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ON THE LONG-TIME STABILITY OF THE IMPLICIT EULER SCHEME FOR THE TWO-DIMENSIONAL NAVIER–STOKES EQUATIONS*

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Abstract. In this paper we study the stability for all positive time of the fully implicit Euler scheme for the two-dimensional Navier–Stokes equations. More precisely, we consider the time discretization scheme and with the aid of the discrete Gronwall lemma and the discrete uniform Gronwall lemma we prove that the numerical scheme is stable.

Key words. Navier–Stokes equations, discrete Gronwall lemmas, implicit Euler scheme

AMS subject classifications. 65M12, 76D05

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1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with boundary $\partial\Omega$ of class C^2. The Navier–Stokes equations of viscous incompressible fluids are

$$u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

where $u = (u_1, u_2)$ is the velocity, p is the pressure, ν is the kinematic viscosity, and f represents body forces applied to the fluid. We complete these equations with the initial condition

$$u(x, 0) = u_0(x), \tag{1.3}$$

with $u_0 : \Omega \to \mathbb{R}^2$ being given, and with the nonslip boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

In the notation described below, system (1.1)–(1.4) can be written as the functional evolution equation

$$u_t + \nu Au + B(u, u) = f, \quad u(0) = u_0. \tag{1.5}$$

In the two-dimensional case under consideration, the solution to the Navier–Stokes equations is known to be smooth for all time (cf. [13]). The velocity u is bounded uniformly for all time by

$$|u(t)|^2_{L^2(\Omega)^2} \le \mathrm{e}^{-\nu\lambda_1 t}|u_0|^2_{L^2(\Omega)^2} + c\big(1 - \mathrm{e}^{-\nu\lambda_1 t}\big)|f|^2_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)}, \tag{1.6}$$

where λ_1 is the first eigenvalue of the Stokes operator A, and we have assumed that $f \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$. Furthermore, using techniques based on the uniform Gronwall

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lemma (cf. [12]), one can bound u uniformly in $H_0^1(\Omega)$ for all $t \geq 0$ by a function which depends on the initial condition

$$|u(t)|^2_{H_0^1(\Omega)^2} \leq K(|u_0|_{H_0^1(\Omega)^2}, |f|_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)}). \tag{1.7}$$

This dependence on the initial data can be dropped when one considers sufficiently large time, $t \geq T_c(|u_0|_{L^2(\Omega)^2}, |f|_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)})$, giving

$$|u(t)|^2_{H_0^1(\Omega)^2} \leq K(|f|_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)}) \quad \forall\, t \geq T_c. \tag{1.8}$$

In this paper we consider a time discretization of (1.5) using the fully implicit Euler scheme

$$\frac{u^n - u^{n-1}}{k} + \nu A u^n + B(u^n, u^n) = f^n, \quad u^0 = u_0, \tag{1.9}$$

where

$$f_n = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} f(t)\, dt, \tag{1.10}$$

and seek to obtain similar bounds on $|u^n|_{H_0^1(\Omega)^2}$.

Before we proceed further, we note that a related result for the linearized implicit Euler scheme

$$\frac{u^n - u^{n-1}}{k} + \nu A u^n + B(u^{n-1}, u^n) = f^n, \quad u^0 = u_0, \tag{1.11}$$

is proved in [7]. A different approach for the linearized implicit Euler scheme for the case without forcing term appears in [3].

Important background information on different computational methods can be found in some of the books and articles available in the literature. On finite elements, see, e.g., [4], [6]; on finite differences and finite elements, [9], [13]; on spectral methods, [1], [5].

For the mathematical setting of the problem, we consider the following spaces:

$$V = \{v \in H_0^1(\Omega)^2,\ \operatorname{div} v = 0\}, \tag{1.12}$$

$$H = \{v \in L^2(\Omega)^2,\ \operatorname{div} v = 0,\ v \cdot n = 0 \text{ on } \partial\Omega\}, \tag{1.13}$$

where n is the unit outward normal on $\partial\Omega$. The space V is endowed with the scalar product

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)\, dx \tag{1.14}$$

and with the corresponding norm

$$\|u\| = ((u, u))^{1/2}, \tag{1.15}$$

and H is endowed with the scalar product and the norm of $L^2(\Omega)^2$, denoted by $(\cdot,\cdot)$ and $|\cdot|$.

