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HIGHER-LEVEL APPELL FUNCTIONS, MODULAR TRANSFORMATIONS, AND CHARACTERS

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Боре Фейгину по случаю его пятидесятилетия

ABSTRACT. We study modular transformation properties of a class of indefinite theta series involved in characters of infinite-dimensional Lie superalgebras. The *level-ℓ Appell functions* $\mathcal{K}_\ell$ satisfy open quasiperiodicity relations with additive theta-function terms emerging in translating by the "period." Generalizing the well-known interpretation of theta functions as sections of line bundles, the $\mathcal{K}_\ell$ function enters the construction of a section of a rank-$(\ell+1)$ bundle $\mathbb{V}_{\ell,\tau}$. We evaluate modular transformations of the $\mathcal{K}_\ell$ functions and construct the action of an $SL(2,\mathbb{Z})$ subgroup that leaves the section of $\mathbb{V}_{\ell,\tau}$ constructed from $\mathcal{K}_\ell$ invariant.

Modular transformation properties of $\mathcal{K}_\ell$ are applied to the affine Lie superalgebra $\widehat{s\ell}(2|1)$ at rational level $k>-1$ and to the $N=2$ super-Virasoro algebra, to derive modular transformations of "admissible" characters, which are not periodic under the spectral flow and cannot therefore be rationally expressed through theta functions. This gives an example where constructing a modular group action involves extensions among representations in a nonrational conformal model.

1. INTRODUCTION

We generalize some elements of the theta-function theory by studying modular transformations of functions that are not doubly quasiperiodic in a variable $\mu\in\mathbb{C}$. Such functions emerge in the study of characters of representations in (nonrational) conformal field theory models based on Lie superalgebras, which motivates investigation of their modular properties.

A modular group representation associated with characters of a suitable set of representations is a fundamental property of conformal field theory models, related to the fusion algebra by the Verlinde formula, via the argument traced to the consistency of gluing a three-punctured sphere into a one-punctured torus — in fact, to consistency of conformal field theory itself [1, 2]. Strictly speaking, this applies to rational conformal field theories, where modular properties of the characters and the structure of the Verlinde formula are known, at least in principle (for a discussion of the modular transformation properties of characters and other quantities and for further references, see [3, 4, 5, 6, 7]). Modular behavior of theta functions can be considered a basic feature underlying good

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modular properties in rational models (in particular, the well-known modular group representation on a class of characters of affine Lie algebras [8]); it is deeply connected with quasiperiodicity of theta functions and hence of the characters in rational models.

But characters that are not quasiperiodic (are not invariant under lattice translations, often called “spectral flows” in that context) cannot be rationally expressed through theta functions. Such characters often occur in nonrational conformal field theory models (an infinite orbit of the spectral flow transform already implies that the theory is nonrational). Modular properties of such characters present a problem both technically (the theta-function theory is of little help) and conceptually (it is unclear what kind of modular invariance is to be expected at all).

On the other hand, the paradigm that any consistent conformal field theory must be related to a modular group representation, even beyond the class of rational theories, motivates studying modular behavior of nonrational characters and, on the technical side, seeking an adequate “replacement” of theta functions with some functions that are not quasiperiodic but nevertheless behave reasonably under modular transformations and can be used as “building blocks” of the characters. Such functions are to be found among indefinite theta series (see [9, 10, 11, 12] and references therein).

In this paper, we study the modular (and other related) properties of *higher-level Appell functions* — a particular instance of indefinite theta series, not-double-quasiperiodic functions involved in the characters of modules of the $\widehat{s\ell}(2|1)$ affine Lie superalgebra [13] and the $N=2$ and $N=4$ superextensions of the Virasoro algebra ([14, 15]). Remarkably, the pattern of modular behavior established for the Appell functions is then reproduced by the characters. For a positive integer ℓ, we define the level-ℓ Appell function as

$$\mathcal{K}_\ell(\tau,\nu,\mu)=\sum_{m\in\mathbb{Z}}\frac{e^{i\pi m^2\ell\tau+2i\pi m\ell\nu}}{1-e^{2i\pi(\nu+\mu+m\tau)}},\qquad \tau\in\mathfrak{h},\qquad \begin{array}{l}\nu,\mu\in\mathbb{C},\\ \mu+\nu\notin\mathbb{Z}\tau+\mathbb{Z}.\end{array}\tag{1.1}$$

Theorem 1.1. *The level-ℓ Appell function $\mathcal{K}_\ell$ satisfies the relations*

$$\mathcal{K}_\ell(\tau+1,\nu,\mu)=\begin{cases}\mathcal{K}_\ell(\tau,\nu+\frac{1}{2},\mu-\frac{1}{2}), & \ell\quad \textit{odd},\\ \mathcal{K}_\ell(\tau,\nu,\mu), & \ell\quad \textit{even},\end{cases}\tag{1.2}$$

and

$$\mathcal{K}_\ell(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau})=\tau e^{i\pi\ell\frac{\nu^2-\mu^2}{\tau}}\,\mathcal{K}_\ell(\tau,\nu,\mu)\\ +\tau\sum_{a=0}^{\ell-1}e^{i\pi\frac{\ell}{\tau}(\nu+\frac{a}{\ell}\tau)^2}\,\Phi(\ell\tau,\ell\mu-a\tau)\vartheta(\ell\tau,\ell\nu+a\tau),\tag{1.3}$$

where

$$\Phi(\tau,\mu)=-\frac{i}{2\sqrt{-i\tau}}-\frac{1}{2}\int_{\mathbb{R}}dx\,e^{-\pi x^2}\,\frac{\sinh\bigl(\pi x\sqrt{-i\tau}(1+2\frac{\mu}{\tau})\bigr)}{\sinh\bigl(\pi x\sqrt{-i\tau}\bigr)}\tag{1.4}$$

(we refer to (1.9)–(1.16) for the theta-function notation).

The simplest, level-1 Appell function $\kappa(x,y;q)=\mathcal{K}_1(q,y^{-1}q,xyq^{-1})$ has appeared in [16, 17], and its S-transformation properties were formulated in [16] as the statement that the difference between κ and its S-transform is divisible by the theta function $\vartheta(\tau,\mu)$. Theorem **1.1** generalizes this to $\ell>1$ and in addition gives an integral representation of the function Φ accompanying the theta-functional terms in the modular transform. This integral representation allows studying the Φ function, which is an important ingredient of the theory of higher-level Appell functions, similarly to Barnes-related functions arising elsewhere [18, 19, 20, 21, 22, 23]. We derive functional equations satisfied by Φ and the formula for its S-transformation.

Already with theta functions, their modular properties are closely related to (and can in fact be derived from) their quasiperiodicity under lattice translations, which in geometric terms means that the theta function $\vartheta(\tau,\,\cdot\,)$ represents a section of a line bundle over the torus determined by the modular parameter τ (*hence* the dependence on the second argument $\nu\in\mathbb{C}$ is doubly quasiperiodic). With the $\mathcal{K}_\ell$ functions, which are no longer doubly quasiperiodic, the geometric counterpart of "open quasiperiodicity" (a[dditive]-quasiperiodicity in [24]) involves sections of rank-$(\ell+1)$ bundles. The simplest Appell function κ in [16, 17] satisfies an open quasiperiodicity relation with an additive *theta-functional* term arising in shifting the argument by the "period,"

$$\kappa(zq,a;q)=a\kappa(z,a;q)+\vartheta(q,z), \tag{1.5}$$

which implies that $(\kappa(\,\cdot\,,a;q),\vartheta(q,\,\cdot\,))$ represents a section of a rank-2 bundle over the elliptic curve [16]. Analogously, the higher-level Appell functions, in the "multiplicative" notation,[1] are quasiperiodic under $x\mapsto xq$ and satisfy an open quasiperiodicity relation with the inhomogeneous terms involving theta functions,

$$\mathcal{K}_\ell(q,x,y\,q)=q^{\frac{\ell}{2}}\,y^\ell\,\mathcal{K}_\ell(q,x,y)+\sum_{a=0}^{\ell-1}x^a\,y^a\,q^a\,\vartheta(q^\ell,x^\ell\,q^a). \tag{1.6}$$

This generalizes (1.5), to which (1.6) reduces for $\ell=1$ (but there also exists a "finer" property for $\ell>1$, see Sec. 2.1). The theta functions occurring in the right-hand side of the modular transform (1.3) are precisely those violating the quasiperiodicity of $\mathcal{K}_\ell$ in (1.6). Together with $\mathcal{K}_\ell$, these theta functions enter the construction of a section of a rank-$(\ell+1)$ bundle,

$$(\mathcal{K}_\ell(\tau,\nu,\mu),\ C_1\vartheta(\ell\tau,\ell\nu),\ \ldots,\ C_\ell\vartheta(\ell\tau,\ell\nu+(\ell-1)\tau)),$$

with C_a such that the entire vector is invariant under a subgroup of lattice translations (see Lemma **3.1** in what follows). Moreover, it turns out that the action of a subgroup of the modular group can be defined on $(\ell+1)$-vectors $F=(f_0(\tau,\nu,\mu),f_1(\tau,\nu),\ldots,f_\ell(\tau,\nu))$ such that the above bundle section is an invariant of this action (Theorem **3.5**, which gives a more "invariant" formulation of the modular properties of the Appell functions). This

[1] We resort to the standard abuse of notation, by freely replacing functional arguments by their exponentials or conversely, by logarithms, via $q=e^{2i\pi\tau}$, $x=e^{2i\pi\nu}$, $y=e^{2i\pi\mu}$, etc.

gives an interesting realization of modular invariance with a *matrix* automorphy factor (cf. [25, 26]).

As noted above, the Appell functions are a specific example of indefinite theta series motivated by the study of characters. We use them to express the characters of "admissible" $\widehat{s\ell}(2|1)$-modules at rational level $k > -1$ and to study modular transformation properties of the characters. In this $\widehat{s\ell}(2|1)$ example, indeed, the higher-level Appell functions prove an adequate substitute for theta functions; their modular behavior described in Theorem **1.1** is essentially "inherited" by the admissible characters, whose S-transform is given by

$$\chi_A(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}) = e^{i\pi k \frac{\nu^2-\mu^2}{2\tau}} \sum_B S_{AB}\chi_B(\tau,\nu,\mu) + \sum_\alpha R_{A\alpha}(\tau,\nu,\mu)\,\Omega_\alpha(\tau,\nu,\mu), \tag{1.7}$$

where S_{AB} is a numerical matrix and the functions $R_{A\alpha}$ are expressed in terms of the above function Φ, and Ω_α are some characters expressed through theta functions, see Theorem **4.1** for the precise statement. This shows a triangular structure of the same type as in (1.3): the additional elements Ω_α occurring in the unconventional S-transform formula of the χ_A (which are not quasiperiodic and hence cannot be rationally expressed through theta functions) are expressible in terms of theta functions and are therefore quasiperiodic under lattice shifts and carry a modular group action.

The theta-functional terms Ω_α turn out to be the characters of certain *extensions* among the admissible $\widehat{s\ell}(2|1)$-representations. A key feature underlying most of the unconventional properties of a number of nonrational conformal field theories is that the irreducible representations allow nontrivial extensions among themselves (by which we mean nonsplittable short exact sequences, or actually the middle modules in such sequences). Such extensions do not occur in rational conformal field theory models.

Behavior of the admissible $\widehat{s\ell}(2|1)$-characters under modular transformations is related to their behavior under *spectral flow transformations*, i.e., a representation of a lattice Λ whose elements $\vec{\theta}$ act via adjoint representation of the group elements $\exp(\vec{\theta}\cdot\vec{h})$, where $\vec{h}$ are Cartan subalgebra elements.[2] The role of spectral flows appears to originate in the fact that the fundamental group of the appropriate moduli space is not just $SL(2,\mathbb{Z})$, but its semidirect product with a lattice. (Lattice translations also require considering the so-called Ramond and Neveu–Schwarz "sectors" and supercharacters.) The admissible $\widehat{s\ell}(2|1)$-characters at the level $k = \frac{\ell}{u} - 1$ acquire additional theta-functional terms under the spectral flow transform $\mathcal{U}_\theta$ with $\theta = u$,

$$(\mathcal{U}_u\chi_A)(\tau,\nu,\mu) = \chi_A(\tau,\nu,\mu) + \sum_\alpha f_\alpha(\tau,\nu,\mu)\,\Omega_\alpha(\tau,\nu,\mu),$$

where $f_\alpha(\tau,\nu,\mu)$ are some trigonometric functions and Ω_α are the same as in the modular transform formula (1.7). This demonstrates an obvious similarity to the properties of the Appell functions (the same theta functions occurring in (1.3) and (1.5)).

[2]The term spectral flow transform is taken over from the $N{=}2$ super-Virasoro algebra [27].

This paper is organized as follows. In Sec. 2, we study the level-ℓ Appell functions $\mathcal{K}_\ell$. The basic quasiperiodicity and some other properties are derived in Sec. 2.1. The $\mathcal{K}_\ell$ functions satisfy several "period multiplication" formulas, which we derive in Sec. 2.2. Formulas relating the higher-level Appell functions to theta functions are given in Sec. 2.3. In Sec. 2.4, we then derive modular transformation properties of the higher-level Appell functions using their *integral* representation through theta functions. The Appell functions of the lower levels are briefly considered in Sec. 2.5. In Sec. 2.6, we next consider the Φ function arising in the modular transformation of $\mathcal{K}_\ell$; its properties are in some respects analogous to the properties of $\mathcal{K}_\ell$. The geometric point of view on the higher-level Appell functions is outlined in Sec. 3. There, we first (in Sec. 3.1) consider how the $\mathcal{K}_\ell$ function and the appropriate theta functions are combined to produce a section of a rank-$(\ell+1)$ bundle. In Sec. 3.2, we then derive the action of a subgroup of $SL(2,\mathbb{Z})$ on these sections (Theorem **3.5**).

In Sec. 4, we use the established properties of the higher-level Appell functions to evaluate modular transformation properties of the class of "admissible" $\widehat{s\ell}(2|1)$-characters. The main result (Theorem **4.1**) is formulated in Sec. 4.1. The characters are expressed through the higher-level Appell functions in Sec. 4.2. Their S-transformation formula is derived in Sec. 4.3. Application of the higher-level Appell functions to $N=2$ and $N=4$ super-Virasoro theories is outlined in Sec. 5.

In Appendix A, we evaluate several useful contour integrals over the torus involving theta and Appell functions. In Appendix B, we recall the $\widehat{s\ell}(2|1)$ affine Lie superalgebra, consider its automorphisms (Sec. B.1) and define some of its modules (Sec. B.2), and finally give the admissible representation characters (Theorem **B.1**, Sec. B.3). The different "sectors" and the corresponding characters are given in Sec. B.4.

Notation. We let $\mathfrak{h}$ denote the upper complex half-plane. The group $SL(2,\mathbb{Z})$ is generated by the two matrices

$$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

with the relations

$$S^2=(ST)^3=(TS)^3=C,$$

where $C^2=\mathbf{1}$. The standard $SL(2,\mathbb{Z})$ action on $\mathfrak{h}\times\mathbb{C}^2$ is

$$\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}:(\tau,\nu,\mu)\mapsto(\gamma\tau,\gamma\nu,\gamma\mu)=\Bigl(\frac{a\tau+b}{c\tau+d},\frac{\nu}{c\tau+d},\frac{\mu}{c\tau+d}\Bigr)\tag{1.8}$$

(where the notation $\gamma\nu$ and $\gamma\mu$ is somewhat loose, because this action depends on τ).

We use the classical theta functions

$$\vartheta_{1,0}(q,z)=\sum_{m\in\mathbb{Z}}q^{\frac{1}{2}(m^2-m)}z^{-m}=\prod_{m\geqslant0}(1+z^{-1}q^m)\prod_{m\geqslant1}(1+zq^m)\prod_{m\geqslant1}(1-q^m),\tag{1.9}$$

$$\vartheta_{1,1}(q,z)=\sum_{m\in\mathbb{Z}}q^{\frac{1}{2}(m^2-m)}(-z)^{-m}=\prod_{m\geqslant 0}(1-z^{-1}q^m)\prod_{m\geqslant 1}(1-zq^m)\prod_{m\geqslant 1}(1-q^m), \tag{1.10}$$

$$\vartheta(q,z)\equiv\vartheta_{0,0}(q,z)=\sum_{m\in\mathbb{Z}}q^{\frac{m^2}{2}}z^m=\vartheta_{1,0}(q,z\,q^{-\frac{1}{2}}). \tag{1.11}$$

Their S transformations are given by

$$\begin{aligned}\vartheta_{1,1}(-\frac{1}{\tau},\frac{\nu}{\tau})&=-i\sqrt{-i\tau}\,e^{i\pi\nu+i\pi\frac{1}{\tau}(\nu-\frac{1}{2})^2+\frac{i\pi}{4}\tau}\,\vartheta_{1,1}(\tau,\nu),\\ \vartheta_{1,0}(-\frac{1}{\tau},\frac{\nu}{\tau})&=\sqrt{-i\tau}\,e^{i\pi\frac{(\nu-\frac{1}{2})^2}{\tau}}\,\vartheta_{1,0}(\tau,\nu+\frac{1}{2}-\frac{\tau}{2}),\end{aligned} \tag{1.12}$$

and

$$\vartheta(-\frac{1}{\tau},\frac{\nu}{\tau})=\sqrt{-i\tau}\,e^{i\pi\frac{\nu^2}{\tau}}\,\vartheta(\tau,\nu). \tag{1.13}$$

The eta function

$$\eta(q)=q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m) \tag{1.14}$$

transforms as

$$\eta(\tau+1)=e^{\frac{i\pi}{12}}\eta(\tau),\qquad\eta(-\frac{1}{\tau})=\sqrt{-i\tau}\,\eta(\tau). \tag{1.15}$$

In a different theta-functional nomenclature, one introduces the higher-level theta functions

$$\theta_{r,\ell}(q,z)=\sum_{j\in\mathbb{Z}+\frac{r}{2\ell}}q^{\ell j^2}z^{\ell j}=z^{\frac{r}{2}}\,q^{\frac{r^2}{4\ell}}\,\vartheta(q^{2\ell},z^\ell q^r). \tag{1.16}$$

Either θ or ϑ turn out to be more convenient depending on circumstances.

For a positive integer p, we use $[x]_p$ to denote $x \bmod p = x - p\left\lfloor\frac{x}{p}\right\rfloor$, where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

2. Higher-Level Appell Functions

2.1. Open quasiperiodicity and other basic properties. For $\ell\in\mathbb{N}$, the level-ℓ Appell function

$$\mathcal{K}_\ell(q,x,y)=\sum_{m\in\mathbb{Z}}\frac{q^{\frac{m^2\ell}{2}}x^{m\ell}}{1-x\,y\,q^m}, \tag{2.1}$$

generalizes the Appell function κ in [16, 17]. Along with (2.1), there is another, double-series representation for $\mathcal{K}_\ell$ (cf. [11, 28, 12])

$$\mathcal{K}_\ell(q,x,y)=\Bigl(\sum_{m\geqslant 0}\sum_{n\geqslant 0}-\sum_{m\leqslant -1}\sum_{n\leqslant -1}\Bigr)q^{\frac{m^2\ell}{2}+mn}x^{m\ell+n}y^n, \tag{2.2}$$

valid for $|q|<|xy|<1$.

The $\mathcal{K}_\ell$ functions satisfy an easily derived quasiperiodicity property in the second argument,

$$\mathcal{K}_\ell(q, xq^n, y) = q^{-\frac{n^2\ell}{2}}\, x^{-n\ell}\mathcal{K}_\ell(q, x, y), \quad n \in \mathbb{Z}, \tag{2.3}$$

and an "open quasiperiodicity" relation along the antidiagonal with respect to the second and third arguments,

$$\mathcal{K}_\ell(q, xq^{-\frac{n}{\ell}}, yq^{\frac{n}{\ell}}) = (xy)^n\mathcal{K}_\ell(q, x, y) + \begin{cases} \displaystyle\sum_{r=1}^{n}(xy)^{n-r}\vartheta(q^\ell, x^\ell q^{-r}), & n \in \mathbb{N},\\ \displaystyle-\sum_{r=n+1}^{0}(xy)^{n-r}\vartheta(q^\ell, x^\ell q^{-r}), & n \in -\mathbb{N}. \end{cases} \tag{2.4}$$

These imply open quasiperiodicity in the third argument,

$$\mathcal{K}_\ell(q, x, yq^n) = q^{\frac{n^2\ell}{2}}y^{n\ell}\mathcal{K}_\ell(q, x, y) + \begin{cases} \displaystyle\sum_{j=0}^{\ell n-1}x^j y^j q^{nj}\vartheta(q^\ell, x^\ell q^j), & n \in \mathbb{N},\\ \displaystyle-\sum_{j=\ell n}^{-1}x^j\, y^j q^{nj}\vartheta(q^\ell, x^\ell q^j), & n \in -\mathbb{N}, \end{cases}$$

or manifestly with only ℓ distinct theta functions in the right-hand side,

$$\begin{aligned}\mathcal{K}_\ell(q, x, yq^n) &= q^{\frac{n^2\ell}{2}}y^{n\ell}\mathcal{K}_\ell(q, x, y)\\ &+ \begin{cases} \displaystyle\sum_{j=0}^{n-1}q^{\frac{j(2n-j)\ell}{2}}y^{j\ell}\sum_{r=0}^{\ell-1}x^r y^r q^{(n-j)r}\vartheta(q^\ell, x^\ell q^r), & n \in \mathbb{N},\\ \displaystyle-\sum_{j=n}^{-1}q^{\frac{j(2n-j)\ell}{2}}y^{j\ell}\sum_{r=0}^{\ell-1}x^r\, y^r q^{(n-j)r}\vartheta(q^\ell, x^\ell q^r), & n \in -\mathbb{N}. \end{cases}\end{aligned} \tag{2.5}$$

There is the easily derived "inversion" property

$$\mathcal{K}_\ell(q, x, y) = -\mathcal{K}_\ell(q, x^{-1}, y^{-1}) + \vartheta(q^\ell, x^\ell) = -y^{-1}x^{-1}\mathcal{K}_\ell(q, x^{-1}q^{\frac{1}{\ell}}, y^{-1}q^{-\frac{1}{\ell}}). \tag{2.6}$$

We also note that in the exponential notation, there are the obvious relations

$$\mathcal{K}_\ell(\tau, \nu + m, \mu) = \mathcal{K}_\ell(\tau, \nu, \mu) = \mathcal{K}_\ell(\tau, \nu, \mu + m), \quad m \in \mathbb{Z}, \tag{2.7}$$

$$\mathcal{K}_\ell(\tau, \nu + \frac{m}{\ell}, \mu - \frac{m}{\ell}) = \mathcal{K}_\ell(\tau, \nu, \mu), \qquad m \in \mathbb{Z}. \tag{2.8}$$

2.2. "Scaling" formulas. The scaling ("period multiplication") formulas in this subsection are useful in studying modular transformations of functions expressed through $\mathcal{K}_\ell$.

