \neq x_{i+1} \quad \text{for } i = 1, 2, \ldots, n-1 \qquad \text{and} \qquad x_n \neq x_1.$$

Let $X \subset \mathbf{R}^{m+1}$ be a smooth closed strictly convex hypersurface, the boundary of the billiard table. Denote by

$$L_X : G(X; A, B, n) \to \mathbf{R}$$

the perimeter length function taken with the minus sign,

$$L_X(x_1, \ldots, x_n) = -\sum_{i=0}^{n} |x_i - x_{i+1}|, \quad i = 0, \ldots, n,$$

where $(x_1, x_2, \ldots, x_n) \in G(X; A, B, n)$ and the distance $|x_i - x_{i+1}|$ is measured in the ambient Euclidean space $\mathbf{R}^{m+1}$. Here we understand that $x_0 = A$ and $x_{n+1} = B$. The function L_X is smooth. The reason for the minus sign becomes clear later.

The following lemma is well known.

LEMMA 2
A point $(x_1, \ldots, x_n) \in G(X; A, B, n)$ is a critical point of L_X if and only if the sequence $A, x_1, \ldots, x_n, B$ determines a billiard trajectory inside X starting at point A and ending at point B.

Proof
An easy calculation shows that a configuration $(x_1, \ldots, x_n) \in G(X; A, B, n)$ is a critical point of L_X if and only if for any $i = 1, 2, \ldots, n$ the vector

$$\frac{x_i - x_{i-1}}{|x_i - x_{i+1}|} + \frac{x_i - x_{i+1}}{|x_i - x_{i+1}|}$$

is orthogonal to the tangent space $T_{x_i}(X)$. The last condition is clearly equivalent to the requirement that the normal to X at x_i bisects the angle between $x_i x_{i-1}$ and $x_i x_{i+1}$. □

Definition 3
The data (X, A, B, n) are called *generic* if the corresponding perimeter length function $L_X : G(X; A, B, n) \to \mathbf{R}$ has only Morse critical points.

Compare [10, §4].

Since X is homeomorphic to S^m, the space $G(X; A, B, n)$ is homeomorphic to $G(S^m; A, B, n)$. The shape of the billiard domain X becomes encoded in the function $L_X : G(S^m; A, B, n) \to \mathbf{R}$, and the problem of estimating the number of billiard trajectories inside X which start at A and end at B turns into a problem of Morse-Lusternik-Schnirelman theory. The difficulty is that we cannot apply the Morse-Lusternik-Schnirelman theory directly to $G(X; A, B, n)$ since this manifold is not compact.

To avoid this difficulty, we replace $G(X; A, B, n)$ by a compact manifold with boundary $G_\varepsilon(X; A, B, n) \subset G(X; A, B, n)$, where $\varepsilon > 0$ is small enough and

$$G_\varepsilon(X; A, B, n) = \left\{(x_1, \dots, x_n) \in X^{\times n} : \prod_{i=0}^{n} |x_i - x_{i+1}| \geq \varepsilon\right\}; \tag{2.1}$$

here $x_0 = A$ and $x_{n+1} = B$. A similar approach can be found in [2] and in [9] and [13] for the two-dimensional case (cf. also [10] for the periodic case).

PROPOSITION 4

If $\varepsilon > 0$ is sufficiently small, then

(a) $G_\varepsilon(X; A, B, n)$ *is a smooth manifold with boundary;*

(b) *the inclusion* $G_\varepsilon(X; A, B, n) \subset G(X; A, B, n)$ *is a homotopy equivalence;*

(c) *all critical points of* $L_X : G(X; A, B, n) \to \mathbf{R}$ *are contained in* $G_\varepsilon(X; A, B, n)$*;*

(d) *at every point of* $\partial G_\varepsilon(X; A, B, n)$*, the gradient of* L_X *has the outward direction.*

This statement is analogous to [10, Proposition 4.1]. The proof given in [10] also applies in this case. The only modification is that in the case $A \neq B$ the arguments of the proof of [10, Proposition 4.1], which explain that a critical configuration cannot lie entirely in a small neighborhood of X, become redundant.

Recall that the Lusternik-Schnirelman category $\mathrm{cat}(Y)$ of a topological space Y is defined as the least integer $k > 0$ such that Y admits an open cover $Y = F_1 \cup \cdots \cup F_k$, such that each inclusion $F_j \to Y$ is null-homotopic.

COROLLARY 5

Let $X \subset \mathbf{R}^{m+1}$ be a smooth strictly convex hypersurface, and let $A, B \in X$ be two fixed points. For any $n \geq 0$, the number of billiard trajectories inside X which start at A, end at B, and make n reflections is at least $\mathrm{cat}(G(S^m; A, B, n))$*, the Lusternik-Schnirelman category of the open string configuration space of the sphere S^m.*

Proof

Choose $\varepsilon > 0$ small enough so that the conclusions of Proposition 4 hold. Since at the points of the boundary $\partial G_\varepsilon(X; A, B, n)$ the gradient of L_X has the outward direction, the critical point theory for manifolds with boundary (see [15]) applies; the conclusion is that the critical points of the restriction $L_X|_{\partial G_\varepsilon(X;A,B,n)}$ should be ignored, and the number of critical pints of L_X lying in the interior of $G_\varepsilon(X; A, B, n)$ is at least the category of $G_\varepsilon(X; A, B, n)$. Since $\operatorname{cat} G_\varepsilon(X; A, B, n) = \operatorname{cat} G(S^m; A, B, n)$ (because of Proposition 4(b)), the number of billiard trajectories inside X which start at A, end at B, and make n reflections is at least $\operatorname{cat}(G(S^m; A, B, n))$. □

In the closed case, that is, when the endpoints are equal ($A = B$), we may use $\mathbf{Z}_2$-symmetry to give a better estimate. This result will appear in Part II.

3. Spectral sequence computing the cohomology of the open string configuration space of a manifold

The following theorem yields a spectral sequence computing the cohomology algebra of the open string configuration space $G(X; A, B, n)$, where X is an arbitrary manifold. It is a Leray spectral sequence associated to the embedding $G(X; A, B, n) \to X^{\times n} = X \times X \times \cdots \times X$ (the nth Cartesian power).

This method was first suggested by B. Totaro [18] for the usual configuration space (i.e., for the space of all configurations $(x_1, \ldots, x_n) \in X^{\times n}$ with $x_i \neq x_j$ for all i, j). In [10] we used a similar spectral sequence for the cyclic configuration space $G(S^m, n)$.

The symbol $\mathbf{k}$ denotes a field.

THEOREM 6

Let X be a connected oriented manifold of dimension $m > 1$, and let $A, B \in X$ be two distinct points.

(A) *There exists a spectral sequence of bigraded differential algebras which converges to $H^*(G(X; A, B, n); \mathbf{k})$ whose E_2-term is the quotient of the bigraded commutative algebra*

$$H^*(X^{\times n}; \mathbf{k})[s_0, s_1, \ldots, s_n],$$

where $H^p(X^{\times n}; \mathbf{k})$ has bidegree $(p, 0)$ and each generator s_i has bidegree $(0, m-1)$, by the relations

$$\begin{aligned}
&s_i^2 = 0 \quad \textit{for } i = 0, 1, \ldots, n,\\
&s_0 s_1 \cdots s_n = 0,\\
&p_1^*(v) s_0 = 0,\\
&p_i^*(v) s_i = p_{i+1}^*(v) s_i \quad \textit{for } i = 1, 2, \ldots, n-1,\\
&p_n^*(v) s_n = 0,
\end{aligned}$$

where $v \in H^*(X;\mathbf{k})$ denotes an arbitrary cohomology class of positive degree and $p_j : X^{\times n} \to X$ denotes the projection onto the jth factor, $j = 1, 2, \dots, n$.

(B) *The first nontrivial differential is d_m, where $m = \dim X$. It acts by*

$$\begin{aligned} d_m(s_0) &= (-1)^m p_1^*([X]), \\ d_m(s_i) &= q_i^*(\Delta), \quad i = 1, 2, \dots, n-1, \\ d_m(s_n) &= p_n^*([X]), \\ d_m\big(H^*(X^{\times n};\mathbf{k})\big) &= 0, \end{aligned}$$

where $q_j : X^{\times n} \to X \times X$ denotes the projection onto the factors j and $j+1$, $[X] \in H^m(X;\mathbf{k})$ is the fundamental class, and $\Delta \in H^m(X \times X;\mathbf{k})$ denotes the cohomology class of the diagonal.

