Permutation Separations and Complete Bipartite Factorisations of $K_{n,n}$

Nigel Martin

Department of Mathematics
University of Durham, Durham, U.K.
nigel.martin@durham.ac.uk

Richard Stong

Department of Mathematics
Rice Univeristy, Houston, TX, USA
stong@math.rice.edu

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Abstract

Suppose $p < q$ are odd and relatively prime. In this paper we complete the proof that $K_{n,n}$ has a factorisation into factors F whose components are copies of $K_{p,q}$ if and only if n is a multiple of $pq(p+q)$. The final step is to solve the "c-value problem" of Martin. This is accomplished by proving the following fact and some variants: For any $0 \leq k \leq n$, there exists a sequence $(\pi_1, \pi_2, \ldots, \pi_{2k+1})$ of (not necessarily distinct) permutations of $\{1, 2, \ldots, n\}$ such that each value in $\{-k, 1-k, \ldots, k\}$ occurs exactly n times as $\pi_j(i) - i$ for $1 \leq j \leq 2k-1$ and $1 \leq i \leq n$.

1 Introduction

This goal of this paper is to complete the study of factorisation of balanced complete bipartite graphs $K_{n,n}$ into factors each of whose components are $K_{p,q}$. This subject began with the study of star-factorisations (where all components are $K_{1,k}$ for some fixed k) of complete bipartite graphs by Ushio [5], Ushio and Tsuruno [6], Wang [7], and Du [1]. The results were extended to factorisations where the components are $K_{p,q}$ by Martin in a sequence of papers [2], [3], and [4]. Specifically we make the following definition.

Definition . *Let F and G be (simple, undirected) graphs. An F-*factor *of G is a spanning subgraph of G whose components are all isomorphic to F. A* (complete) F-factorisation *of G is a decomposition of G as a union of edge-disjoint F-factors.*

The first paper in the sequence [2] derives necessary conditions for a $K_{p,q}$-factorisation of $K_{m,n}$ called the Basic Arithmetic Conditions (BAC). The natural BAC Conjecture states that these BAC conditions are also sufficient for a $K_{p,q}$-factorisation. In addition [2], shows that it suffices to consider the case where p and q are coprime and resolves the BAC Conjecture when p and q are coprime and either p or q is even. For the special case of factorisations of balanced complete bipartite graphs $K_{n,n}$ and odd, relatively prime $p < q$, the BAC conditions reduce to just that n must be a multiple of $pq(p+q)$ [2, Theorem 2.5] and it suffices to consider the case $n = pq(p+q)$. The final paper in this sequence [4] reduces the question of whether a factorisation exists for odd, relatively prime $p < q$ to a much simpler question called the "c-value problem". Martin [4] shows that the c-value problem is solvable provided $\frac{1}{2}p^2+O(p) > q > p$. In this paper we will rephrase the c-value problem as a question involving permutations. With the greater flexibility provided by permutations we will give a complete positive solution to the c-value problem and thus will conclude:

Balanced Factorisation Theorem. $K_{n,n}$ has a $K_{p,q}$-factorisation if and only if the BAC conditions hold.

Despite the fact that the goal of this paper is to prove that $K_{p,q}$-factorisations exist, we will not be concerned with graphs directly since we can tie in to results in [4] instead. Specifically, Martin [4] makes the following definition.

Definition . *A* cross-section *of a sequence $(X_i)_{i=1}^t$ of subsets of the integers is a sequence $(x_i)_{i=1}^t$ such that $x_i \in X_i$ for all i. A cross-section $(x_i)_{i=1}^t$ is called* consistent *if for all $i \neq j$ we have $x_i - x_j \neq i - j$.*

(This definition of consistency actually differs slightly from that in [4]. However by [4, Lemma 14] this simpler definition is equivalent in our context.) Using this terminology Martin [4, Theorem 1, Theorem 2 and Lemma 14] proves the following result.

Theorem(Martin [4]). Given coprime odd integers p and q with $3 \leq p < q$ let $n = pq(p+q)$, $s = (p-1)/2$ and $t = (q-1)/2$. If $p+q \equiv 0 \pmod 4$, then define $S = \{x| -s \leq x \leq s\}$ and if $p + q \equiv 2 \pmod 4$, then define $S = \{x| -(s+1) \leq x \leq s+1, x \neq \pm 1\}$. Define sequences of sets $(X_i)_{i=1}^t$ and $(Y_i)_{i=1}^{t+1}$ by $X_i = S \cap \{x|i-t \leq x \leq i-1\}$ and $Y_i = S \cap \{x|i-t-1 \leq x \leq i-1\}$. Suppose there exist p consistent cross-sections of $(X_i)_{i=1}^t$ and p consistent cross-sections of $(Y_i)_{i=1}^{t+1}$ so that in aggregate each number in S occurs q times in the cross-sections, then $K_{n,n}$ admits a $K_{p,q}$-factorisation.

We will refer to the problem of whether two such collections of consistent cross-sections as required above exist for (p, q) as the "c-value problem" for p and q. (Again this terminology differs slightly from [4]. In [4] the "c-value problem" is a more elaborate statement and existence of these cross-sections is sufficient but not necessary to solve the c-value problem. However since we will show the desired cross-sections always exist this distinction will become moot.)

Thus the real content of this paper will be the construction of the desired cross-sections. In Section 2, we will rephrase the c-value problem as a question involving permutations.

This provides a slightly cleaner statement, allows us to bring in the convenient notation for permutations, and enables us to use some geometric insight. In Section 3, we will develop some lemmas for building useful sequences of permutations. In Section 4, we will prove that the c-value problem has a solution for $p+q \equiv 0 \pmod 4$ by giving an inductive construction of the desired cross-sections. This inductive argument is basically a strengthening of the approach given in [4, Section 8]. (A more complicated direct construction is also possible.) In Section 5, we adapt the arguments from Section 4 to solve most cases of the c-value problem for $p+q \equiv 2 \pmod 4$. This case is slightly harder and uses the case $p+q \equiv 0 \pmod 4$ as a building block in the construction. Finally in Section 6 we solve the few remaining cases of the c-value problem for $p+q \equiv 2 \pmod 4$.

2 A permutation interpretation of the c-value problem

Suppose throughout the rest of this paper that p and q are odd, relatively prime integers with $q > p$. Let $n = pq(p+q)$, $t = (q-1)/2$ and $s = (p-1)/2$. If $p+q \equiv 0 \pmod 4$, let $S = \{x| -s \leq x \leq s\}$ and if $p+q \equiv 2 \pmod 4$, let $S = \{x| -s-1 \leq x \leq s+1, \ \ x \neq \pm 1\}$. For $1 \leq i \leq t$ we define $X_i = S \cap \{x|i-t \leq x \leq i-1\}$ and for $1 \leq i \leq t+1$ we define $Y_i = S \cap \{x|i-t-1 \leq x \leq i-1\}$. Recall that the c-value problem for p and q is to find $p = 2s+1$ consistent cross-sections of $(X_1, \ldots, X_t)$ and $p = 2s+1$ consistent cross-sections of $(Y_1, \ldots, Y_{t+1})$ so that in aggregate each element of S occurs exactly $q = 2t+1$ times in the cross-sections.

Suppose $(x_1, \ldots, x_t)$ is a consistent cross-section for $(X_1, \ldots, X_t)$. Then we can define $\sigma(i)$ for $0 \leq i \leq t-1$ by $\sigma(i) = i - x_{i+1}$. Note that $\sigma(i) \leq i - (i+1-t) = t-1$, $\sigma(i) \geq i - i = 0$, and by consistency the $\sigma(i)$ are distinct. Thus σ is a permutation of $\{0, 1, \ldots, t-1\}$. Further we have $\sigma(i) - i = -x_{i+1} \in S$. Conversely, given such a permutation σ we can construct a consistent cross-section by $x_i = i-1-\sigma(i-1)$.

Similarly, suppose $(y_1, \ldots, y_{t+1})$ is a consistent cross-section for $(Y_1, \ldots, Y_{t+1})$. Then we can define $\sigma(i)$ for $0 \leq i \leq t$ by $\sigma(i) = i - y_{i+1}$. As above $\sigma(i) \leq i-(i+1-t-1) = t$, $\sigma(i) \geq i - i = 0$, and by consistency the $\sigma(i)$ are distinct. Thus σ is a permutation of $\{0, 1, \ldots, t\}$. Further we have $\sigma(i) - i = -y_{i+1} \in S$. Conversely, given such a permutation σ we can construct a consistent cross-section by $y_i = i-1-\sigma(i-1)$.

Thus the c-value problem can be rephrased entirely in terms of permutations giving the following lemma.