We first recall the elementary theta-function identity

$$\vartheta(q, z) = \sum_{s=0}^{p-1}q^{\frac{s^2}{2}}\, z^s\vartheta(q^{p^2}, z^p q^{ps}) \tag{2.9}$$

and its version for $p = 2\ell u$ with *coprime* ℓ and u,

$$\vartheta(q,z) = \sum_{r''=1}^{2\ell}\sum_{s''=1}^{u} q^{\frac{1}{2}(ur''-\ell(s''-1))^2}\, z^{ur''-\ell(s''-1)}\vartheta\big(q^{(2u\ell)^2}, z^{2u\ell}q^{u\ell(ur''-\ell(s''-1))}\big) \tag{2.10}$$

(where we use double-primed variables in order to help identifying them in more complicated formulas below).

Similarly, there is an elementary identity expressing $\mathcal{K}_\ell(q,x,y)$ through $\mathcal{K}_\ell$ with the "period" q^{u^2} for an arbitrary positive integer u,

$$\mathcal{K}_\ell(q,x,y) = \sum_{a=0}^{u-1}\sum_{b=0}^{u-1} q^{\frac{a^2\ell}{2}+ab}x^{a\ell+b}y^b\mathcal{K}_\ell(q^{u^2}, x^u q^{ua+b\frac{u}{\ell}}, y^u q^{-b\frac{u}{\ell}}),\ u\in\mathbb{N}. \tag{2.11}$$

Whenever u *is coprime with* ℓ, a formula relating the $\mathcal{K}_\ell$ functions with the "periods" q and q^{u^2} differently from (2.11) is

$$\mathcal{K}_\ell(q,x,y) = \sum_{s=0}^{u-1}\sum_{\theta=0}^{u-1} x^{\ell s}\, y^{\ell\theta}\, q^{\frac{s^2-\theta^2}{2}\ell}\, \mathcal{K}_\ell(q^{u^2}, x^u\, q^{su}, y^u\, q^{-\theta u}) \\ + \sum_{\substack{r=1\\ ur-\ell s\geqslant 1}}^{\ell-1}\sum_{s=1}^{u-1} x^{ur}\, y^{ur-\ell s}\, q^{urs-\frac{\ell s^2}{2}}\vartheta(q^\ell, x^\ell\, q^{ur}),$$

which is shown with the help of the identity

$$\frac{1-q^{\ell u}}{(1-q^\ell)(1-q^u)} - \frac{1}{1-q} = -\sum_{\substack{r=1\\ ur-\ell s\geqslant 1}}^{\ell-1}\sum_{s=1}^{u-1} q^{ur-\ell s} \tag{2.12}$$

for coprime positive integers u and ℓ.

The identity in the next lemma is crucial in Sec. 4. For $n\in\mathbb{Z}$, let

$$\boldsymbol{r}_{\ell,u}[n] = \left\lfloor \frac{\ell n}{u} \right\rfloor.$$

Lemma 2.1. *For coprime positive integers ℓ and u,*

$$\begin{aligned}
&\mathcal{K}_{2\ell}(q^{\frac{1}{u}}, x^{\frac{1}{u}}, y^{\frac{1}{u}}) - \mathcal{K}_{2\ell}(q^{\frac{1}{u}}, x^{-\frac{1}{u}}, y^{\frac{1}{u}}) = \\
&\quad = \sum_{s'=1}^{u}\sum_{b=0}^{u-1} x^{\frac{\ell}{u}(s'-1)} y^{\frac{\ell}{u}(s'+1+2b)}\, q^{-\frac{\ell}{u}(b+1)(b+s')}\big(xyq^{-b-1}\big)^{-\boldsymbol{r}_{\ell,u}[s'+2b+1]} \\
&\qquad \times \Big(\mathcal{K}_{2\ell}(q^u, xq^{\frac{s'-1}{2}-\frac{u}{2\ell}\boldsymbol{r}_{\ell,u}[s'+2b+1]}, yq^{-\frac{s'+1}{2}-b+\frac{u}{2\ell}\boldsymbol{r}_{\ell,u}[s'+2b+1]}) \\
&\qquad\quad - x^{2\boldsymbol{r}_{\ell,u}[s'+2b+1]}q^{(s'-1)\boldsymbol{r}_{\ell,u}[s'+2b+1]} \\
&\qquad\qquad \times \mathcal{K}_{2\ell}(q^u, x^{-1}q^{-\frac{s'-1}{2}-\frac{u}{2\ell}\boldsymbol{r}_{\ell,u}[s'+2b+1]}, yq^{-\frac{s'+1}{2}-b+\frac{u}{2\ell}\boldsymbol{r}_{\ell,u}[s'+2b+1]})\Big).
\end{aligned} \tag{2.13}$$

PROOF. The formula in the Lemma is equivalently rewritten as

$$\mathcal{K}_{2\ell}(q^{\frac{1}{u}}, x^{\frac{1}{u}}, y^{\frac{1}{u}}) - \mathcal{K}_{2\ell}(q^{\frac{1}{u}}, x^{-\frac{1}{u}}, y^{\frac{1}{u}}) =$$

$$
\begin{aligned}
&= \sum_{s'=1}^{u}\sum_{b=0}^{u-1} x^{-2\frac{\ell}{u}b+\frac{[\ell(s'+2b-1)]_u}{u}}\, y^{\frac{[\ell(s'+2b-1)]_u}{u}}\, q^{\frac{b}{u}(\ell b-[\ell(s'+2b-1)]_u)}\\
&\quad\times \bigl(\mathcal{K}_{2\ell}(q^u, xq^{-b+\frac{[\ell(s'+2b-1)]_u}{2\ell}}, yq^{-\frac{[\ell(s'+2b-1)]_u}{2\ell}})\\
&\qquad - x^{2\boldsymbol{r}_{\ell,u}[s'-1+2b]} q^{(s'-1)\boldsymbol{r}_{\ell,u}[s'-1+2b]}\\
&\qquad\qquad\times \mathcal{K}_{2\ell}(q^u, x^{-1}q^{-b-(s'-1)+\frac{[\ell(s'+2b-1)]_u}{2\ell}}, yq^{-\frac{[\ell(s'+2b-1)]_u}{2\ell}})\bigr).
\end{aligned}
$$

Indeed, the summand here is mapped into that in (2.13) by a redefinition of the b variable. This changes the b summation limits, but Eq. (2.3) shows that the summand actually depends on b only $\bmod u$, and hence the interval of u consecutive values of b can be translated arbitrarily. But the last equation can be shown directly using the definition (2.1) and the fact that for coprime ℓ and u, $[\ell s']_u$ takes all the values in $[0,\dots,u-1]$ as s' ranges over the set of any u sequential values. ■

2.3. Relations to theta functions. Some special combinations of the Appell functions can be expressed through theta functions. We first note an identity showing that the higher-level Appell functions are expressible through $\mathcal{K}_1$ modulo a ratio of theta functions (cf. a more general statement in [12]).

Lemma 2.2. *For $\ell \geqslant 2$,*

$$
\begin{aligned}
&\vartheta(q^\ell, x)\mathcal{K}_\ell(q,z,y) - \sum_{r=0}^{\ell-1} z^r y^r \vartheta(q^\ell, z^\ell q^r)\mathcal{K}_1(q^\ell, x^{-1}, y^\ell q^{-r}) =\\
&\qquad\qquad = -\vartheta(q^\ell, y^\ell z^\ell x^{-1})\,\frac{\vartheta_{1,1}(q, zy^{1-\ell}x)\, q^{-\frac{1}{8}}\eta(q)^3}{\vartheta_{1,1}(q,zy)\vartheta_{1,1}(q,xy^{-\ell})}.
\end{aligned}
$$

This can be proved either directly (using (2.5) and (2.2), via resummations similar to those in Eqs. (2.17)–(2.19) below) or by noting that in view of the open quasiperiodicity formulas, the left-hand side is in fact *quasiperiodic* in y (and obviously, in x and z), and is therefore expressible as a ratio of theta functions; the actual theta functions in this ratio are found by matching the quasiperiodicity factors, and the remaining q-dependent factor is then fixed by comparing the residues of both sides.

Lemma 2.3. *For an even level 2ℓ,*

$$
\sum_{b=0}^{\ell-1} x^{2b} q^{\frac{b^2}{\ell}}\bigl(\mathcal{K}_{2\ell}(q, xq^{\frac{b}{\ell}}, y) - \mathcal{K}_{2\ell}(q, x^{-1}q^{-\frac{b}{\ell}}, y)\bigr) = -\frac{\vartheta_{1,1}(q^{\frac{1}{\ell}}, x^2) q^{-\frac{1}{8\ell}}\,\eta(q^{\frac{1}{\ell}})^3}{\vartheta_{1,1}(q^{\frac{1}{\ell}}, xy)\vartheta_{1,1}(q^{\frac{1}{\ell}}, xy^{-1})}. \tag{2.14}
$$

To prove this, we use the same strategy as above, the crucial point being quasiperiodicity, which is shown as follows. With $\Delta_\ell f(q,x,y)$ used to temporarily denote $f(q,x,yq) - q^{\frac{\ell}{2}} y^\ell f(q,x,y)$, it follows from Eq. (2.5) that

$$
\Delta_\ell\bigl(\mathcal{K}_{2\ell}(q, xq^{\frac{b}{\ell}}, y) - \mathcal{K}_{2\ell}(q, x^{-1}q^{-\frac{b}{\ell}}, y)\bigr) = \tag{2.15}
$$

$$= \sum_{a=1}^{\ell-1} x^{-a} q^{-\frac{ab}{\ell}+a} y^a (y^{2\ell-2a} q^{\ell-a} - 1) \vartheta(q^{2\ell}, x^{2\ell} q^{2b-a}) \\ + \sum_{a=1}^{\ell-1} x^a q^{\frac{ab}{\ell}+a} y^a (1 - y^{2\ell-2a} q^{\ell-a}) \vartheta(q^{2\ell}, x^{2\ell} q^{2b+a}).$$

This also shows that $x^{2b}\, q^{\frac{b^2}{\ell}}\, \Delta_\ell \mathcal{K}_{2\ell}(q, x^{-1} q^{-\frac{b}{\ell}}, y)$ depends on b only modulo ℓ. In applying $\sum_{b=0}^{\ell-1} x^{2b}\, q^{b^2/\ell}$ to the second term in the right-hand side of (2.15), we can therefore make the shift $b \mapsto b - a$ without changing the summation limits for b. This readily implies that the left-hand side of (2.14) is quasiperiodic in y.

2.4. Integral representation and the S-transform of $\mathcal{K}_\ell$. Although the Appell functions cannot be rationally expressed through theta functions, they admit an *integral* representation through a ratio of theta functions. This integral representation proves to be a useful tool, in particular in finding modular transformations of $\mathcal{K}_\ell$. We give this representation in (2.16) and then use it in the calculation leading to Theorem **1.1**.

Lemma 2.4. *The Appell function admits the integral representation*

$$\mathcal{K}_\ell(\tau, \nu, \mu) = -e^{-\frac{i\pi}{4}\tau} \int_0^1 d\lambda\, \vartheta(\ell\tau, \ell\nu - \lambda)\, \frac{\vartheta_{1,1}(\tau, \nu+\mu+\lambda)\, \eta(\tau)^3}{\vartheta_{1,1}(\tau, \nu+\mu)\vartheta_{1,1}(\tau, \lambda+i0)}, \tag{2.16}$$

where $+i0$ specifies the contour position to bypass the singularities.

PROOF. Starting with the easily derived identity

$$\sum_{m\in\mathbb{Z}} \mathcal{K}_\ell(q, z, yq^m) x^m = \vartheta(q^\ell, z^\ell x^{-1}) \sum_{m\in\mathbb{Z}} \frac{x^m}{1 - yzq^m} \tag{2.17}$$

and combining it with the identity [11, 14]

$$\sum_{m\in\mathbb{Z}} \frac{x^m}{1 - yq^m} = -\frac{\vartheta_{1,1}(q, xy)\, \prod_{i\geqslant 1}(1-q^i)^3}{\vartheta_{1,1}(q, y)\vartheta_{1,1}(q, x)}, \tag{2.18}$$

we obtain

$$\sum_{m\in\mathbb{Z}} \mathcal{K}_\ell(q, z, yq^m) x^m = -\vartheta(q^\ell, z^\ell x^{-1})\, \frac{\vartheta_{1,1}(q, zyx)\, q^{-\frac{1}{8}} \eta(q)^3}{\vartheta_{1,1}(q, zy)\vartheta_{1,1}(q, x)}. \tag{2.19}$$

The right-hand side of (2.18) is a meromorphic function of x with poles at $x = q^n$, $n \in \mathbb{Z}$, but the identity holds in the annulus $\boldsymbol{A}_1 = \{x \bigm| |q| < |x| < 1\}$, where the left-hand side converges. We therefore temporarily assume that $x \in \boldsymbol{A}_1$ and then analytically continue the final result. Integrating over a closed contour inside this annulus yields

$$\mathcal{K}_\ell(q, z, y) = -\frac{1}{2i\pi} \oint \frac{dx}{x}\, \vartheta(q^\ell, z^\ell x^{-1})\, \frac{\vartheta_{1,1}(q, zyx)\, q^{-\frac{1}{8}} \eta(q)^3}{\vartheta_{1,1}(q, zy)\vartheta_{1,1}(q, x)}. \tag{2.20}$$

In the exponential notation $z = e^{2i\pi\nu}$, $y = e^{2i\pi\mu}$, $x = e^{2i\pi\lambda}$, the annulus $\boldsymbol{A}_1$ is mapped into any of the parallelograms $\boldsymbol{P}_n$, $n \in \mathbb{Z}$, with the vertices $(n, n+1, n+1+$

$\tau, n + \tau)$. We choose $n = 0$ in what follows. In the exponential notation, the integration contour is then mapped into a contour in the interior of $\boldsymbol{P}_0$ connecting the points in a close vicinity of 0 and 1 respectively. Equation (2.16) thus follows. ■

The integral representation in the Lemma allows us to find the S-transform of $\mathcal{K}_\ell$. (As regards the T transformation, it readily follows that Eq. (1.2) holds for $\mathcal{K}_\ell$.) For this, we use the known S-transformation properties of the η and ϑ functions entering (2.16), with the result

$$\mathcal{K}_\ell(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}) = -\sqrt{\frac{-i\tau}{\ell}}\,\tau \int_0^1 d\lambda\, e^{i\pi(\frac{\ell\nu^2}{\tau} + \frac{\lambda^2\tau}{\ell} + 2\lambda\mu)} \\ \times \sum_{r=0}^{\ell-1} e^{2i\pi r(\nu - \frac{\tau}{\ell}\lambda) + i\pi\frac{r^2}{\ell}\tau} \vartheta(\ell\tau, \ell\nu - \tau\lambda + r\tau) \frac{\vartheta_{1,1}(\tau, \nu+\mu+\tau\lambda)\, e^{-\frac{i\pi\tau}{4}}\eta(\tau)^3}{\vartheta_{1,1}(\tau, \nu+\mu)\vartheta_{1,1}(\tau, \tau\lambda-\varepsilon)}, \tag{2.21}$$

where we also used (2.9) (with $u = \ell$) to rewrite the theta function $\vartheta(\frac{\tau}{\ell}, \cdot)$ occurring in the S-transform of (2.16) (and where infinitesimal positive ε specifies the contour position). Equation (2.21) allows applying Lemma **2.2**. As a result, after some additional simple transformations involving (2.4) and (2.3), the second line in (2.21) becomes

$$\sum_{r=0}^{\ell-1} e^{i\pi\frac{r^2}{\ell}\tau - 2i\pi r\mu - 2i\pi\frac{r}{\ell}\tau\lambda} \Big(\vartheta(\ell\tau, \tau\lambda + \ell\mu - r\tau)\,\mathcal{K}_\ell(\tau, \nu, \mu) \\ - \sum_{a=0}^{\ell-1} e^{2i\pi a(\nu+\mu)} \vartheta(\ell\tau, \ell\nu + a\tau)\,\mathcal{K}_1(\ell\tau, -\tau\lambda - \ell\mu + r\tau, \ell\mu - a\tau)\Big).$$

In the first term here, we next use (2.9), which gives the integral in (A.1), and in integrating the second term, we change the integration variable as $\lambda \to \frac{\lambda+r}{\tau}$, which then allows us to do the r summation explicitly. This gives

$$\mathcal{K}_\ell(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}) = \tau\, e^{i\pi\ell\frac{\nu^2-\mu^2}{\tau}} \mathcal{K}_\ell(\tau, \nu, \mu) \\ - \sqrt{\frac{-i\tau}{\ell}} \sum_{a=0}^{\ell-1} \vartheta(\ell\tau, \ell\nu + a\tau)\, e^{i\pi\ell\frac{(\nu+\frac{a}{\ell}\tau)^2}{\tau}} \int\limits_{-\tau-a\tau}^{(\ell-1-a)\tau} d\lambda\, e^{i\pi\frac{\lambda^2}{\ell\tau} - 2i\pi\frac{\lambda(\ell\mu-a\tau)}{\ell\tau}} \\ \times \mathcal{K}_1(\ell\tau, \lambda - i0 - (\ell\mu - a\tau), \ell\mu - a\tau), \tag{2.22}$$

showing that the remaining integral is the one in (A.5). This leads to the sought equation expressing the S-transform of $\mathcal{K}_\ell$ through a single $\mathcal{K}_\ell$ function and ℓ theta functions, Eq. (1.3), with

$$\Phi(\tau, \mu) = \phi(\tau, \mu) - \frac{i}{2\sqrt{-i\tau}},$$

$$\phi(\tau, \mu) = \frac{i}{\sqrt{-i\tau}} \int_0^\tau d\lambda\, e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}} \mathcal{K}_1(\tau, \lambda - \mu, \mu)$$

involved in the theta-function terms (see Appendix A for the principal-value integral). Equivalently, the ϕ function can be rewritten as

$$\phi(\tau,\mu) = -\frac{1}{2}\int_{\mathbb{R}} dx\, e^{-\pi x^2}\,\frac{\sinh\bigl(\pi x\sqrt{-i\tau}(1+2\frac{\mu}{\tau})\bigr)}{\sinh\bigl(\pi x\sqrt{-i\tau}\bigr)} \tag{2.23}$$

and Φ as

$$\Phi(\tau,\mu) = -\int\limits_{\mathbb{R}-i0} dx\, e^{-\pi x^2}\,\frac{e^{-2i\pi x\frac{\mu}{\sqrt{-i\tau}}}}{1-e^{-2\pi x\sqrt{-i\tau}}}. \tag{2.24}$$

This proves the formula for the S-transform of $\mathcal{K}_\ell$ in Theorem **1.1**. The integral is to be analytically continued from $\tau = it$ with $t\in\mathbb{R}_{>0}$.