Proof

Consider the inclusion $\psi : G(X; A, B, n) \to X^{\times n}$ and the Leray spectral sequence (see [4]) of the continuous map ψ,

$$E_2^{p,q} = H^p(X^{\times n}; R^q\psi_*\mathbf{k}) \Rightarrow H^{p+q}\big(G(X; A, B, n);\mathbf{k}\big),$$

where $R^q\psi_*\mathbf{k}$ is the sheaf on $X^{\times n}$ associated with the presheaf

$$U \mapsto H^q\big(U \cap G(X; A, B, n);\mathbf{k}\big).$$

To describe the sheaves $R^q\psi_*\mathbf{k}$, consider partitions of the set $\{0, 1, \dots, n, n+1\}$ into intervals, that is, subsets of the form $\{i, i+1, i+2, \dots, i+j\}$. For any such partition J, we denote by X_J the subset of $X^{\times n}$, consisting of all configurations $c = (x_1, x_2, \dots, x_n) \in X^{\times n}$, satisfying the following conditions:

$x_i = x_j$	if i and j lie in the same interval of the partition J;
$x_i = A$	if index i lies in the same interval as 0;
$x_i = B$	if index i lies in the same interval as $n+1$.

Given two interval partitions I and J, we say that J *refines* I and write $I \prec J$ if the intervals of I are unions of the intervals of J. We denote by $|J|$ the number of intervals in the partition J. Note that $I \prec J$ implies $X_I \subset X_J$ and $|I| \le |J|$. For the partition J with $|J| = 1$, $X_J = \emptyset$ holds (since we assume that $A \ne B$). If $|J| = 2$, then X_J is a single point. For $|J| > 2$ the space X_J is homeomorphic to the Cartesian power $X^{\times(|J|-2)}$.

As in [10], we denote by $D(X, n)$ the subset of $X^{\times n}$ satisfying the conditions $x_i \ne x_{i+1}$ for $i = 1, \dots, n-1$. The configuration space $D(\mathbf{R}^m, n)$ is homotopy equivalent to the product of spheres $(S^{m-1})^{\times(n-1)}$. A homotopy equivalence

$D(\mathbf{R}^m, n) \to (S^{m-1})^{\times(n-1)}$ is given by the map

$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1 - x_2}{|x_1 - x_2|}, \dots, \frac{x_{n-1} - x_n}{|x_{n-1} - x_n|}\right). \tag{3.1}$$

Fixing an orientation of the sphere $[S^{m-1}] \in H^{m-1}(S^{m-1}; \mathbf{k})$ determines a *canonical top-dimensional class* in $H^{(n-1)(m-1)}(D(\mathbf{R}^m, n); \mathbf{k})$ which is the pullback of the product $[S^{m-1}] \times \cdots \times [S^{m-1}]$ under (3.1).

If $A, B \in X$ are two points, we denote by $G(X; A, \emptyset, n)$ the subspace of $D(X, n)$ consisting of configurations $(x_1, \dots, x_n)$ with $x_1 \neq A$; similarly, we denote by $G(X; \emptyset, B, n) \subset D(X, n)$ the subspace of configurations with $x_n \neq B$.

Let J be a partition of $\{0, 1, 2, \dots, n+1\}$ on intervals of lengths $j_1, \dots, j_r$, and let

$$c = (x_1, x_2, \dots, x_n) \in X_J, \quad c \notin \bigcup_{I \prec J,\, I \neq J} X_I.$$

We claim that the stalk of the sheaf $R^q\psi_*\mathbf{k}$ at c equals

$$(R^q\psi_*\mathbf{k})_c = H^q\big(D(\mathbf{R}^m, j_1) \times \cdots \times D(\mathbf{R}^m, j_r); \mathbf{k}\big).$$

Indeed, by definition, this stalk is $H^q(U \cap G(X; A, B, n); \mathbf{k})$, where U is a small open ball around c. If $c = (x_1, x_2, \dots, x_n)$, then we may choose points

$$y_1 = A, y_2, \dots, y_{r-1}, y_r = B \in X,$$

one for each interval of J, so that $x_i = y_{j_s}$ if i belongs to the sth interval. Let $U_j \subset X$ be a small open neighborhood of y_j, so that each U_j is diffeomorphic to $\mathbf{R}^m$ and the sets U_j and $U_{j'}$ are disjoint when the points y_j and $y_{j'}$ are distinct. Then we may take $U = U_1^{\times j_1} \times U_2^{\times j_2} \times \cdots \times U_r^{\times j_r}$, and our claim follows.

We see that $R^q\psi_*\mathbf{k}$ vanishes unless q is a multiple of $m-1$ and

$$\dim(R^{s(m-1)}\psi_*\mathbf{k})_c = \begin{cases} 0 & \text{for } s > n+2-|J|, \\ \binom{n+2-|J|}{s} & \text{for } s \leq n+2-|J|. \end{cases}$$

For an interval partition J of $\{0, 1, 2, \dots, n+1\}$ with $|J| > 1$, denote by ε_J the constant sheaf with stalk $\mathbf{k}$ and support X_J. We claim the following.

For any $r = 2, 3, \dots, n+2$, the sheaf $R^{(n+2-r)(m-1)}\psi_*\mathbf{k}$ is isomorphic to the direct sum of sheaves

$$R^{(n+2-r)(m-1)}\psi_*\mathbf{k} \simeq \bigoplus_{|J|=r} \varepsilon_J, \tag{3.2}$$

the sum taken over all interval partitions J with $|J| = r$.

To prove the claim, let I be an interval partition of $\{0, 1, \dots, n+1\}$ into intervals of length $i_1, i_2, \dots, i_s$, where $s = |I| > 1$. Then for any interval partition J into intervals of length $j_1, j_2, \dots, j_r$ such that $I \prec J$, we have the canonical inclusion

$$\nu_{JI} : D(\mathbf{R}^m, i_1) \times \cdots \times D(\mathbf{R}^m, i_s) \to D(\mathbf{R}^m, j_1) \times \cdots \times D(\mathbf{R}^m, j_r).$$

The target space of map ν_{JI} has a canonical nonzero $((n+2-r)(m-1))$-dimensional cohomology class (cf. above). The induced map ν_{JI}^* on $((n+2-r)(m-1))$-dimensional cohomology with $\mathbf{k}$ coefficients is a monomorphism. Let z_{JI} denote the image of the top-dimensional canonical class under the induced map ν_{JI}^*. Then (in a manner similar to [18, Lemma 3]) for a fixed I, the classes $\{z_{JI}\}$ form a linear basis of the cohomology $H^{(n+2-r)(m-1)}(D(\mathbf{R}^m, i_1) \times \cdots \times D(\mathbf{R}^m, i_s); \mathbf{k})$, where J runs over all partitions with $I \prec J$ and $|J| = r$.

Indeed, using the map into the product of spheres (3.1), we see that a linear basis of the cohomology $H^{(n+2-r)(m-1)}(D(\mathbf{R}^m, i_1) \times \cdots \times D(\mathbf{R}^m, i_s); \mathbf{k})$ is formed by monomials $s_{a_1} s_{a_2} \cdots s_{a_{n+2-r}}$ with $0 \le a_1 < a_2 < \cdots < a_{n+2-r} \le n+1$, such that for any $p = 1, \dots, n+2-r$ the indices a_p and $a_p + 1$ belong to the same interval of partition I. Let J be the partition determined by the equivalence relation on $\{0, 1, \dots, n+1\}$, where $a_p \sim a_p + 1$. Then $I \prec J$ and the above monomial coincides with the class z_{JI}.

Given a partition J of the set $\{0, 1, \dots, n+1\}$ on intervals having lengths $j_1 + 1, j_2, \dots, j_{r-1}, j_r + 1$, where $r > 1$, consider the commutative diagram

$$\begin{array}{ccc} G(X; A, B, n) & \xrightarrow{\ \psi\ } & X^{\times n} \\ \downarrow & & \downarrow \text{id} \\ G(X; A, \emptyset, j_1) \times D(X, j_2) \times \cdots \times G(X; \emptyset, B, j_r) & \xrightarrow{\ g_J\ } & X^{\times n} \end{array}$$

formed by the natural inclusions. Define sheaf $\varepsilon'_J = R^{(n+2-r)(m-1)} g_{J*}(\mathbf{k})$ over $X^{\times n}$. We want to show that ε'_J is isomorphic to ε_J, that is, that it is the constant sheaf with stalk $\mathbf{k}$ and support X_J. First, ε'_J vanishes outside X_J (since we are considering the cohomology of the top dimension). Let U be a small open neighborhood of a point $c \in X_J \subset X^{\times n}$, such that $U = \prod U_i$, where all U_i are small open disks and $U_i = U_j$ if i and j lie in the same interval of J. Then

$$\varepsilon'_J(U) = H^{s(m-1)}\big(D(U_{i_1}, j_1 + 1) \times D(U_{i_2}, j_2) \times \cdots \times D(U_{i_r}, j_r + 1); \mathbf{k}\big) \simeq \mathbf{k}$$

(where $s = n + 2 - r$) has a canonical element (cf. above). This gives a continuous section of ε'_J over X_J, and hence $\varepsilon'_J \simeq \varepsilon_J$.