Lemma 1. *The c-value problem for (p,q) is equivalent to finding a sequence $(\sigma_i)_{i=1}^{2s+1}$ of permutations of $\{0, 1, \ldots, t-1\}$ and a sequence $(\pi_i)_{i=1}^{2s+1}$ of permutations of $\{0, 1, \ldots, t\}$ such that in aggregate each value in S occurs exactly $2t+1$ times as $\sigma_j(i) - i$ or $\pi_j(i) - i$.*

Note that the lemma accounts for all $pq = (2s+1)(2t+1) = |S| \cdot (2t+1)$ values of $\sigma_j(i) - i$ and $\pi_j(i) - i$, hence neither $\sigma_j(i) - i$ nor $\pi_j(i) - i$ can achieve values outside of S.

For a permutation σ, we will refer to the values of $\sigma(i)-i$ as the *separations* achieved by σ. Note that the separations achieved by σ^{-1} are exactly the negatives of those achieved by σ. We will call a permutation σ *value-symmetric* if for all m, $\sigma(i)-i=m$ and $\sigma(i)-i=-m$ have the same number of solutions. The arguments below will use mostly value-symmetric permutations. (Otherwise we will use a permutation and its inverse together, thus achieving symmetry of values from the pair.) Note that permutations of order two are always value-symmetric.

One advantage to working with value-symmetric permutations (or combinations of permutations which achieve symmetry) is that we can focus on only the nonnegative separations. To keep track of these we will use partition notation. Specifically, suppose σ is a value-symmetric permutation (or more generally a symmetric collection of permutations) which achieves n_i separations of i for $0 \le i \le t-1$. Then we will say σ achieves $(t-1)^{n_{t-1}}(t-2)^{n_{t-2}}\cdots 1^{n_1}0^{n_0}$.

This reinterpreted c-value problem asks for two sets of permutations which in aggregate achieve every separation in S a total of $2t+1$ times. One might be optimistic and try to achieve a stronger version of the c-value problem, where the first set $(\sigma_i)_{i=1}^{t}$ achieve each separation in S exactly t times and the second set $(\pi_i)_{i=1}^{t+1}$ achieve each separation in S exactly $t+1$ times. For $p+q \equiv 0 \pmod 4$, this prompts the following family of claims.

Claim (t,s)**.** For $s<t$ there is a sequence $(\sigma_1,\dots,\sigma_{2s+1})$ of (not necessarily distinct) permutations of $\{0,\dots,t-1\}$ such that in aggregate each value in $\{-s,1-s,\dots,s\}$ occurs t times as $\sigma_j(i)-i$.

For $p+q \equiv 0 \pmod 4$, a positive solution to Claim (t,s) would supply the desired set of (σ_i) and a positive solution to Claim $(t+1,s)$ would supply the desired set of (π_i). In Section 4, we will prove that Claim (t,s) holds for $0 \le s < t$ and thus solve the c-value problem for $p+q \equiv 0 \pmod 4$.

For the case $p+q \equiv 2 \pmod 4$ a similar optimism prompts looking at the following family of guesses.

Guess (t,s)**.** For $s+1<t$ there is a sequence $(\sigma_1,\dots,\sigma_{2s+1})$ of (not necessarily distinct) permutations of $\{0,\dots,t-1\}$ such that in aggregate every value in $S=\{-s-1,-s,\dots,-2,0,2,\dots,s+1\}$ occurs t times as $\sigma_j(i)-i$.

For $p+q \equiv 2 \pmod 4$, a positive solution to Guess (t,s) would supply the desired set of (σ_i) and a positive solution to Guess $(t+1,s)$ would supply the desired set of (π_i). Unfortunately, these Guesses are not always true. In Section 5, we will prove that Guess (t,s) is false for $s=t-2$. However we will show that Guess (t,s) holds for $0 \le s < t-4$. This will solve the c-value problem for $p+q \equiv 2 \pmod 4$ unless $q=p+4$. For this last case we cannot split the problem into two disjoint pieces, but we will solve it in Section 6 using the techniques we will develop in the earlier sections.

3 Constructions of sequences of permutations

There are several advantages to rephrasing the c-value problem in terms of permutations. One of these is that we can think of permutations geometrically. Specifically, consider a $t \times t$ square divided into t^2 unit squares labelled by pairs (i, j) with $0 \le i, j \le t-1$. Then we can view a permutation σ of $\{0, 1, \ldots, t-1\}$ as a collection of t unit squares with one square in each row and one in each column by taking the squares $(i, \sigma(i))$. The separations $\sigma(i) - i$ correspond to the diagonal on which these unit squares lie, with a separation of zero corresponding to the main diagonal $\{(i, i)\}$. For future reference, we will refer to the collection of squares $\{(i, t-i-1)\}$ as the anti-diagonal. This geometric picture allows new permutations to be built from old permutations in a variety of ways. We will usually describe these constructions by formulas below, but considering the geometric picture may help the reader understand some of the constructions better.

If σ is a permutation of $\{0, \ldots, t-1\}$ and τ is a permutation of $\{0, \ldots, u-1\}$, then we will define the concatenation $\sigma * \tau$ to be the permutation of $\{0, \ldots, t+u-1\}$ obtained by setting $\sigma * \tau(i) = \sigma(i)$ if $0 \le i \le t-1$ and $\sigma * \tau(i) = \tau(i-t) + t$ if $t \le i \le t+u-1$. Note that the set of values achieved by $\sigma * \tau$ is the union of the sets of the values achieved by σ and by $\tau(i)$. Thus we have the following easy lemma.

Lemma 2. *(a) If Claims (t, s) and (u, s) are true, then Claim $(t+u, s)$ is also true.*

(b) If Guesses (t, s) and (u, s) are true, then Guess $(t+u, s)$ is also true.

Proof. Let $(\sigma_1, \ldots, \sigma_{2s+1})$ and $(\tau_1, \ldots, \tau_{2s+1})$ be solutions to Claims (t, s) and (u, s) (resp. Guesses (t, s) and (u, s)), then $(\sigma_1 * \tau_1, \ldots, \sigma_{2s+1} * \tau_{2s+1})$ solves Claim $(t+u, s)$ (resp. Guess $(t+u, s)$). □

Lemma 3. *(a) For any odd $k \ge 1$ there exists a value-symmetric permutation τ of $\{0, 1, \ldots, k-1\}$ such that for all $0 \le i \le k-1$ we have $\tau(i) - i \ne \pm 1$ and every value in $\{1-k, \ldots, k-1\}$ occurs at most once as $\tau(i) - i$.*

(b) For any even $k \ge 2$ there exists a value-symmetric permutation τ of $\{0, 1, \ldots, k-1\}$ such that every non-zero value in $\{1-k, \ldots, k-1\}$ occurs at most once as $\tau(i) - i$ and zero does not occur.

(c) For any even $k \ge 2$ there exists a value-symmetric permutation τ of $\{0, 1, \ldots, k-1\}$ such that every non-zero value in $\{1-k, \ldots, k-1\}$ occurs at most once as $\tau(i) - i$, zero occurs at most twice and ± 1 do not occur.

Proof. For (a) and (b) take $\tau(i) = k-1-i$. For (c) take $\tau(i) = k-2-i$ for $0 \le i \le k-2$ and $\tau(k-1) = k-1$. □

Using Lemma 3, we can give a greedy algorithm for constructing permutations that in aggregate exhaust a desired set of values of $\sigma(i) - i$. We will exploit this greedy algorithm by dealing with some values of $\sigma(i) - i$ by more direct means, then using the greedy argument to fill in the gaps. The gaps that are left can be viewed as being filled by permutations of $\{0, \ldots, t'-1\}$ for some $t' \le t$. Thus we will need to produce permutations of various intervals. As a result we get the technical conditions below.

Lemma 4. *Suppose we are given a sequence* $(t_1, \ldots, t_k)$ *of positive integers (ordered in ascending order), a sequence* $(n_0, \ldots, n_{s-1})$ *of nonnegative integers, and an integer* $n_s > 0$ *such that:*

(i) $n_0 + 2\sum_{i=1}^{s} n_i = \sum_{j=1}^{k} t_j$*;*

(ii) $n_i \geq s + k - i - 1$ *for* $2 \leq i \leq s-1$*; and*

(iii) let m *be the number of even integers among the* t_j*, then* $2n_1 + n_0 \geq k + m$ *and* $n_0 \geq k - m$*.*

Then there exist $(\sigma_j)_{j=1}^{k}$ *where* σ_j *is a permutation of* $\{0, 1, \ldots, t_j - 1\}$ *such that in aggregate* i *and* $-i$ *each occur* n_i *times as* $\sigma_j(i) - i$*.*

Proof. The proof is by induction on $\sum_{j=1}^{k} t_j$ and starts trivially with this sum being 1 when the data require that $s = 0$ and $k = 1$. The inductive step is attacked by a detailed case analysis which is best broken down into a series of cases and sublemmas.