2.5. The lowest-level Appell functions. Appell functions of levels 1 and 2 have some special or simplified properties. For $\mathcal{K}_1$, Eq. (1.3) becomes (cf. [16])

$$\mathcal{K}_1(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau}) = \tau e^{i\pi\frac{\nu^2-\mu^2}{\tau}}\,\mathcal{K}_1(\tau,\nu,\mu) + \tau\, e^{i\pi\frac{\nu^2}{\tau}}\,\Phi(\tau,\mu)\vartheta(\tau,\nu). \tag{2.25}$$

For $\ell=1$, the formula in Lemma **2.2** becomes an identity in [16],

$$\vartheta(q,z)\,\mathcal{K}_1(q,x,y) - \vartheta(q,x)\,\mathcal{K}_1(q,z,y) = \frac{\vartheta(q,xyz)\,\vartheta_{1,1}(q,x^{-1}z)\,q^{-\frac{1}{8}}\,\eta(q)^3}{\vartheta_{1,1}(q,x^{-1}y^{-1})\,\vartheta_{1,1}(q,yz)}.$$

For $\mathcal{K}_2$, Eq. (2.14) simplifies to

$$\mathcal{K}_2(q,x,y) - \mathcal{K}_2(q,x^{-1},y) = -\frac{\vartheta_{1,1}(q,x^2)\,q^{-\frac{1}{8}}\,\eta(q)^3}{\vartheta_{1,1}(q,xy)\vartheta_{1,1}(q,xy^{-1})}. \tag{2.26}$$

2.6. The Φ function and its properties. We now study the properties of the function Φ appearing in the S transform of higher-level Appell functions. These properties include open quasiperiodicity relations — which can be viewed as functional equations satisfied by Φ — and a modular transformation formula. They already follow from Eq. (2.25), or alternatively, can be derived from the integral representation (2.23), similarly to the study of Barnes-like special functions arising in various problems [18, 19] (see also [20, 21, 22, 23]). Unlike Barnes-like functions, however, the Φ function cannot be evaluated as a sum of residues (tentatively, at $x_n = i\frac{n}{\sqrt{-i\tau}}$, $n\in\mathbb{Z}_{\geqslant 0}$) of the integral (2.23), because the Gaussian exponential causes the sum to diverge.

2.6.1. Open quasiperiodicity and related properties. First, a simple calculation allows explicitly evaluating $\phi(\tau,\mu)$ in (2.23) for $\mu=\frac{m\tau}{2}$, $m\in\mathbb{Z}$:

$$\begin{aligned} \phi(\tau,\frac{m\tau}{2}) &= -\frac{1}{2}\sum_{j=0}^{m} e^{-i\pi\tau\frac{(m-2j)^2}{4}}, \qquad m\geqslant 0,\\ \phi(\tau,-\frac{m\tau}{2}) &= \frac{1}{2}\sum_{j=1}^{m-1} e^{-i\pi\tau\frac{(m-2j)^2}{4}}, \qquad m\geqslant 1. \end{aligned} \tag{2.27}$$

In particular, $\phi(\tau,-\tau/2)=0$.

Next, elementary transformations with the integral representation, involving the identity

$$\frac{e^{2\pi x m\sqrt{-i\tau}}-1}{2\sinh(\pi x\sqrt{-i\tau})}=\sum_{j=0}^{m-1}e^{\pi x\sqrt{-i\tau}(2j+1)},\qquad m\in\mathbb{N},$$

show that Φ satisfies the equations

$$\begin{aligned}\Phi(\tau,\mu{+}m\tau)&=\Phi(\tau,\mu)-\sum_{j=1}^{m}e^{-i\pi\frac{(\mu+j\tau)^2}{\tau}},\\ \Phi(\tau,\mu{-}m\tau)&=\Phi(\tau,\mu)+\sum_{j=0}^{m-1}e^{-i\pi\frac{(\mu-j\tau)^2}{\tau}},\end{aligned}\qquad m\in\mathbb{N}.\tag{2.28}$$

Similarly to the equations for $\mathcal{K}_\ell$, these are open quasiperiodicity relations. They can be alternatively derived from (2.25) and the corresponding property (2.5) of the Appell functions. For this, we evaluate the commutator of the S transform of $\mathcal{K}_\ell(\tau,\nu,\mu)$ and the translation of the μ argument by elements of the lattice generated by $(1,\tau)$; because modular transformations act on lattice translations (thus forming the semidirect product), this results in equations for Φ, equivalent to (2.28).

A slightly more involved calculation with the integral representation leads to the "dual" open quasiperiodicity relations

$$\begin{aligned}\Phi(\tau,\mu{+}m)&=e^{-i\pi\frac{m^2}{\tau}-2i\pi m\frac{\mu}{\tau}}\,\Phi(\tau,\mu)+\frac{i}{\sqrt{-i\tau}}\sum_{j=1}^{m}e^{i\pi\frac{j(j-2m)}{\tau}-2i\pi j\frac{\mu}{\tau}},\\ \Phi(\tau,\mu{-}m)&=e^{-i\pi\frac{m^2}{\tau}+2i\pi m\frac{\mu}{\tau}}\,\Phi(\tau,\mu)-\frac{i}{\sqrt{-i\tau}}\sum_{j=0}^{m-1}e^{i\pi\frac{j(j-2m)}{\tau}+2i\pi j\frac{\mu}{\tau}},\end{aligned}\tag{2.29}$$

where $m\in\mathbb{N}$. To show this, we recall the analytic continuation prescription and write $\Phi(it,\mu)$ with $t\in\mathbb{R}_+$ as in (2.24). We then consider $\Phi(it,\mu+m)$ with $m\in\mathbb{N}$ and change the integration variable as $x=x'-i\frac{m}{\sqrt{t}}$. This gives

$$\Phi(it,\mu{+}m)=e^{-\pi\frac{m^2}{t}-2\pi\frac{m\mu}{t}}\,\Phi_{\frac{im}{\sqrt{t}}}(it,\mu),$$

where $\Phi_{\frac{im}{\sqrt{t}}}(it,\mu)$ is given by the integral along $\mathbb{R}+i\frac{m}{\sqrt{t}}-i0$ of *the same integrand* as for $\Phi(it,\mu)$. A residue calculation in accordance with

$$\Phi_{\frac{im}{\sqrt{t}}}(it,\mu)=\Phi(it,\mu)-2i\pi\sum_{n=0}^{m-1}\operatorname*{res}_{x=i\frac{n}{\sqrt{t}}}\left(-e^{-\pi x^2}\,\frac{e^{-2i\pi x\frac{\mu}{\sqrt{t}}}}{1-e^{-2\pi x\sqrt{t}}}\right)$$

(see Fig. 1) then yields the first equation in (2.29). Alternatively, Eqs. (2.29) can also be deduced from (2.25) and the corresponding Appell function property in Eq. (2.7).

Next, a "reflection property" follows from (2.6) (or can be directly derived from (2.23)),

$$\Phi(\tau,-\mu)=\frac{-i}{\sqrt{-i\tau}}-e^{-i\pi\frac{\mu^2}{\tau}}-\Phi(\tau,\mu),\tag{2.30}$$

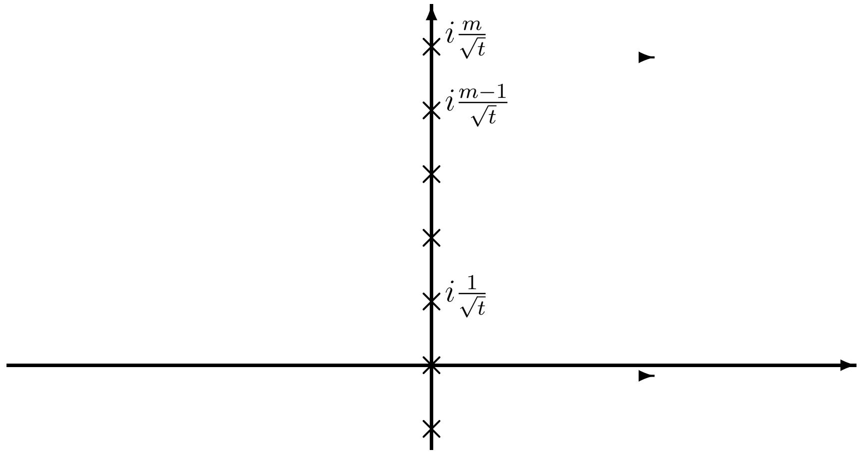

FIGURE 1. Integration contours for $\Phi(it,\mu)$ (the lower dashed line) and $\Phi_{\frac{im}{\sqrt{t}}}(it,\mu)$ (the upper dashed line) in the complex x plane and poles of the integrand (crosses).

or equivalently,

$$\Phi(\tau,\mu+1) = -e^{-2i\pi\frac{\mu+\frac{1}{2}}{\tau}}\Phi(\tau,-\mu-\tau). \tag{2.31}$$

A simple "scaling law" for Φ follows from (2.11) and (2.25) (or directly from integral representation (2.23)),

$$\Phi(\tau,\mu) = \sum_{b=0}^{p-1}\Phi(p^2\tau, p\mu - bp\tau), \qquad p\in\mathbb{N}. \tag{2.32}$$

In the case of "scaling" with an even factor, we have the following two formulas. As before, $[x]_p = x \bmod p$.

Lemma 2.5. *For any $\ell\in\mathbb{N}$ and $m\in\mathbb{Z}$,*

$$\sum_{a=0}^{2\ell-1} e^{i\pi\frac{am}{\ell}}\Phi(2\ell\tau, 2\ell\mu - a\tau) = e^{i\pi\frac{[m]_{2\ell}^2}{2\ell\tau}+2i\pi\frac{\mu}{\tau}[m]_{2\ell}}\Phi(\frac{\tau}{2\ell},\mu+\frac{[m]_{2\ell}}{2\ell}), \tag{2.33}$$

$$\frac{1}{2\ell}\sum_{a=0}^{2\ell-1} e^{2i\pi\frac{a\mu}{\tau}-i\pi\frac{am}{\ell}+i\pi\frac{a^2}{2\ell\tau}}\Phi(\frac{\tau}{2\ell},\mu+\frac{a}{2\ell}) = \Phi(2\ell\tau, 2\ell\mu - [m]_{2\ell}\,\tau). \tag{2.34}$$

For $m=0$, Eq. (2.33) reduces to (2.32), and Eq. (2.34) becomes

$$\frac{1}{2\ell}\sum_{a=0}^{2\ell-1} e^{2i\pi\frac{a\mu}{\tau}+i\pi\frac{a^2}{2\ell\tau}}\Phi(\frac{\tau}{2\ell},\mu+\frac{a}{2\ell}) = \Phi(2\ell\tau, 2\ell\mu). \tag{2.35}$$

PROOF. Both formulas (2.32) and (2.34) follow from manipulations with the integral representation, but as we see shortly, these two equations are S-dual to each other, and it therefore suffices to prove any one of them. We actually show (2.35), from which the rest (all of the m dependence) follows via (2.28). For this, we evaluate the left-hand side using the integral representation and change the integration variable x in the ath term as $x = x' - i\frac{a}{2\ell\sqrt{-i\tau}}$; as before, we analytically continue from the positive part of the imaginary axis $\tau = it$. Then (omitting the prime at x)

$$\frac{1}{2\ell}\sum_{a=0}^{2\ell-1} e^{i\pi\frac{a\mu}{\ell\tau}+i\pi\frac{a^2}{4\ell^2\tau}}\Phi(\tau,\mu+\frac{a}{2\ell}) = -\frac{1}{2\ell}\sum_{a=0}^{2\ell-1}\int\limits_{\mathbb{R}+\frac{ia}{2\ell\sqrt{t}}-i0} dx\, e^{-\pi x^2}\frac{e^{\pi x\sqrt{t}\,\frac{2\mu}{\tau}}}{1-e^{i\pi\frac{a}{\ell}-2\pi x\sqrt{t}}}.$$

But the integration contour can be deformed to $\mathbb{R}-i0$ in *all* terms, which allows us to evaluate the sum over a as

$$\frac{1}{2\ell}\sum_{a=0}^{2\ell-1}\frac{1}{1-e^{i\pi\frac{a}{\ell}}q} = \frac{1}{1-q^{2\ell}},$$

which shows (2.35). ■

We also note that combining Eqs. (2.9), (2.32), and (2.29) allows expressing the S-transform of $\mathcal{K}_\ell$, Eq. (1.3), as

$$\begin{aligned}(2.36)\quad \mathcal{K}_\ell(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau}) &= \tau\, e^{i\pi\ell\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_\ell(\tau,\nu,\mu)\\ &\quad+\frac{\tau}{\ell}\sum_{a=0}^{\ell-1} e^{i\pi\frac{\ell}{\tau}\nu^2+i\pi\frac{a^2}{\ell\tau}+2i\pi a\frac{\mu}{\tau}}\Phi(\frac{\tau}{\ell},\mu+\frac{a}{\ell})\,\vartheta(\frac{\tau}{\ell},\nu-\frac{a}{\ell}).\end{aligned}$$

With some care, Eq. (2.36) can be derived directly via the Poisson resummation formula.

2.6.2. Modular transformations of the Φ function. Modular group relations impose constraints on the function Φ appearing in the S-transform of higher-level Appell functions. With the action of $C = S^2$ given by Eq. (2.30), we act with both sides of the relation $(ST)^3 = C$ on $\mathcal{K}_1$. Comparing the results gives the identity

$$\begin{aligned}(2.37)\quad i\sqrt{-i\tau}\,e^{i\pi\frac{\mu^2}{\tau}}\Phi(\tau,\mu) + e^{i\pi\frac{(\mu+\frac{1}{2})^2}{\tau-1}}\Phi(1-\frac{1}{\tau},\frac{\mu}{\tau}+\frac{1}{2})&\\ - i\sqrt{-i\tau}\,e^{i\pi\frac{(\mu+\frac{1}{2})^2}{\tau-1}}\Phi(\tau-1,\mu+\frac{1}{2}) &= 0.\end{aligned}$$

Next, with the action of S given by Eq. (2.25), the $SL(2,\mathbb{Z})$ relation $S^2 = C$ results in the S-transformation formula for Φ,

$$(2.38)\qquad \Phi(-\frac{1}{\tau},\frac{\mu}{\tau}) = -i\sqrt{-i\tau}\big(e^{i\pi\frac{\mu^2}{\tau}}\Phi(\tau,\mu)+1\big) = -i\sqrt{-i\tau}e^{i\pi\frac{\mu^2}{\tau}}\Phi(\tau,\mu-\tau).$$

It now follows that identities (2.33) and (2.34) are the S-transform of each other.

It is instructive to verify (the first equality in) (2.38) by comparing the asymptotic expansions of the integral in (2.23) as $-i\tau\searrow 0$ and $\tau\to\infty$. We first find the asymptotic form of $\phi(it,\mu)$ for large positive t. Writing

$$\phi(it,iy) \asymp -\int_{\epsilon}^{+\infty} dx\, e^{-\pi x^2}\,\frac{\sinh\big(\pi x\sqrt{t}(1+2\frac{y}{t})\big)}{\sinh\big(\pi x\sqrt{t}\big)}$$

$$\overset{t\to+\infty}{\asymp} -2\sum_{m=0}^{\infty}\int_{\epsilon}^{+\infty}dx\,e^{-\pi x^2}\,\sinh\Bigl(\pi x(\sqrt{t}+2\frac{y}{\sqrt{t}})\Bigr)\,e^{-\pi x(2m+1)\sqrt{t}},$$

we readily obtain

$$\phi(it,iy)\overset{t\to\infty}{\asymp}\frac{1}{2}\sum_{m=0}^{\infty}e^{\pi\frac{((m+1)t+y)^2}{t}}\,\mathrm{erfc}\Bigl(\sqrt{\frac{\pi}{t}}((m+1)t+y+\sqrt{t}\epsilon)\Bigr)\\ -\frac{1}{2}\sum_{m=0}^{\infty}e^{\pi\frac{(y-mt)^2}{t}}\,\mathrm{erfc}\Bigl(\sqrt{\frac{\pi}{t}}(mt-y+\sqrt{t}\epsilon)\Bigr),$$

with the complementary error function

$$\mathrm{erfc}(z)=1-\mathrm{erf}(z)=1-\frac{2}{\sqrt{\pi}}\int_0^z dte^{-t^2}.$$

We isolate the $m=0$ term in the first sum (the only term where the argument of erfc is small as $t\to\infty$), rearrange the remaining series, and use the asymptotic expansion

$$\mathrm{erfc}(z)\asymp\frac{1}{\sqrt{\pi}}\,e^{-z^2}\sum_{n\geqslant 0}(-1)^n\frac{(2n-1)!!}{2^n\,z^{2n+1}}$$

for large positive z. We can then set $\epsilon=0$, which gives

$$\phi(it,iy)\overset{t\to\infty}{\asymp}-\frac{1}{2}\,e^{\pi\frac{y^2}{t}}\,\mathrm{erfc}(-\sqrt{\frac{\pi}{t}}y)+\frac{1}{2}\bigl(\cot(\frac{\pi y}{t})-\frac{t}{\pi y}\bigr)\\ +\sum_{n=1}^{\infty}\frac{(-1)^n\,(2n-1)!!}{(2\pi)^{n+1}}\,\frac{\zeta(2n+1,1+\frac{y}{t})-\zeta(2n+1,1-\frac{y}{t})}{t^{n+\frac{1}{2}}},$$

where

$$\zeta(s,a)=\sum_{m=0}^{\infty}(m+a)^{-s}.$$

Using

$$\zeta(n,1+x)=\sum_{i\geqslant 0}(-1)^i\binom{n+i-1}{n-1}x^i\zeta(n+i),$$

where $\zeta(s)=\sum_{m\geqslant 1}m^{-s}$, and expressing the ζ-function values at even positive integers through the Bernoulli numbers B_{2n} as

$$\zeta(2n)=\pi^{2n}\,\frac{2^{n-1}(-1)^{n+1}}{n!\,(2n-1)!!}\,B_{2n},$$

we then obtain the large-t asymptotic expansion

$$\phi(it,iy)\overset{t\to\infty}{\asymp}-\frac{1}{2}\,e^{\pi\frac{y^2}{t}}\,\mathrm{erfc}(-\sqrt{\frac{\pi}{t}}y)-\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-4)^j\pi^{2j+n+1}B_{2(j+n+1)}}{(j+n+1)\,(2j+1)!\,n!}\,\frac{y^{2j+1}}{t^{\frac{3}{2}+2j+n}}. \tag{2.39}$$

The erfc function in (2.39) can be expanded further, with the result

$$\phi(it,iy)\overset{t\to\infty}{\asymp}-\frac{1}{2}\,e^{\pi\frac{y^2}{t}}-\sum_{n=0}^{\infty}\frac{\pi^n B_{2n}}{n!\,t^{\frac{1}{2}+n}}\,y\,{}_1F_1\bigl(1-n,\frac{3}{2},\frac{\pi y^2}{t}\bigr).$$

We next find the small-t expansion.[3] Writing

$$\phi(it,iy) = -\frac{1}{2}\int_{-\infty}^{+\infty} dx\, e^{-\pi x^2}\Bigl(\cosh\bigl(2\pi x\frac{y}{\sqrt{t}}\bigr) + \coth\bigl(\pi x\sqrt{t}\bigr)\sinh\bigl(2\pi x\frac{y}{\sqrt{t}}\bigr)\Bigr), \tag{2.40}$$

we calculate the $\cosh$ integral and expand $\coth$, with the result

$$\phi(it,iy) \overset{t\to 0}{\asymp} -\frac{1}{2}\,e^{\frac{\pi y^2}{t}} - \sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}\int_0^{\infty} dx\, e^{-\pi x^2}(\pi x\sqrt{t})^{2j-1}\sinh\bigl(2\pi x\frac{y}{\sqrt{t}}\bigr).$$

This involves the integrals

$$\int_0^{\infty} dx\, e^{-\pi x^2}x^{-1}\sinh(\beta x) = -\frac{i\pi}{2}\,\mathrm{erf}\bigl(\frac{i\beta}{2\sqrt{\pi}}\bigr),$$
$$\int_0^{\infty} dx\, e^{-\pi x^2}x^{2j-1}\sinh(\beta x) = (-1)^j\,\frac{i\sqrt{\pi}}{(4\pi)^j}\,e^{\frac{\beta^2}{4\pi}}\,H_{2j-1}\bigl(\frac{i\beta}{2\sqrt{\pi}}\bigr), \quad j\geqslant 1,$$

where H_m are the Hermite polynomials. They can be written as

$$H_m(x) = (2x)^m\left(1-\binom{m}{2}\frac{1}{2x^2}+1\cdot 3\binom{m}{4}\frac{1}{(2x^2)^2}-1\cdot 3\cdot 5\binom{m}{6}\frac{1}{(2x^2)^3}+\dots\right),$$

which gives the small-t expansion

$$\begin{aligned}\phi(it,iy) \overset{t\to 0}{\asymp} -\frac{1}{2}\,e^{\pi\frac{y^2}{t}} + \frac{i}{2\sqrt{t}}\,\mathrm{erf}\Bigl(i\sqrt{\frac{\pi}{t}}\,y\Bigr)\\ - e^{\pi\frac{y^2}{t}}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{B_{2(j+n+1)}\,4^j\pi^{2j+n+1}}{(j+n+1)\,n!\,(2j+1)!}\,y^{2j+1}\,t^n.\end{aligned} \tag{2.41}$$

To verify consistency with the S-transform formula (2.38), we rewrite it as

$$\phi(\frac{i}{t},\frac{y}{t}) + i\sqrt{t}\,e^{-\pi\frac{y^2}{t}}\,\phi(it,iy) = -\frac{i}{2}\,\sqrt{t} - \frac{1}{2}\,e^{-\pi\frac{y^2}{t}}.$$

With the above asymptotic expansions, we then find that indeed,

$$(2.41)\Bigg|_{\substack{t\to\frac{1}{t}\\ y\to -i\frac{y}{t}}} + i\sqrt{t}\,e^{-\pi\frac{y^2}{t}}\cdot(2.39) = -\frac{i}{2}\,\sqrt{t} - \frac{1}{2}\,e^{-\pi\frac{y^2}{t}},$$

showing that the asymptotic expansions of the integral in (1.4) obey the S transformation formula (2.38).