The commutative diagram gives a map of sheaves $\varepsilon_J \to R^{(n+2-r)(m-1)}\psi_*(\mathbf{k})$, and, summing, we obtain a map of sheaves

$$\bigoplus_{|J|=r} \varepsilon_J \to R^q\psi_*(\mathbf{k}), \quad \text{where } q = (n+2-r)(m-1),$$

which, as we have seen above, is an isomorphism on stalks; hence it is an isomorphism, and the claim (3.2) follows.

We arrive at the following description of the term E_2 of the Leray spectral sequence:

$$E_2^{p,r(m-1)} = \bigoplus_{|J|=n+2-r} H^p(X_J; \mathbf{k}),$$

where J runs over all partitions of $\{0, 1, \dots, n+1\}$ with $|J| > 1$. In order to identify this description with the one given in the statement of the theorem, assign to a monomial $s_{i_1} \cdots s_{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r \le n$ the equivalence relation on the set of indices $\{0, 1, \dots, n+1\}$ generated by

$$i_1 \sim i_1 + 1, \ \ i_2 \sim i_2 + 1, \dots, i_r \sim i_r + 1.$$

This equivalence relation defines a partition J of the set $\{0, 1, \dots, n+1\}$ on $n+2-r$ intervals. In view of the relations

$$\begin{aligned} p_1^*(v)s_0 &= 0, \\ p_i^*(v)s_i &= p_{i+1}^*(v)s_i, \quad \text{where } i = 1, \dots, n-1, \\ p_n^*(v)s_n &= 0, \end{aligned} \tag{3.3}$$

the term $H^p(X^{\times n}; \mathbf{k})s_{i_1} \cdots s_{i_r}$ is isomorphic to $H^p(X_J; \mathbf{k})$.

The monomial $s_0 s_1 \cdots s_n$ corresponds to the partition $|J| = 1$, which we should ignore since $X_J = \emptyset$; this explains the relation $s_0 s_1 \cdots s_n = 0$.

Now we prove Theorem 6(B) concerning the differentials of the spectral sequence. The first nontrivial differential is d_m. To find d_m it is enough to find the cohomology classes $d_m(s_i) \in H^m(X^{\times n}; \mathbf{k})$, where $i = 0, 1, \dots, n$.

We use functoriality of the Leray spectral sequence and the following well-known property. Let Y be a manifold, and let $Z \subset Y$ be a submanifold of codimension $m > 1$ with oriented normal bundle. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y; R^q\phi_*\mathbf{k}) \Rightarrow H^{p+q}(Y - Z; \mathbf{k})$$

of the inclusion $\phi : (Y - Z) \to Y$. The sheaf $R^{m-1}\phi_*\mathbf{k}$ is the constant sheaf with support Z and stalk $\mathbf{k}$ for $q = m - 1$, and it vanishes for all other values $q > 0$. The only nonzero differential, $d_m : E_2^{0,m-1} \to E_2^{m,0}$, acts as follows: the class $1 \in H^0(Z; \mathbf{k}) = E_2^{0,m-1}$ is mapped into $d_m(1) = [Z] \in H^m(Y; \mathbf{k})$, the class dual to Z, where the same orientation of the normal bundle to Z is used to trivialize the sheaf $R^{m-1}\phi_*\mathbf{k}$ and to define the dual class $[Z]$.

In order to show the first relation $d_m(s_0) = (-1)^m p_1^*([X])$, consider the diagram

$$\begin{array}{ccc} G(X;A,B,n) & \xrightarrow{\subset} & X^{\times n} \\ \downarrow & & \downarrow \text{id} \\ (X-A)\times X^{\times(n-1)} & \xrightarrow{\subset} & X^{\times n} \end{array}$$

and apply the previous remark to the bottom row with $Y = X^{\times n}$ and $Z = A \times X^{\times(n-1)}$. The sign $(-1)^m$ appears as the degree of the antipodal map $S^{m-1} \to S^{m-1}$; the framing of the normal bundle to $A \subset X$, which we use to define the fundamental class $[X] \in H^m(X;\mathbf{k})$, is antipodal to the framing determined by (3.1), which we use to trivialize the derived sheaf.

To obtain relations $d_m(s_i) = q_i^*(\Delta)$ with $i = 1, \ldots, n-1$, we use the commutative diagram

$$\begin{array}{ccc} G(X;A,B,n) & \xrightarrow{\subset} & X^{\times n} \\ \downarrow & & \downarrow \text{id} \\ X^{\times(i-1)} \times (X\times X - \Delta) \times X^{\times(n-i-1)} & \xrightarrow{\subset} & X^{\times n} \end{array}$$

and apply the remark above with $Y = X^{\times n}$ and $Z = X^{\times(i-1)} \times \Delta \times X^{\times(n-i-1)}$. The last relation $d_m(s_n) = p_n^*([X])$ follows similarly. Theorem 6 is proven. □

We apply Theorem 8 and [10, Proposition 2.2] to compute the integral cohomology of the configuration space $G(\mathbf{R}^m; A, B, n)$. For any $i = 0, 1, \ldots, n$ we have the map

$$\phi_i : G(\mathbf{R}^m; A, B, n) \to S^{m-1}, \qquad (x_1, \ldots, x_n) \mapsto \frac{x_i - x_{i+1}}{|x_i - x_{i+1}|} \in S^{m-1},$$

where we understand that $x_0 = A$ and $x_{n+1} = B$. Define the cohomology classes

$$s_i \in H^{m-1}\big(G(\mathbf{R}^m; A, B, n); \mathbf{Z}\big) \quad \text{as } s_i = \phi_i^*([S^{m-1}]),\ i = 0, 1, \ldots, n.$$

PROPOSITION 7

For $m > 1$ the algebra $H^*(G(\mathbf{R}^m; A, B, n); \mathbf{Z})$ (where $A \neq B$) is generated by cohomology classes

$$s_0, s_1, \ldots, s_n \in H^{m-1}\big(G(\mathbf{R}^m; A, B, n); \mathbf{Z}\big),$$

and all relations between the classes s_i are consequences of

$$s_0^2 = s_1^2 = \cdots = s_n^2 = 0, \qquad s_i s_j = (-1)^{m-1} s_j s_i, \qquad s_0 s_1 \cdots s_n = 0.$$

Proof

If we replace $\mathbf{Z}$ by a field $\mathbf{k}$, the result follows directly from Theorem 8. In particular, we see that the dimension of the cohomology of $G(\mathbf{R}^m; A, B, n)$ does not depend on the field of coefficients. We conclude that the integral cohomology of $G(\mathbf{R}^m; A, B, n)$ has no torsion and is nonzero only in dimensions divisible by $m-1$.

Consider the cyclic configuration space $G(\mathbf{R}^m, n+2)$ (cf. [10]) and the fibration

$$G(\mathbf{R}^m, n+2) \to G(\mathbf{R}^m, 2) \simeq S^{m-1}, \qquad (x_1, \dots, x_{n+2}) \mapsto (x_{n+2}, x_1), \tag{3.4}$$

which has $G(\mathbf{R}^m; A, B, n)$ as the fiber. The nonzero rows of the Serre spectral sequence have numbers divisible by $m-1$; also, the spectral sequence has only two columns, $p = 0$ and $p = m-1$. We obtain that all differentials of the spectral sequence vanish and that the cohomology of the fiber $H^*(G(\mathbf{R}^m; A, B, n); \mathbf{Z})$ is isomorphic to the factor of the ring $H^*(G(\mathbf{R}^m, n+2); \mathbf{Z})$ with respect to the ideal generated by class s_{n+2} (the pullback of the fundamental class of the base). Comparing the above information with the structure of the ring $H^*(G(\mathbf{R}^m, n+2); \mathbf{Z})$, described in [10, Proposition 2.2], proves Proposition 7. □

4. Cohomology of open string configuration spaces of spheres

In this section we state a theorem describing the cohomology of the configuration space $G(S^m; A, B, n)$, assuming that the points A and B are distinct. We see that the additive structure of the cohomology algebra $H^*(G(S^m; A, B, n); \mathbf{k})$ is similar for all m, but the multiplication depends on the parity of the dimension m. Also, the case $m = 1$ is special since the space $G(S^1; A, B, n)$ consists of $n+1$ path-connected components and each is contractible (cf. Section 7); in this case only a zero-dimensional cohomology exists.

Let $\mathbf{k}$ be a field.