We denote by A the linear continuous operator from V into V' such that

$$\langle Au, v\rangle_{V',V} = ((u, v)) \quad \forall\, u, v \in V. \tag{1.16}$$

The domain of A in H is denoted by $D(A)$ and, using the regularity theory for the Stokes equation (see, for instance, [13]), one can show that

$$D(A) = H^2(\Omega)^2 \cap V. \tag{1.17}$$

We have the following inclusions:

$$D(A) \subset V \subset H, \tag{1.18}$$

and the so-called Poincaré inequality holds true:

$$|u| \le \frac{1}{\sqrt{\lambda_1}}\|u\| \quad \forall\, u \in V, \tag{1.19}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator A.

As is well known, the form (1.5) of the Navier–Stokes equations was derived by Leray [8], using the weak formulation of the Navier–Stokes equations. The latter is obtained by multiplying (1.1) by a test function $v \in V$ and integrating by parts over Ω, using Green's formula, viz.,

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = (f(t), v) \quad \forall\, v\, \in V, \tag{1.20}$$

where

$$b(u, v, w) = \sum_{i,j=1,2} \int_\Omega u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x)\, dx. \tag{1.21}$$

The form b is trilinear continuous on $H^1(\Omega)^2$ and enjoys the following properties:

$$|b(u, v, w)| \le c_b |u|^{1/2} |Au|^{1/2} \|v\| |w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H, \tag{1.22}$$

$$|b(u, v, w)| \le c_b |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2} \quad \forall\, u, v, w \in V, \tag{1.23}$$

$$b(u, v, v) = 0 \quad \forall\, u, v \in V, \tag{1.24}$$

the last equation implying

$$b(u, v, w) = -b(u, w, v) \quad \forall\, u, v, w \in V. \tag{1.25}$$

Using b, we define the bilinear operator B from $V \times V$ into V' by

$$\langle B(u, v), w\rangle_{V',V} = b(u, v, w) \quad \forall\, u, v, w \in V. \tag{1.26}$$

For more details about the functional spaces $D(A)$, V, and H as well as the operators A, B, and b, the reader is referred to, e.g., [2], [11], and [13].

2. H^1 stability and the main result. Throughout the paper, we assume that $f \in L^\infty(\mathbb{R}_+; H)$ and we set $|f|_\infty := |f|_{L^\infty(\mathbb{R}_+;H)}$. We adopt the following convention: c_i denotes constants that depend only on the parameters such as λ_1, ν, etc.; K_i depend in addition on $u(t_*)$ at some specified time t_* and on the forcing f; κ_i are bounds on the timestep k and may depend on u_0 and f.

In proving the main result, we will need a couple of preliminary lemmas. We begin with an analogue of (1.6), proved in almost the same way (see, e.g., [12, p. 109]).

LEMMA 2.1. *For every $k > 0$, we have*

$$|u^n|^2 \le (1+\nu\lambda_1 k)^{-n}|u_0|^2 + [1-(1+\nu\lambda_1 k)^{-n}]\frac{|f|_\infty^2}{\nu^2\lambda_1^2} \quad \forall\, n \ge 0, \tag{2.1}$$

and there exists $K_1 = K_1(|u_0|, |f|_\infty)$ such that

$$|u^n|^2 \le K_1 \quad \forall\, n \ge 0, \tag{2.2}$$

and

$$\nu \sum_{j=i}^{n} k\|u^j\|^2 \le K_1 + (n-i+1)k\,\frac{|f|_\infty^2}{\nu\lambda_1} \quad \forall\, i = 1, \dots, n. \tag{2.3}$$

Proof. Taking the scalar product of (1.9) with $2ku^n$ in H and using the relation

$$2(\varphi-\psi,\varphi) = |\varphi|^2 - |\psi|^2 + |\varphi-\psi|^2 \quad \forall \varphi, \psi \in H, \tag{2.4}$$

and the skew property (1.24), we obtain

$$|u^n|^2 - |u^{n-1}|^2 + |u^n - u^{n-1}|^2 + 2\nu k\|u^n\|^2 = 2k(f^n, u^n). \tag{2.5}$$

Using the Cauchy–Schwarz inequality and the Poincaré inequality (1.19), we majorize the right-hand side of (2.5) by

$$2k|f^n||u^n| \le \frac{2k}{\sqrt{\lambda_1}}|f^n|\|u^n\| \le \nu k\|u^n\|^2 + \frac{k}{\nu\lambda_1}|f^n|^2. \tag{2.6}$$

Relations (2.5) and (2.6) imply

$$|u^n|^2 - |u^{n-1}|^2 + |u^n - u^{n-1}|^2 + \nu k\|u^n\|^2 \le \frac{k}{\nu\lambda_1}|f^n|^2. \tag{2.7}$$