Case 1. $s = 0$. In this case take all the permutations as the identity.

From now assume $s > 0$.

Case 2. $t_1 \leq s+1$ is odd. In this case, let σ_1 be the permutation from Lemma 3(a). The effect is to reduce k by 1, m remains the same and every n_i reduces by 1 for i even. Conditions (i) - (iii) clearly remain satisfied.

Case 3. $t_1 \leq s+1$ is even. In this case, let σ_1 be the appropriate permutation from Lemma 3(b) or 3(c). The choice between the permutation provided by Lemma 3(b) and 3(c) is determined by whether n_1 is zero or not. In either case it is clear that the conditions (i) - (iii) remain satisfied.

A form of this construction will be used at various other points in the proof and at these points similar arguments about preservation of the conditions will apply. From now assume that $t_1 > s+1$.

Sublemma 4.1. *Let* $0 < u \leq t_k$ *and suppose that* τ *is a value symmetric permutation of* $(0, \ldots, u-1)$ *achieving the values* $\prod_{i=0}^{s}(i)^{r_i}$ *and so that the sequences* $(t_1, \ldots, t_{k-1}, t_k - u)$ *and* $(n_0 - r_0, \ldots, n_s - r_s)$ *(after re-ordering, if necessary) still satisfy the hypotheses of Lemma 4, then a solution* $\{\sigma_i\}_{i=1}^{k}$ *to this latter problem extends to a solution of the original by replacing* σ_k *with* $\sigma_k * \tau$*. (Note that if* $u = t_k$ *then this is interpreted by* σ_k *being the empty permutation.)*

Proof. Simply apply the induction. □

Case 4. $s = 1$. In this case, by assumption $t_k \geq t_1 \geq 2$ and $n_1 > 0$. Let $u = 2$ and $\tau = (0\ \ 1)$ and apply Sublemma 4.1. Conditions (i) - (iii) are easily satisfied in both cases.

Case 5. $s = 2$. In this case, by assumption $t_k \geq t_1 \geq 3$. If $t_1 = 3$ then take $\sigma_1 = (0\ \ 2)(1)$. This reduces k by 1, leaves m unchanged and reduces both n_0 and n_2 by 1. Conditions (i) - (iii) remain satisfied. Otherwise all $t_j \geq 4$. If $n_1 > 0$ then let $u = 4$ and $\tau = (0\ \ 1\ \ 3\ \ 2)$. If $n_1 = 0$ let $u = 4$ then if $n_2 \geq 2$ let $\tau = (0\ \ 2)(1\ \ 3)$ and if $n_2 = 1$, let $\tau = (0\ \ 2)(1)(3)$. In each case conditions (i) - (iii) remain satisfied.

From now we assume $s \geq 3$.

Sublemma 4.2. $t_k \geq 2s - 2$.

Proof. From condition (ii), $\sum_{i=2}^{s-1} n_i \geq \frac{1}{2}(s-2)(s-3) + k(s-2)$. Hence

$$\sum_{j=1}^{k} t_j = n_0 + 2\sum_{i=1}^{s} n_i > (s-2)(s-3) + 2k(s-2) + k + m.$$

Thus the average value of the t_j is more than $2s - 3$. □

Sublemma 4.3. *If* $n_s \geq \min\{t_k, 2s\} - s$ *the inductive step proceeds.*

Proof. If $t_k \geq 2s$ then let $u = 2s$, $\tau = \prod_{r=0}^{s-1}(r \;\; r+s)$ and apply Sublemma 4.1. n_s reduces by s.

If $t_k = 2s-1$, then let $\sigma_k = (s-1)\prod_{r=0}^{s-2}(r \;\; r+s)$. As $2s-1$ is odd k reduces by 1, m is unchanged, n_0 reduces by 1 and n_s reduces by $s-1$. It is clear that the conditions remain satisfied.

If $t_k = 2s-2$, then let $\sigma_k = \alpha \prod_{r=0}^{s-3}(r \;\; r+s)$ where α is either $(s-2)(s-1)$ if $n_1 = 0$ or $(s-2 \;\; s-1)$ if $n_1 > 0$. Again it is simple to check that the conditions remain satisfied and n_s reduces by $s-2$.

In each case the hypothesis ensures that n_s is reducible by the amount required. □

From here we can now also assume that $n_s < s$ since otherwise Sublemma 4.3 allows a further reduction.

Case 6. $s = 3$. Note that in this case we have $n_3 < 3$, and, since $t_1 > 4$, we must have $t_k \geq 5 = 2s - 1$.

If $n_3 = 2$ and $t_k \geq 6$, let $u = 6$ and $\tau = (0 \;\; 2 \;\; 5 \;\; 3)(1 \;\; 4)$ which provides separations $(3)^2(2)^1$ whence we can apply Sublemma 4.1 since $n_1 \geq k$. If $n_3 = 2$ and $t_k = 5$, then we are covered by Sublemma 4.3.

If $n_3 = 1$, $n_2 \geq 2$ and $t_k \geq 6$ let $u = 6$ and $\tau = (0 \;\; 2)(1 \;\; 4)(3 \;\; 5)$ and apply Sublemma 4.1. If $n_3 = 1$, $n_2 = 1$ or 2, and $t_k = 5$, let $\sigma_k = (0 \;\; 3)(2 \;\; 4)(1)$ and since 5 is odd, the conditions still apply.

Thus we are left with a final case: $n_3 = 1$, $n_2 = 1$ and $t_k \geq 6$. As $n_2 \geq k$ this means that $k = 1$ and $t = t_k = 2 + 2n_1 + n_0 \geq 6$. If $n_1 \geq 1$ then let $u = 6$ and $\tau = (0 \;\; 3 \;\; 1)(2 \;\; 4 \;\; 5)$ which has separations $3^1 2^1 1^1$. Now apply Sublemma 4.1 and the conditions remain satisfied since s reduces by 2. If $n_1 = 0$, let $u = 5$ and $\tau = (0 \;\; 3)(2 \;\; 4)(1)$ and apply Sublemma 4.1.

From here we assume that $s \geq 4$, $t_k \geq 2s-2$, $t_1 \geq s+1$, and $n_s < s$. Let p be the smallest integer such that $n_s + n_{s-1} + \cdots + n_{s-p} + p \geq s$. As $s \geq 4$ and $\sum_{i=0}^{s-2} n_{s-i} + s - 2 \geq s$, such a p exists and is less than $s-1$. Also $n_s + 0 < s$, hence $p > 0$. For ease of notation, let $a_r = \sum_{i=0}^{r} n_{s-i}$, so $a_0 = n_s$ and $a_i - a_{i-1} = n_{s-i}$. Conventionally let $a_{-1} = 0$.

Case 7. $p > 1$. If p is even, take $u = 2s - p$ and let τ be

$$\prod_{i=0}^{p-1}\left(\prod_{r=a_{i-1}+i}^{a_i+i-1}(r \;\; r+s-i)\right)\prod_{r=a_{p-1}+p}^{s-1}(r \;\; r+s-p)$$
$$\times \prod_{r=0}^{(p-2)/2}(a_{2r}+2r \;\; a_{2r+1}+2r+1).$$

If $p > 1$ is odd, take $u = 2s - p + 1$ and let τ be

$$\prod_{i=0}^{p-1}\left(\prod_{r=a_{i-1}+i+1}^{a_i+i}(r \;\; r+s-i)\right)\prod_{r=a_{p-1}+p+1}^{s}(r \;\; r+s-p)$$
$$\times \prod_{r=0}^{(p-1)/2}(a_{2r-1}+2r \;\; a_{2r+1}+2r+1).$$

In each case, u is even and τ is a complete set of disjoint transpositions achieving separations $\pm(s-i)$ n_{s-i} times for $0 \le i < p$ and separations $\pm(s-p)$ at most n_{s-p} times.

To be sure of this we need to examine the separations produced in the final product set in each case to ensure that these do not over-contribute to separations of these sizes. These additional separations come in sizes $\pm(n_{s-i}+1)$ for $0 \le i < p$.

If $p > 1$ then, by the minimality of p we know that $\sum_{i=0}^{p-1} n_{s-i} + (p-1) < s$. But each of the values of the n_{s-i} are at least 1, so this inequality manipulates to $n_{s-i}+1 \le s-2p+2 < s-p$ unless $p = 2$ and we have equality here. But if that is the case, then $n_s + n_{s-1} = s-2$ and the penultimate product in the expression for τ is in fact empty so that we have not found any separation yet of size $\pm(s-2)$ and there is space for this extra one. Also note that these extra separations all have size at least 2 so cannot affect the counting of n_0 and n_1.

Again, if $p > 1$ the value for u is always even and is no more than $2s-2 \le t_k$. So Sublemma 4.1 can be applied.

So we are left with the case $p = 1$.