THEOREM 8

The cohomology $H^*(G(S^m; A, B, n); \mathbf{k})$ of the open string configuration space (where $A \neq B$) has additive generators

$$\sigma_i \in H^{i(m-1)}\big(G(S^m; A, B, n); \mathbf{k}\big), \quad i = 0, 1, \dots, n;$$

the Poincaré polynomial of $G(S^m; A, B, n)$ equals $1 + t^{m-1} + t^{2(m-1)} + \cdots + t^{n(m-1)}$.

For $m \geq 3$ odd, the multiplication is given by

$$\sigma_i \sigma_j = \begin{cases} \frac{(i+j)!}{i! \cdot j!} \cdot \sigma_{i+j} & \text{if } i + j \leq n, \\ 0 & \text{if } i + j > n. \end{cases} \tag{4.1}$$

For $m \geq 2$ even, the multiplication is given by

$$\sigma_i \sigma_j = \begin{cases} \frac{[(i+j)/2]!}{[i/2]! \cdot [j/2]!} \cdot \sigma_{i+j} & \text{if } i + j \leq n \text{ and } i \text{ or } j \text{ is even}, \\ 0 & \text{if either } i + j > n \text{ or both } i \text{ and } j \text{ are odd}. \end{cases} \tag{4.2}$$

Recall that $[x]$ denotes the integer part of x.

The proof of Theorem 8 is given in Sections 5 and 6.

Remark 9

Choosing an arbitrary point $C \in S^m$, where $C \neq A$ and $C \neq B$, we obtain an inclusion $\phi : G(S^m - C; A, B, n) \to G(S^m; A, B, n)$; here we may identify $S^m - C$ with $\mathbf{R}^m$. From the proof of Theorem 8 it is clear that the induced homomorphism

$$\begin{aligned} \phi^* : H^*\big(G(S^m; A, B, n); \mathbf{k}\big) \to H^*\big(G(S^m - C; A, B, n); \mathbf{k}\big) \\ = H^*\big(G(\mathbf{R}^m; A, B, n); \mathbf{k}\big) \end{aligned}$$

is injective and that its image may easily be described. For example, for m odd, ϕ^* maps each generator σ_r to the degree r symmetric function of classes s_i:

$$\phi^*(\sigma_r) = \sum_{0 \leq i_1 < i_2 < \cdots < i_r \leq n} s_{i_1} s_{i_2} \cdots s_{i_r}, \quad r = 1, \ldots, n, \tag{4.3}$$

where $s_0, \ldots, s_n \in H^{m-1}(G(S^m - C; A, B, n); \mathbf{k}) = H^{m-1}(G(\mathbf{R}^m; A, B, n); \mathbf{k})$ are the generators given by Proposition 7.

For even m, the classes $\phi^*(\sigma_r)$ may also be described. Such a description may easily be extracted from the proof of Theorem 8. For instance,

$$\phi^*(\sigma_1) = s_n - s_{n-1} + \cdots + (-1)^n s_0 \qquad \text{and} \qquad \phi^*(\sigma_2) = -\sum_{0 \leq i < j \leq n} (-1)^{i+j} s_i s_j,$$

as follows from formulae (6.3). More generally,

$$\phi^*(\sigma_r) = (-1)^{[r/2]+nr} \cdot \sum_{0 \leq i_1 < i_2 < \cdots < i_r \leq n} (-1)^{i_1 + i_2 + \cdots + i_r} s_{i_1} s_{i_2} \cdots s_{i_r}, \tag{4.4}$$

where the sum is taken over all increasing sequences $0 \leq i_1 < i_2 < \cdots < i_r \leq n$. This follows using

$$\beta_j \beta_{j+1} = \beta_j s_{j+1} = \sum_{i=0}^{j} (-1)^{i+j} s_i s_{j+1}$$

from our definition (6.4).

5. Proof of Theorem 8 for *m* odd

Theorem 6 gives a spectral sequence of bigraded algebras which converges to the cohomology algebra $H^*(G(S^m; A, B, n); \mathbf{k})$; the initial term $E_2 = E_m$ has generators $u_1, \dots, u_n$, having bidegree $(m, 0)$, which satisfy

$$u_i u_j = -u_j u_i, \qquad u_i^2 = 0,$$

and also generators $s_0, s_1, \dots, s_n$, having bidegree $(0, m-1)$, which satisfy

$$\begin{aligned} s_i s_j &= s_j s_i, \qquad s_i^2 = 0, \\ s_i u_j &= u_j s_i, \\ u_1 s_0 &= 0, \\ (u_i - u_{i+1}) s_i &= 0 \quad \text{for } i = 1, 2, \dots, n-1, \\ u_n s_n &= 0, \\ s_0 s_1 \cdots s_n &= 0. \end{aligned} \tag{5.1}$$

Here u_i denotes $1 \times \cdots \times u \times 1 \times \cdots \times 1 \in H^m((S^m)^{\times n}; \mathbf{k})$, where u is the fundamental class of the sphere S^m which appears on the place number i.

The differential $d = d_m : E_m \to E_m$ acts by

$$\begin{aligned} du_j &= 0, \\ ds_0 &= -u_1, \\ ds_i &= u_i - u_{i+1} \quad \text{for } i = 1, 2, \dots, n-1, \\ ds_n &= u_n. \end{aligned}$$

We introduce new variables $v_0, v_1, \dots, v_n$:

$$\begin{aligned} v_0 &= -u_1, \\ v_i &= u_i - u_{i+1} \quad \text{for } i = 1, \dots, n-1, \\ v_n &= u_n. \end{aligned}$$

We have the following relations:

$$\begin{aligned} &\text{(i)} \quad v_i v_j = -v_j v_i, \qquad v_i^2 = 0, \\ &\text{(ii)} \quad v_0 + \cdots + v_n = 0, \\ &\text{(iii)} \quad v_i s_i = 0 \quad \text{for } i = 0, 1, \dots, n, \\ &\text{(iv)} \quad s_i s_j = s_j s_i, \qquad s_i v_j = v_j s_i, \\ &\text{(v)} \quad s_i^2 = 0, \\ &\text{(vi)} \quad s_0 s_1 \cdots s_n = 0, \\ &\text{(vii)} \quad ds_i = v_i \quad \text{for } i = 0, 1, \dots, n, \\ &\text{(viii)} \quad dv_i = 0. \end{aligned} \tag{5.2}$$

Denote by $\sigma_k \in E_m$ the kth symmetric function in variables $s_0, s_1, \ldots, s_n$; that is,

$$\sigma_0 = 1 \qquad \text{and} \qquad \sigma_k = \sum_{0\le i_1<\cdots<i_k\le n} s_{i_1}s_{i_2}\cdots s_{i_k} \quad \text{for } k = 1, 2, \ldots, n.$$

It is clear (because of (v) in (5.2)) that

$$\sigma_i\sigma_j = \begin{cases} \binom{i+j}{i}\sigma_{i+j} & \text{for } i+j \le n, \\ 0 & \text{for } i+j > n. \end{cases} \tag{5.3}$$

We have

$$d\sigma_1 = d(s_0 + \cdots + s_n) = v_0 + \cdots + v_n = 0$$

and, similarly,

$$d(\sigma_i) = (v_0 + v_1 + \cdots + v_n)\sigma_{i-1} = 0$$

for any i. Hence we have found nonzero cycles $\sigma_0, \sigma_1, \ldots, \sigma_n$ which (for obvious geometric reasons) cannot belong to the image of d. Our purpose is to show that these classes additively generate the whole cohomology $H^*(E_m, d)$.

Denote by (A_n, d) the graded differential algebra with generators $v_0, \ldots, v_n$ and $s_0, \ldots, s_n$ which satisfy relations (i), (iii), (iv), (v), and (vi) of (5.2). The differential $d : A_n \to A_n$ is given by formulae (vii) and (viii) of (5.2). We consider A_n with the total grading, where each s_i has degree $m-1$ (even) and each v_i has degree m (odd).

We claim that $H^j(A_n, d) = 0$ for $j > 0$. The proof uses induction on n. For $n = 0$, the claim is obvious. We have a natural inclusion $A_{n-1} \to A_n$ which identifies A_{n-1} with the subalgebra of A_n generated by $s_0, \ldots, s_{n-1}$ and $v_0, \ldots, v_{n-1}$. We show that the factor A_n/A_{n-1} is acyclic which clearly implies an induction step. Any element $a \in A_n/A_{n-1}$ can be uniquely represented in the form $a = s_n x + v_n y$, where $x, y \in A_{n-1}$. If $da = 0$, then

$$d(a) = v_n x + s_n d(x) - v_n d(y) = s_n d(x) + v_n[x - d(y)] = 0$$

and hence $x = d(y)$ and $a = d(s_n y)$. The claim follows.