3. Geometry and Further Properties of the Appell Functions

In this section, we consider some elements of the geometric interpretation of higher-level Appell functions and then formulate their modular properties in terms of the action of subgroups of $SL(2,\mathbb{Z})$. As noted in [16], $(\kappa(\,\cdot\,,a;q),\vartheta(q,\,\cdot\,))$ is a section of a rank-2 vector bundle over the torus. For $\mathcal{K}_\ell$ with $\ell>1$, we unify $\mathcal{K}_\ell$ and the theta functions arising in the open quasiperiodicity formula for $\mathcal{K}_\ell$ into a vector K_ℓ representing

[3] We are grateful to V.I. Ritus for the elegant derivation in Eqs. (2.40)–(2.41).

a section of a rank-$(\ell+1)$ bundle. In the space of $(\ell+1)$-vectors, sections of this bundle are selected by the *invariance* condition with respect to the action of a lattice in $\mathbb{C}^2$ (Lemma **3.1**). Moreover, there is a representation of a (ℓ-dependent) subgroup of $SL(2,\mathbb{Z})$ on $(\ell+1)$-vectors such that the section K_ℓ is also invariant under this subgroup action (Theorem **3.5**). This description of the action of (a subgroup of) $SL(2,\mathbb{Z})$ as an invariance statement is in the spirit of the well-known result for classic theta functions, which we quote in Lemma **3.3** below. The "essence" of the modular group action is then hidden in an automorphy factor involved in defining this action. With K_ℓ, similarly, the invariance statement in Theorem **3.5** below involves a judiciously chosen automorphy *matrix*.

3.1. Lattice translations and bundle sections. We begin with constructing a vector bundle $\mathbb{V}_{\ell,\tau}$, of the rank determined by the number of terms in the right-hand side of the open quasiperiodicity formula for $\mathcal{K}_\ell$. This bundle

$$\begin{array}{c}\mathbb{C}^2 \times \mathbb{C}^{\ell+1}/\mathcal{R}\\ \Big\downarrow\\ \mathbb{T}_{\ell,\tau}\end{array} \tag{3.1}$$

is defined as follows. We take the 4-dimensional torus $\mathbb{T}_{\ell,\tau} = \mathbb{C}^2/\mathbb{L}_{\ell,\tau}$, where $\mathbb{L}_{\ell,\tau} \subset \mathbb{C}^2$ is the lattice generated by the vectors

$$\gamma_1 = \boldsymbol{e}, \quad \gamma_2 = \frac{1}{\ell}(\boldsymbol{e} - \boldsymbol{f}), \quad \gamma_3 = \tau\boldsymbol{e}, \quad \gamma_4 = \frac{1}{\ell}\tau(\boldsymbol{e} - \boldsymbol{f}),$$

with $\boldsymbol{e}$ and $\boldsymbol{f}$ being the standard basis in $\mathbb{C}^2$. With ν and μ denoting the corresponding coordinates and $(\nu,\mu,v) \in \mathbb{C}^2 \times \mathbb{C}^{\ell+1}$, the relations $\mathcal{R}$ are given by

$$\begin{aligned}\mathcal{R} = \big\{&(\nu+1,\mu,v) \sim (\nu,\mu,v),\ (\nu+\tfrac{1}{\ell},\mu-\tfrac{1}{\ell},v) \sim (\nu,\mu,A_2^{-1}v),\\ &(\nu+\tau,\mu,v) \sim (\nu,\mu,A_3(\tau,\nu)^{-1}v),\ (\nu+\tfrac{\tau}{\ell},\mu-\tfrac{\tau}{\ell},v) \sim (\nu,\mu,A_4(\tau,\nu,\mu)^{-1}v)\big\}.\end{aligned}$$

where the matrices A_2, $A_3(\tau,\nu)$, and $A_4(\tau,\nu,\mu)$ are

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & e^{-2i\pi\frac{1}{\ell}} & 0 & \dots & 0\\ 0 & 0 & 0 & e^{-2i\pi\frac{2}{\ell}} & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & e^{-2i\pi\frac{\ell-1}{\ell}}\end{pmatrix},$$

$$A_3(\tau,\nu) = e^{i\pi\ell\tau + 2i\pi\ell\nu}\mathbf{1}_{(\ell+1)\times(\ell+1)},$$

$$A_4(\tau,\nu,\mu) = e^{2i\pi\nu + i\pi\frac{1}{\ell}\tau}\begin{pmatrix} e^{2i\pi\mu - i\pi\frac{1}{\ell}\tau} & \boldsymbol{v}\\ \mathbf{0} & B\end{pmatrix},$$

where the $\ell \times \ell$-matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

cyclically permutes the standard basis vectors, and $\boldsymbol{v} = (1, \underbrace{0, 0, \dots, 0}_{\ell-1})$. The projection in (3.1) is given by $(\nu, \mu, v) \mapsto (\nu, \mu, 0)$.

We introduce the ℓ-dimensional vector $\boldsymbol{\theta}_\ell(\tau, \nu) = \bigl(\theta_{2r,\ell}(\frac{\tau}{2}, \nu)\bigr)_{0 \leqslant r \leqslant \ell-1}$ (these are the same theta functions that appear in the open quasiperiodicity formula for $\mathcal{K}_\ell$ and in (1.3)).

Lemma 3.1. *The* $(\ell + 1)$*-vector*

$$\mathsf{K}_\ell(\tau, \nu, \mu) = \begin{pmatrix} \mathcal{K}_\ell(\tau, \nu, \mu) \\ \boldsymbol{\theta}_\ell(\tau, \nu) \end{pmatrix}$$

is a section of the bundle $\mathbb{V}_{\ell,\tau}$.

The Lemma is almost tautological in view of Eqs. (2.7), (2.8), (2.3), and (2.4) and quasiperiodicity of the theta functions. We nevertheless note that the assertion is conveniently formalized using the operators U_i, $i = 1, 2, 3, 4$, acting on functions $f(\tau, \,\cdot\,, \,\cdot\,) : \mathbb{C}^2 \to \mathbb{C}^{\ell+1}$ as

$$\begin{aligned} U_1 f(\tau, \nu, \mu) &= f(\tau, \nu + 1, \mu), \\ U_2 f(\tau, \nu, \mu) &= A_2 f(\tau, \nu + \frac{1}{\ell}, \mu - \frac{1}{\ell}), \\ U_3 f(\tau, \nu, \mu) &= A_3(\tau, \nu) f(\tau, \nu + \tau, \mu), \\ U_4 f(\tau, \nu, \mu) &= A_4(\tau, \nu, \mu) f(\tau, \nu + \frac{\tau}{\ell}, \mu - \frac{\tau}{\ell}). \end{aligned}$$

Lemma **3.1** is then a reformulation of the following easily verified fact.

Lemma 3.2. *The operators* U_i, $i = 1, 2, 3, 4$*, pairwise commute, and hence* $\gamma_i \mapsto U_i$ *is a multiplicative representation of the Abelian group* $\mathbb{L}_{\ell,\tau}$*. Sections of* $\mathbb{V}_{\ell,\tau}$ *can therefore be identified with invariants of* $\mathbb{L}_{\ell,\tau}$ *in this representation.*

3.2. The action of a subgroup of the modular group. We now identify a subgroup in $SL(2, \mathbb{Z})$ and construct its action on functions $f : \mathfrak{h} \times \mathbb{C}^2 \to \mathbb{C}^{\ell+1}$ (where $\mathfrak{h}$ is the upper half-plane) such that the vector K_ℓ defined in Lemma **3.1** is *invariant*. The aim of this subsection is to prove Theorem **3.5**. This action involves a *matrix* automorphy factor, which can be considered an Appell-function analogue of the automorphy factor involved in the classic statement in Lemma **3.3**.

3.2.1. Automorphy factors. We begin with recalling the $SL(2,\mathbb{Z})$ action on $\mathfrak{h}\times\mathbb{C}^2$, Eq. (1.8). Possible $SL(2,\mathbb{Z})$ actions (actually, *anti*representations) on functions $f:\mathfrak{h}\times\mathbb{C}\to\mathbb{C}$ are given by

$$(3.2)\qquad (\gamma\,\boldsymbol{.}\, f)(\tau,\nu)=j(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right);\tau,\nu)f(\gamma\tau,\gamma\nu),$$

where j is an *automorphy factor* satisfying the standard cocycle condition. More generally, $SL(2,\mathbb{Z})$ actions on functions $f:\mathfrak{h}\times\mathbb{C}^2\to\mathbb{C}^{\ell+1}$ are given by

$$(3.3)\qquad \gamma\cdot f(\tau,\nu,\mu)=\mathsf{J}_\ell(\gamma;\tau,\nu,\mu)f(\gamma\tau,\gamma\nu,\gamma\mu),$$

where $\mathsf{J}_\ell(\gamma;\tau,\nu,\mu)$ is the matrix automorphy factor, a $(\ell+1)\times(\ell+1)$-matrix satisfying the cocycle condition

$$(3.4)\qquad \mathsf{J}_\ell(\alpha\beta;\tau,\nu,\mu)=\mathsf{J}_\ell(\beta;\tau,\nu,\mu)\mathsf{J}_\ell(\alpha;\beta\tau,\beta\nu,\beta\mu),\qquad \mathsf{J}_\ell(\mathbf{1};\tau,\nu,\mu)=\mathbf{1}.$$

Let $\Gamma_{1,2}$ be the subgroup in $SL(2,\mathbb{Z})$ consisting of matrices $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ with $ab\in2\mathbb{Z}$ and $cd\in2\mathbb{Z}$. We recall the following result about the invariance of theta functions under $\Gamma_{1,2}$.

Lemma 3.3 ([29])**.** *The theta function $\vartheta(\tau,\nu)$ is invariant under the action of $\Gamma_{1,2}$ (see (3.2)) with the automorphy factor*

$$j(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right);\tau,\nu)=\zeta_{c,d}^{-1}(c\tau+d)^{-\frac{1}{2}}e^{-i\pi\frac{c\nu^2}{c\tau+d}}.$$

Here,

$$\zeta_{c,d}=\begin{cases}e^{i\pi\frac{d-1}{4}}\left(\frac{c}{|d|}\right), & c\text{ even},\\ e^{-i\pi\frac{c}{4}}\left(\frac{d}{c}\right), & c\text{ odd}\end{cases}$$

and $\left(\frac{c}{d}\right)$ is defined as the quadratic residue for odd positive prime d and multiplicatively extended to all d; see [30] for the details (we note that c and d are coprime because of the determinant condition in $SL(2,\mathbb{Z})$, and we assume $\left(\frac{0}{\pm1}\right)=1$). As a simple corollary of the Lemma, we have the formula

$$(3.5)\qquad \eta\Big(\frac{a\tau+b}{c\tau+d}\Big)^3=\zeta_{c,d}\,(c\tau+d)^{\frac{3}{2}}\,e^{\frac{i\pi}{4}(a+c)(b+d)+\frac{i\pi}{4}-i\pi\frac{a+c}{2}}\eta(\tau)^3,\quad \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_{1,2},$$

to be used in what follows.

3.2.2. Matrix automorphy factors. The classic invariance statement in Lemma **3.3** extends to the theta-vector $\boldsymbol{\theta}_\ell(\tau,\nu)$ introduced before Lemma **3.1**. This is shown in the next Lemma, but we first set the necessary notation.

Let $\Gamma_{1,2\ell}$ be the subgroup of $SL(2,\mathbb{Z})$ consisting of matrices $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ such that $ab\equiv 0\bmod 2\ell$ and $cd\equiv0\bmod2\ell$. For $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_{1,2\ell}$, let $\gcd(\ell,a)=\ell_a$ and $\gcd(\ell,c)=\ell_c$ (hence, $\ell_a\,\ell_c=\ell$).

We also need an ℓ-dimensional representation $\boldsymbol{D}'_\ell$ of $\Gamma_{1,2\ell}$ defined as

$$\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \mapsto \boldsymbol{D}'_\ell(\gamma) = (d_{s,n})_{\substack{0\leqslant s\leqslant\ell-1,\\ 0\leqslant n\leqslant\ell-1}}$$

where

$$d_{s,n} = \begin{cases} \frac{1}{\sqrt{\ell_a}}\, e^{-2i\pi\frac{bcrs}{\ell}} & \text{if } n\equiv(sa-rc)\bmod\ell\\ & \text{for some integer } 0\leqslant r\leqslant\ell_a-1,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 3.4. *The vector $\boldsymbol{\theta}_\ell(\tau,\nu)$ is invariant under the action of $\Gamma_{1,2\ell}$ given by*

$$\gamma\,\centerdot\, f(\tau,\nu,\mu) = \boldsymbol{J}(\gamma;\tau,\nu,\mu)f(\gamma\tau,\gamma\nu,\gamma\mu),$$

with the $\ell\times\ell$ automorphy factor

$$\boldsymbol{J}(\gamma;\tau,\nu,\mu) = k_\ell(\gamma;\tau,\nu)\boldsymbol{E}_\ell(\gamma;\tau),$$

where

$$k_\ell(\gamma;\tau,\nu) = e^{-i\pi\frac{c\ell\nu^2}{c\tau+d}},\qquad \boldsymbol{E}_\ell(\gamma;\tau) = \zeta^{-1}_{\frac{c}{\ell_c},\ell_a d}(c\tau+d)^{-\frac12}\,\boldsymbol{D}_\ell(\gamma), \tag{3.6}$$

and $\boldsymbol{D}_\ell(\gamma) = \boldsymbol{D}'_\ell(\gamma)^{-1}$.

The proof essentially reduces to the formula

$$\theta^{(\ell)}_s\Big(\frac{a\tau+b}{c\tau+d},\frac{\nu}{c\tau+d}\Big) = \zeta_{\frac{c}{\ell_c},\ell_a d}\,\frac{\sqrt{c\tau+d}}{\sqrt{\ell_a}}\, e^{i\pi\frac{c\ell\nu^2}{c\tau+d}}\sum_{r=0}^{\ell_a-1} e^{-2i\pi\frac{bcrs}{\ell}}\theta^{(\ell)}_{sa-rc}(\tau,\nu), \tag{3.7}$$

which can be verified for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_{1,2\ell}$ by direct calculation.

Example. To illustrate the structure of the automorphy factor in the Lemma, we consider two examples with the matrix γ chosen as $S=\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $C=S^2=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}$. For S, the matrix elements D_{nm} of $\boldsymbol{D}_\ell(S)$ are given by

$$D_{nm} = \frac{1}{\sqrt{\ell}}\, e^{2i\pi\frac{n\,m}{\ell}},\qquad n,m=0,1,\dots,\ell-1.$$

For C, we have

$$\boldsymbol{D}_\ell(C) = (\boldsymbol{D}_\ell(S))^2 = \begin{pmatrix} 1 & 0 & \dots\dots & 0\\ 0 & \dots\dots\dots & & 1\\ 0 & \dots\dots & 1 & 0\\ \dots & \dots & \dots & \dots\\ 0 & 1 & 0\ \dots & 0\end{pmatrix}.$$

3.2.3. Modular behavior of K_ℓ. With these ingredients, we now formulate an analogue of Lemma **3.3** for the vector K_ℓ in Lemma **3.1**. As a final preparation, we define the automorphy factor before formulating the result, because the corresponding formulas are somewhat bulky in view of several cases that must be considered (the reader may first concentrate on the "basic" case $c>0$).

For $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_{1,2\ell}$, let $\mathsf{J}_\ell(\gamma;\tau,\nu,\mu)$ be the $(\ell+1)\times(\ell+1)$ matrix defined as

$$\mathsf{J}_\ell(\gamma;\tau,\nu,\mu) = \begin{cases} k_\ell(\gamma;\tau,\nu)\begin{pmatrix} l_\ell(\gamma;\tau,\mu) & \boldsymbol{F}_\ell(\gamma;\tau,\mu)\boldsymbol{E}_\ell(\gamma;\tau)\\ 0 & \boldsymbol{E}_\ell(\gamma;\tau)\end{pmatrix}, & c>0,\\ \begin{pmatrix} -1 & \boldsymbol{v}\\ \boldsymbol{0} & \boldsymbol{D}_\ell(C)\end{pmatrix}, & c=0 \text{ and } \gamma = C = \left(\begin{smallmatrix} -1 & 0\\ 0 & -1\end{smallmatrix}\right),\\ \boldsymbol{1}, & c=0 \text{ and } \gamma = \left(\begin{smallmatrix} 1 & 2\ell b\\ 0 & 1\end{smallmatrix}\right),\\ \mathsf{J}_\ell(C;\tau,\nu,\mu), & c=0 \text{ and } \gamma = \left(\begin{smallmatrix} -1 & 2\ell b\\ 0 & -1\end{smallmatrix}\right),\\ \mathsf{J}_\ell(-\gamma;\tau,\nu,\mu)\mathsf{J}(C;\tau,\nu,\mu), & c<0,\end{cases}$$

where $k_\ell(\gamma;\tau,\nu)$ and $\boldsymbol{E}_\ell(\gamma;\tau)$ are given in (3.6),

$$l_\ell(\gamma;\tau,\nu) = (c\tau+d)^{-1} e^{i\pi\frac{c\ell\nu^2}{c\tau+d}},$$

and $\boldsymbol{F}_\ell(\gamma;\tau,\mu) = (F^{(\ell)}_1, F^{(\ell)}_2, \dots, F^{(\ell)}_\ell)$ is the vector with the components

$$F^{(\ell)}_r = -\zeta_{\frac{c}{\ell_c},\ell_a d}\, i\sqrt{-i}\,\sqrt{\frac{\ell_c}{c}}\, e^{i\pi\frac{c\ell\mu^2}{c\tau+d}}\, e^{i\pi\frac{rd(2\ell\mu-r\tau)}{\ell(c\tau+d)}} \times \sum_{s=0}^{\frac{c}{\ell_c}-1} e^{2i\pi(\ell\mu-r\tau)\frac{sd}{c\tau+d}-i\pi\tau\frac{s^2 d}{c\tau+d}}\,\Phi(\ell\tau+\ell\frac{d}{c}, \ell\mu+s\ell\frac{d}{c}-r\tau).$$

Theorem 3.5. *The section of $\mathbb{V}_{\ell,\tau}$ given by $\mathsf{K}_\ell(\tau,\nu,\mu)$ is invariant under the action of $\Gamma_{1,2\ell}$ given by Eqs.* (3.3) *with the matrix automorphy factor J_ℓ defined above.*

Example: the S transformation. For the S matrix, the automorphy factor in the Theorem becomes

$$\mathsf{J}_\ell(S;\tau,\nu,\mu) = e^{-i\pi\ell\frac{\nu^2}{\tau}}\begin{pmatrix} \tau^{-1}e^{i\pi\ell\frac{\mu^2}{\tau}} & -(-i\tau)^{-\frac{1}{2}}e^{i\pi\ell\frac{\mu^2}{\tau}}\boldsymbol{\Psi}\boldsymbol{D}_\ell(S)\\ \boldsymbol{0} & (-i\tau)^{-\frac{1}{2}}\boldsymbol{D}_\ell(S)\end{pmatrix},$$

where $\boldsymbol{D}_\ell(S)$ is given above and $\boldsymbol{\Psi} = (\Psi_a)_{a=0,1,\dots,\ell-1}$ is the row vector with the components $\Psi_a = \Phi(\ell\tau, \ell\mu - a\tau)$.