Case 8. $p = 1$, $t_k \ge 2s$, and $n_s \le s-3$. In this case we take proceed as in the previous case. The only difference is that because $p = 1$ and $t_k \ge 2s$, we are concerned about the separation $n_s + 1$ introduced by the last product in τ. If $n_s \le s-3$ then this is at most $s-2 < s-1 = s-p$ and the same argument as above applies.

Case 9. $n_s = s-1$. If $t_k \ge 2s$, take $u = 2s$ and $\tau = (0 \;\; s-1 \;\; 2s-1 \;\; s)\prod_{i=1}^{s-2}(i \;\; i+s)$. This gives separations $s^{s-1}(s-1)^1$ and Sublemma 4.1 applies. If $t_k < 2s$ then by Sublemma 4.3 the situation is reducible also.

Case 10. $n_s = s-2$, $n_{s-1} \ge 2$ and $t_k \ge 2s$. Let $u = 2s$ and $\tau = (0 \;\; s-1)(s \;\; 2s-1)\prod_{i=1}^{s-2}(i \;\; i+s)$ which gives separations $s^{s-2}(s-1)^2$ and Sublemma 4.1 applies.

Note that if $n_s = s-2$, by condition (ii), this case can only fail to be applicable when $k=1$. Further in this case from the proof of Sublemma 4.2 and the fact that $s > 3$ it follows that $t_k \geq 2s$. Thus we are left with the following situations:

(a) $n_s \leq s-3$, $2s-2 \leq t_k \leq 2s-1$ and $n_s + n_{s-1} + 1 \geq s$.

(b) $n_s = s-2$, $n_{s-1} = 1$, $k=1$ and $t_k \geq 2s$, and

Case 11. $n_s \leq s-3$, $t_k = 2s-1$ and $n_s + n_{s-1} + 1 \geq s$. In this case, let

$$\omega_k = (n_s) \prod_{i=0}^{n_s-1} (i \;\; i+s) \prod_{i=n_s+1}^{s-1} (i \;\; i+s-1).$$

This uses all n_s separations of size s together with $s-n_s-1 \leq n_{s-1}$ separations of size $s-1$ and has one fixed point. But $2k-1$ is odd so m is unchanged and k reduces by 1 so conditions (i) - (iii) remain satisfied.

Case 12. $n_s \leq s-3$, $t_k = 2s-2$ and $n_s + n_{s-1} + 1 \geq s$. In this case, let

$$\omega_k = (n_s \;\; s-1) \prod_{i=0}^{n_s-1} (i \;\; i+s) \prod_{i=n_s+1}^{s-2} (i \;\; i+s-1).$$

This uses all n_s separations of size s together with $s-n_s-2 \leq n_{s-1}$ separations of size $s-1$ together with one of size $s-1-n_s$ where $s-2 \geq s-1-n_s \geq 2$. Thus we can reduce s by 1 and the inductive step can proceed.

Case 13. $n_s = s-2$, $k = n_{s-1} = 1$ and $t = t_k \geq 2s$. Since we have $s > 3$, we must also have $n_{s-2} \geq 2$. Now let $u = 2s$ and $\tau = (0 \;\; (s-2) \;\; (2s-2) \;\; s \;\; (2s-1) \;\; (s-1)) \prod_{i=1}^{s-3} (i \;\; i+s)$. This has separations $s^{n-2}(s-1)^1(s-2)^1$ and we can apply Sublemma 4.1. □

Lemma 4 assumes that the desired separations include all values from 2 up to s. However, we will also want to apply Lemma 4 in the situation where there are a relatively small number of larger separations and then 2 up to s. We will do this by first invoking Lemma 5 below. This will give us permutations that achieve the desired larger separations and have contiguous blocks of fixed points. A block of b consecutive fixed points can be replaced by a translate of a permutation of $\{0, \ldots, b-1\}$ to give other separations. Thus Lemma 4 can be used to build permutations to replace these blocks and give any further permutations. This is one reason why Lemma 4 was phrased to build different lengths of permutations.

Lemma 5. *Suppose we are given $t > r \geq 1$. Write $t = ar + e$, where $0 \leq e < r$. If a is even let $N = ar/2$ and if a is odd let $N = (a-1)r/2 + e$. Note that in either case $N \geq (t-r)/2$.*

(a) There is a value-symmetric permutation π of $\{0, \ldots, t-1\}$ which achieves separations $r^N 0^{t-2N}$ and for which the fixed points form a contiguous block.

(b) For any $1 \leq n \leq N$, there is a value-symmetric permutation π of $\{0, \ldots, t-1\}$ which achieves separations $r^n 0^{t-2n}$ and for which the fixed points form at most two contiguous blocks.

Proof. Consider the infinite product of transpositions

$$(0\ \ r)(1\ \ r+1)\cdots(r-1\ \ 2r-1)(2r\ \ 3r)\cdots(3r-1\ \ 4r-1)(4r\ \ 5r)\cdots.$$

For (a) take π to be all the transpositions on this list which only involve points in $\{0,\ldots,t-1\}$. For (b) take the first n transpositions in this product. □

Lemmas 4 and 5 will give us a way of completing a set of permutations to a solution to Claim (t,s). We also need to get started by producing a useful set of permutations. One method for producing these is given by the following lemma.

Lemma 6. *Suppose $v>b\geq 0$ and Claim (v,b) and $(v+1,b)$ are both true. Then*

(a) There is a sequence $(\sigma_j)_{j=1}^{2b+1}$ of $\{0,\ldots,4v-1\}$ such that the σ_j and their inverses achieve the separations $(v+b)^{4v}(v+b-1)^{4v}\cdots(v-b)^{4v}$.

(b) There is a sequence $(\sigma_j)_{j=1}^{2b+1}$ of $\{0,\ldots,4v\}$ such that the σ_j and their inverses achieve the separations $(v+b+1)^{v}(v+b)^{4v+1}(v+b-1)^{4v+1}\cdots(v-b+1)^{4v+1}(v-b)^{3v+1}$.

(c) There is a sequence $(\sigma_j)_{j=1}^{2b+1}$ of $\{0,\ldots,4v+1\}$ such that the σ_j and their inverses achieve the separations $(v+b+1)^{2v+2}(v+b)^{4v+2}(v+b-1)^{4v+2}\cdots(v-b+1)^{4v+2}(v-b)^{2v}$.

(d) There is a sequence $(\sigma_j)_{j=1}^{2b+1}$ of $\{0,\ldots,4v+2\}$ such that the σ_j and their inverses achieve the separations $(v+b+1)^{3v+2}(v+b)^{4v+3}(v+b-1)^{4v+3}\cdots(v-b+1)^{4v+3}(v-b)^{v+1}$.

Proof. Let $(\tau_j)_{j=1}^{2b+1}$ be a solution to Claim (v,b) and let $(\phi_j)_{j=1}^{2b+1}$ be a solution to Claim $(v+1,b)$. For (a) define permutations σ_j by

$$\sigma_j(i)=\begin{cases}\tau_j(i)+v & (0\leq i\leq v-1)\\ \tau_j(i-v) & (v\leq i\leq 2v-1)\\ \tau_j(i-2v)+3v & (2v\leq i\leq 3v-1)\\ \tau_j(i-3v)+2v & (3v\leq i\leq 4v-1)\end{cases}.$$

For (b) define permutations σ_j by

$$\sigma_j(i)=\begin{cases}\tau_j(i)+v & (0\leq i\leq v-1)\\ \tau_j(i-v) & (v\leq i\leq 2v-1)\\ \phi_j(i-2v)+3v & (2v\leq i\leq 3v)\\ \tau_j(i-3v-1)+2v & (3v+1\leq i\leq 4v)\end{cases}.$$

For (c) define permutations σ_j by

$$\sigma_j(i)=\begin{cases}\tau_j(i)+v & (0\leq i\leq v-1)\\ \tau_j(i-v) & (v\leq i\leq 2v-1)\\ \phi_j(i-2v)+3v+1 & (2v\leq i\leq 3v)\\ \phi_j(i-3v-1)+2v & (3v+1\leq i\leq 4v+1)\end{cases}.$$

For (d) define permutations σ_j by

$$\sigma_j(i)=\begin{cases}\phi_j(i)+v+1 & (0\leq i\leq v)\\ \phi_j(i-v-1) & (v+1\leq i\leq 2v+1)\\ \tau_j(i-2v-2)+3v+3 & (2v+2\leq i\leq 3v+1)\\ \phi_j(i-3v-2)+2v+2 & (3v+2\leq i\leq 4v+2)\end{cases}.$$

These permutations are each composed of four blocks. The diagonals of these blocks are either v or $v+1$ above or below the main diagonal. A τ (resp. ϕ) block K above the main diagonal contributes all separations $K-b, \ldots, K+b$ exactly v (resp. $v+1$) times. (Taking $K<0$ is equivalent to assuming below the diagonal.) Combining these remarks we see that the σ_j and their inverses combine to give the desired separations. $\square$

4 The case $p+q \equiv 0 \pmod 4$

We now turn to using the tools of the previous section to prove Claim (t, s). The proof will be by induction on $s+t$, however the inductive step will require us to construct permutations meeting the following extra criterion.
$(*)$ If $s \geq 1$ and $(t, s) \neq (3,1)$, then $\sigma_{2s}(s) = \sigma_{2s+1}(s) = 0$ and $\sigma_{2s}(t-1) = \sigma_{2s+1}(t-1) = t-s-1$.