Introduce a new differential $\delta_n : A_n \to A_n$ of degree m:

$$\delta_n(x) = \Big(\sum_{i=0}^{n} v_i\Big)x.$$

Clearly, $\delta_n^2 = 0$ and $\delta_n d = -d\delta_n$; however, δ_n does not obey the Leibnitz rule. We claim that

$$H^i(A_n, \delta_n) = \begin{cases} \mathbf{k} & \text{if } i = (n+1)(m-1), \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

and that a nontrivial cohomology class is represented by the product $s_0s_1s_2\cdots s_n$. Indeed, each element of A_n can be written as a sum of monomials in s_i, v_i. For $I\subset\{0,1,2,\dots,n\}$, denote by s_I the product of all s_i for $i\in I$. Similarly, we label the monomials $v_{i_1}\cdots v_{i_r}$ with $i_1<i_2<\cdots<i_r$ as v_J, where J is a subset $J=\{i_1,\dots,i_r\}\subset\{0,1,2,\dots,n\}$. Note that the product $s_Iv_J\in A_n$ is nontrivial if and only if I and J are disjoint subsets of $\{0,1,\dots,n\}$. Note also that

$$\delta_n(s_Iv_J)=\sum_{i\notin I\cup J}\epsilon_i s_I v_{J\cup\{i\}},$$

where ϵ_i is ±1 depending on whether J contains an even or odd number of members less than i. We see that application of δ_n does not change the multi-index I. Hence the complex (A_n,δ_n) splits into a direct sum over different multi-indices I. Fix a set I, and denote by k the cardinality of the set $\{0,1,\dots,n\}-I$. Then the respective part of the complex (A_n,δ_n) is isomorphic to the standard cochain complex of the simplex with k vertices; the differential of an r-dimensional face (i.e., set J) is the sum of $(r+1)$-dimensional faces that contain the given one (sets $J\cup\{i\}$). Note that empty set J is also allowed. This complex has zero cohomology unless $k=0$ (empty simplex), in which case the cohomology is $\mathbf{k}$. This exceptional case corresponds to $I=\{0,1,\dots,n\}$, and (5.4) follows.

Let $\mathscr{I}_n\subset A_n$ and $\mathscr{K}_n\subset A_n$ denote the image and the kernel of $\delta_n:A_n\to A_n$. Note that $\mathscr{I}_n\subset\mathscr{K}_n$ and that the factor $\mathscr{K}_n/\mathscr{I}_n$ is one-dimensional generated by the product $s_0s_1\cdots s_n$. Hence we obtain

$$H^j(\mathscr{I}_n,d)\simeq H^j(\mathscr{K}_n,d),\qquad j\neq(n+1)(m-1),$$

and $H^{(n+1)(m-1)}(\mathscr{I}_n,d)=0$.

Since we know that $H^j(A_n,d)=0$ for $j>0$, the short exact sequence

$$0\to\mathscr{K}_n\to A_n\xrightarrow{\delta_n}\mathscr{I}_n\to0$$

gives isomorphisms

$$H^{j+m-1}(\mathscr{I}_n,d)\simeq H^j(\mathscr{K}_n,d)$$

for all $j>1$. This leads to periodicity

$$H^j(\mathscr{I}_n,d)\simeq H^{j+m-1}(\mathscr{I}_n,d)\quad\text{for all } j\neq1,\ j\neq(n+1)(m-1).$$

On the other hand, it is obvious that for $1<j<2m-1$ the cohomology $H^j(\mathscr{I}_n,d)$ vanishes unless $j=m$ and that for $j=m$ it is one-dimensional (generated by the class $v_0+v_1+\cdots+v_n$). This shows that

$$\dim H^j(\mathscr{I}_n,d)=\begin{cases}1 & \text{for } j=i(m-1)+1,\ 1\le i\le n+1,\\ 0 & \text{otherwise.}\end{cases}$$

Using $H^j(A_n, d) = 0$, we get

$$\dim H^j(A_n/\mathscr{I}_n, d) = \begin{cases} 1 & \text{for } j = i(m-1),\ 0 \le i \le n+1, \\ 0 & \text{otherwise.} \end{cases}$$

The term E_m is obtained from $A_n/\mathscr{I}_n$ by factoring with respect to the ideal generated by the product $s_0 s_1 \cdots s_n$, which carries the top-dimensional cohomology space $H^{(n+1)(m-1)}(A_n/\mathscr{I}_n, d)$. Hence

$$\dim H^j(E_m, d) = \begin{cases} 1 & \text{for } j = i(m-1),\ 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

This proves that the classes $\sigma_0, \sigma_1, \dots, \sigma_n \in H^*(E_m, d)$ (which were described at the beginning of the proof) span the cohomology.

6. Proof of Theorem 8 for m even

6.1

Theorem 6 gives a spectral sequence of bigraded algebras converging to $H^*(G(S^m; A, B, n); \mathbf{k})$, with the initial term $E_2 = E_m$ described below.

E_m has generators $u_1, \dots, u_n$, having bidegree $(m, 0)$, which satisfy

$$u_i^2 = 0, \qquad u_i u_j = u_j u_i,$$

and also generators $s_0, s_1, \dots, s_n$, having bidegree $(0, m-1)$, which satisfy

$$\begin{aligned} & s_i s_j = -s_j s_i, \qquad s_i^2 = 0, \\ & s_i u_j = u_j s_i, \\ & u_1 s_0 = 0, \\ & (u_i - u_{i+1}) s_i = 0 \quad \text{for } i = 1, 2, \dots, n-1, \\ & u_n s_n = 0, \\ & s_0 s_1 \cdots s_n = 0. \end{aligned} \tag{6.1}$$

Here, as in the previous section, u_i denotes $1 \times \cdots \times u \times 1 \times \cdots \times 1 \in H^m((S^m)^{\times n}; \mathbf{k})$, where u is the fundamental class of the sphere S^m and appears in the position number i.

The differential $d = d_m : E_m \to E_m$ is given by

$$\begin{aligned} & du_j = 0, \\ & ds_0 = u_1, \\ & ds_i = u_i + u_{i+1} \quad \text{for } i = 1, 2, \dots, n-1, \\ & ds_n = u_n. \end{aligned}$$

Our purpose is to compute the cohomology of (E_m, d); from the answer we obtain, it is clear that all further differentials d_r, $r > m$, vanish and thus $H^*(E_m^{*,*}, d) = E_\infty^{*,*}$.

6.2

In this section we describe nontrivial classes

$$\sigma_i \in E_m^{i(m-1),0}, \quad i = 1, 2, \ldots, n,$$

which are cocycles $d\sigma_i = 0$. They appear in the first column of the spectral sequence and hence cannot belong to the image of d. Later we show that the cohomology classes of cocycles σ_i span the whole cohomology of (E_m, d).

Let us denote

$$\beta_i = s_i - s_{i-1} + \cdots + (-1)^i s_0 \in E_m^{m-1,0} \quad \text{for } i = 0, 1, \ldots, n.$$

We may express s_i as $\beta_i + \beta_{i-1}$ for $i \geq 1$ and $s_0 = \beta_0$. We have

$$\begin{aligned} \beta_i\beta_j &= -\beta_j\beta_i, \qquad \beta_i^2 = 0, \\ d\beta_i &= u_{i+1} \quad \text{for } i = 0, 1, \ldots, n-1, \\ d\beta_n &= 0. \end{aligned}$$

Relations (6.1) give

$$\begin{aligned} \beta_0 u_1 &= 0, \\ \beta_{i-1}u_i - \beta_i u_{i+1} + d(\beta_{i-1}\beta_i) &= 0 \quad \text{for } i = 1, \ldots, n-1, \\ \beta_{n-1}u_n + d(\beta_{n-1}\beta_n) &= 0. \end{aligned} \tag{6.2}$$

Now we set

$$\sigma_1 = \beta_n \in E_m^{m-1,0}, \qquad \sigma_2 = \sum_{i=0}^{n-1} \beta_i\beta_{i+1} \in E_m^{2(m-1),0}. \tag{6.3}$$

Then $d\sigma_1 = 0$, and (using (6.2)) we obtain $d\sigma_2 = 0$.