Example: $\ell=1$. The above formulas become somewhat more transparent in the simplest case $\ell=1$. For $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_{1,2}$ with $c>0$, we then have

$$\mathcal{K}_1(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\tau, \frac{\nu}{c\tau+d}, \frac{\mu}{c\tau+d}) = (c\tau+d)\, e^{i\pi\frac{c(\nu^2-\mu^2)}{c\tau+d}}\,\mathcal{K}_1(\tau,\nu,\mu) + \zeta_{c,d}\, i\sqrt{-i}\,\frac{c\tau+d}{\sqrt{c}} e^{i\pi\frac{c\nu^2}{c\tau+d}} \sum_{\alpha=0}^{c-1} e^{2i\pi\mu\frac{\alpha d}{c\tau+d} - i\pi\tau\frac{\alpha^2 d}{c\tau+d}}\,\Phi(\tau+\frac{d}{c}, \mu+\alpha\frac{d}{c})\,\vartheta(\tau,\nu).$$

For the section $\mathsf{K}_1(\tau,\nu,\mu)$ in Lemma **3.1**, we therefore have

$$\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \,\boldsymbol{.}\, \mathsf{K}_1 = \mathsf{K}_1, \qquad \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_{1,2},$$

where the action is defined as above with the 2×2 automorphy factor (for $c>0$)

$$\mathsf{J}_1(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right);\tau,\nu,\mu)=(c\tau+d)^{-\frac{1}{2}}e^{-i\pi\frac{c\nu^2}{c\tau+d}}\\
\times\begin{pmatrix}(c\tau{+}d)^{-\frac{1}{2}}e^{i\pi\frac{c\mu^2}{c\tau+d}} & -\frac{i\sqrt{-i}}{\sqrt{c}}e^{i\pi\frac{c\mu^2}{c\tau+d}}\sum\limits_{\alpha=0}^{c-1}e^{i\pi\alpha d\frac{2\mu-\tau\alpha}{c\tau+d}}\Phi(\tau{+}\frac{d}{c},\mu{+}\alpha\frac{d}{c})\\ 0 & \zeta_{c,d}^{-1}\end{pmatrix}.$$

3.2.4. Proof of Theorem 3.5. We first evaluate the integral representation (2.16) with (τ,ν,μ) transformed by an element $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_{1,2}$; using that $a+c$ and $b+d$ are then odd and applying (3.5), we obtain

$$\mathcal{K}_\ell(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\tau,\frac{\nu}{c\tau{+}d},\frac{\mu}{c\tau{+}d})=\int\limits_0^{c\tau+d}d\lambda\,\vartheta\Big(\ell\frac{a\tau{+}b}{c\tau{+}d},\frac{\ell\nu{-}\lambda}{c\tau{+}d}\Big)\;e^{2i\pi\frac{c}{c\tau+d}(\nu+\mu)\lambda}\\
\times\frac{e^{-i\frac{\pi\tau}{4}}\,\vartheta(\tau,\nu{+}\mu{+}\lambda{+}\frac{\tau+1}{2})\eta(\tau)^3}{\vartheta(\tau,\nu{+}\mu{+}\frac{\tau+1}{2})\vartheta(\tau,\lambda{+}\frac{\tau+1}{2})}.\tag{3.8}$$

We next assume that ac is divisible by ℓ. Equation (3.7) then gives

$$\mathcal{K}_\ell(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\tau,\frac{\nu}{c\tau+d},\frac{\mu}{c\tau+d})=\\
=\zeta_{\frac{c}{\ell_c},\ell_a d}\,\frac{\sqrt{c\tau+d}}{\sqrt{\ell_a}}\,e^{i\pi\frac{c\ell\nu^2}{c\tau+d}}\sum_{r=0}^{\ell_a-1}\int\limits_0^{c\tau+d}d\lambda\,e^{i\pi\frac{c(\lambda^2+2\ell\lambda\mu)}{\ell(c\tau+d)}+2i\pi rc(\mu+\frac{\lambda}{\ell})+i\pi\frac{r^2c^2}{\ell}\tau}\\
\times\vartheta(\ell\tau,\ell\nu{-}(\lambda{+}rc\tau))\frac{e^{-i\frac{\pi\tau}{4}}\,\vartheta(\tau,\nu{+}\mu{+}\lambda{+}rc\tau{+}\frac{\tau+1}{2})\eta(\tau)^3}{\vartheta(\tau,\nu{+}\mu{+}\frac{\tau+1}{2})\vartheta(\tau,\lambda{+}rc\tau{+}\frac{\tau+1}{2})},$$

where we also shifted the theta function arguments using quasiperiodicity in order to have $\lambda+rc\tau$ in the ratio of the theta functions. This allows us to apply Lemma **2.2**, with the result

$$\mathcal{K}_\ell(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\tau,\frac{\nu}{c\tau+d},\frac{\mu}{c\tau+d})=\\
=\zeta_{\frac{c}{\ell_c},\ell_a d}\,\frac{\sqrt{c\tau+d}}{\sqrt{\ell_a}}\,e^{i\pi\frac{c\ell\nu^2}{c\tau+d}}\sum_{r=0}^{\ell_a-1}\int\limits_0^{c\tau+d}d\lambda\,e^{i\pi\frac{c(\lambda^2+2\ell\lambda\mu)}{\ell(c\tau+d)}+2i\pi rc(\mu+\frac{\lambda}{\ell})+i\pi\frac{r^2c^2}{\ell}\tau}\\
\times\Big(\vartheta(\ell\tau,\ell\mu{+}\lambda{+}rc\tau)\mathcal{K}_\ell(\tau,\nu,\mu)\\
-\sum_{k=0}^{\ell-1}e^{2i\pi k(\nu+\mu)}\vartheta(\ell\tau,\ell\nu+k\tau)\mathcal{K}_1(\ell\tau,-\ell\mu{-}(\lambda{+}rc\tau),\ell\mu{-}k\tau)\Big).\tag{3.9}$$

This therefore consists of two terms, with the integration variable λ involved in the argument of the theta function in the first term and in the argument of $\mathcal{K}_1$ in the second term. In the first term, we change the variable as $\lambda\to\lambda-r(c\tau+d)-\ell\mu$ and use (A.3).

In the second term, we change the variable as $\lambda \to \lambda - r(c\tau + d)$ and then use (A.7) with $c \to \frac{c}{\ell_c}$, $d \to \ell_a d$, $\tau \to \ell\tau$, and $\mu \to \ell\mu - k\tau$. This gives

$$\mathcal{K}_\ell\big(\big(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\big)\,\tau, \frac{\nu}{c\tau+d}, \frac{\mu}{c\tau+d}\big) = (c\tau+d)e^{i\pi\frac{c\ell(\nu^2-\mu^2)}{c\tau+d}}\,\mathcal{K}_\ell(\tau,\nu,\mu)$$
$$+ \zeta_{\frac{c}{\ell_c},\ell_a d}\, i\sqrt{-i}\sqrt{\frac{\ell_c}{c}}\,(c\tau+d)\, e^{i\pi\frac{c\ell\nu^2}{c\tau+d}}\sum_{k=0}^{\ell-1} e^{2i\pi k\nu + i\pi\frac{k^2}{\ell}\tau}\,\vartheta(\ell\tau, \ell\nu+k\tau)$$
$$\times\, e^{i\pi\frac{kd(2\ell\mu-k\tau)}{\ell(c\tau+d)}}\sum_{\alpha=0}^{\frac{c}{\ell_c}-1} e^{2i\pi(\ell\mu-k\tau)\frac{\alpha d}{c\tau+d} - i\pi\tau\frac{\alpha^2 d}{c\tau+d}}\,\Phi(\ell\tau+\ell\frac{d}{c}, \ell\mu+\alpha\ell\frac{d}{c}-k\tau).$$

This shows the desired behavior of the first element in $\mathsf{K}_\ell(\tau,\nu,\mu)$. The rest of the calculation leading to the statement of Theorem **3.5** involves only theta functions and is therefore standard.

4. Modular Transformations of $\widehat{s\ell}(2|1)$ Characters

As an application of the higher-level Appell functions, we consider the "admissible" representations of the affine Lie superalgebra $\widehat{s\ell}(2|1)$ at the level $k = \frac{\ell}{u} - 1$ with coprime positive integers ℓ and u. For $\ell \geqslant 2$, neither these representations nor their characters are periodic under the spectral flow (B.5), the characters cannot therefore be rationally expressed through theta functions, and Appell functions enter the game.

The spectral flow is an action of the $\mathbb{Z}$ lattice, and because it acts on the admissible representation characters freely, there are infinitely many representations involved and the theory is certainly nonrational. It might then be expected that defining the modular group action would also require infinitely many characters. But the actual situation turns out to be somewhat closer to the case of rational conformal field theories: if extensions among the representations are taken into account, the spectral flow and the modular group action can be defined on a finite number of characters. For the spectral flow, this is shown by elementary manipulations, but the calculation of the action of $S \in SL(2,\mathbb{Z})$ is more complicated. The resulting formula for the S-transform of the characters in Theorem **4.1** resembles that for the Appell functions: although the characters are not closed under the S transformation, the offending terms are given by Φ times theta-functional terms, whose modular properties are already standard.

We have to consider a number of facts pertaining to the $\widehat{s\ell}(2|1)$ representation theory. We follow [13],[4] with most of the representation-theory part collected in Appendix B. Calculations with the Appell functions are given in Secs. 4.2 and 4.3 below.

4.1. Formulation of the main result. In Theorem **B.1**, we find the characters $\chi_{r,s,\ell,u;\theta}$ of the admissible $\widehat{s\ell}(2|1)$-representation $\mathcal{L}_{r,s,\ell,u;\theta}$ in the Ramond sector. The $_{;\theta}$ notation

[4] See also [31, 32] for aspects of the $\widehat{s\ell}(2|1)$-representation theory at fractional level.

is for the spectral flow transform, see (B.5) [13, 33]. The four different sectors (Ramond, Neveu–Schwarz, super-Ramond, and super-Neveu–Schwarz) are mapped under the S and T transformations as indicated in (B.35). Any of the S-arrows in (B.35) allows reconstructing any other, and it is therefore a matter of taste which of these to evaluate explicitly. The super-Ramond sector is chosen in the next theorem. With theta-functional terms inevitably occurring in the S-transform of $\chi_{r,s,\ell,u;\theta}$, such terms can be added to the χ characters from the start (as we see in Sec. 4.2, the theta-functional terms in question are actually the characters defined in (B.30)). It then turns out that to avoid redundancy, we can label the admissible characters by (s, θ) with $s = 1, \dots, u$ and $\theta = 0, \dots, u-1$ (see (4.7)). We then have the following result.

Theorem 4.1. *At the level $k = \frac{\ell}{u} - 1$ with coprime positive integers ℓ and u, the S-transform of the super-Ramond admissible $\widehat{s\ell}(2|1)$-characters $\chi^{\mathrm{sR}}_{(s;\theta)}$ is given by*

$$\begin{aligned}(4.1)\quad \chi^{\mathrm{sR}}_{(s;\theta)}\Big(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\Big) = e^{i\pi k\frac{\nu^2-\mu^2}{2\tau}} \sum_{s'=1}^{u}\sum_{\theta'=0}^{u-1} S^{\ell,u}_{(s,\theta),(s',\theta')}\chi^{\mathrm{sR}}_{(s';\theta')}(\tau,\nu,\mu)\\ - e^{i\pi k\frac{\nu^2-\mu^2}{2\tau}} \sum_{r'=1}^{\ell-1}\sum_{s'=1}^{u} R_{s,\theta,r',s'}(\tau,\mu)\Omega_{r',s'}(\tau,\nu,\mu+1),\end{aligned}$$

where the characters $\Omega_{r',s'}$ are defined in (B.30)*,*

$$S^{\ell,u}_{(s,\theta),(s',\theta')} = \frac{1}{u}\, e^{2i\pi\frac{\ell}{u}(s+s'+\theta+\theta'+s'\theta+s\theta'+2\theta\theta')} e^{i\pi\frac{[\ell(s'+1+2\theta')]_u-[\ell(s+2\theta+1)]_u}{u}},$$

and

$$\begin{aligned}R_{s,\theta,r',s'}(\tau,\mu) &= \frac{(-1)^{r's}}{u}\, e^{i\pi\frac{\ell}{2u\tau}\left(\mu+\tau(s'+1-\frac{u}{\ell}r')\right)^2 - i\pi\frac{[\ell(s+2\theta+1)]_u}{u} + i\pi\frac{\ell}{u}(2s'+s-ss')}\\ &\times\sum_{b=0}^{u-1}\Big(e^{i\pi(s+2\theta+1)\frac{2\ell b+[ur'+\ell(s'-1)]_{2\ell}}{u}}\Phi(2u\ell\tau, -\ell\mu-2b\ell\tau-[ur'-\ell(s'-1)]_{2\ell}\tau)\\ &- e^{i\pi(s+2\theta+1)\frac{2\ell(b+1)-[ur'+\ell(s'-1)]_{2\ell}}{u}}\Phi(2u\ell\tau, -\ell\mu-2(b+1)\ell\tau+[ur'-\ell(s'-1)]_{2\ell}\tau)\Big).\end{aligned}$$

We recall that $[x]_u$ denotes $x \bmod u$. The theorem is proved by a calculation based on the properties of the Appell functions established above. The several-step derivation is given in Sec. 4.3.

The S-transformation formula in the theorem has a triangular structure similar to that for the Appell functions: the $\chi_{(s;\theta)}$ characters are transformed through themselves *and* the additional characters $\Omega_{r',s'}$, while the latter, being expressed through theta functions, transform through themselves. The Ω characters in the right-hand side of (4.1) are multiplied with the Φ functions, defined in (1.4) and studied in Sec. 2.6. We note that the arguments of Φ above depend on s' "weakly," in fact only on $s' \bmod 2$.

4.2. Relation to the Appell functions and open quasiperiodicity. We first express the admissible representation characters (B.22) and (B.26) through the higher-level Appell functions: the "nontrivial" part of the characters, Eq. (B.23), is expressed through the Appell functions with the *even* level 2ℓ as

$$\begin{aligned}\psi_{r,s,\ell,u}(\tau,\nu,\mu) = {}&\mathcal{K}_{2\ell}(u\tau, -\tfrac{\nu}{2} - \tfrac{u(r-1)}{2\ell}\tau + \tfrac{s-1}{2}\tau, \tfrac{1}{2} - \tfrac{\mu}{2} + \tau\tfrac{u(r-1)}{2\ell} - \tau\tfrac{s+1}{2})\\ &- e^{2i\pi(s-1)(r-1)\tau - 2i\pi(r-1)\nu}\\ &\quad\times \mathcal{K}_{2\ell}(u\tau, \tfrac{\nu}{2} - \tfrac{u(r-1)}{2\ell}\tau - \tfrac{s-1}{2}\tau, \tfrac{1}{2} - \tfrac{\mu}{2} + \tau\tfrac{u(r-1)}{2\ell} - \tau\tfrac{s+1}{2}),\end{aligned}\tag{4.2}$$

and hence the characters are given by

$$\begin{aligned}\chi_{r,s,\ell,u;\theta}(\tau,\nu,\mu) = {}&\boldsymbol{\Theta}(\tau,\nu,\mu)\\ &\times e^{2i\pi(\theta+1)(r-1-\frac{\ell}{u}(s+\theta))\tau + i\pi(r-1-\frac{\ell}{u}(s-1))\nu + i\pi(r-1-\frac{\ell}{u}(s+1+2\theta))\mu}\\ &\times \Bigl(\mathcal{K}_{2\ell}(u\tau, -\tfrac{\nu}{2} - \tfrac{u(r-1)}{2\ell}\tau + \tfrac{s-1}{2}\tau, \tfrac{1-\mu}{2} + \tau\tfrac{u(r-1)}{2\ell} - \tau\tfrac{s+1+2\theta}{2})\\ &\quad - e^{2i\pi(s-1)(r-1)\tau - 2i\pi(r-1)\nu}\\ &\quad\times \mathcal{K}_{2\ell}(u\tau, \tfrac{\nu}{2} - \tfrac{u(r-1)}{2\ell}\tau - \tfrac{s-1}{2}\tau, \tfrac{1-\mu}{2} + \tau\tfrac{u(r-1)}{2\ell} - \tau\tfrac{s+1+2\theta}{2})\Bigr).\end{aligned}\tag{4.3}$$

We now discuss the range of the labels in these characters. First, $1 \leqslant s \leqslant u$. Second, the twist θ takes all integer values in principle, but modulo addition of theta-functional terms to the characters, the θ parameter can be restricted to u consecutive values, because quasiperiodicity of $\mathcal{K}_\ell$ implies an open quasiperiodicity property relating the admissible $\widehat{s\ell}(2|1)$ characters with their spectral-flow transform by $\theta = u$ (see (B.5)),

$$\begin{aligned}\chi_{r,s,\ell,u;\theta+u} = \chi_{r,s,\ell,u;\theta} &- \sum_{a=1}^{r-1}(-1)^{a+r+1}\Omega_{a,s;\theta}\\ &+ \sum_{a=1}^{\ell-1}(-1)^{a+r+1}\overline{\Omega}_{a,s;\theta+s+1} - \sum_{a=r}^{\ell-1}(-1)^{a+r+1}\overline{\Omega}_{a,s;\theta+u},\end{aligned}\tag{4.4}$$

where $\Omega_{r,s,j}$ are the characters (B.28), expressible in terms of theta functions. Thus, although the admissible $\widehat{s\ell}(2|1)$-characters are not invariant under the spectral flow (B.5) with any θ (i.e., are mapped into nonisomorphic representations), their characters are invariant modulo theta-functional terms.[5]

[5] This is somewhat similar to the case with the admissible $\widehat{s\ell}(2)$ representations, which are not invariant under any spectral flow transformations, whereas their characters, given by quasiperiodic functions, are invariant under a certain sublattice of spectral flow translations. (That is not a contradiction because the admissible representation characters are *meromorphic* functions of the variable that is translated under the spectral flow, cf. [14], and we are actually speaking of *analytic continuation* of characters.)

Remark: $\boldsymbol{\ell=1}$. For $\ell=1$, Eq. (2.26) allows us to express the characters in terms of theta functions, which immediately shows that the characters are periodic under the spectral flow with the period u and readily leads to their modular transformation properties (the modular transformations for $\ell=1$ were derived in [34]). This originates in the fact that $\mathcal{U}_\theta$ with $\theta=u$ acts as an isomorphism on the representations $\mathcal{L}_{1,s,1,u}$ (the representations are invariant under the β automorphism (B.4), and accordingly, only one of the two values of r remains for $\ell=1$; for the characters, this can be easily verified using formulas in Sec. 2.1). There remain u^2 representations $\mathcal{L}_{1,s,1,u;\theta}$ with $1\leqslant s\leqslant u$ and $0\leqslant\theta\leqslant u-1$ [33].

We assume $\ell\geqslant 2$ in what follows and often abbreviate the notation $\chi_{r,s,\ell,u}$, $\psi_{r,s,\ell,u}$, etc. to $\chi_{r,s}$, $\psi_{r,s}$, etc.

Next, it follows from formulas in Sec. 2.1 that the dependence of the characters on r also amounts to additive theta-functional terms; for $r\geqslant 2$, $\psi_{r,s}$ is related to $\psi_{1,s}$ as

$$\psi_{r,s}(q,x,y)=(-x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^{-1})^{r-1}\Bigl(\psi_{1,s}(q,x,y)+\Theta_{r,s}(q,x,y)\Bigr), \tag{4.5}$$

where

$$\Theta_{r,s}(q,x,y)=\sum_{r'=1}^{r-1}(-1)^{r'}y^{\frac{r'}{2}}q^{-\frac{ur'^2}{4\ell}+\frac{r'(s+1)}{2}}\left(\theta_{r',\ell}(q^u,xq^{-(s-1)})-\theta_{-r',\ell}(q^u,xq^{-(s-1)})\right).$$

Consequently, the corresponding formula relating $\chi_{r,s}$ to $\chi_{r',s}$ involves the $\Omega_{r,s,h}$ characters (B.28):

$$\begin{aligned}\chi_{r,s;\theta}(q,x,y)&=(-1)^{r-r'}\chi_{r',s;\theta}(q,x,y)\\&+\begin{cases}\displaystyle\sum_{a=0}^{r-r'-1}(-1)^a\Omega_{r-1-a,s,\frac{r-1-a}{2}-\frac{\ell}{u}\frac{s+1+2\theta}{2}}(q,x,y), & r'<r,\\ \displaystyle\sum_{a=0}^{r'-r-1}(-1)^a\Omega_{r+a,s,\frac{r+a}{2}-\frac{\ell}{u}\frac{s+1+2\theta}{2}}(q,x,y), & r'>r.\end{cases}\end{aligned} \tag{4.6}$$

This is another instance of a triangular structure, with the characters being invariant only up to theta-functional terms, which are already invariant.

To avoid redundancy, therefore, any fixed value of r can be used, for example only the characters $\chi_{1,s,\ell,u;\theta}$ can be considered. More generally, we can choose a unique r for each (s,θ). As we see in what follows, a useful such choice is to set $r=\bigl\lfloor\frac{\ell}{u}(s+2\theta+1)\bigr\rfloor+1$. We use the special notation for these characters,

$$\chi_{(s;\theta)}=\chi_{\lfloor\frac{\ell}{u}(s+2\theta+1)\rfloor+1,s,\ell,u;\theta},\qquad 1\leqslant s\leqslant u,\ 0\leqslant\theta\leqslant u-1. \tag{4.7}$$

4.3. Evaluating the S-transform of $\chi_{r,s,\ell,u;\theta}$. The S-transform of $\chi_{r,s;\theta}(\tau,\nu,\mu)\equiv\chi_{r,s,\ell,u;\theta}(\tau,\nu,\mu)$ is found in several steps.