Using again the Poincaré inequality (1.19), we find from (2.7)

$$|u^n|^2 \le \frac{1}{\alpha}|u^{n-1}|^2 + \frac{k}{\alpha\nu\lambda_1}|f^n|^2, \tag{2.8}$$

where

$$\alpha = 1 + \nu\lambda_1 k. \tag{2.9}$$

Using (2.8) recursively, we find

$$\begin{aligned} |u^n|^2 &\le \frac{1}{\alpha^n}|u^0|^2 + \frac{k}{\nu\lambda_1}\sum_{i=1}^{n}\frac{1}{\alpha^i}|f^{n+1-i}|^2 \\ &\le (1+\nu\lambda_1 k)^{-n}\,|u_0|^2 + \frac{|f|_\infty^2}{\nu^2\lambda_1^2}[1-(1+\nu\lambda_1 k)^{-n}], \end{aligned} \tag{2.10}$$

which proves (2.1); (2.1) easily implies (2.2) with

$$K_1(|u_0|, |f|_\infty) := |u_0|^2 + \frac{1}{\nu^2\lambda_1^2}|f|_\infty^2. \tag{2.11}$$

Now adding up (2.7) with n from i to m and dropping some terms, we find

$$\begin{aligned}\nu k \sum_{j=i}^{m} \|u^j\|^2 &\le |u^{i-1}|^2 + \frac{k}{\nu\lambda_1}\sum_{j=i}^{m}|f^j|^2 \\ &\le K_1 + \frac{|f|_\infty^2}{\nu\lambda_1}(m-i+1)k,\end{aligned} \tag{2.12}$$

which is just (2.3) with n in place of m. □

Corollary 2.2. *If*

$$0 < k \le \frac{1}{\nu\lambda_1} =: \kappa_1, \tag{2.13}$$

then

$$|u^n|^2 \le 2\rho_0^2 \quad \forall\, nk \ge T_0(|u_0|, |f|_\infty) := \frac{4}{\nu\lambda_1}\ln\left(\frac{|u_0|}{\rho_0}\right), \tag{2.14}$$

where $\rho_0 := |f|_\infty/(\nu\lambda_1)$.

Proof. From the bound (2.1) on $|u^n|^2$, we infer that

$$|u^n|^2 \le \big(1+\nu\lambda_1 k\big)^{-n}|u_0|^2 + \rho_0^2,$$

and using assumption (2.13) on k and the fact that $1+x \ge \exp(x/2)$ if $x \in (0,1)$, we obtain

$$|u^n|^2 \le \exp\Big(-nk\frac{\nu\lambda_1}{2}\Big)|u_0|^2 + \rho_0^2.$$

For $nk \ge T_0$, the above inequality implies conclusion (2.14) of the corollary. □

We now seek to obtain uniform bounds on u^n in V similar to those obtained in H (see (2.2)). To this end, we first derive bounds on a finite interval of time (see Proposition 2.5). We then repeatedly use these together with (a discrete uniform Gronwall) Lemma 2.6 on successive intervals to arrive at the desired uniform bounds.

We begin with some preliminary inequalities. Taking the scalar product of (1.9) with $2kAu^n$ in H, we obtain

$$\begin{aligned}&\|u^n\|^2 - \|u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2 + 2\nu k|Au^n|^2 \\ &\quad + 2kb(u^n, u^n, Au^n) = 2k(f^n, Au^n).\end{aligned} \tag{2.15}$$

Using property (1.22) of the trilinear form b and recalling (2.2), we have the following bound of the nonlinear term:

$$\begin{aligned}2kb(u^n, u^n, Au^n) &\le 2\, c_b\, k\, |u^n|^{1/2}\|u^n\||Au^n|^{3/2} \\ &\le \frac{\nu k}{2}|Au^n|^2 + \frac{27c_b^4}{2\nu^3}K_1 k\|u^n\|^4.\end{aligned} \tag{2.16}$$

We bound the right-hand side of (2.15) by Cauchy–Schwarz,

$$\begin{aligned}2k(f^n, Au^n) &\le 2k|f^n||Au^n| \\ &\le \frac{\nu k}{2}|Au^n|^2 + \frac{2}{\nu}k|f^n|^2.\end{aligned} \tag{2.17}$$

Relations (2.15)–(2.17) imply

$$\|u^n\|^2 - \|u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2 + \nu k|Au^n|^2 \le \frac{27c_b^4}{2\nu^3}K_1k\|u^n\|^4 + \frac{2}{\nu}k|f^n|^2, \tag{2.18}$$

from which we obtain

$$0 \le c_2K_1k\|u^n\|^4 - \|u^n\|^2 + \|u^{n-1}\|^2 + c_3k|f|_\infty^2\,, \tag{2.19}$$

where

$$c_2 = \frac{27c_b^4}{2\nu^3} \quad\text{and}\quad c_3 = \frac{2}{\nu}. \tag{2.20}$$

Unlike (2.7), (2.19) does not (directly) provide a useful bound for $\|u^n\|$, so we proceed to show that (2.19) does give a proper bound for $\|u^n\|$ if the timestep k is sufficiently small.