For any $k \leq n/2$ we define

$$\sigma_{2k} = \sum \beta_{i_1}\beta_{i_1+1}\beta_{i_2}\beta_{i_2+1}\cdots\beta_{i_k}\beta_{i_k+1} \in E_m^{2k(m-1),0}, \tag{6.4}$$

where

$$i_r + 1 < i_{r+1}, \quad 0 \leq i_r < n \text{ for } r = 1, \ldots, k.$$

For $2k+1 \leq n$ we define

$$\sigma_{2k+1} = \sigma_1 \cdot \sigma_{2k} \in E_m^{(2k+1)(m-1),0}.$$

It is clear that the classes $\sigma_1, \sigma_2, \dots, \sigma_n$ are nonzero and

$$(\sigma_2)^k = k! \cdot \sigma_{2k} \qquad \text{and} \qquad \sigma_1^2 = 0. \tag{6.5}$$

Hence classes σ_i satisfy the following multiplication law:

$$\sigma_i \sigma_j = \begin{cases} 0 & \text{if either } i + j > n \text{ or both } i \text{ and } j \text{ are odd,} \\ \frac{[(i+j)/2]!}{[i/2]! \cdot [j/2]!} \cdot \sigma_{i+j} & \text{if } i + j \le n \text{ and } i \text{ or } j \text{ is even.} \end{cases} \tag{6.6}$$

We have

$$\begin{aligned} d(\sigma_{2k}) &= \sum \beta_{i_1}\beta_{i_1+1} \cdots \beta_{i_{k-1}}\beta_{i_{k-1}+1} d(\beta_j \beta_{j+1}) \\ &= \sigma_{2(k-1)} \cdot \sum_{j=0}^{n-1} d(\beta_j \beta_{j+1}) = \sigma_{2(k-1)} \cdot d(\sigma_2) = 0. \end{aligned}$$

In the first sum, j runs over the set $\{0, \dots, n-1\}$ and indices $i_1, \dots, i_{k-1}$ satisfy $0 \le i_r < n$ and $i_r + 1 < i_{r+1}$. Thus we have

$$d(\sigma_i) = 0 \quad \text{for all } i = 1, \dots, n.$$

6.3

Next we show that $H^*(E_m, d)$ contains no nontrivial cohomology classes except linear combinations of $\sigma_1, \dots, \sigma_n$. More precisely, we show that the cohomology $H^j(E_m, d)$ (considered with respect to the total grading) vanishes if $j > n(m-1)$ or if j is not divisible by $m-1$, and is one-dimensional otherwise.

We introduce new variables v_j, where $j = 0, 1, \dots, n$, given by

$$\begin{aligned} v_0 &= u_1, \\ v_i &= u_{i+1} - u_i \quad \text{for } i = 1, 2, \dots, n-1, \\ v_n &= -u_n. \end{aligned}$$

The new variables commute $v_i v_j = v_j v_i$ and satisfy the following:

$$\begin{aligned} &\text{(i)} \quad v_0^2 = 0, \\ &\text{(ii)} \quad v_i^2 + 2v_i(v_0 + v_1 + \cdots + v_{i-1}) = 0 \quad \text{for } i = 1, 2, \dots, n, \\ &\text{(iii)} \quad v_0 + v_1 + \cdots + v_n = 0, \\ &\text{(iv)} \quad v_i s_i = 0 \quad \text{for } i = 0, 1, 2, \dots, n, \\ &\text{(v)} \quad s_i s_j = -s_j s_i, \\ &\text{(vi)} \quad s_0 s_1 \cdots s_n = 0, \\ &\text{(vii)} \quad ds_i = 2v_0 + 2v_1 + \cdots + 2v_{i-1} + v_i \quad \text{for } i = 0, 1, \dots, n, \\ &\text{(viii)} \quad dv_i = 0. \end{aligned} \tag{6.7}$$

Let us denote by (A_n, d) the graded differential algebra with generators $v_0, \dots, v_n$ and $s_0, \dots, s_n$ satisfying relations (i), (ii), (iv), and (v). Thus we simply ignore relations (iii) and (vi).

The differential $d : A_n \to A_n$ is given by formulae (vii) and (viii). Note that the ideal generated by the relations (i), (ii), (iv), and (v) is invariant under the differential d; for example, $d(v_i s_i) = v_i(2v_0 + \cdots + 2v_{i-1} + v_i)$ belongs to the ideal because of relation (ii). Thus $d : A_n \to A_n$ is well defined.

LEMMA 10
$H^j(A_n, d) = 0$ for all $j > 0$.

Proof
Using relations (i), (ii), (iv), and (v), we see that the additive basis of A_n is given by monomials of the form $v_I s_J$, where

$$I,\ J \subset \{0, 1, \dots, n\}, \quad I \cap J = \emptyset,$$

are disjoint multi-indices. Hence it is clear that for $j < n$ the differential algebra A_j can be embedded into A_n; in fact, A_j may be identified with the subalgebra generated by $s_0, \dots, s_j$ and $v_0, \dots, v_j$.

The factor A_j/A_{j-1} has a very simple structure. Each element $a \in A_j/A_{j-1}$ has a unique representation of the form $a = s_j x + v_j y$, where $x, y \in A_{j-1}$. From formula (v) we obtain that the differential of A_j/A_{j-1} acts as follows: $da = v_j x - s_j\, d(x) + v_j\, d(y)$. Hence $da = 0$ is equivalent to $x + dy = 0$, which implies that $a = d(s_j y)$. Thus we obtain that each factor A_j/A_{j-1} is acyclic.

The statement of the lemma now follows by induction. □

Consider now the homomorphism $\delta_n : A_n \to A_n$ given by multiplication by $v_0 + v_1 + \cdots + v_n$; that is,

$$\delta_n(x) = (v_0 + v_1 + \cdots + v_n)x, \quad x \in A_n.$$

Using relations (i) and (ii), one obtains $\delta_n^2 = 0$; that is, δ_n may be viewed as a new differential on A_n. Note that δ_n increases the total grading by m.

LEMMA 11
One has

$$H^j(A_n, \delta_n) = \begin{cases} 0 & \text{for } j \neq (n+1)(m-1), \\ \mathbf{k} & \text{for } j = (n+1)(m-1), \end{cases}$$

and the product $s_0 s_1 \cdots s_n \in A_n$ is a cocycle (with respect to δ_n) representing a nontrivial cohomology class.

Proof
We use induction on n. The statement is trivial when $n = 0$. Let us assume that it is true for $n-1$. Consider the homomorphism

$$\phi : A_{n-1} \to A_n, \qquad \phi(x) = v_n \cdot x, \quad x \in A_{n-1}.$$

It is clear that ϕ is injective and increases the total degree by m. Using relation (ii), one finds

$$\begin{aligned}\delta_n(\phi(x)) &= (v_0 + \cdots + v_n)v_n x \\ &= -v_n(v_0 + \cdots + v_{n-1}) \cdot x \\ &= -\phi\big(\delta_{n-1}(x)\big).\end{aligned}$$

Hence we obtain a short exact sequence

$$0 \to A_{n-1} \xrightarrow{\phi} A_n \to A_n/\phi(A_{n-1}) \to 0$$

and a long homological sequence

$$\xrightarrow{\phi} H^j(A_n, \delta_n) \to H^j\big(A_n/\phi(A_{n-1}), \delta_n\big) \xrightarrow{\kappa} H^j(A_{n-1}, \delta_{n-1}) \to \cdots .$$

We show that the connecting homomorphism

$$\kappa : H^j\big(A_n/\phi(A_{n-1}), \delta_n\big) \to H^j(A_{n-1}, \delta_{n-1})$$

is an isomorphism for all $j \neq (n+1)(m-1)$ and that is an epimorphism with one-dimensional kernel for $j = (n+1)(m-1)$. This clearly implies the statement of the lemma.

Any element $a \in A_n/\phi(A_{n-1})$ has a unique representation of the form

$$a = x + s_n y, \quad x, y \in A_{n-1}.$$

Then $\delta_{n-1}(a) \in A_n/\phi(A_{n-1})$ equals $\delta_n(x) - s_n\delta_{n-1}(y)$, and hence we obtain

$$H^j\big(A_n/\phi(A_{n-1}), \delta_n\big) \simeq H^j(A_{n-1}, \delta_{n-1}) \oplus H^{j-m+1}(A_{n-1}, \delta_{n-1}),$$

where the first summand corresponds to the class of x and the second summand corresponds to the class of y.

Suppose that a is a cycle of the relative complex $A_n/\phi(A_{n-1})$. In order to calculate $\kappa(a)$, the image under the connecting homomorphism, we have to view $a = x + s_n y$ as a chain in A_n and compute $\delta_n(a) \in A_n$. We obtain $\delta_n(a) = \phi(a)$, which shows that κ is always an epimorphism and that it is an isomorphism if and only if $H^{j-m+1}(A_{n-1}, \delta_{n-1}) = 0$; by our induction hypothesis, this holds if $j - m + 1 \neq n(m-1)$.

This completes the proof of Lemma 11. □

Let $\mathscr{I}_n \subset A_n$ and $\mathscr{K}_n \subset A_n$ denote the image and the kernel of $\delta_n : A_n \to A_n$. Note that $\mathscr{I}_n \subset \mathscr{K}_n$ and that by Lemma 11 the factor $\mathscr{K}_n/\mathscr{I}_n$ is one-dimensional (generated by the product $s_0 s_1 \cdots s_n$). Hence we obtain

$$H^j(\mathscr{I}_n, d) \simeq H^j(\mathscr{K}_n, d), \quad j \neq (n+1)(m-1),$$

and $H^{(n+1)(m-1)}(\mathscr{I}_n, d) = 0$.