With the aim to use Eq. (4.5), which allows doing the most difficult part of the calculation for $r=1$ only, we first rewrite the character as

$$\chi_{r,s;\theta}(\tau,\nu,\mu)=e^{-2i\pi\frac{\ell}{u}(\theta+1)(s+\theta)\tau-i\pi\frac{\ell}{u}(s-1)\nu-i\pi\frac{\ell}{u}(s+1+2\theta)\mu}\\ \times(-1)^{r-1}\big(\psi_{1,s}+\Theta_{r,s}\big)(\tau,\nu,\mu{+}2\theta\tau)\,\boldsymbol{\Theta}(\tau,\nu,\mu).$$

In evaluating $\chi_{r,s;\theta}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})$, we then use Eqs. (1.9)–(1.15) to find the S-transform of $\boldsymbol{\Theta}$, which gives

$$\chi_{r,s;\theta}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})=(-1)^{r-1}\frac{1}{\tau}\,e^{2i\pi(\theta+1)\frac{\ell}{u}(s+\theta)\frac{1}{\tau}}\,e^{-i\pi(s-1)\frac{\ell}{u}\frac{\nu}{\tau}}\,e^{-i\pi\frac{\ell}{u}(s+1+2\theta)\frac{\mu}{\tau}}\\ \times\big(\psi_{1,s}+\Theta_{r,s}\big)(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu-2\theta}{\tau})\,e^{i\pi\frac{(\mu-\tau)^2-\nu^2}{2\tau}}\boldsymbol{\Theta}(\tau,\nu,\mu{-}\tau{+}1),$$

where it remains to find the S-transform of $\psi_{1,s}+\Theta_{r,s}$. For $\Theta_{r,s}$, the calculation is again standard, based on

$$\theta_{a,\ell}(-\frac{u}{\tau},\frac{\nu}{\tau})-\theta_{-a,\ell}(-\frac{u}{\tau},\frac{\nu}{\tau})=\sqrt{\frac{-i\tau}{2\ell u}}\,e^{i\pi\frac{\ell\nu^2}{2u\tau}}\sum_{r'=1}^{2\ell-1}e^{-i\pi\frac{ar'}{\ell}}\big(\theta_{r',\ell}(\frac{\tau}{u},\frac{\nu}{u})-\theta_{-r',\ell}(\frac{\tau}{u},\frac{\nu}{u})\big).$$

For $\psi_{1,s}$, we express through the Appell functions as in (4.2) and use the S-transform formula (1.3), which we rewrite for the level 2ℓ,

$$\mathcal{K}_{2\ell}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})=\tau e^{2i\pi\ell\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_{2\ell}(\tau,\nu,\mu)+\tau e^{2i\pi\ell\frac{\nu^2}{\tau}}\sum_{r'=0}^{2\ell-1}\Phi(2\ell\tau,2\ell\mu{-}r'\tau)\theta_{r',\ell}(\tau,2\nu).$$

This gives

$$\psi_{1,s}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})=\frac{\tau}{u}\,e^{i\pi\ell\frac{(\nu+s-1)^2-(\mu-\tau-s-1)^2}{2u\tau}}\\ \times\big(\mathcal{K}_{2\ell}(\frac{\tau}{u},-\frac{\nu{+}s{-}1}{2u},-\frac{\mu{-}\tau{-}s{-}1}{2u})-\mathcal{K}_{2\ell}(\frac{\tau}{u},\frac{\nu{+}s{-}1}{2u},-\frac{\mu{-}\tau{-}s{-}1}{2u})\big)\\ -\frac{\tau}{u}\,e^{i\pi\ell\frac{(\nu+s-1)^2}{2u\tau}}\sum_{r'=0}^{2\ell-1}\Phi(\frac{2\ell\tau}{u},-\ell\,\frac{\mu{-}\tau{-}s{-}1}{u}-r'\frac{\tau}{u})\\ \times\big(\theta_{r',\ell}(\frac{\tau}{u},\frac{\nu{+}s{-}1}{u})-\theta_{-r',\ell}(\frac{\tau}{u},\frac{\nu{+}s{-}1}{u})\big).$$

Putting the ψ and Θ parts together, we use the second identity in (2.29) to obtain

$$(\psi_{1,s}+\Theta_{r,s})(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})=\frac{\tau}{u}\,e^{i\pi\ell\frac{(\nu+s-1)^2-(\mu-\tau-s-1)^2}{2u\tau}}\\ \times\Big(\mathcal{K}_{2\ell}(\frac{\tau}{u},-\frac{\nu{+}s{-}1}{2u},-\frac{\mu{-}\tau{-}s{-}1}{2u})-\mathcal{K}_{2\ell}(\frac{\tau}{u},\frac{\nu{+}s{-}1}{2u},-\frac{\mu{-}\tau{-}s{-}1}{2u})\\ -\boldsymbol{\mathfrak{F}}_{r,s}(\tau,\nu,\mu{-}\tau)\Big), \tag{4.8}$$

where

$$\mathfrak{F}_{r,s}(\tau,\nu,\mu) = e^{i\pi\frac{\ell}{2u\tau}(\mu-s-1+\frac{u}{\ell}(r-1))^2}\sum_{r'=1}^{2\ell-1} e^{i\pi r'\frac{r-1}{\ell}} \times \Phi(\frac{2\ell\tau}{u},-\frac{\ell}{u}(\mu-s-1)-\frac{r'\tau}{u}-r+1)\big(\theta_{r',\ell}(\frac{\tau}{u},\frac{\nu+s-1}{u})-\theta_{-r',\ell}(\frac{\tau}{u},\frac{\nu+s-1}{u})\big). \tag{4.9}$$

Therefore, $\chi_{r,s;\theta}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})$ is evaluated as

$$\begin{aligned}\chi_{r,s;\theta}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau}) &= \frac{(-1)^{r-1}}{u}\, e^{i\pi k\frac{\nu^2-(\mu-\tau)^2}{2\tau}\,-\,i\pi\frac{\ell}{u}(s+1+2\theta)}\,\boldsymbol{\Theta}(\tau,\nu,\mu-\tau+1)\\ &\times\Big(\mathcal{K}_{2\ell}(\frac{\tau}{u},-\frac{\nu+s-1}{2u},-\frac{\mu-\tau-s-2\theta-1}{2u})-\mathcal{K}_{2\ell}(\frac{\tau}{u},\frac{\nu+s-1}{2u},-\frac{\mu-\tau-s-2\theta-1}{2u})\Big)\\ &+\frac{(-1)^r}{u}\, e^{i\pi k\frac{\nu^2-(\mu-\tau)^2}{2\tau}\,-\,i\pi\frac{\ell}{u}(s+1+2\theta)}\mathfrak{F}_{r,s}(\tau,\nu,\mu-\tau-2\theta)\,\boldsymbol{\Theta}(\tau,\nu,\mu-\tau+1),\end{aligned} \tag{4.10}$$

where $k=\frac{\ell}{u}-1$ is the $\widehat{s\ell}(2|1)$ level.

The next step is to show that the $\mathcal{K}_{2\ell}$-terms in (4.10) are expressible through the χ characters and the theta-functional terms are expressible through the Ω characters. We first show that the term involving $\mathfrak{F}_{r,s}$, which has arisen in form (4.9), can be expressed through the characters $\Omega_{r',s'}$ introduced in (B.30). Elementary manipulations show that

$$\begin{aligned}\mathfrak{F}_{r,s}(\tau,\nu,\mu) &= e^{i\pi\frac{\ell}{2u\tau}(\mu-s-1+\frac{u}{\ell}(r-1))^2}\\ &\times\frac{1}{2\ell}\sum_{a=r-2\ell}^{r-1}\sum_{r'=1}^{2\ell-1} e^{i\pi r'\frac{r-1-a}{\ell}}\,\Phi(\frac{2\ell\tau}{u},-\frac{\ell}{u}(\mu-s-1)-\frac{r'\tau}{u}-r+1)\\ &\times\big(\vartheta(\frac{\tau}{2\ell u},\frac{\nu+s-1}{2u}+\frac{a}{2\ell})-\vartheta(\frac{\tau}{2\ell u},\frac{\nu+s-1}{2u}-\frac{a}{2\ell})\big).\end{aligned}$$

Here, the range of the a summation (2ℓ consecutive values) can be shifted arbitrarily, and it was chosen such that (2.33) becomes applicable with no remanining $[\cdot]_{2\ell}$. Applying (2.33) to Φ and rearranging the ϑ-part in accordance with (2.10), we then have

$$\begin{aligned}\mathfrak{F}_{r,s}(\tau,\nu,\mu) &= \frac{1}{2\ell}\sum_{a=r-2\ell}^{r-1} e^{i\pi\frac{\ell}{2u\tau}(\mu-s-1+\frac{u}{\ell}a)^2}\Phi(\frac{\tau}{2\ell u},-\frac{\mu-s-1}{2u}-\frac{a}{2\ell})\\ &\times\sum_{r''=1}^{2\ell-1}\sum_{s''=1}^{u} e^{i\pi\frac{u}{\ell}ar''-i\pi\frac{\ell}{u}(s-1)(s''-1)+i\pi a(s''-1)+i\pi(s-1)r''}\,\varpi_{r'',s''}(\tau,\nu),\end{aligned}$$

where

$$\varpi_{r,s}(q,x) = x^{-\frac{\ell}{u}\frac{s-1}{2}}q^{\frac{\ell}{4u}(s-1)^2}\big(\theta_{r,\ell}(q^u,xq^{-(s-1)})-\theta_{-r,\ell}(q^u,xq^{-(s-1)})\big)$$

accumulates the y- and h- independent factors in the character (B.28). From identity (2.35), we now have

$$\mathfrak{F}_{r,s}(\tau,\nu,\mu) = e^{i\pi\frac{u\ell}{2\tau}(\frac{\mu-s-1}{u}+\frac{r-1}{\ell})^2}\sum_{r'=1}^{2\ell-1}\sum_{s'=1}^{u} e^{i\pi\frac{u}{\ell}\left(r-1-\frac{\ell}{u}(s-1)\right)\left(r'+\frac{\ell}{u}(s'-1)\right)} \\ \times\Phi\Big(\frac{2\ell\tau}{u},-\frac{\ell}{u}(\mu-s-1)-r+1-\frac{[ur'-\ell(s'-1)]_{2\ell}}{u}\tau\Big)\varpi_{r',s'}(\tau,\nu).$$

Next, identity (2.32) (with $p=u$) allows us to express the Φ functions involved here through $\Phi(2u\ell\tau,\,\ldots-2\ell b\tau)$. This produces the integer $\ell(s+1)-u(r-1)$ in the argument of each Φ, to which we can further apply (2.29), with the result

$$\mathfrak{F}_{r,s}(\tau,\nu,\mu) = e^{i\pi\frac{\ell}{u}\frac{\mu^2}{2\tau}}\sum_{r'=1}^{2\ell-1}\sum_{s'=1}^{u}\sum_{b=0}^{u-1} e^{-i\pi(s-1)(r'+\frac{\ell}{u}(s'-1))+2i\pi b\frac{\ell}{u}(s+1)} \\ \times e^{i\pi(s+1)\frac{[ur'+\ell(s'-1)]_{2\ell}}{u}}\Phi(2u\ell\tau,-\ell\mu-[ur'-\ell(s'-1)]_{2\ell}\tau-2b\ell\tau)\varpi_{r',s'}(\tau,\nu) \\ + e^{i\pi\frac{u\ell}{2\tau}\left(\frac{\mu-s-1}{u}+\frac{r-1}{\ell}\right)^2}\sum_{r'=1}^{2\ell-1}\sum_{s'=1}^{u} e^{i\pi\frac{u}{\ell}\left(r-1-\frac{\ell}{u}(s-1)\right)\left(r'+\frac{\ell}{u}(s'-1)\right)} \\ \times H_{\ell(s+1)-u(r-1),ur'-\ell(s'-1)}(\tau,\mu)\varpi_{r',s'}(\tau,\nu),$$

where

$$H_{m,m'}(\tau,\mu) = \begin{cases} \dfrac{iu}{\sqrt{-2iu\ell\tau}}\displaystyle\sum_{\substack{n\geqslant 1\\ nu\leqslant m}} e^{i\pi\frac{n^2u-2nm}{2\ell\tau}+i\pi n(\frac{\mu}{\tau}+\frac{m'}{\ell})}, & m\geqslant 1,\\ -\dfrac{iu}{\sqrt{-2iu\ell\tau}}\displaystyle\sum_{\substack{n\leqslant 0\\ nu\geqslant m+1}} e^{i\pi\frac{n^2u-2nm}{2\ell\tau}+i\pi n(\frac{\mu}{\tau}+\frac{m'}{\ell})}, & m\leqslant -1.\end{cases}$$

We observe that the first (triple-sum) term in $\mathfrak{F}_{r,s}$ is actually independent of r. The second term does depend on r, but $H_{m,m'}$ vanishes whenever $1\leqslant m\leqslant u-1$, and because we consider such m in what follows (more precisely, r such that the corresponding m is within this interval), we drop the second (double-sum) term in $\mathfrak{F}_{r,s}$. More precisely, Eq. (4.10) actually involves $\mathfrak{F}_{r,s}$ with the argument shifted by -2θ with integer θ. The integer in the argument of Φ mentioned above, and hence the m label in $H_{m,m'}$, is then $\ell(s+2\theta-1)-u(r-1)$. With r chosen such that $1\leqslant m\leqslant u-1$, it then follows that $\mathfrak{F}_s(\tau,\nu,\mu-2\theta)\equiv\mathfrak{F}_{r,s}(\tau,\nu,\mu-2\theta)$ is given by

$$\mathfrak{F}_s(\tau,\nu,\mu-2\theta) = e^{i\pi\frac{\ell}{u}\frac{\mu^2}{2\tau}}\sum_{r'=1}^{2\ell-1}\sum_{s'=1}^{u}\sum_{b=0}^{u-1} e^{-i\pi(s-1)(r'+\frac{\ell}{u}(s'-1))+2i\pi b\frac{\ell}{u}(s+2\theta+1)} \\ \times e^{i\pi(s+2\theta+1)\frac{[ur'+\ell(s'-1)]_{2\ell}}{u}}\Phi(2u\ell\tau,-\ell\mu-[ur'-\ell(s'-1)]_{2\ell}\tau-2b\ell\tau)\varpi_{r',s'}(\tau,\nu).$$

Using (B.29) and the fact that $[-a]_n = n - [a]_n$ for $a > 0$, we obtain

$$(4.11)\quad \mathfrak{F}_s(\tau,\nu,\mu{-}2\theta) = e^{i\pi\frac{\ell}{u}\frac{\mu^2}{2\tau}} \sum_{r'=1}^{\ell-1}\sum_{s'=1}^{u}\sum_{b=0}^{u-1} e^{-i\pi(s-1)(r'+\frac{\ell}{u}(s'-1))}$$
$$\times\bigl(e^{i\pi(s+2\theta+1)\frac{[ur'+\ell(s'-1)]_{2\ell}+2\ell b}{u}}\Phi(2u\ell\tau,-\ell\mu{-}[ur'{-}\ell(s'{-}1)]_{2\ell}\tau{-}2b\ell\tau)$$
$$-\,e^{i\pi(s+2\theta+1)\frac{-[ur'+\ell(s'-1)]_{2\ell}+2\ell(b+1)}{u}}\Phi(2u\ell\tau,-\ell\mu{+}[ur'{-}\ell(s'{-}1)]_{2\ell}\tau{-}2(b{+}1)\ell\tau)\bigr)$$
$$\times\,\varpi_{r',s'}(\tau,\nu).$$

To finish the calculation, it remains to recognize the χ characters in the first term in (4.10). Lemma **2.1** gives

$$(4.12)\quad \chi_{r,s;\theta}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) = \frac{(-1)^{r-1}}{u}\,e^{i\pi k\frac{\nu^2-(\mu-\tau)^2}{2\tau}\;-\;i\pi\frac{\ell}{u}(s+1+2\theta)}$$
$$\times\Bigl(\sum_{s'=1}^{u}\sum_{\theta'=0}^{u-1}e^{i\pi\frac{\ell}{u}(s'+1+2\theta')(s+1+2\theta)-i\pi\frac{\ell}{u}(s'-1)(s-1)+i\pi\frac{[\ell(s'+1+2\theta')]_u}{u}}\chi_{(s';\theta')}(\tau,\nu,\mu{-}\tau{+}1)$$
$$-\,\mathfrak{F}_{r,s}(\tau,\nu,\mu{-}\tau{-}2\theta)\,\boldsymbol{\Theta}(\tau,\nu,\mu{-}\tau{+}1)\Bigr).$$

The characters arising in the right-hand side are those in (4.7), and we can therefore restrict to the same characters in the left-hand side, i.e., choose $\chi_{(s;\theta)}$ as "representatives" of the $\chi_{r,s,\ell,u;\theta}(q,x,y)$ characters with different r (see (4.6)). With $\chi_{(s;\theta)}$ in the left-hand side, it is then easy to see that the above $H_{m,m'}$ terms indeed vanish.

The S-transform formula (4.12) applies to the characters $\chi_{r,s;\theta}\equiv\chi_{r,s,\ell,u;\theta}$ in the Ramond sector, which are related to super-Neveu–Schwarz characters under S, see (B.33). Accordingly, we can rewrite the above formula with the super-Neveu–Schwarz characters in the right-hand side,

$$(4.13)\quad \text{1st term in the rhs of (4.12)} = \frac{(-1)^{\boldsymbol{r}_{\ell,u}[s+1+2\theta]+1}}{u}\,e^{i\pi k\frac{\nu^2-\mu^2}{2\tau}\;-\;i\pi\frac{\ell}{u}(s+2+2\theta)}$$
$$\times\sum_{s'=1}^{u}\sum_{\theta'=0}^{u-1}e^{i\pi\frac{\ell}{u}(s'+1+2\theta')(s+1+2\theta)-i\pi\frac{\ell}{u}(s'-1)(s-1)+i\pi\frac{[\ell(s'+1+2\theta')]_u}{u}}\chi^{\mathrm{sNS}}_{(s';\theta')}(\tau,\nu,\mu).$$

The result given at the beginning of this section is for the super-Ramond sector, which is mapped into itself. Using (B.34) and recalling (4.11), we immediately rewrite (4.13) as (4.1).

5. Super-Virasoro characters

The higher-level Appell functions also arise in superconformal extensions of the Virasoro algebra.

5.1. $N=2$ **characters.** Following [14], we consider the admissible representations of the $N=2$ [super-Virasoro] algebra with central charge $c=3(1-\frac{2}{t})$, $t=\frac{u}{\ell}$. As in the $\widehat{s\ell}(2|1)$ case, the spectral flow transform acts freely on the representations, and therefore the theory is nonrational. The admissible representation characters are given by

$$\omega_{r,s,u,\ell}(q,z) = z^{-\frac{\ell}{u}(r-1)+s-1}\,\varphi_{r,s,u,\ell}(q,z)\,\frac{\vartheta_{1,0}(q,z)}{q^{-\frac{1}{8}}\eta(q)^3},$$
$$1\leqslant r\leqslant u-1,\quad 1\leqslant s\leqslant \ell,$$

where

$$\begin{aligned}\varphi_{r,s,u,\ell}(q,z) = \mathcal{K}_{2\ell}(q^u, q^{\frac{r}{2}-\frac{u}{2\ell}(s-1)}, -z^{-1}\,q^{-\frac{r}{2}+\frac{u}{2\ell}(s-1)})&\\ - q^{r(s-1)}\,\mathcal{K}_{2\ell}(q^u, q^{-\frac{r}{2}-\frac{u}{2\ell}(s-1)}, -z^{-1}q^{-\frac{r}{2}+\frac{u}{2\ell}(s-1)}).&\end{aligned}$$

The $N=2$ spectral flow acts on the character of any $N=2$-module $\mathfrak{D}$ as

$$\omega^{\mathfrak{D}}_{;\theta}(q,z) = z^{-\frac{c}{3}\theta}q^{\frac{c}{6}(\theta^2-\theta)}\omega^{\mathfrak{D}}(q,zq^{-\theta}),$$

with $\theta\in\mathbb{Z}$. For $\omega_{r,s,u,\ell}$ above, *open quasiperiodicity* occurs for the spectral flow transform with $\theta=u$ [14].

The modular transformations of $\omega_{r,s,u,\ell;\theta}$ can be derived either by repeating the calculations in Sec. 4 in the $N=2$ context or by noting that the $N=2$ characters follow by taking residues of the appropriate $\widehat{s\ell}(2|1)$-characters, and hence Theorem **4.1** implies the $N=2$ modular transformation formula. Taking the residues amounts to using the formulas

$$\vartheta_{1,1}(q,q^n)=0,\qquad \frac{\partial\vartheta_{1,1}(q,z)}{\partial z}\Big|_{z=q^n} = (-1)^n\,q^{-\frac{1}{8}}\eta(q)^3\,q^{-\frac{n^2}{2}-\frac{3n}{2}},\quad n\in\mathbb{Z}.$$

Noting that

$$\psi_{r,s,\ell,u}(q,q^n,z^2\,q^{2\theta}) = \varphi_{s-n-1,r,u,\ell}(q,z\,q^{\theta+1+\frac{n}{2}}),$$

we then immediately obtain that for even n,

$$\operatorname*{res}_{x=q^n}\chi_{r,s,\ell,u;\theta}(q,x,z^2) = z^{-k}\,q^{-k\frac{n^2}{4}+n}\,\frac{\vartheta_{1,0}(q,z)}{q^{-\frac{1}{8}}\eta(q)^3}\,\omega_{s-n-1,r,u,\ell;-(\theta+\frac{n}{2}+1)}(q,z).$$

The spectral flow transform by $-\theta$ (rather than θ) in the right-hand side is due to oppositely chosen conventions for the $N=2$ algebra in [14], which we reproduce for ease of comparison, and for $\widehat{s\ell}(2|1)$ in [13], which we follow here. The $\widehat{s\ell}(2|1)$-characters then correspond to $N=2$ characters in accordance with the relation between the $\widehat{s\ell}(2|1)$ and $N=2$ *representations* [35, 36] under the Hamiltonian reduction [37, 38, 39].

5.2. $N=4$ **characters.** Another application of the Appell functions is to models of the $N=4$ super-Virasoro algebra. To avoid lengthening an already sufficiently long paper, we only note that the unitary irreducible $N=4$ characters at central charge $c=6k$,

$k \in \mathbb{N}$, derived in [15], can be expressed through the level-$2(k+1)$ Appell functions as[6]

$$\begin{aligned}\mathrm{Char}^0_{k,j}(q,z,y) &= \frac{q^{j-\frac{2k-1}{8}}}{y^{-1}-y}\,\frac{\vartheta(q,zy)\vartheta(q,zy^{-1})}{\vartheta_{1,1}(q,z^2)\eta(q)^3}\\ &\times\Big(yz^{2j}\,\mathcal{K}_{2(k+1)}(q,zq^{\frac{2j+1}{2(k+1)}},-yq^{\frac{k-2j}{2(k+1)}}) - y^{-1}z^{2j}\,\mathcal{K}_{2(k+1)}(q,zq^{\frac{2j+1}{2(k+1)}},-y^{-1}q^{\frac{k-2j}{2(k+1)}})\\ &\quad - yz^{-2j}q^{-1}\,\mathcal{K}_{2(k+1)}(q,zq^{\frac{-2j+1}{2(k+1)}},-yq^{\frac{2j-k-2}{2(k+1)}})\\ &\qquad + y^{-1}z^{-2j}q^{-1}\,\mathcal{K}_{2(k+1)}(q,zq^{\frac{-2j+1}{2(k+1)}},-yq^{\frac{2j-k-2}{2(k+1)}})\Big),\quad j=0,\frac{1}{2},\dots,\frac{k}{2},\end{aligned}$$

which reduces evaluation of the modular transform of the characters to a calculation with Appell functions based on Theorem **1.1**.