Note that as the order of the sequence of permutations is not significant, it will be sufficient to demonstrate that two permutations satisfying $(*)$ exist within the construction as we can then simply reorder the permutations. Note that when $t = s+1$ the two conditions in $(*)$ are the same. The power of the condition $(*)$ is illustrated by the following two lemmas.

Lemma 7. *Let $\{\tau_j\}_{j=1}^{2a+1}$ be a solution to Claim (u, a) satisfying $(*)$, then there is a set $\{\sigma_j\}_{j=1}^{2a+1}$ of permutations of $\{0, \ldots, 2u-1\}$ providing separations of the form $\prod_{x=-a}^{a}(u+x)^u$ also satisfying $(*)$ in the context of $t = 2u$ and $s = u+a$.*

Proof. Define $\sigma_j(i) = \tau_j(i) + u$ for $0 \leq i \leq u-1$ and $\sigma_j(i) = \tau_j(i-u)$ for $u \leq i \leq 2u-1$.

Then in the former case, the separations defined by the σ_j, cover the range from $u-a$ to $u+a$, and, in the latter case, since $\sigma_j(i) - i = \tau_j(i-u) - (i-u) - u$, the separations cover the range $-u-a$ to $-u+a$ as required.

Now as $s = u+a \geq u$, $\sigma_{2a}(s) = \sigma_{2a+1}(s) = \tau_{2a}(s-u) = \tau_{2a}(a) = 0$ from $(*)$. Also $\sigma_{2a}(2u-1) = \sigma_{2a+1}(2u-1) = \tau_{2a}(2u-1-u) = \tau_{2a}(u-1) = u-a-1 = 2u-(u+a)-1 = t-s-1$. Thus $(*)$ is satisfied by the (σ_j) $\square$

Lemma 8. *Let $u > a$ and let $\{\tau_j\}_{j=1}^{2a-1}$ be a solution to Claim $(u, a-1)$ satisfying $(*)$ and $\{\phi_j\}_{j=1}^{2a+1}$ be a solution to Claim $(u+1, a)$ also satisfying $(*)$, then then there is a set $\{\sigma_j\}_{j=1}^{2a+1}$ of permutations of $\{0, \ldots, 2u\}$ such that they, together with their inverses, provide separations $(\prod_{x=-a+2}^{a}(u+x)^{2u+1})(u-a+1)^{u+1}(u-a)^{u-3}0^6$ also satisfying $(*)$ in the context of $t = 2u+1$ and $s = u+a$.*

If $u = a$ the same hypothesis leads to the same conclusion except that the terms $(u-a)^{u-3}0^6$ need to be replaced by 0^{2u}.

Proof. For $1 \leq j \leq 2a-1$, define permutations σ_j of $\{0, \ldots, t-1\}$ by $\sigma_j(i) = \phi_j(i) + u$ for $0 \leq i \leq u$ and $\sigma_j(i) = \tau_j(i-u-1)$ for $u+1 \leq i \leq 2u$.

This uses every τ_j and every ϕ_j except ϕ_{2a} and ϕ_{2a+1} which satisfy $\phi_{2a}(a) = \phi_{2a+1}(a) = 0$ and $\phi_{2a}(u) = \phi_{2a+1}(u) = u-a$ by $(*)$. Also note here that the same argument as in

Lemma A above, shows that the fact that the τ_j satisfy $(*)$ implies that the σ_j defined to this point also do.

Define σ_{2a} by $\sigma_{2a}(i) = \phi_{2a}(i) + u$ for $0 \le i \le u-1$ and $i \ne a$, $\sigma_{2a}(i) = \phi_{2a+1}^{-1}(i-u)$ for $u+1 \le i \le 2u$ and $i \ne 2u-a$, $\sigma_{2a}(a) = a$, $\sigma_{2a}(u) = u$, and $\sigma_{2a}(2u-a) = 2u-a$. Note that if $u = a$, this defines $\sigma_{2a}(a)$ (identically) three times.

The case $u = a$ is slightly different because σ_{2a} gets only one fixed point instead of three from these equations as a, u and $2u-a$ coincide.

In aggregate the τ_j achieve separations $\prod_{x=0}^{a-1} x^u$, therefore their contribution to the σ_j *and their inverses* is $\prod_{x=-a+2}^{a}(u+x)^u$ (the inverses deliver the positive values).

In aggregate the ϕ_j achieve separations $\prod_{x=0}^{a} x^{u+1}$. But, in constructing the σ_j, the special values in the definition of σ_{2a} mean that we miss four separations of $-a$ (for $u = a$ we miss only two) and gain six fixed points (for $u = a$ we gain only two fixed points). Therefore their contribution to the σ_j *and their inverses* is $(\prod_{x=-a+1}^{a}(u+x)^{u+1})(u-a)^{u-3}0^6$ (for $u = a$, the terms $(u-a)^{u-3}0^6$ become 0^{2u}).

Counting up, the total contribution of the σ_j and their inverses is as required. □

Theorem 9. *Claim (t,s) holds for all $t > s \ge 0$, moreover the solutions can be chosen to satisfy $(*)$.*

Proof. We will proceed by induction on $s+t$.

Case 1. $s = 0$. In this case, for any t we take a single permutation σ_1 as the identity.

Case 2. $s = 1$. For $(t,s) = (2u,1)$ take $\sigma_2 = \sigma_3 = \prod_{i=0}^{u-1}(2i\ \ 2i+1)$. If $u = 1$ and $u = 2$ take σ_1 to be the identity and for $u \ge 2$ let $\sigma_1 = (0\ \ 1)(2u-2, 2u-1)$

For $(t,s) = (2u+1,1)$, and $u \ge 3$ let $\sigma_1 = (0\ \ 1)(2u-1\ \ 2u)$, $\sigma_2 = (2u-1\ \ 2u)\prod_{i=0}^{u-2}(2i\ \ 2i+1)$ and $\sigma_3 = (2u-1\ \ 2u)\prod_{i=0}^{u-3}(2i\ \ 2i+1)$.

For $(3,1)$, we let $\sigma_1 = \sigma_2 = \sigma_3 = (0\ \ 1)$. For $(5,1)$ we let $\sigma_1 = (0\ \ 1)$ and $\sigma_2 = \sigma_3 = (0\ \ 1)(3\ \ 4)$.

Case 3. $t = s+1$. We consider the cases t even and t odd separately. Suppose $t = 2u$ and $s = 2u-1$. Then apply Lemma 7 using solutions to Claim $(u, u-1)$ (which exist by induction) to produce $2u-1$ permutations delivering the separations $\prod_{x=1}^{2u-1}(x)^u$ and satisfying $(*)$. Duplicating these and adding the identity permutation completes the construction.

If $t = 2u+1$ and $s = 2u$, then apply Lemma 8 using solutions to Claims $(u, u-1)$ and $(u+1, u)$ (which again exist by induction) to provide $4u$ permutations with separations $0^{2u}1^{u+1}\prod_{x=2}^{2u}(x)^{2u+1}$. The final permutation $\prod_{i=0}^{u-1}(2i\ \ 2i+1)$ gives the remaining separations 1^u0^1.

From here on we assume that $t > s+1$ and $s > 1$.

Case 4. $t \ge 2s+2$. Here we can write $t = u+v$ where both $u, v \ge s+1$. So, by induction Claims (u,s) and (v,s) hold with $(*)$. Lemma 2 gives a concatenated solution to Claim (t,s) and it is easy to check that this construction provides a solution also satisfying $(*)$.

From here on we assume that $2s+2 > t > s+1$ and $s > 1$.

Case 5. $s = 2$. The only cases left are $(t, s) = (4, 2)$ and $(5, 2)$. For $(4, 2)$, let σ_1 be the identity, $\sigma_2 = \sigma_3 = (0\ \ 1)(2\ \ 3)$ and $\sigma_4 = \sigma_5 = (0\ \ 2)(1\ \ 3)$.

For $(5, 2)$, let σ_1 be the identity, $\sigma_2 = (0\ \ 2\ \ 1)(3\ \ 4)$, $\sigma_3 = (0\ \ 2\ \ 4\ \ 3\ \ 1)$ and $\sigma_4 = \sigma_5 = (0\ \ 1\ \ 3\ \ 4\ \ 2)$.