From Lemma 10 and the short exact sequence

$$0 \to \mathscr{K}_n \to A_n \xrightarrow{\delta_n} \mathscr{I}_n \to 0,$$

we obtain isomorphisms

$$H^{j+m-1}(\mathscr{I}_n, d) \simeq H^j(\mathscr{K}_n, d)$$

for all $j > 1$. This gives periodicity

$$H^j(\mathscr{I}_n, d) \simeq H^{j+m-1}(\mathscr{I}_n, d) \quad \text{for all } j \neq 1, \;\; j \neq (n+1)(m-1).$$

On the other hand, it is obvious that for $1 < j < 2m-1$ the cohomology $H^j(\mathscr{I}_n, d)$ vanishes unless $j = m$ and, for $j = m$, it is one-dimensional (generated by the class $v_0 + v_1 + \cdots + v_n$). This shows that

$$\dim H^j(\mathscr{I}_n, d) = \begin{cases} 1 & \text{for } j = i(m-1)+1, \;\; 1 \leq i \leq n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 10, we get

$$\dim H^j(A_n/\mathscr{I}_n, d) = \begin{cases} 1 & \text{for } j = i(m-1), \;\; 0 \leq i \leq n+1, \\ 0 & \text{otherwise.} \end{cases}$$

6.4. End of the proof of Theorem 8 for m even

The differential algebra (E_m, d) is obtained from (A_n, d) by adding relations (iii) and (vi) of (6.7); therefore (E_m, d) is obtained from $(A_n/\mathscr{I}_n, d)$ by adding relation (vi) of (6.7). We know that algebra $H^*(A_n/\mathscr{I}_n, d)$ is generated by $\sigma_1, \ldots, \sigma_n$, where $\deg(\sigma_i) = i(m-1)$. It is clear that the product $s_0 s_1 \cdots s_n$ is a nontrivial cycle of $A_n/\mathscr{I}_n$ having degree $(n+1)(m-1)$. Comparing all this information, we conclude that the classes $\sigma_1, \ldots, \sigma_n$ form an additive basis of $H^*(E_m, d) = E_{m+1}$. All further differentials d_r with $r > m$ vanish.

This clearly concludes the proof of Theorem 8 for m even, and Theorem 8 is completely proven. □

7. Proof of Theorem 1

For $m > 1$, Theorem 1 follows from Corollary 5 and Theorem 8. If $m > 1$ is odd, we obtain a nonzero power

$$\sigma_1^n = n! \cdot \sigma_n \neq 0 \in H^{n(m-1)}\big(G(S^m; A, B, n); \mathbf{k}\big),$$

where $\mathbf{k}$ is a field of characteristic zero. Hence the cohomological cup length of $G(S^m, n)$ with $\mathbf{k}$ coefficients is at least n, and hence the Lusternik-Schnirelman category of $G(S^m, n)$ is at least $n+1$.

For m even we use Theorem 8. It gives a nontrivial cup product

$$\begin{cases} \sigma_2^{n/2} = \left(\frac{n}{2}\right)! \cdot \sigma_n & \text{if } n \text{ is even,} \\ \sigma_1 \sigma_2^{(n-1)/2} = \left[\frac{n}{2}\right]! \cdot \sigma_n & \text{if } n \text{ is odd.} \end{cases}$$

Hence we obtain that the Lusternik-Schnirelman category of $G(S^m; A, B, n)$ is at least $[(n+1)/2] + 1$.

In the case $m = 1$, we may use a direct argument. We may identify S^1 with the unit circle on the complex plane $\mathbf{C}$. Then a configuration $(x_1, \dots, x_n) \in G(S^1; A, B, n)$ (where we assume that $A \neq B$) can be described by a point of the open n-dimensional unit cube $(\phi_1, \dots, \phi_n) \in (0,1)^n$, such that

$$x_1 = A \exp(2\pi i \phi_1) \qquad \text{and} \qquad x_j = x_{j-1} \exp(2\pi i \phi_j) \quad \text{for } j = 2, \dots, n.$$

If $\psi \in (0,1)$ is such that $B = A \exp(2\pi i \psi)$, then a point $(\phi_1, \dots, \phi_n) \in (0,1)^n$ corresponds to a configuration of the open string configuration space $G(S^1; A, B, n)$ if and only if $\sum_{j=1}^n \phi_j - \psi$ is not an integer. The hyperplanes

$$\sum_{j=1}^n \phi_j = \psi + k, \quad \text{where } k = 0, 1, \dots, n-1,$$

divide the cube $(0,1)^n$ into $n+1$ connected components, each being convex and hence contractible. We obtain that the configuration space $G(S^1; A, B, n)$ has $n+1$ path-connected components and each is contractible. This gives

$$\operatorname{cat}\big(G(S^1; A, B, n)\big) = n + 1,$$

and our statement follows from Corollary 5. □

8. Cyclic configuration spaces of spheres and loop spaces

In this section we show that the open string configuration space of the sphere S^m is homotopy equivalent to an appropriate skeleton of the space of based loops on S^m. Hence the configuration space $G(S^m; A, B, n)$ serves as a finite-dimensional approximation to ΩS^m.

Let $\Omega(S^m; A, B)$ denote the space of all H^1-paths $\gamma : [0,1] \to S^m$ starting at a point $A \in S^m$ and ending at a point $B \in S^m$. We refer to [12, Chapter 1] and to [11, Chapter 5] for definitions. For a path $\gamma \in \Omega(S^m; A, B)$, we denote by $\ell(\gamma)$ the length of γ, that is, $\ell(\gamma) = \int_0^1 |\dot\gamma(\xi)|\, d\xi$.

Given points $A, B \in S^m$ and an integer $n > 0$, we denote by $\Omega_n \subset \Omega(S^m; A, B)$ the subspace of all paths having length less than $n\pi$.

G_n denotes $G(S^m; A, B', n-1)$, where $B' = (-1)^n B$.

We assume below that $A \neq B$ and $A \neq -B$.

THEOREM 12
There is a homotopy equivalence $G_n \simeq \Omega_n$.

Proof
First we describe a continuous map $\psi : G_n \to \Omega_n$. Let $c = (x_1, \dots, x_{n-1}) \in G_n$ be a cyclic configuration. Define a sequence $y_0, y_1, \dots, y_n$ of points of S^m, where $y_0 = A$, $y_n = B$, and $y_i = (-1)^i x_i$ for $i = 1, \dots, n-1$. Let $l_i < \pi$ denote the length of the shortest arc between y_i and y_{i+1}. Combining these arcs, we obtain a broken geodesic curve of length $L = l_0 + l_1 + \cdots + l_{n-1}$ starting at A and ending at B. Note that $L \neq 0$ thanks to our assumption $A \neq \pm B$. The path $\psi(c) \in \Omega_n$ is obtained by passing this curve with constant velocity L^{-1}. In particular,

$$\psi(c)\big((l_0 + l_1 + \cdots + l_{i-1})L^{-1}\big) = y_i.$$

Now we describe a map $\phi : \Omega_n \to G_n$. Let $\gamma \in \Omega_n$, $\gamma : [0,1] \to S^m$. There exist numbers $t_0 = 1 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that the length of γ between the points $\gamma(t_i)$ and $\gamma(t_{i+1})$ equals $\ell(\gamma)/n$. The numbers t_i may be nonunique since there could be intervals where the velocity $\dot\gamma$ is identically zero. However, the points $\gamma(t_i) \in S^m$ of the sphere are uniquely determined by path γ; moreover, $\gamma(t_i)$ depends continuously on γ. We define

$$\phi(\gamma) = (x_1, \dots, x_{n-1}) \in G_n,$$

where

$$x_i = (-1)^i \gamma(t_i), \quad i = 1, \dots, n-1.$$

Condition $x_i \neq x_{i+1}$ follows since we assume that $\ell(\gamma) < n\pi$, and hence the length of the partial curve $\gamma|_{[t_i, t_{i+1}]}$ is less than π.

Let us show that the composition $\phi \circ \psi : G_n \to G_n$ is homotopic to the identity map. Let $c = (x_1, \dots, x_{n-1}) \in G_n$ be a configuration. Then $\psi(c)$ is a curve with a constant velocity which combines the geodesic arcs between the points $(-1)^i x_i$ and $(-1)^{i+1} x_{i+1}$. A homotopy $h_\tau : G_n \to G_n$, where $\tau \in [0,1]$, may be defined by

$$h_\tau(x_1, \dots, x_{n-1}) = \big(z_1(\tau), \dots, z_{n-1}(\tau)\big),$$

where $(-1)^i z_i$ is the point on the path $\psi(c)$ which is

$$(1-\tau)\cdot(l_0+\cdots+l_{i-1})+\tau\cdot i\cdot\ell\big(\psi(c)\big)\cdot n^{-1}$$

distance away from A along $\psi(c)$. Here l_i denotes the length of the shortest arc between x_i and $-x_{i+1}$, and $\ell(\psi(c)) = l_0+\cdots+l_{n-1}$ is the length of $\psi(c)$. It is clear that the distance between $z_i(\tau)$ and $-z_{i+1}(\tau)$ along $\psi(c)$ is less than π and hence that these points are not antipodal. This shows that $z_i(\tau)\neq z_{i+1}(\tau)$ for all $i=0,1,\dots,n-1$. Clearly, $h_0=\mathrm{id}$ and $h_1=\phi\circ\psi$.