6. Conclusions

We have investigated the modular properties of the Appell functions and used these to calculate modular transformations of characters in some nonrational conformal models. Expressing representation characters through higher-level Appell functions can be viewed as going one step up in functional complexity compared with the characters expressed through theta functions: while the characters are not quasiperiodic, the quasiperiodicity-violating terms are still given by theta functions. Efficient manipulations with the $\mathcal{K}_\ell$ functions, as in the study of $\widehat{s\ell}(2|1)$-characters, require using properties of the Φ function defined by integral representation (1.4) (which is at the same time the $\boldsymbol{b}$-period integral of $\mathcal{K}_\ell$, Eq. (A.6)). We have studied the properties of Φ in some detail.[7]

There are many rational models of conformal field theory, but nonrational models are also interesting. The theory of nonrational models is still in its infancy, however. The axioms of rational conformal field theory can be relaxed to different degrees, which in some cases gives "almost rational" theories whose structure may be worth studying, but difficulties in treating them in the same spirit as truly rational theories emerge at full scale in calculating the modular group representation on characters. The characters of nonrational models are usually not expressible in terms of theta functions; going beyond rational conformal field theories requires an adequate replacement of theta functions with some functions that are not quasiperiodic but nevertheless behave reasonably under modular transformations.

In the examples in this paper, the spectral flow transform action leads to infinite proliferation of representations, and at first sight also of characters to be involved in modular transform formulas. But the deviation from rational theories may be expected to be "soft" because the spectral-flow-transformed representations, although nonisomorphic,

[6]An alternative form of the character follows by applying Eq. (2.4) to each Appell function.

[7]The function $f(\tau,\mu) = -\Phi(-\frac{1}{\tau},\frac{\mu}{\tau})$ has appeared in [40], where the role of the integral representation was to give a solution of finite-difference equations (2.28) and (2.29), in a context not unrelated to the present one.

have "the same" structure. It turns out that at the expense of including extensions among the representations, a modular group action can be defined on a finite number of characters. Technically, this was achieved by first studying modular properties of the Appell functions, which demonstrate a triangular structure in their behavior under both lattice translations and modular transformations.

More specifically, we investigated the properties of $\widehat{s\ell}(2|1)$ models based on the set of admissible representations at rational level. A crucial property of these representations is that they allow nontrivial extensions among themselves. Such extensions do not occur in rational theories but are typical of logarithmic conformal field theories (see [41, 42, 26, 43, 44] and extensive bibliography therein; such extensions of representations have been known to play an important role in the derivation of the modular transformations of $N = 4$ superconformal characters since [15], where the corresponding characters are called "massive at the unitarity bound."). It might therefore be expected that the theory can be consistently formulated as a logarithmic one (i.e., further extension of modules results in modules where L_0 and/or some Cartan generators act nondiagonally). There also arises a very general problem of defining a reasonable class of nonrational conformal field theory models, where by "reasonable" we mean that the properties known in the rational case are *modified*, but *not dropped* in going beyond the rational models. Good examples are the $(1, p)$ Virasoro models [26] and, probably, the logarithmic extensions of all the (p', p) models. We hope that some features of this class of nonrational models have also been captured in this paper.

We have found the S-transform (4.1) of admissible characters in the sense that we expressed $\chi(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau})$ in terms of the $\chi(\tau, \nu, \mu)$ and $\Omega(\tau, \nu, \mu)$ characters. At the next step, we face the "$S(\tau)$ problem," which is a typical difficulty encountered in nonrational conformal field theories: the matrix representing the action of $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ on the characters acquires dependence on coordinates on the moduli space,

$$\boldsymbol{\chi}(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) \centerdot x) = S(x)\boldsymbol{\chi}(x),$$

where $\boldsymbol{\chi}(x)$ is a vector whose entries are the characters and x denotes coordinates on the moduli space ($x = (\tau, \nu, \dots)$, with $\gamma \centerdot (\tau, \nu, \dots) = (\frac{a\tau+b}{c\tau+d}, \frac{c\nu}{c\tau+d}, \dots)$ for $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in SL(2, \mathbb{Z})$). The problem is that the matrix $S(x)$ depends on x in general, making the Verlinde formula in its standard form inapplicable.

The general strategy to deal with the "$S(x)$ problem" was outlined in [26]. The modular group action on the characters is to be redefined as

$$\gamma * \boldsymbol{\chi}(x) = J(\gamma; x)\boldsymbol{\chi}(\gamma \centerdot x),$$

with a matrix automorphy factor $J(\gamma; x)$ such that the matrix

$$S = J(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right); x)S(x) \tag{6.1}$$

is numerical (independent of the coordinates on the moduli space).[8] Then S defined in (6.1) can be used in a Verlinde-like formula (an example of successful application of this ideology is given in [26]). For the $\widehat{s\ell}(2|1)$-characters, with the vector $\boldsymbol{\chi}$ composed of the χ *and* Ω characters, the most essential part of the "$S(x)$ problem" is the τ- and μ-dependence in the Φ functions arising in the S-transform. The Φ functions, studied in Sec. 2.6, are a characteristic element of the Appell function theory. We leave this problem for a future work.

We finally note that the derivation of modular transformation properties of $\mathcal{K}_\ell$ and the characters given above may not be "optimized" — apart from technical improvements, a more "conceptual" derivation must exist, possibly applicable to more general indefinite theta series.

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APPENDIX A. THE $\boldsymbol{a}$- AND $\boldsymbol{b}$-CYCLE INTEGRALS ON THE TORUS

We first evaluate the integral along the $\boldsymbol{b}$ cycle on the torus,

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\,\vartheta(\tau,\lambda) = \int_0^\tau d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\,\vartheta(\tau,\lambda) = \sum_{m\in\mathbb{Z}}\int_{m\tau}^{(m+1)\tau} d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}},$$

where we shifted the integration variable as $\lambda \to \lambda - m\tau$ in each term of the ϑ-series. For $\Im\tau > 0$, the integrals are defined by analytic continuation from $\tau = it$ with $t \in \mathbb{R}_{>0}$, and therefore

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\,\vartheta(\tau,\lambda) = i\int_{\mathbb{R}} dx\, e^{-\pi\frac{x^2}{t}}\,\Big|_{t=-i\tau} = i\sqrt{-i\tau}. \tag{A.1}$$

The "dual" integral is, obviously,

$$\oint_{\boldsymbol{a}} d\lambda\,\vartheta(\tau,\lambda) \stackrel{\text{def}}{=} \int_0^1 d\lambda\,\vartheta(\tau,\lambda) = 1. \tag{A.2}$$

[8] We note that the (τ,ν,μ)-dependence through the standard scalar factor $e^{i\pi k\frac{\nu^2-\mu^2}{2\tau}}$ is in fact eliminated similarly, with the standard scalar automorphy factor. Any automorphy factor must satisfy the cocycle equation; also see [25] for the matrix case.

Somewhat more generally than in (A.1), we can evaluate the integral

$$\int_{\mu}^{\mu+c\tau+d} d\lambda\, e^{i\pi\frac{c\lambda^2}{c\tau+d}}\,\vartheta(\tau,\lambda),\qquad c\in\mathbb{N},\quad d\in\mathbb{Z},\quad cd\in 2\mathbb{Z},$$

(with an arbitrary μ) by shifting the integration variable as $\lambda\to\lambda-m(\tau+\frac{d}{c})$ in each term of the theta series and then summing over m as $\sum_{m\in\mathbb{Z}}f(m)=\sum_{\alpha=1}^{c}\sum_{m\in\mathbb{Z}}f(cm+\alpha)$. With even cd, it then follows that m drops from the exponentials, and we readily obtain

$$\int\limits_{\mu}^{\mu+c\tau+d} d\lambda\, e^{i\pi\frac{c\lambda^2}{c\tau+d}}\,\vartheta(\tau,\lambda)=\sum_{m\in\mathbb{Z}}\int\limits_{\alpha(\tau+\frac{d}{c})+m(c\tau+d)}^{\alpha(\tau+\frac{d}{c})+(m+1)(c\tau+d)} d\lambda\, e^{i\pi\frac{c\lambda^2}{c\tau+d}}\cdot\sum_{\alpha=1}^{c}e^{-i\pi\alpha^2\frac{d}{c}}$$
$$=i\sqrt{\frac{-i(c\tau+d)}{c}}\cdot\sqrt{-ic}\,\zeta_{c,d}^{-1}=(c\tau+d)^{\frac{1}{2}}\,\zeta_{c,d}^{-1},\tag{A.3}$$

where the two factors are a Gaussian integral, calculated by analytic continuation (for τ such that $\Im\tau>0$) from the integral over $\mathbb{R}$, and a Gaussian sum, see, e.g., [29].

Remarkably, much similarity is preserved if theta functions are replaced with Appell functions in the above integrals. We first consider the corresponding analogue of (A.1),

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}}\,\mathcal{K}_1(\tau,\lambda-\mu,\mu)\stackrel{\text{def}}{=}\int_0^{\tau} d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}}\,\mathcal{K}_1(\tau,\lambda+\varepsilon-\mu,\mu),$$

where an infinitesimal positive real ε specifies the prescription to bypass the singularities. Again continuing from $\tau=it$ and $\mu=iy$ with positive real t and real y, we have

$$=i\sum_{m\in\mathbb{Z}}\int_0^{t}dx\, e^{-\pi\frac{x^2-2xy}{t}}\,\frac{e^{-\pi t^2m^2-2\pi m(x-y)}}{1-e^{-2\pi(x+mt)}-i0}\Bigg|_{\substack{t=-i\tau\\ y=-i\mu}}.$$

Making the same substitution $\lambda\to\lambda-m\tau$ as above, or $x\to x-mt$, and using that

$$\int_{-\infty}^{+\infty}dx\,\frac{f(x)}{x-i0}=\fint_{-\infty}^{+\infty}dx\,\frac{f(x)}{x}+i\pi\operatorname*{res}_{x=0}f(x),$$

we obtain

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}}\,\mathcal{K}_1(\tau,\lambda-\mu,\mu)=-i\sqrt{-i\tau}\,\Phi(\tau,\mu),\tag{A.4}$$

with Φ defined in (1.4). The derivation shows that the same result is valid for the "$\boldsymbol{b}$"-integral with a translated contour:

$$\int_{\alpha\tau}^{\tau+\alpha\tau} d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}}\,\mathcal{K}_1(\tau,\lambda+\varepsilon-\mu,\mu)=-i\sqrt{-i\tau}\,\Phi(\tau,\mu),\qquad\alpha\in\mathbb{R}.\tag{A.5}$$

The version of (A.4) for $\mathcal{K}_\ell$ is given by

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\ell\frac{\lambda^2-2\lambda\mu}{\tau}}\,\mathcal{K}_\ell(\tau,\lambda-\mu,\mu)=-i\sqrt{\frac{-i\tau}{\ell}}\,\Phi(\tfrac{\tau}{\ell},\mu).\tag{A.6}$$

More generally than in (A.4), we can evaluate the integral

$$\text{(A.7)}\quad \int_{\alpha\tau}^{c\tau+d+\alpha\tau} d\lambda\, e^{i\pi\frac{\lambda^2+2\lambda\mu}{\tau+\frac{d}{c}}}\, \mathcal{K}_1(\tau,-\lambda+\varepsilon-\mu,\mu)$$
$$= -i\sqrt{-i(\tau+\frac{d}{c})}\sum_{r=0}^{c-1} e^{2i\pi\mu\frac{rd}{c\tau+d}-i\pi\tau\frac{r^2d}{c\tau+d}}\,\Phi(\tau+\frac{d}{c},\mu+r\frac{d}{c}),\quad \alpha\in\mathbb{R}$$

for $c\in\mathbb{N}$, $d\in\mathbb{Z}$, and $cd\in 2\mathbb{Z}$.

Similarly to (A.2), we have the dual, $\boldsymbol{a}$-cycle integral

$$\text{(A.8)}\qquad \oint_{\boldsymbol{a}} d\lambda\mathcal{K}_1(\tau,\lambda-\mu,\mu)\stackrel{\text{def}}{=}\int_0^1 d\lambda\mathcal{K}_1(\tau,\lambda-\mu+i0,\mu)=1.$$

APPENDIX B. $\widehat{s\ell}(2|1)$ ALGEBRA, ITS AUTOMORPHISMS AND MODULES

B.1. The algebra and automorphisms. The affine Lie superalgebra $\widehat{s\ell}(2|1)$ is spanned by four bosonic currents E^{12}, H^-, F^{12}, and H^+, four fermionic ones, E^1, E^2, F^1, and F^2, and the central element (which we identify with its eigenvalue k). The $\widehat{s\ell}(2)$ subalgebra is generated by E^{12}, H^-, and F^{12}, and it commutes with the $u(1)$ subalgebra generated by H^+. The fermions E^1 and F^2 on one hand and F^1 and E^2 on the other hand form $s\ell(2)$ doublets. The nonvanishing commutation relations are

$$\text{(B.1)}\qquad \begin{aligned}
&[H^-_m,E^{12}_n]=E^{12}_{m+n}, && [H^-_m,F^{12}_n]=-F^{12}_{m+n},\\
&[E^{12}_m,F^{12}_n]=m\delta_{m+n,0}k+2H^-_{m+n}, && [H^\pm_m,H^\pm_n]=\mp\tfrac{1}{2}m\delta_{m+n,0}k,\\
&[F^{12}_m,E^2_n]=F^1_{m+n}, && [E^{12}_m,F^2_n]=-E^1_{m+n},\\
&[F^{12}_m,E^1_n]=-F^2_{m+n}, && [E^{12}_m,F^1_n]=E^2_{m+n},\\
&[H^\pm_m,E^1_n]=\tfrac{1}{2}E^1_{m+n}, && [H^\pm_m,F^1_n]=-\tfrac{1}{2}F^1_{m+n},\\
&[H^\pm_m,E^2_n]=\mp\tfrac{1}{2}E^2_{m+n}, && [H^\pm_m,F^2_n]=\pm\tfrac{1}{2}F^2_{m+n},\\
&[E^1_m,F^1_n]=-m\delta_{m+n,0}k+H^+_{m+n}-H^-_{m+n}, &&\\
&[E^2_m,F^2_n]=m\delta_{m+n,0}k+H^+_{m+n}+H^-_{m+n}, &&\\
&[E^1_m,E^2_n]=E^{12}_{m+n}, && [F^1_m,F^2_n]=F^{12}_{m+n}.
\end{aligned}$$

The Sugawara energy-momentum tensor is given by

$$\text{(B.2)}\qquad T_{\text{Sug}}=\frac{1}{k+1}\big(H^-H^--H^+H^++E^{12}F^{12}+E^1F^1-E^2F^2\big).$$

There are involutive algebra automorphisms

$$\text{(B.3)}\qquad \alpha:\ \begin{array}{lll}
E^1_n\mapsto F^2_n, & E^2_n\mapsto F^1_n, & E^{12}_n\mapsto F^{12}_n,\\
F^1_n\mapsto E^2_n, & F^2_n\mapsto E^1_n, & F^{12}_n\mapsto E^{12}_n,\\
H^+_n\mapsto H^+_n, & H^-_n\mapsto -H^-_n, &
\end{array}$$

and

$$\beta:\ \begin{array}{lll} E^1_n\mapsto E^2_n, & E^2_n\mapsto E^1_n, & E^{12}_n\mapsto E^{12}_n,\\ F^1_n\mapsto -F^2_n, & F^2_n\mapsto -F^1_n, & F^{12}_n\mapsto F^{12}_n,\\ H^+_n\mapsto -H^+_n, & H^-_n\mapsto H^-_n, & \end{array}\tag{B.4}$$

and a $\mathbb{Z}$ subgroup of automorphisms called the spectral flow,

$$\mathcal{U}_\theta:\ \begin{array}{ll} E^1_n\mapsto E^1_{n-\theta}, & E^2_n\mapsto E^2_{n+\theta},\\ F^1_n\mapsto F^1_{n+\theta}, & F^2_n\mapsto F^2_{n-\theta}, \end{array}\quad H^+_n\mapsto H^+_n+k\theta\delta_{n,0},\tag{B.5}$$

where $\theta\in\mathbb{Z}$ (and the $\widehat{s\ell}(2)$ subalgebra remains invariant). We note the relations

$$\alpha^2=1,\qquad \beta^2=1,\qquad (\alpha\beta)^4=1,\qquad \alpha\mathcal{U}_\theta=\mathcal{U}_\theta\alpha,\qquad (\beta\mathcal{U}_\theta)^2=1.\tag{B.6}$$

Another $\mathbb{Z}$ algebra of automorphisms (a spectral flow affecting the $\widehat{s\ell}(2)$ subalgebra, cf. [14]) acts as

$$\mathcal{A}_\eta:\ \begin{array}{lll} E^1_n\mapsto E^1_{n+\eta}, & E^2_n\mapsto E^2_{n+\eta}, & E^{12}_n\mapsto E^{12}_{n+2\eta},\\ F^1_n\mapsto F^1_{n-\eta}, & F^2_n\mapsto F^2_{n-\eta}, & F^{12}_n\mapsto F^{12}_{n-2\eta},\\ & H^-_n\mapsto H^-_n+k\eta\delta_{n,0}, & H^+_n\mapsto H^+_n \end{array}\tag{B.7}$$

There also exists the automorphism $\gamma=\mathcal{U}_{\frac{1}{2}}\circ\mathcal{A}_{-\frac{1}{2}}$ (while $\mathcal{U}_{\frac{1}{2}}$ and $\mathcal{A}_{-\frac{1}{2}}$ are not automorphisms, but rather mappings into an isomorphic algebra, their composition is). For $\theta\in\mathbb{Z}$, its powers $\mathcal{T}_\theta=\gamma^\theta$ map the generators as

$$\mathcal{T}_\theta:\ \begin{array}{llll} E^1_n\mapsto E^1_{n-\theta}, & E^2_n\mapsto E^2_n, & E^{12}_n\mapsto E^{12}_{n-\theta}, & H^-_n\mapsto H^-_n-\frac{k}{2}\theta\delta_{n,0},\\ F^1_n\mapsto F^1_{n+\theta}, & F^2_n\mapsto F^2_n, & F^{12}_n\mapsto F^{12}_{n+\theta}, & H^+_n\mapsto H^+_n+\frac{k}{2}\theta\delta_{n,0}. \end{array}\tag{B.8}$$

Spectral flow transform (B.5), affecting the fermions and leaving the $\widehat{s\ell}(2)$ subalgebra invariant, plays an important role in the study of $\widehat{s\ell}(2|1)$-representations [13]. We use the notation

$$\mathcal{P}_{;\theta}\equiv\mathcal{U}_\theta\mathcal{P}$$

for the action of spectral flow transform on any $\widehat{s\ell}(2|1)$-module $\mathcal{P}$. Obviously, $\mathcal{P}_{;0}\equiv\mathcal{P}$. For a module $\mathcal{P}$, we let

$$\chi[\mathcal{P}](q,x,y)=\mathrm{Tr}_{\mathcal{P}}\big(q^{L_0}\,x^{H^-_0}\,y^{H^+_0}\big)\tag{B.9}$$

be its character. The character $\chi[\mathcal{P}]_{;\theta}$ of the spectral-flow transformed module $\mathcal{P}_{;\theta}$ is expressed through the character of $\mathcal{P}$ as

$$\chi[\mathcal{P}]_{;\theta}(q,x,y)=y^{-k\theta}\,q^{-k\theta^2}\,\chi[\mathcal{P}](q,x,y\,q^{2\theta}).\tag{B.10}$$

B.2. Highest-weight conditions and modules. A significant role in the $\widehat{s\ell}(2|1)$ representation theory is played by the spectral flow transform (B.5), which is a family of $\widehat{s\ell}(2|1)$ automorphisms. Applying algebra automorphisms to modules gives nonisomorphic modules in general. The (upper) triangular subalgebra is also mapped under the action of automorphisms, and the annihilation conditions satisfied by highest-weight vectors change accordingly. Thus, the existence of an automorphism group leads to a freedom in choosing the type of annihilation conditions imposed on highest-weight vectors in highest-weight representations (in particular, Verma modules). We consider the family of annihilation conditions

$$\begin{aligned} &E^1_{\geqslant -\theta} \approx 0, \qquad E^2_{\geqslant \theta} \approx 0, \\ &F^1_{\geqslant \theta+1} \approx 0, \qquad F^2_{\geqslant 1-\theta} \approx 0, \qquad F^{12}_{\geqslant 1} \approx 0, \end{aligned} \qquad \theta \in \mathbb{Z} \tag{B.11}$$

that are an orbit of $\mathcal{U}_\theta$ (the $\approx$ sign means that the left-hand sides must be applied to a vector; at the moment, we are interested in the list of annihilation operators, rather than in the vector, hence the notation). These annihilation conditions are called the *twisted highest-weight conditions* in general. By the $\widehat{s\ell}(2|1)$ commutation relations, the conditions explicitly written in (B.11) imply that $E^{12}_{\geqslant 0} \approx 0$ and $H^{\pm}_{\geqslant 1} \approx 0$. This is understood in similar relations in what follows.