Now we assume that $s \geq 3$.

Case 6. $t = 2s + 1$. There are two cases depending whether s is even or odd. Suppose first that $s = 2r \geq 4$. Then we define σ_1 by

$$\sigma_1(i) = \begin{cases} 2r - 1 - i & (0 \leq i \leq r - 1) \\ 3r & (i = r) \\ 2r - i & (r + 1 \leq i \leq 2r) \\ 6r + 1 - i & (2r + 1 \leq i \leq 3r) \\ 6r - i & (3r + 1 \leq i \leq 4r) \end{cases}$$

Taking two copies of σ_1 and of σ_1^{-1} provides 4 permutations satisfying $(*)$ and with aggregate separations $s^6 \prod_{x=1}^{s-1}(x)^4$. Now take $2s - 5$ copies of $\prod_{i=0}^{r-1}(i\ \ s - i) \prod_{i=1}^{r}(s + i\ \ 2s + 1 - i)$ together with two copies of $\prod_{i=2}^{r}(i\ \ s + 2 - i) \prod_{i=1}^{r+1}(s + i\ \ 2s + 1 - i)$ to complete the construction.

The case $s = 2r + 1 \geq 5$ is similar. Define σ_1 by

$$\sigma_1(i) = \begin{cases} 2r - i & (0 \leq i \leq r - 1) \\ 3r + 1 & (i = r) \\ 2r + 1 - i & (r + 1 \leq i \leq 2r + 1) \\ 6r + 4 - i & (2r + 2 \leq i \leq 3r + 2) \\ 6r + 3 - i & (3r + 3 \leq i \leq 4r + 2) \end{cases}$$

Taking two copies of σ_1 and of σ_1^{-1} provides 4 permutations satisfying $(*)$ and with aggregate separations $s^6 1^2 0^4 \prod_{x=2}^{s-1}(x)^4$. Now take $2s - 5$ copies of $\prod_{i=0}^{r}(i\ \ s - i) \prod_{i=1}^{r+1}(s + i\ \ 2s + 1 - i)$ together with two copies of $(0\ \ 1) \prod_{i=2}^{r+1}(i\ \ s + 2 - i) \prod_{i=1}^{r}(s + i\ \ 2s + 1 - i)$ to complete the construction.

Case 7. $t = 2s$. Apply Lemma 7 with $u = s$ and $a = 0$ to construct one permutation σ with aggregate separations s^s. Note that σ satisfies $(*)$ Thus two copies of σ and a solution to Claim $(t, s - 1)$ (by induction) complete the construction.

Case 8. $s = 3$. The only case not covered by the previous results is Claim $(5, 3)$. A solution to this is: $\sigma_1 = (0\ \ 2\ \ 4\ \ 3\ \ 1)$, $\sigma_2 = (0\ \ 1\ \ 3\ \ 4\ \ 2)$, $\sigma_3 = (0\ \ 2)(1\ \ 3)$, $\sigma_4 = (0\ \ 1)(2\ \ 3)$, $s_5 = (0\ \ 3)(1\ \ 2)$ and $\sigma_6 = \sigma_7 = (0\ \ 3)(1\ \ 4)$.

So from now we assume that $s \geq 4$ and $2s > t > s + 1$.

Case 9. t even. In this case $t = 2u \leq 2s - 2$. Let $s = u + a$ so that $1 \leq a \leq u - 2$ (the extremes being when $t = 2s - 2$ and $t = s + 2$ respectively). Applying Lemma 7 (using Claim (u, a)) delivers $2a + 1$ permutations with aggregate separations $\prod_{x=u-a}^{u+a}(x)^u$. We take two copies of each together with a solution to Claim $(t, u - a - 1)$ to achieve all the required separations.

Lemma 7 ensures that $(*)$ holds except in the case that $(u,a) = (3,1)$ which corresponds to $(t,s) = (6,4)$. Here a specific solution is: σ_1 is the identity, $\sigma_2 = \sigma_3 = (0\ \ 3)(1\ \ 4)(2\ \ 5)$, $\sigma_4 = \sigma_5 = \sigma_6 = (0\ \ 2)(1\ \ 3)(4\ \ 5)$ and $\sigma_7 = \sigma_8 = \sigma_9 = (0\ \ 4)(1\ \ 5)(2\ \ 3)$.

Case 10. $s = 4$ or $s = 5$. The above working covers all cases except Claims $(7,4)$, $(7,5)$ and $(9,5)$ for which we can give explicit solutions.

A solution for the case $(7,4)$ is: $\sigma_1 = (0\ \ 1\ \ 2)(3\ \ 5)(4\ \ 6)$, $\sigma_2 = (0\ \ 2\ \ 1)(3\ \ 5)(4\ \ 6)$, $\sigma_3 = \sigma_4 = (0\ \ 3)(1\ \ 4)(2\ \ 6)$, $\sigma_5 = (0\ \ 3)(1\ \ 2)(4\ \ 5)$, $\sigma_6 = (0\ \ 4)(1\ \ 3)(5\ \ 6)$, $\sigma_7 = (0\ \ 2)(3\ \ 4)(5\ \ 6)$ and $\sigma_8 = \sigma_9 = (0\ \ 4)(2\ \ 6)(1\ \ 5)$.

A solution for the case $(7,5)$ is: $\sigma_1 = \sigma_2 = \sigma_3 = (0\ \ 5)(1\ \ 4)(2\ \ 6)$, $\sigma_4 = (0\ \ 4\ \ 1\ \ 5\ \ 2\ \ 6\ \ 3)$, $\sigma_5 = s_4^{-1}$, $\sigma_6 = (0\ \ 2\ \ 1)(3\ \ 5)(4\ \ 6)$, $\sigma_7 = (0\ \ 1\ \ 2)(3\ \ 5)(4\ \ 6)$, $\sigma_8 = (0\ \ 4)(1\ \ 2)(5\ \ 6)$, $\sigma_9 = (0\ \ 1)(2\ \ 3)(4\ \ 5)$ and $\sigma_{10} = \sigma_{11} = (0\ \ 5)(1\ \ 6)(2\ \ 4)$.

A solution for the case $(9,5)$ is: $\sigma_1 = (0\ \ 5\ \ 1\ \ 6\ \ 2\ \ 7\ \ 3\ \ 8\ \ 4)$, $\sigma_2 = \sigma_1^{-1}$, $\sigma_3 = \sigma_4 = \sigma_5 = (0\ \ 3)(1\ \ 4)(2\ \ 5)(6\ \ 8)$, $\sigma_6 = (0\ \ 5)(2\ \ 6)(3\ \ 7)(4\ \ 8)$, $\sigma_7 = (0\ \ 4)(1\ \ 3)(5\ \ 7)(6\ \ 8)$, $\sigma_8 = (0\ \ 2)(1\ \ 3)(4\ \ 6)(7\ \ 8)$, $\sigma_9 = (0\ \ 1)(2\ \ 3)(4\ \ 5)(6\ \ 7)$ and $\sigma_{10} = \sigma_{11} = (0\ \ 5)(3\ \ 8)(1\ \ 2)(6\ \ 7)$

So now we assume that $s > 5$, $2s > t > s+1$ and t odd. This is the point at which the more complex work begins and we need to apply Lemma 4, 5 and 6. Note that we have now dealt with all cases where $t < 9$, hence let $t = 2u+1 \geq 9$.

Sublemma 9.1. *Let $t = 2u+1 \geq 9$. Then there is a set of $4(s-u)$ permutations which deliver the separations $0^6(2u-s)^{u-3}(2u-s+1)^{u+1}\prod_{x=2u-s+2}^{s}(x)^t$. The inductive step then proceeds so long as $t \leq \min\{2s-5, \frac{4}{3}s+3\}$ and in particular when $t = s+2$.*

Proof. Apply Lemma 8 with $a = s-u$ to get the set of permutations. This leaves us requiring $4u-2s+1$ permutations to deliver the remaining separations, which amount to $(2u-s+1)^u(2u-s)^{u+4}0^{t-6}\prod_{x=1}^{2u-s-1}(x)^t$. Note that, as $t \geq s+2$, $2u-s \geq 1$ with equality only if $t = s+2$.

Apply Lemma 4. All the permutations are of odd length. If $t > s+2$ so that $2u-s \geq 2$, and to apply Lemma 4 we require $t-6 \geq 4u-2s+1$ which simplifies to $t \leq 2s-5$, $t \geq (4u-2s+1)+(2u-s+1)-1-2$ which simplifies to $t \leq \frac{1}{2}(3s+5)$ and $u+4 \geq (4u-2s+1)+(2u-s+1)-1-(2u-s)$ which simplifies to $t \leq \frac{4}{3}s+3$. On the other hand, for positive s, $\frac{1}{2}(3s+5) \geq \frac{4}{3}s+3$.