LEMMA 2.3. *Suppose that $0 < k \le \kappa_1$ and assume that, for some n, we have*

$$c_2K_1k\left(K_2\|u^{n-1}\|^2 + c_4|f|_\infty^2\right) \le \frac{1}{5}, \tag{2.21}$$

where $K_2(|u_0|, |f|_\infty) = 2 + 4c_b^2K_1/\nu^2$ and $c_4 = 4/(\nu^2\lambda_1)$. Then (2.19) implies

$$\|u^n\|^2 \le \|u^{n-1}\|^2\left[1 + c_5K_1k\left(\|u^{n-1}\|^2 + k|f|_\infty^2\right)\right] + c_6k|f|_\infty^2 \tag{2.22}$$

for some constants c_5 and c_6.

Proof. Relation (2.19) implies either

$$\|u^n\|^2 \le \frac{1-\sqrt{\Delta_{n-1}}}{2c_2K_1k} \tag{2.23}$$

or

$$\|u^n\|^2 \ge \frac{1+\sqrt{\Delta_{n-1}}}{2c_2K_1k}, \tag{2.24}$$

where

$$\Delta_{n-1} = 1 - 4c_2K_1k(\|u^{n-1}\|^2 + c_3k|f|_\infty^2) > 0 \quad\text{by (2.13) and (2.21).} \tag{2.25}$$

We now show that (2.21) excludes (2.24). Indeed, taking the scalar product of (1.9) with $2k(u^n - u^{n-1})$ in H, we obtain

$$2|u^n - u^{n-1}|^2 + \nu k\|u^n\|^2 - \nu k\|u^{n-1}\|^2 + \nu k\|u^n - u^{n-1}\|^2 + 2k\,b(u^n, u^n, u^n - u^{n-1}) = 2k\,(f^n, u^n - u^{n-1}). \tag{2.26}$$

Using properties (1.23), (1.24), and (1.25) of the trilinear form b and recalling (2.2), we bound the nonlinear term as

$$\begin{aligned} 2kb(u^n, u^n, u^n - u^{n-1}) &= -2kb(u^n, u^n, u^{n-1}) \\ &\le 2c_bk|u^n|\|u^n\|\|u^{n-1}\| \\ &\le \frac{\nu}{2}k\|u^n\|^2 + \frac{2c_b^2}{\nu}K_1k\|u^{n-1}\|^2. \end{aligned} \tag{2.27}$$

We bound the right-hand side of (2.26) using Cauchy–Schwarz,

$$
\begin{aligned}
2k(f^n, u^n - u^{n-1}) &\le 2k|f^n||u^n - u^{n-1}| \\
&\le \frac{2}{\sqrt{\lambda_1}} k|f^n|\|u^n - u^{n-1}\| \\
&\le \frac{\nu}{2} k\|u^n - u^{n-1}\|^2 + \frac{2}{\nu\lambda_1} k|f^n|^2.
\end{aligned} \tag{2.28}
$$

Relations (2.26)–(2.28) imply

$$
\begin{aligned}
&2|u^n - u^{n-1}|^2 + \frac{\nu}{2}k\|u^n\|^2 - \left(\nu + \frac{2c_b^2}{\nu}K_1\right)k\|u^{n-1}\|^2 \\
&\quad + \frac{\nu}{2}k\|u^n - u^{n-1}\|^2 \le \frac{2}{\nu\lambda_1}k|f^n|^2,
\end{aligned} \tag{2.29}
$$

from which we obtain

$$
\|u^n\|^2 \le K_2\|u^{n-1}\|^2 + c_4|f|_\infty^2, \tag{2.30}
$$

and using hypothesis (2.21) we find

$$
2c_2K_1k\|u^n\|^2 \le 2c_2K_1k\left(K_2\|u^{n-1}\|^2 + c_4|f|_\infty^2\right) < 1, \tag{2.31}
$$