Accordingly, a twisted Verma module $\mathcal{P}_{h_-,h_+,k;\theta}$ over the level-k $\widehat{s\ell}(2|1)$ algebra is freely generated by $E^1_{\leqslant -\theta-1}$, $E^2_{\leqslant \theta-1}$, $E^{12}_{\leqslant -1}$, $F^1_{\leqslant \theta}$, $F^2_{\leqslant -\theta}$, $F^{12}_{\leqslant 0}$, $H^-_{\leqslant -1}$, and $H^+_{\leqslant -1}$ from the *twisted highest-weight state* $|h_-, h_+, k; \theta\rangle$ satisfying annihilation conditions (B.11) and additionally fixed by the eigenvalue relations

$$H^-_0 \, |h_-, h_+, k; \theta\rangle = h_- \, |h_-, h_+, k; \theta\rangle, \tag{B.12}$$

$$H^+_0 \, |h_-, h_+, k; \theta\rangle = (h_+ - k\theta) \, |h_-, h_+, k; \theta\rangle. \tag{B.13}$$

With the parameterization of the H^+_0 eigenvalue chosen in (B.13), we have

$$\mathcal{U}_{\theta'} |h_-, h_+, k; \theta\rangle = |h_-, h_+, k; \theta + \theta'\rangle \tag{B.14}$$

and, obviously, $\mathcal{U}_{\theta'} \mathcal{P}_{h_-,h_+,k;\theta} = \mathcal{P}_{h_-,h_+,k;\theta+\theta'}$ for the Verma modules. This simple behavior of $|h_-, h_+, k; \theta\rangle$ under the spectral flow explains the subtraction of $k\theta$ in (B.13).

The character of $\mathcal{P}_{h_-,h_+,k;\theta}$ is given by

$$\chi[\mathcal{P}]_{h_-,h_+,k;\theta}(q, x, y) = x^{h_-} \, y^{h_+ - (k+1)\theta} \, q^{\frac{h_-^2 - h_+^2}{k+1} + 2\theta h_+ - (k+1)\theta^2} \, \boldsymbol{\Theta}(q, x, y), \tag{B.15}$$

where

$$\boldsymbol{\Theta}(q, x, y) = \frac{\vartheta_{1,0}(q, x^{\frac{1}{2}} y^{\frac{1}{2}}) \vartheta_{1,0}(q, x^{\frac{1}{2}} y^{-\frac{1}{2}})}{\vartheta_{1,1}(q, x) \, q^{-\frac{1}{8}} \, \eta(q)^3}. \tag{B.16}$$

We let $|h_-, k; \theta\rangle^-$ denote a state satisfying the highest-weight conditions

$$\begin{aligned} &E^1_{\geqslant -\theta} \approx 0, \qquad E^2_{\geqslant \theta} \approx 0, \\ &F^1_{\geqslant \theta} \approx 0, \qquad F^2_{\geqslant 1-\theta} \approx 0 \end{aligned} \tag{B.17}$$

and the eigenvalue relations

$$H_0^-|h_-,k;\theta\rangle^- = h_-|h_-,k;\theta\rangle^-. \tag{B.18}$$

Conditions (B.17) are stronger than (B.11), and as a result, the eigenvalue of H_0^+ is no longer an independent parameter. We let $\mathcal{N}^-_{h_-,k;\theta}$ denote the (twisted) *narrow Verma module* — the module freely generated by $E^1_{\leqslant-\theta-1}$, $E^2_{\leqslant\theta-1}$, $E^{12}_{\leqslant-1}$, $F^1_{\leqslant\theta-1}$, $F^2_{\leqslant-\theta}$, $F^{12}_{\leqslant0}$, $H^-_{\leqslant-1}$, and $H^+_{\leqslant-1}$ from $|h_-,k;\theta\rangle^-$.[9]

Similarly, let $|h_-,k;\theta\rangle^+$ denote the states satisfying a different set of the highest-weight conditions

$$\begin{aligned} E^1_{\geqslant-\theta} &\approx 0, & E^2_{\geqslant\theta} &\approx 0,\\ F^1_{\geqslant\theta+1} &\approx 0, & F^2_{\geqslant-\theta} &\approx 0 \end{aligned} \tag{B.19}$$

(which are again stronger than (B.11)) and the eigenvalue relations

$$H_0^-|h_-,k;\theta\rangle^+ = h_-|h_-,k;\theta\rangle^+. \tag{B.20}$$

We write $\mathcal{N}^+_{h_-,k;\theta}$ for the corresponding (twisted) narrow Verma module freely generated by $E^1_{\leqslant-\theta-1}$, $E^2_{\leqslant\theta-1}$, $E^{12}_{\leqslant-1}$, $F^1_{\leqslant\theta}$, $F^2_{\leqslant-\theta-1}$, $F^{12}_{\leqslant0}$, $H^-_{\leqslant-1}$, and $H^+_{\leqslant-1}$ from $|h_-,k;\theta\rangle^+$.

The characters of $\mathcal{N}^-_{h_-,k;\theta}$ and $\mathcal{N}^+_{h_-,k;\theta}$ are given by

$$\begin{aligned} \chi[\mathcal{N}^-]_{h_-,k;\theta}(q,x,y) &= \frac{\chi[\mathcal{P}]_{h_-,h_-,k;\theta}(q,x,y)}{1+q^{-\theta}x^{-\frac{1}{2}}y^{-\frac{1}{2}}},\\ \chi[\mathcal{N}^+]_{h_-,k;\theta}(q,x,y) &= \frac{\chi[\mathcal{P}]_{h_-,-h_-,k;\theta}(q,x,y)}{1+q^{\theta}x^{-\frac{1}{2}}y^{\frac{1}{2}}}, \end{aligned}$$

where $\chi[\mathcal{P}]$ is defined in (B.15). The twisted narrow Verma modules are convenient in constructing a resolution of the admissible representations $\mathcal{L}_{r,s}$, see the next subsection.

B.3. Admissible $\widehat{s\ell}(2|1)$ representations $\mathcal{L}_{r,s,\ell,u;\theta}$. The admissible $\widehat{s\ell}(2|1)$-representations, which belong to the class of irreducible highest-weight representations characterized by the property that the corresponding Verma modules are *maximal* elements with respect to the (appropriately defined) Bruhat order, have arisen in a vertex-operator extension of two $\widehat{s\ell}(2)$ algebras with the "dual" levels k and k' such that $(k+1)(k'+1) = 1$ [33]; via this extension $\widehat{s\ell}(2)_k \oplus \widehat{s\ell}(2)_{k'} \to \widehat{s\ell}(2|1)_k$, the admissible $\widehat{s\ell}(2|1)$ representations are related to the admissible $\widehat{s\ell}(2)$ representations [45]. We fix the $\widehat{s\ell}(2|1)$ level as

$$k = \frac{\ell}{u} - 1$$

with coprime positive integers ℓ and u.

[9]The name *Verma* module is a (very convenient) abuse of terminology. The $\mathcal{N}^-$ modules, as well as $\mathcal{N}^+$ introduced momentarily, occur as submodules generated from a charged singular vector in the proper Verma modules $\mathcal{P}$ [13]. The modules are called *narrow* for the reason explained in [13] (essentially because they *are* narrow compared with the proper Verma modules).

$$
\begin{tikzcd}[column sep=small]
& \mathcal{N}^{-}_{j-\frac{1}{2},k} & \\
\mathcal{N}^{+}_{-j-\frac{\ell}{u}(s-1),k;s-1} \arrow[ur] & & \mathcal{N}^{+}_{\ell-j-\frac{\ell}{u}(s-1),k;s-u-1} \arrow[ul] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
\mathcal{N}^{-}_{j-n\ell-\frac{1}{2},k;-nu} \arrow[u] \arrow[urr] & & \mathcal{N}^{-}_{j+n\ell-\frac{1}{2},k;nu} \arrow[u] \arrow[ull] \\
\mathcal{N}^{+}_{-j-\frac{\ell}{u}(s-1)-n\ell,k;s+nu-1} \arrow[u] \arrow[urr] & & \mathcal{N}^{+}_{-j-\frac{\ell}{u}(s-1)+(n+1)\ell,k;s-1-(n+1)u} \arrow[u] \arrow[ull] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull]
\end{tikzcd}
$$

FIGURE 2. Mappings between narrow Verma modules.

For $1 \leqslant r \leqslant \ell$ and $1 \leqslant s \leqslant u$, the admissible $\widehat{s\ell}(2|1)$ representations $\mathcal{L}_{r,s,\ell,u;\theta}$ is the irreducible quotient of the Verma module $\mathcal{P}_{\frac{r}{2}-\frac{\ell}{u}\frac{s-1}{2},\frac{r}{2}-\frac{\ell}{u}\frac{s+1}{2},\frac{\ell}{u}-1;\theta}$. We omit the level $k = \frac{\ell}{u}-1$ in $\mathcal{P}_{h_-,h_+,k;\theta}$ and similar notation in what follows. In this Verma module, there is the charged singular vector given by $E^1_{-\theta-1}F^2_{-\theta}$ acting on the twisted highest-weight vector $|\frac{r}{2} - \frac{\ell}{u}\frac{s-1}{2}, \frac{r}{2} - \frac{\ell}{u}\frac{s+1}{2}; \theta\rangle$. The corresponding submodule is the narrow Verma module $\mathcal{N}^{-}_{\frac{r}{2}-\frac{s-1}{2}\frac{\ell}{u};\theta+1}$ and the quotient is the narrow Verma module $\mathcal{N}^{-}_{\frac{r-1}{2}-\frac{\ell}{u}\frac{s-1}{2};\theta+1}$,

$$0 \to \mathcal{N}^{-}_{\frac{r}{2}-\frac{s-1}{2}\frac{\ell}{u};\theta+1} \to \mathcal{P}_{\frac{r}{2}-\frac{\ell}{u}\frac{s-1}{2},\frac{r}{2}-\frac{\ell}{u}\frac{s+1}{2};\theta} \to \mathcal{N}^{-}_{\frac{r-1}{2}-\frac{\ell}{u}\frac{s-1}{2},\frac{\ell}{u}-1;\theta+1} \to 0. \tag{B.21}$$

The admissible representation $\mathcal{L}_{r,s,\ell,u;\theta}$ is therefore the irreducible quotient of the latter narrow Verma module. Combining the canonical mapping $\mathcal{N}^{-}_{\frac{r-1}{2}-\frac{\ell}{u}\frac{s-1}{2};\theta+1} \to \mathcal{L}_{r,s,\ell,u;\theta}$ with the mappings in Fig. 2, taken from [13], with $j - \frac{1}{2} = \frac{r-1}{2} - \frac{\ell}{u}\frac{s-1}{2}$, we obtain a resolution of the admissible representation. The resolution readily implies a character formula.

Theorem B.1. *For $1 \leqslant r \leqslant \ell$, $1 \leqslant s \leqslant u$, and $\theta \in \mathbb{Z}$, the character of $\mathcal{L}_{r,s,\ell,u;\theta}$ is given by*

$$\chi_{r,s,\ell,u;\theta}(q,x,y) = q^{(\theta+1)(r-1-\frac{\ell}{u}(s+\theta))}\, x^{\frac{r-1}{2}-\frac{s-1}{2}\frac{\ell}{u}}\, y^{\frac{r-1}{2}-\frac{s+1}{2}\frac{\ell}{u}-\theta\frac{\ell}{u}} \\ \times \psi_{r,s,\ell,u}(q,x,yq^{2\theta})\, \boldsymbol{\Theta}(q,x,y), \tag{B.22}$$

where

$$\text{(B.23)}\quad \psi_{r,s,\ell,u}(q,x,y)=\sum_{m\in\mathbb{Z}}q^{m^2u\ell-mu(r-1)}\Bigl(\frac{q^{m\ell(s-1)}x^{-m\ell}}{1+y^{-\frac{1}{2}}x^{-\frac{1}{2}}q^{mu-1}} - q^{(s-1)(r-1)}\,x^{1-r}\frac{q^{-m\ell(s-1)}x^{m\ell}}{1+y^{-\frac{1}{2}}x^{\frac{1}{2}}q^{mu-s}}\Bigr)$$

(*and* $\boldsymbol{\Theta}$ *is defined in* (B.16)).

For $r<p$, this is proved by a straightforward summation of the characters of the twisted narrow Verma modules involved in the resolution. For $r=p$, the above character formula follows from a somewhat different resolution, with a special role played by one of the charged singular vectors (see [13]).

Unless $\ell=1$, the discrete $\widehat{s\ell}(2|1)$ automorphism β (see Appendix B) maps the $\mathcal{L}_{r,s,\ell,u;\theta}$ representations into nonisomorphic ones, $\overline{\mathcal{L}}_{r,s,\ell,u}=\beta\,\mathcal{L}_{r,s,\ell,u}$, which gives the second half of the admissible $\widehat{s\ell}(2|1)$-representations. It follows that the character of $\overline{\mathcal{L}}_{r,s,\ell,u}$ can be written as

$$\text{(B.24)}\qquad \overline{\chi}_{r,s,\ell,u}(q,x,y)=\chi_{r,s,\ell,u}(q,x,y^{-1}),$$

which after a simple calculation gives

$$\text{(B.25)}\qquad \overline{\chi}_{r,s,\ell,u}=-\chi_{1-r,s,\ell,u;-s-1},$$

and hence $\overline{\chi}_{r,s,\ell,u;\theta}=-\chi_{1-r,s,\ell,u;\theta-s-1}$,

$$\text{(B.26)}\quad \overline{\chi}_{r,s,\ell,u;\theta}(q,x,y)=-q^{(\theta-s)(-r-\frac{\ell}{u}(\theta-1))}\,x^{-\frac{r}{2}-\frac{s-1}{2}\frac{\ell}{u}}\,y^{-\frac{r}{2}+\frac{s+1}{2}\frac{\ell}{u}-\theta\frac{\ell}{u}} \times\psi_{1-r,s,\ell,u}(q,x,yq^{2(\theta-s-1)})\,\boldsymbol{\Theta}(q,x,y).$$

We finally consider the Verma modules $\mathcal{P}_{h_-,h_+;\theta}$ with the same $h_-=\frac{r}{2}-\frac{\ell}{u}\frac{s-1}{2}$ as in the Verma module involved in the construction of $\mathcal{L}_{r,s,\ell,u;\theta}$. Let $\mathcal{M}_{r,s,h_+;\theta}\equiv\mathcal{M}_{r,s,h_+,\ell,u;\theta}$ be the quotient with respect to the MFF singular vectors defined in [13]. In the case where no charged singular vectors exist in the above $\mathcal{P}_{h_-,h_+;\theta}$, i.e., for

$$\text{(B.27)}\qquad \frac{ur}{2\ell}-\frac{s-1}{2}\pm\frac{u}{\ell}\,h_+\notin\mathbb{Z},$$

the modules $\mathcal{M}_{r,s,h_+;\theta}$ are irreducible and $\mathcal{M}_{r,s,h_+;\theta}\simeq\mathcal{M}_{r,s,h_+-\frac{\ell}{u}\theta;0}$. Another straightforward calculation shows that the character of $\mathcal{M}_{r,s,h_+;\theta}$ is given by

$$\text{(B.28)}\quad \Omega_{r,s,h}(q,x,y)=y^h\,x^{-\frac{\ell}{u}\frac{s-1}{2}}\,q^{\frac{\ell}{4u}(s-1)^2-\frac{u}{\ell}h^2} \times\bigl(\theta_{r,\ell}(q^u,xq^{-(s-1)})-\theta_{-r,\ell}(q^u,xq^{-(s-1)})\bigr)\,\boldsymbol{\Theta}(q,x,y),$$

where $h=h_+-(k+1)\theta$. It follows that

$$\text{(B.29)}\qquad \begin{gathered}\Omega_{r+2n\ell,s,h}(q,x,y)=\Omega_{r,s,h}(q,x,y),\qquad \Omega_{n\ell,s,h}(q,x,y)=0,\quad n\in\mathbb{Z},\\ \Omega_{-r,s,h}(q,x,y)=-\Omega_{r,s,h}(q,x,y).\end{gathered}$$

We use the special notation for the *reducible* modules $\mathcal{M}_{r,s,h_+}$ with h_+ such that the left-hand side of (B.27) *is* an integer, namely with $h_+ = \pm(\frac{r}{2} - \frac{\ell}{u}\frac{s+1}{2})$: $\mathcal{M}_{r,s} = \mathcal{M}_{r,s,\frac{r}{2}-\frac{\ell}{u}\frac{s+1}{2}}$ and $\overline{\mathcal{M}}_{r,s} = \mathcal{M}_{r,s,-\frac{r}{2}+\frac{\ell}{u}\frac{s+1}{2}}$. The respective characters of $\mathcal{M}_{r,s}$ and $\overline{\mathcal{M}}_{r,s}$ are given by

$$\begin{aligned}\Omega_{r,s}(q,x,y) &= \Omega_{r,s,\frac{r}{2}-\frac{\ell}{u}\frac{s+1}{2}}(q,x,y),\\ \overline{\Omega}_{r,s}(q,x,y) &= \Omega_{r,s,-\frac{r}{2}+\frac{\ell}{u}\frac{s+1}{2}}(q,x,y).\end{aligned} \tag{B.30}$$

Reducibility of these modules can be expressed as the exact sequences

$$\begin{aligned}&0\to\mathcal{L}_{r+1,s}\to\mathcal{M}_{r,s}\to\mathcal{L}_{r,s}\to 0,\\ &0\to\overline{\mathcal{L}}_{r+1,s}\to\overline{\mathcal{M}}_{r,s}\to\overline{\mathcal{L}}_{r,s}\to 0,\end{aligned}\qquad 1\leqslant r\leqslant \ell-1.$$

Consequently,

$$\begin{aligned}\chi_{r+1,s}+\chi_{r,s} &= \Omega_{r,s},\\ \overline{\chi}_{r+1,s}+\overline{\chi}_{r,s} &= \overline{\Omega}_{r,s},\end{aligned}\qquad 1\leqslant r\leqslant \ell-1, \tag{B.31}$$

which can also be easily verified directly using (4.3) and (2.4). We note that at the same time, $\Omega_{r,s}$ and $\overline{\Omega}_{r,s}$ are essentially (apart from their y-dependence) the admissible characters of a level-$(k-1)$ $\widehat{s\ell}(2)$ algebra obtained from $\widehat{s\ell}(2|1)_k$ by the *reduction* with respect to only the fermionic generators; this reduction therefore sends reducible $\widehat{s\ell}(2|1)$-modules into irreducible $\widehat{s\ell}(2)$-ones, and the cohomology of the complex associated with the reduction is certainly not concentrated at one term.

B.4. Ramond and Neveu–Schwarz characters and supercharacters. The $\widehat{s\ell}(2|1)$-characters introduced above are in the so-called Ramond sector. We also introduce Neveu–Schwarz characters and supercharacters in both sectors as follows. The Neveu–Schwarz characters are simply the $\theta = -\frac{1}{2}$ spectral flow transformations of the above (Ramond) characters: for the (twisted) character χ of any $\widehat{s\ell}(2|1)$-module in the Ramond sector, the corresponding (twisted) Neveu–Schwarz character is $\chi^{\mathrm{NS}}_{;\theta} = \chi_{;\theta-\frac{1}{2}}$, and therefore (see (B.10))

$$\chi^{\mathrm{NS}}_{;\theta}(\tau,\nu,\mu) = e^{i\pi k\mu - i\pi k\frac{\tau}{2}}\chi_{;\theta}(\tau,\nu,\mu-\tau). \tag{B.32}$$

With the supercharacter of a module $\mathcal{P}$ defined as

$$\sigma[\mathcal{P}](\tau,\nu,\mu) = \mathrm{Tr}_{\mathcal{P}}\bigl(e^{2i\pi L_0\tau + 2i\pi H_0^-\nu + 2i\pi H_0^+(\mu+1)}\bigr),$$

we then have the Ramond supercharacters $\chi^{\mathrm{sR}}_{;\theta}\equiv\sigma^{\mathrm{R}}_{;\theta}$ and the Neveu–Schwarz supercharacters $\chi^{\mathrm{sNS}}_{;\theta}\equiv\sigma^{\mathrm{NS}}_{;\theta}$ (for $\theta\in\mathbb{Z}$)

$$\chi^{\mathrm{sNS}}_{;\theta}(\tau,\nu,\mu) = e^{i\pi k\mu - i\pi\frac{k}{2}\tau + i\pi k}\chi_{;\theta}(\tau,\nu,\mu-\tau+1), \tag{B.33}$$

$$\chi^{\mathrm{sR}}_{;\theta}(\tau,\nu,\mu) = \chi_{;\theta}(\tau,\nu,\mu+1), \tag{B.34}$$

expressed in terms of the Ramond character χ.

Schematically, behavior of characters in the different sectors under the S and T modular transformations can be summarized in the diagram

$$\begin{tikzcd} \mathrm{R} \arrow[loop above, "T"] \arrow[r, bend left, "S"] & \mathrm{sNS} \arrow[l, bend left, "S"] \arrow[d, bend left, "T"] \\ \mathrm{sR} \arrow[loop above, "T"] \arrow[loop below, "S"] & \mathrm{NS} \arrow[u, bend left, "T"] \arrow[loop below, "S"] \end{tikzcd} \tag{B.35}$$

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