If $t = s+2$, then $2u-s = 1$ and we are looking for 3 permutations with separations $2^u1^{u+4}0^{t-6}$. Lemma 4 applies so long as $t-6 \geq 3$, i.e. $t \geq 9$ which is the case here. □

At this point we have to separate into two strands depending whether u is even or odd.

Sublemma 9.2. *(a) If $u = 2v \geq 4$ and $3v \geq s+1$, then there is a set of $6u-4s-2$ permutations delivering separations $(2u-s)^v(s-u+1)^{3v+1}\prod_{x=s-u+2}^{2u-s-1}(x)^t$. In this case the problem reduces to finding $2s-2u+3$ permutations to deliver separations $(2u-s+1)^{2v}(2u-s)^{v+4}(s-u+1)^v0^{t-6}\prod_{x=1}^{s-u}(x)^t$.*
(b) If $u = 2v+1 \geq 5$ and $3v \geq s$ then there is a set of $6u-4s-4$ permutations delivering separations $(2u-s-1)^{3v+2}(s-u+1)^{v+1}\prod_{x=s-2v+2}^{2u-s-2}(x)^t$. In this case the problem reduces

to finding $2s-2u+5$ permutations to deliver separations

$$(2u-s+1)^{2v+1}(2u-s)^{2v+5}(2s-u-1)^{v+1}(s-u+1)^{3v+2}0^{t-6}\prod_{x=1}^{s-u}(x)^t.$$

Proof. These are applications of Lemma 6(b) with $b=3v-s-1$ and of Lemma 6(d) with $b=3v-s$ respectively. □

Case 11. $t=2s-1$ and $t\neq 11$. Suppose first $u=s-1=2v$ is even. Then $t=4v+1$ and as $s>5$, $v\geq 3$. From Sublemma 9.2 we need 5 permutations to achieve the separations $u^u(u-1)^{v+4}2^v1^t0^{t-6}$. From Lemma 5, there is one permutation with separations u^u0^1.

Again, using Lemma 5, we can achieve the separations $(u-1)^{v+4}$ with either one or two permutations. If $v>5$ this needs 1 permutation with one odd and one even gap. If $v<5$ this needs 2 permutations with two odd and one even gap. If $v=5$ we do it with one permutation and one odd gap.

Thus we are left to achieve the separations $2^v1^t0^{t-7}$ using Lemma 4. In terms of the inequalities required, the worst case arises when $v=3$ when $k=6$, $m=1$, $n_0=t-7=6$ and $n_1=13$ and the inequalities are clearly satisfied.

Next suppose $u=s-1=2v+1$ is odd. Then $t=4v+3$ and as $t\neq 11$, $v\geq 3$. From Sublemma 9.2 we need 7 permutations to achieve the separations $u^u(u-1)^{u+4}(u-2)^{v+1}2^{3v+2}1^t0^{t-6}$. As above, Lemma 5 gives one permutation with separations u^u0^1.

As $v\geq 3$, $u\geq 7$ and so $u+4\leq 2(u-1)$ and $v+1<u-2$. Then Lemma 5 gives two permutations with separations $(u-1)^{u+4}$ and three gaps, two odd and one even. It also gives one permutation with separations $(u-2)^{v+1}$ and two gaps one odd and one even. So we apply Lemma 4 to find the remaining separations $2^{3v+2}1^t0^{t-7}$ from 8 permutations, two of which have even length. The worst case situation arises when $v=3$ and $t=15$ when it is clear that the inequalities of Lemma 4 are satisfied.

Case 12. $t=11$ and $s=6$. If we start at the point we got to in Sublemma 9.1, we need to look for 9 permutations delivering $5^54^93^{11}2^{11}1^{11}0^5$. The following achieve this:

(0 4 8 3 7 2 6 1 5)(9 10) and its inverse give $5^44^51^2$.

(0 5)(1 3)(2 4)(6 8)(7 9) gives $5^12^40^1$.

(0 4)(1 5)(2 6)(3 7)(8 9) gives $4^41^10^1$.

(0 3 1 4 2)(5 8)(6 9)(7 10) and its inverse gives 3^82^3.

This leaves us requiring separations $3^32^41^80^3$ in 3 10-permutations. Lemma 4 says this can be done.

Sublemma 9.3. *If $u=2v\geq 4$ then the inductive step proceeds when $2s-1>t\geq\max\{s+4,\frac{1}{3}(4s+7)\}$.*

Proof. Let $r=2u-s$ then $t\geq\frac{1}{3}(4s+7)$ corresponds to $r\geq(u+2)/2=v+1$ and $3v\geq s+1$. So we can apply Sublemma 9.2(a) and we are left searching for $s-r+3$ permutations delivering separations $(r+1)^{2v}r^{v+4}(u-r+1)^v0^{t-6}\prod_{x=1}^{u-r}(x)^t$. As $t\leq 2s-2$ it follows that $t\geq 2r+4>2(r+1)$. We now apply Lemma 5 to deal with the separations of size r and $r+1$.

As $2s-1>t>2(r+1)$, $2v>r+1$. Now Lemma 5 implies that we can obtain the separations $(r+1)^{2v}$ in two permutations with 3 gaps of fixed points two of odd length and one of even length. We use Lemma 5 also to obtain the separations r^{v+4} in two permutations with 4 gaps, two odd and two even (if r is large enough relative to v this might be possible in a single permutation with fewer gaps, but we deliberately choose otherwise to cut down the case analysis).

Now we are searching for $2s-2u+6$ permutations, three of them of even length to find separations $(s+1-u)^v 0^{t-6}\prod_{x=1}^{s-u}(x)^t$. We test the inequalities of Lemma 4.

We require $(s+1-u)+(2s-2u+6)-1-3\leq t$. This is equivalent to $5t\geq 6s+5$ which is certainly true from the hypothesis we assume.

We require $3t-6\geq(2s-2u+6)+3$ which is equivalent to $2t\geq s+8$ which is true as $t\geq s+4$ and $s>5$.

Finally we require $t-6\geq(2s-2u+6)-3$ which is equivalent to $t\geq s+4$.

Thus these are all satisfied and we are done. □

Combining Sublemmas 9.1 and 9.3 leads to

Case 13. $t=4v+1$. From Sublemma 9.1 we are done if $t\leq\min\{2s-5,\frac{4}{3}s+3\}$. From Sublemma 9.3 we are done if $t\geq\max\{s+4,\frac{1}{3}(4s+7)\}$. Clearly we are done unless either $2s-5$ or $s+4$ are the preferred limits. The latter case only occurs if $s<6$ which we have already dealt with. The former case can occur when $s\leq 11$. A detailed analysis of the small number cases shows that the only difficulty occurs when $(t,s)=(9,6)$.

Starting at the point we got to in Sublemma 9.1, we are left needing separations $3^4 2^8 1^9 0^3$ in 9 permutations which may be:

$(0\ 3\ 1\ 4\ 2)(5\ 7)(6\ 8)$ and its inverse which give separations $3^2 2^7$.

$(0\ 3)(1\ 2)(4\ 6)(5\ 8)$ giving separations $3^2 2^1 1^1 0^1$.

$(0\ 1)(2\ 3)(4\ 5)(6\ 7)$ twice giving separations $1^8 0^2$.

Note that, incidentally, by this point we have proved the theorem for $s=6$ also. We finally have to deal with the $t=4v+3$ case.

Sublemma 9.4. *If $u=2v+1\geq 5$ then the inductive step proceeds when $2s-1>t\geq\max\{s+7,\frac{4}{3}s+1)\}$.*

Proof. Let $r=2u-s$ then $t\geq\frac{4}{3}s+1$ corresponds to $r\geq(u+3)/2=v+2$ and $3v\geq s-1$. We want to apply Sublemma 9.2(b) and there seems to be a problem when $3v=s-1$. But this translates to $t=\frac{1}{3}(4s+5)$ which case is covered by Sublemma 9.1 so long as t is not also greater than $2s-5$. But this would require $s<10$. The only occurrence would in fact be $v=2$, $s=7$ and $t=11$. But our assumption is that $t\geq s+7$ also, so this case is excluded.

So we assume that $3v\geq s$ and we can apply Sublemma 9.2(b). This leaves us searching for $s-r+5$ permutations delivering separations $(r+1)^{2v+1}r^{2v+5}(r-1)^{v+1}(u-r+1)^{3v+2}0^{t-6}\prod_{x=1}^{u-r}(x)^t$. As $t\leq 2s-2$ it again follows that $t\geq 2r+4>2r+3$. We now apply Lemma 5 to deal with the separations of size $r-1$, r and $r+1$.

As $2s-1>t>2r+3$, $2v+1>r+1$. Now Lemma 5 implies that we can obtain the separations $(r+1)^{2v+1}$ in two permutations with 3 gaps of fixed points two of odd length and one of even length. An exactly similar argument deals with the separations r^{2v+5}.