which contradicts (2.24). Therefore, (2.19) implies (2.23) and hence

$$
\begin{aligned}
\|u^n\|^2 &\le \frac{1 - \left[1 - 4c_2K_1k\left(\|u^{n-1}\|^2 + c_3k|f|_\infty^2\right)\right]^{1/2}}{2c_2K_1k} \\
&= 2\frac{\|u^{n-1}\|^2 + c_3k|f|_\infty^2}{1 + \sqrt{1-x}},
\end{aligned} \tag{2.32}
$$

where

$$
x = 4c_2K_1k(\|u^{n-1}\|^2 + c_3k|f|_\infty^2).
$$

Since $x \le 4/5$ by (2.21) and

$$
\frac{2}{1+\sqrt{1-x}} \le 1 + \frac{x}{2} \quad \text{if} \quad 0 \le x \le \frac{4}{5},
$$

relation (2.32) implies, under assumption (2.21), that

$$
\|u^n\|^2 \le \left(\|u^{n-1}\|^2 + c_3k|f|_\infty^2\right)\left[1 + 2c_2K_1k\left(\|u^{n-1}\|^2 + c_3k|f|_\infty^2\right)\right]. \tag{2.33}
$$

Using (2.21) once again, (2.33) immediately implies (2.22). ∎

In order to obtain estimates on a finite interval of time, we will inductively use Lemma 2.3, together with the following result, which was proved in [10] and which we repeat here for convenience.

LEMMA 2.4. *Given $k > 0$, an integer $n_* > 0$, and positive sequences ξ_n, η_n, and ζ_n such that*

$$
\xi_n \le \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n \quad \textit{for}\ \ n = 1, \ldots, n_*, \tag{2.34}
$$

we have, for any $n \in \{2, \dots, n_*\}$,

$$\xi_n \le \xi_0 \exp\left(\sum_{i=0}^{n-1} k\eta_i\right) + \sum_{i=1}^{n-1} k\zeta_i \exp\left(\sum_{j=i}^{n-1} k\eta_j\right) + k\zeta_n\,. \tag{2.35}$$

Proof. Using (2.34) recursively, we derive

$$\xi_n \le \xi_0 \prod_{i=0}^{n-1}(1+k\eta_i) + \sum_{i=1}^{n} k\zeta_i \prod_{j=i}^{n-1}(1+k\eta_j)$$

with the convention that $\prod_{j=\alpha}^{\beta} r_j = 1$ for $\beta < \alpha$. Using the fact that $1 + x \le \mathrm{e}^x$ for all $x \in \mathbb{R}$, the conclusion of the lemma follows. □

PROPOSITION 2.5 (estimates on a finite interval). *Let* $T > 0$ *and let* $K_3(\cdot,\cdot,\cdot)$ *be the function, monotonically increasing in all its arguments, given in* (2.47). *Suppose the timestep* k *is such that*

$$k \le \min\{\kappa_1, \kappa_2(|u_0|, |f|_\infty), \kappa_3(\|u^0\|, |f|_\infty, T)\}, \tag{2.36}$$

where κ_1 *is given by* (2.13), *and*

$$\kappa_2(|u_0|, |f|_\infty) = \frac{1}{10c_2c_4K_1|f|_\infty^2}, \tag{2.37}$$

$$\kappa_3(\|u^0\|, |f|_\infty, T) = \frac{1}{10c_2K_1K_2K_3(\|u^0\|, |f|_\infty, T)}. \tag{2.38}$$

Then (i) *relation* (2.22) *holds for all* $n = 1, \dots, N := \lfloor T/k \rfloor$, *and* (ii)

$$\|u^n\|^2 \le K_3(\|u^0\|, |f|_\infty, nk) \quad \forall\, n = 1, \dots, N := \lfloor T/k \rfloor. \tag{2.39}$$

Proof. Let $T > 0$ and k be such that hypothesis (2.36) is satisfied. We will use induction on n.

Since $\|u^0\|^2 \le K_3(\|u^0\|, |f|_\infty, 0)$, (2.37) and (2.38) imply that condition (2.21) of Lemma 2.3 is satisfied for $n = 1$,

$$c_2K_1k(K_2\|u^0\|^2 + c_4|f|_\infty^2) \le \frac{1}{10} + \frac{1}{10} \le \frac{1}{5}. \tag{2.40}$$

By the same lemma, we have

$$\|u^1\|^2 \le \|u^0\|^2\left[1 + c_5K_1k\left(\|u^0\|^2 + k|f|_\infty^2\right)\right] + c_6k|f|_\infty^2. \tag{2.41}$$