We are left to show existence of a homotopy $\psi\circ\phi\simeq 1:\Omega_n\to\Omega_n$. We construct it in three steps. Given a path $\gamma\in\Omega_n$, denote by $s_\gamma:[0,1]\to[0,1]$ its length function

$$s_\gamma(t)=\ell(\gamma)^{-1}\cdot\int_0^t|\dot\gamma(\xi)|\,d\xi.$$

There is a unique path $r_\gamma:[0,1]\to S^m$ such that $r_\gamma(s_\gamma(t))=\gamma(t)$ for all $t\in[0,1]$. Formally, we may write $r_\gamma=\gamma\circ s_\gamma^{-1}$; although the inverse function s_γ^{-1} may be multivalued, the path r_γ is single-valued and satisfies the Lipschitz condition with constant $\ell(\gamma)$. Hence r_γ belongs to H^1. Geometrically, the curve r_γ is the same curve γ viewed with the natural parametrization. It has been proven by D. Anosov (cf. [1, Theorems 2 and 3]) that

(1) *the map $\Omega_n\to\Omega_n$, sending γ to r_γ, is continuous*;

(2) *there exists a homotopy*

$$\Pi_\tau:\Omega_n\to\Omega_n,\qquad \tau\in[0,1],$$

where Π_0 is the identity and $\Pi_1(\gamma)=r_\gamma$ for all $\gamma\in\Omega_n$.

Paper [1] deals with closed curves, but all the arguments of the proof (see [1, §§6 and 7]) apply without modifications to the case of curves with fixed endpoints. Observe also that the homotopy of [1, Theorem 3] (described in [1, §7])) preserves the lengths of the curves.

The path $\psi\circ\phi(\gamma)$ is a broken-line geodesic with constant velocity connecting the points

$$y_i=r_\gamma\Big(\frac{i}{n}\Big),\quad i=0,\dots,n,$$

and

$$y_i=\psi\circ\phi(\gamma)\Big(\frac{l_0+\cdots+l_{i-1}}{L}\Big).$$

Here l_j denotes the length of the shortest arc between y_j and y_{j+1}, and L denotes $l_0+\cdots+l_{n-1}$. We use the following well-known claim.

CLAIM

Let $p,q\in S^m$ be two points of the sphere with $\mathrm{dist}(p,q)<\pi$*. Consider the space P*

of all H^1-smooth paths $\gamma : [a, b] \to S^m$ starting at p, ending at q, and having the length less than π. Then there exists a homotopy $h_\tau : P \to P$, where $\tau \in [0, 1]$, such that $h_0 = \mathrm{id}$ and, for any $\gamma \in P$, the path $h_1(\gamma)$ is the geodesic arc of shortest length from p to q.

Applying this homotopy to the restrictions of r_γ on intervals $[i/n, (i+1)/n]$, where $i = 0, \dots, n-1$, we obtain a homotopy between Π_1 and the map $G : \Omega_n \to \Omega_n$, where for $\gamma \in \Omega_n$ the path $G(\gamma) : [0, 1] \to S^m$ is the broken geodesic with vertices at the points $G(\gamma)(i/n) = r_\gamma(i/n)$.

In the third and final step we describe a homotopy $H_\tau : \Omega_n \to \Omega_n$ between the maps G and $\psi \circ \phi$. It may be defined by setting $H_\tau(\gamma)(t) = G(\gamma)(\sigma_\tau(t))$, where $\sigma_\tau : [0, 1] \to [0, 1]$ is the piecewise linear homeomorphism given by the formula

$$\sigma_\tau(t) = (1-\tau)t + \tau\big[l_0 + l_1 + \cdots + l_{i-1} + l_i(tn - i)\big] \cdot L^{-1}$$

for $i/n \le t \le (i+1)/n$, and $\tau \in [0, 1]$. Then $H_0 = G$ and $H_1 = \psi \circ \phi$. Theorem 12 is proven. □

Remark 13

Theorem 12 leads to a different proof of Theorem 8. Indeed, the space $\Omega(S^m; A, B)$ is homotopy equivalent to the space of based loops ΩS^m, and the Morse theory (see, e.g., [5]) shows that Ω_n is homotopy equivalent to the $((n-1)(m-1))$-dimensional skeleton of ΩS^m. Combining this with Serre's famous calculation (see [16]) of the cohomology of ΩS^m gives Theorem 8.

This approach does not, however, give the result of Remark 9 relating the generators of the cohomology of the cyclic configuration space of the sphere with the standard generators of the cohomology algebra of cyclic configuration spaces of the Euclidean space. This result will be used in the second part of this paper.

Acknowledgments. I would like to thank S. Tabachnikov for useful discussions.

I am grateful to the anonymous referee for deep remarks and help. The referee's remarks were essential in finding the statement of Theorem 12 and its proof. Therefore I consider Theorem 12 a joint result with the referee.

References

[1] D. V. ANOSOV, *Some homotopies in a space of closed curves,* Math. USSR-Izv. **17** (1981), 423–453. MR 83i:58028 583

[2] I. BABENKO, *Periodic trajectories of three-dimensional Birkhoff billiards,* Math. USSR-Sb. **71**, no. 1 (1992), 1–13. MR 91m:58128 559, 563

[3] G. BIRKHOFF, *Dynamical Systems*, Amer. Math. Soc. Colloq. Publ. **9**, Amer. Math. Soc., Providence, 1966. MR 35:1

[4] A. BOREL, *Seminar on Transformation Groups*, Ann. of Math. Stud. **46**, Princeton Univ. Press, Princeton, 1960. MR 22:7129 565

[5] R. BOTT, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313 – 337. MR 22:987 584

[6] ———, *Lectures on Morse theory, old and new*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 331 – 358. MR 84m:58026a

[7] F. COHEN, "Artin's braid groups, classical homotopy theory, and sundry other curiosities" in *Braids (Santa Cruz, Calif., 1986)*, Contemp. Math. **78**, Amer. Math. Soc., Providence, 1988, 167 – 206. MR 90a:20077

[8] ———, *On configuration spaces, their homology, and Lie algebras*, J. Pure Appl. Algebra **100** (1995), 19 – 42. MR 96d:55005

[9] H. CROFT and H. SWINNERTON-DYER, *On the Steinhaus billiard table problem*, Proc. Cambridge Philos. Soc. **59** (1963), 37 – 41. MR 26:2925 563

[10] M. FARBER and S. TABACHNIKOV, *Topology of cyclic configuration spaces and periodic trajectories of multi-dimensional billiards*, Topology **41** (2002), 553 – 589. CMP 1 910 041 559, 560, 562, 563, 564, 565, 569, 570

[11] J. JOST, *Riemannian Geometry and Geometric Analysis*, Universitext, Springer, Berlin, 1995. MR 96g:53049 582

[12] W. KLINGENBERG, *Lectures on Closed Geodesics*, Grundlehren Math. Wiss. **230**, Springer, Berlin, 1978. MR 57:17563 582

[13] V. V. KOZLOV and D. V. TRESHCHEV, *Billiards: A Genetic Introduction to the Dynamics of Systems with Impacts*, Transl. Math. Monogr. **89**, Amer. Math. Soc., Providence, 1991. MR 93k:58094a 563

[14] J. MILNOR, *Morse Theory*, Ann. of Math. Stud. **51**, Princeton Univ. Press, Princeton, 1973. MR 29:634

[15] M. MORSE and E. BAIADA, *Homotopy and homology related to the Schoenflies problem*, Ann. of Math. (2) **58** (1953), 142 – 165. MR 15:146g 564

[16] J.-P. SERRE, *Homologie singulière des espaces fibrés: Applications*, Ann. of Math. (2) **54** (1951), 425 – 505. MR 13:574g 584

[17] S. TABACHNIKOV, *Billiards*, Panor. Synth. **1**, Soc Math. France, Montrouge, 1995. MR 96c:58134

[18] B. TOTARO, *Configuration spaces of algebraic varieties*, Topology **35** (1996), 1057 – 1067. MR 97g:57033 564, 567

Department of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel; mfarber@post.tau.ac.il