Now assume that (2.21) holds for $n = 1, \dots, m$ for some $m \le N$. Then by Lemma 2.3, (2.22) holds for $n = 1, \dots, m$; furthermore, we can bound $\|u^m\|$ as follows. We write the stepwise bound (2.22) in Lemma 2.3 in the form

$$\xi_n \le \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta, \tag{2.42}$$

where

$$\xi_n = \|u^n\|^2, \quad \eta_n = c_5K_1(\|u^n\|^2 + k|f|_\infty^2), \quad \text{and} \quad \zeta = c_6|f|_\infty^2. \tag{2.43}$$

Our intention is to apply (the discrete Gronwall) Lemma 2.4. So we compute for $i > 0$, using (2.3),

$$\sum_{j=i}^{m-1} k\eta_j = c_5 K_1 \sum_{j=i}^{m-1} k\left(\|u^j\|^2 + k|f|_\infty^2\right) \le c_7 K_1\big[K_1 + (m-i)k|f|_\infty^2\big]; \tag{2.44}$$

similarly, for $i = 0$,

$$\sum_{j=0}^{m-1} k\eta_j = c_5 K_1 \sum_{j=0}^{m-1} k\left(\|u^j\|^2 + k|f|_\infty^2\right) \le c_7 K_1\big(K_1 + mk|f|_\infty^2\big) + c_5 K_1 k\|u^0\|^2. \tag{2.45}$$

We note that, using (2.38) and recalling that $K_2 \ge 2$, the last term can be bounded as

$$c_5 K_1 k\|u^0\|^2 \le \frac{c_5\|u^0\|^2}{10c_2 K_2 K_3(\|u^0\|, |f|_\infty, T)} \le \frac{c_5}{10c_2K_2}\frac{\|u^0\|^2}{K_3(\|u^0\|, |f|_\infty, 0)} \le \frac{c_5}{20c_2}. \tag{2.46}$$

The middle term in (2.35) here is

$$\sum_{i=1}^{m-1} k\zeta \exp\left(\sum_{j=i}^{m-1} k\eta_j\right) \le c_6|f|_\infty^2 \sum_{i=1}^{m-1} k\exp(c_7K_1^2 + c_7K_1(m-i)k|f|_\infty^2) \le c_6|f|_\infty^2 \exp(c_7K_1^2)mk\ \exp(c_7K_1\, mk\,|f|_\infty^2).$$

The following bound on $\|u^m\|^2$ then follows from (2.35):

$$\begin{aligned}\|u^m\|^2 &\le \|u^0\|^2 \exp(c_7K_1\,|f|_\infty^2\, mk)\ \exp(c_7K_1^2 + c_5/(20c_2)) \\ &\quad + 2c_6|f|_\infty^2 \exp(c_7K_1^2)\, mk\ \exp(c_7K_1\,|f|_\infty^2\, mk) \\ &=: K_3(\|u^0\|, |f|_\infty, mk).\end{aligned} \tag{2.47}$$

We note that the bound K_3 depends on the initial discrete value through its norm $\|u^0\|$ and also on m, but this latter dependence is only through the time mk. We also note the dependence of K_3 on $|u_0|$ through K_1, but K_1 bounds all $|u^n|^2$.

It is now clear that, given the hypothesis of the proposition, the timestep k satisfies condition (2.21) as long as $m \le \lfloor T/k \rfloor$, completing the proof. ∎

Now, since Proposition 2.5 gives a bound on $\|u^n\|^2$ that is valid on a finite time interval only, we are going to extend the result to infinite time by repeatedly applying it and the following (discrete uniform Gronwall) lemma, which is a slightly more general version of the discrete uniform Gronwall lemma of Shen [10].

Lemma 2.6. *Given $k > 0$, positive integers n_1, n_2, n_* such that $n_1 < n_*$, $n_1 + n_2 + 1 \le n_*$, positive sequences ξ_n, η_n, and ζ_n such that*

$$\xi_n \le \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n \quad \textit{for } n = n_1, \dots, n_*, \tag{2.48}$$

and given the bounds

$$\sum_{n=n'}^{n'+n_2} k\eta_n \le a_1(n_1,n_*), \qquad \sum_{n=n'}^{n'+n_2} k\zeta_n \le a_2(n_1,n_*), \qquad \sum_{n=n'}^{n'+n_2} k\xi_n \le a_3(n_1,n_*) \tag{2.49}$$

for any n' satisfying $n_1 \le n' \le n_* - n_2$, we have

$$\xi_n \le \left(\frac{a_3(n_1,n_*)}{kn_2} + a_2(n_1,n_*)\right) \mathrm{e}^{a_1(n_1,n_*)} \tag{2.50}$$

for any n such that $n_1 + n_2 + 1 \le n \le n_*$.

Proof. Let n_3 and n_4 be such that $n_1 \le n_3 - 1 \le n_4 \le n_2 + n_3 - 1 \le n_* - 1$. Using (2.48) recursively, we derive

$$\xi_{n_2+n_3} \le \xi_{n_4} \prod_{i=n_4}^{n_3+n_2-1} (1 + k\eta_i) + \sum_{i=n_4+1}^{n_3+n_2} k\zeta_i \prod_{j=i}^{n_2+n_3-1} (1 + k\eta_j) \tag{2.51}$$

with the convention that $\prod_{j=\alpha}^{\beta} r_j = 1$ for $\beta < \alpha$. Using the fact that $1 + x \le \mathrm{e}^x$ for all $x \in \mathbb{R}$, and recalling the first two assumptions in (2.49), we obtain

$$\xi_{n_2+n_3} \le (\xi_{n_4} + a_2)e^{a_1}.$$

Multiplying this inequality by k, summing n_4 from $n_3 - 1$ to $n_2 + n_3 - 2$, and using the third assumption in (2.49) gives the conclusion (2.50) of the lemma. ∎

We are now in a position to give the main result, that is, to derive a uniform bound for $\|u^n\|$ for all $n \ge 1$.

THEOREM 2.7. *Let $u_0 \in V$, $f \in L^\infty(\mathbb{R}_+; H)$, and u^n be the solution of the numerical scheme* (1.9). *Also, let $r \ge 4\kappa_1$ be arbitrarily fixed and let k be such that*

$$k \le \min\{\kappa_1, \kappa_2(|u_0|, |f|_\infty), \kappa_3(\|u_0\|, |f|_\infty, T_0 + r), \kappa_3(\rho_1, |f|_\infty, r)\}, \tag{2.52}$$

where $\kappa_1 = 1/(\nu\lambda_1)$ was defined in (2.13)*, $\kappa_2(\cdot,\cdot)$ and $\kappa_3(\cdot,\cdot,\cdot)$ are given in Proposition 2.5, T_0, the time of entering an absorbing ball for $|u^n|$, is given by* (2.14)*, and $\rho_1(|f|_\infty, r)$ is given in* (2.57).

Then we have

$$\|u^n\|^2 \le K_5(\|u_0\|, |f|_\infty) \quad \forall\, n \ge 1, \tag{2.53}$$

where $K_5(\cdot,\cdot)$ is a continuous function defined on $\mathbb{R}_+^2$, increasing in both arguments. Moreover,

$$\|u^n\|^2 \le K_4(|f|_\infty) \quad \forall\, n \ge N_0 + N_r := \lfloor T_0/k \rfloor + \lfloor r/k \rfloor, \tag{2.54}$$

i.e., $\|u^n\|$ is bounded independently of u_0 beyond $N_0 + N_r$.

Proof. Let $r \ge 4\kappa_1$ be arbitrarily fixed and let k be such that (2.52) holds.

The idea for deriving a uniform bound for $\|u^n\|^2$ for all $n \ge 1$ is as follows:

(i) Applying first Proposition 2.5 on $(0, T_0 + r)$ (that is, for $n = 1, \dots, N_0 + N_r$), we get an upper bound for $\|u^n\|$ for $n = 1, \dots, N_0 + N_r$; applying Lemma 2.6, we show that $\|u^{N_0+N_r}\|^2 \le \rho_1^2$, where $\rho_1(|f|_\infty, r)$ is defined in (2.57).

(ii) Iterating Proposition 2.5 and Lemma 2.6, at each step $i \ge 2$, we show that for all $n = N_0 + (i-1)N_r + 1, \dots, N_0 + iN_r$, $\|u^n\|^2$ is bounded by $K_3(\|u^{N_0+(i-1)N_r}\|,$

$|f|_\infty, r)$; using the estimate on $\|u^{N_0+(i-1)N_r}\|$ from the previous step, we obtain that $\|u^n\|^2$ is bounded independently of the initial value for all $n = N_0 + (i-1)N_r + 1, \ldots, N_0 + iN_r$ for every $i \geq 2$ (and thus for all $n \geq N_0 + N_r$).

We now proceed to give a rigorous proof of the theorem.

Noting that, by hypothesis, k satisfies condition (2.36) of Proposition 2.5 with $T = T_0 + r$, we first apply Proposition 2.5 and obtain that (2.22) holds for all $n = 1, \ldots, N_0 + N_r$, and

$$\|u^n\|^2 \leq K_3(\|u^0\|, |f|_\infty, nk) \quad \forall\, n = 1, \ldots, N_0 + N_r. \tag{2.55}$$

At this point we know that for k satisfying hypothesis (2.52),

$$\|u^n\|^2 \leq \|u^{n-1}\|^2[1 + c_5 K_1 k(\|u^{n-1}\|^2 + k|f|_\infty^2)] + c_6 k|f|_\infty^2 \quad \forall\, n = 1, \ldots, N_0 + N_r, \tag{2.56}$$

and we apply (the discrete uniform Gronwall) Lemma 2.6 with $\xi_n = \|u^n\|^2$, $\eta_n = c_5 K_1(\|u^n\|^2 + k|f|_\infty^2)$, $\zeta_n = c_6|f|_\infty^2$, $n_1 = N_0 + 1$, $n_2 = N_r - 2$, and $n_* = N_0 + N_r$ to obtain a bound for $\|u^{N_0+N_r}\|$. In computing the sums $a_1(n_1, n_*)$, $a_2(n_1, n_*)$, and $a_3(n_1, n_*)$ that appear there, we note that since all those sums are taken for $n \geq N_0$ and since, by hypothesis, k satisfies condition (2.13) of Corollary 2.2, we can replace K_1, the bound on $|u^n|^2$, by $2\rho_0^2$, whenever the former appears. For every $n' = N_0 + 1, N_0 + 2$, we compute, using (2.3) and (2.14) for the first and last lines,

$$2c_5\rho_0^2 \sum_{n=n'}^{n'+n_2} \left(k\|u^n\|^2 + k^2|f|_\infty^2\right) \leq c_8\rho_0^2(\rho_0^2 + r|f|_\infty^2),$$

$$c_6 \sum_{n=n'}^{n'+n_2} k|f|_\infty^2 \leq c_6\, r|f|_\infty^2,$$

$$\sum_{n=n'}^{n'+n_2} k\|u^n\|^2 \leq c_9(\rho_0^2 + r|f|_\infty^2).$$

Using the conclusion (2.50) of Lemma 2.6 and the fact that $r \geq 4\kappa_1$, we obtain

$$\begin{aligned}\|u^{N_0+N_r}\|^2 &\leq \left[2c_9\left(\rho_0^2/r + |f|_\infty^2\right) + c_6\, r|f|_\infty^2\right]\exp\left(c_8\rho_0^2(\rho_0^2 + r|f|_\infty^2)\right)\\ &=: \rho_1(|f|_\infty; r)^2.\end{aligned} \tag{2.57}$$

Now, since by hypothesis $k \leq \kappa_3(\rho_1, |f|_\infty, r)$ and since $\kappa_3(\cdot,\cdot,\cdot)$ is a decreasing function of its arguments, we can regard $u^{N_0+N_r}$ as our initial data and apply Proposition 2.5 with $T = r$. We obtain that relation (2.22) holds for all $n = N_0 + N_r + 1, \ldots, N_0 + 2N_r$, and

$$\|u^n\|^2 \leq K_3(\|u^{N_0+N_r}\|, |f|_\infty, N_r k) \quad \forall\, n = N_0 + N_r + 1, \ldots, N_0 + 2N_r. \tag{2.58}$$

Thanks to (2.57) and to the fact that $K_3(\cdot,\cdot,\cdot)$ is an increasing function of all its arguments, we have

$$\|u^n\|^2 \leq K_3(\rho_1, |f|_\infty, N_r k) \quad \forall\, n = N_0 + N_r + 1, \ldots, N_0 + 2N_r. \tag{2.59}$$

Applying again Lemma 2.6 with $n_1 = N_0 + N_r + 1$, $n_2 = N_r - 2$, and $n_* = N_0 + 2N_r$, we obtain

$$\|u^{N_0+2N_r}\|^2 \leq \rho_1^2\,. \tag{2.60}$$

Iterating Proposition 2.5 and Lemma 2.6 and reasoning as above, we arrive at

$$\|u^n\|^2 \le K_3(\rho_1, |f|_\infty, r) =: K_4(|f|_\infty) \quad \forall\, n \ge N_0 + N_r, \tag{2.61}$$

and recalling (2.55), we conclude

$$\begin{aligned} \|u^n\|^2 &\le \max\{K_3(\|u_0\|, |f|_\infty, T_0 + r), K_4(|f|_\infty)\} \\ &=: K_5(\|u_0\|, |f|_\infty) \quad \forall\, n \ge 1, \end{aligned} \tag{2.62}$$

thus proving the theorem. ∎

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