We use Lemma 5 also to obtain the separations $(r-1)^{v+1}$ in two permutations with 4 gaps, two odd and two even (if r is large enough relative to v this might be possible in a single permutation with fewer gaps, but we again deliberately choose otherwise to cut down the case analysis).

This leaves us trying to find $2s-2u+9$ gaps, four of them of even length to find separations $(s+1-u)^{3v+2}0^{t-6}\prod_{x=1}^{s-u}(x)^t$. We test the inequalities of Lemma 4.

We require $(s+1-u)+(2s-2u+9)-1-3\leq t$. This is equivalent to $5t\geq 6s+15$ which is certainly true from the hypothesis we assume.

We require $3t-6\geq(2s-2u+9)+4$ which is equivalent to $2t\geq s+8$ which is true as $t\geq s+7$ and $s>6$.

Finally we require $t-6\geq(2s-2u+9)-3$ which is equivalent to $t\geq s+\frac{13}{2}$ which is guaranteed by the hypothesis.

Thus these are all satisfied and we are done. □

Combining Sublemmas 9.1 and 9.4 leads to

Case 14. $t=4v+3$. From Sublemma 9.1 we are done if $t\leq\min\{2s-5,\frac{4}{3}s+3\}$. From Sublemma 9.4 we are done if $t\geq\max\{s+7,\frac{4}{3}s+1\}$. Clearly we are done unless either $2s-5$ or $s+7$ are the preferred limits. The latter case only occurs if $s<18$. The former case can only occur when $s\leq 11$. A detailed analysis of the small number cases shows that the only difficulty occurs when $(t,s)=(11,7)$ or $(15,9)$.

For the $(11,7)$ case, starting at the point we got to in Sublemma 9.1, we are left needing separations $4^5 3^9 2^{11} 1^{11} 0^5$ in 7 permutations which may be:

(0 4 1 5 2 6 3)(7 9)(8 10) and its inverse which give separations $4^3 3^4 2^4$

(0 4 1 5 2)(3 6)(7 9)(8 10) and its inverse giving separations $4^2 3^4 2^5$.

(0 3)(1 2)(4 6)(5 7)(8 9) giving separations $3^1 2^2 1^2 0^1$.

(0 1)(2 3)(4 5)(6 7)(8 9) giving separations $1^5 0^1$.

(0 1)(2 3)(4 5)(6 7) giving separations $1^5 0^3$.

For the $(15,9)$ case, starting at the point we got to in the middle of Sublemma 9.4 just before we tried to apply Lemma 5, we are left needing separations $6^7 5^{11} 4^4 3^{11} 2^{15} 1^{15} 0^9$ in 9 permutations. Here we have enough control to move to using Lemma 4. Thus we can achieve the separations 6^7 in two permutations with 3 gaps, odd and one even. We can do the same for the separations 5^{11}. The separations 4^4, however, may be achieved in a single permutation with one odd gap. So we are left to try to apply Lemma 4 with 11 permutations, two of even length to achieve the separations $3^{11} 2^{15} 1^{15} 0^9$. But the inequalities are indeed satisfied for Lemma 4 and we are finally done. □

5 The case $p+q\equiv 2 \pmod 4$

The case $p+q\equiv 2 \pmod 4$ is slightly harder than the case $p+q\equiv 0 \pmod 4$ just completed. One reason for this is that the analogous Guesses (t,s) are not always true.

Lemma 10. *Guess* $(t, t-2)$ *is false for all* $t \geq 4$.

Proof. For any permutation σ of $\{0, 1, \ldots, t-1\}$, let $n_k(\sigma)$ be the number of inputs i for which $\sigma(i) - i = k$. View σ as a diagram as follows. Consider a $t \times t$ grid of unit squares, labelled (x, y) for $0 \leq x, y \leq t-1$. We will think of $(0, 0)$ as the lower left hand square, $(0, t-1)$ as upper left, etc.. View a permutation σ as a choice of t of these squares with one in each row and column by choosing the squares $(x, \sigma(x))$.

Assign scores to the first k columns, with column 0 scoring k, column 1 scoring $k-1$, etc.. Similarly assign scores $k, k-1, \ldots, 1$ to the top k rows (rows $t-1, t-2, \ldots, t-k$). For any square $(i, \sigma(i))$ used by σ assign a score which is the sum of the scores on its row and column. Note that if $\sigma(i) - i = t - j$, then $(i, \sigma(i))$ scores at least $2k+1-j$. Summing the scores in two different ways (either according to values of $\sigma(i) - i$ or directly) gives

$$2kn_{t-1}(\sigma) + (2k-1)n_{t-2}(\sigma) + \cdots + n_{t-2k}(\sigma) \leq k(k+1).$$

Now suppose Guess $(t, t-2)$ were true, thus there are $(\sigma_1, \ldots, \sigma_{2t-3})$ such that $\sum_{j=1}^{2t-3} n_k(\sigma_j) = t$ for $k \in \{-s-1, -s, \ldots, -2, 0, 2, \ldots, s+1\}$. For $t \geq 4$ let $k = \lfloor (t-2)/2 \rfloor$. Summing the inequality above gives

$$k(2k+1)t \leq k(k+1)(2t-3).$$

Hence rearranging $t \geq 3(k+1) \geq 3(t-1)/2$. For $t \geq 4$ this is a contradiction. Thus Guess $(t, t-2)$ is false. □

The fact that Guess (t, s) is not always true will complicate things in two ways. Obviously it will mean that there are a few cases of the c-value problem which will require a different argument. These additional arguments are given in the next section. Also since our proof is inductive, the missing cases will complicate the inductive argument. The one minor simplification is that we will already have the solutions to the Claims (t, s) available as building blocks. Adapting the proof of Theorem 9 gives the following theorem.

Theorem 11. *Guess* (t, s) *holds for all* $t - 4 > s \geq 0$. *In addition, Guess* $(s+3, s)$ *holds when* s *is odd in which case one of the desired permutations may be taken as the identity.*

Proof. We will proceed by induction on $s + t$.

Case 1. $s = 0$. In this case, for any t we take a single permutation σ_1 as the identity.

Case 2. $s = 1$. For $t = 4$, take $\sigma_1 = \sigma_2 = (0\ \ 2)(1\ \ 3)$ and σ_3 as the identity. Otherwise, Lemma 4 gives a direct solution.

So from here we assume that $s \geq 2$.

Case 3. $t \geq 2s + 8$. Here we can write $t = u + v$ where both u, $v \geq s + 4$. So, by induction Guesses (u, s) and (v, s) are true. Lemma 2 then gives a concatenated solution to Guess (t, s).

So from here on we assume $t < 2s + 8$.

Case 4. $s = 2$. In this case we are looking for 5 permutations delivering separations $3^t 2^t 0^t$. If t is odd then Lemma 4 applies directly. This leaves only the cases $(10, 2)$, $(8, 2)$ and $(6, 2)$ left to deal with.

Guess $(6, 2)$ is solved with 3 copies of $(0\ \ 3)(1\ \ 4)(2\ \ 5)$ and 2 copies of $(0\ \ 2)(1\ \ 3)$.

Guess $(8, 2)$ is solved with 2 copies of $(0\ \ 2)(1\ \ 3)(4\ \ 6)(5\ \ 7)$, 2 copies of $(0\ \ 3)(1\ \ 4)(2\ \ 5)$ and one copy of $(0\ \ 3)(1\ \ 4)$.

Guess $(10, 2)$ is solved with 2 copies of $(0\ \ 2)(1\ \ 3)(4\ \ 6)(5\ \ 7)$, 2 copies of $(0\ \ 3)(1\ \ 4)(2\ \ 5)(6\ \ 9)$ and one copy of $(0\ \ 2)(1\ \ 4)(3\ \ 6)(5\ \ 7)$.

Now we assume that $s \geq 3$ always.

Case 5. $2s + 3 \leq t \leq 2s + 7$. These cases will require a combination of Lemmas 4, 5 and 6. As we have $s \geq 3$, $t \geq 8$. Let $v = \lfloor \frac{t}{4} \rfloor \geq 2$.

So we have $t = 4v + d$ where $0 \leq d \leq 3$. Putting these into the inequality $2s + 3 \leq t \leq 2s + 7$, we see that there are 10 separate subcases to look at, which unfortunately have to be tackled separately.

(a) $(t, s) = (4v, 2v - 3)$: Let $b = v - 2$ and apply Lemma 6(a). This provides $4v - 6$ permutations and gives all separations except 0^t which is achieved by a final identity permutation.

(b) $(t, s) = (4v, 2v - 2)$: Let $b = v - 2$ and apply Lemma 6(a). This provides $4v - 6$ permutations and gives all separations except $(2v - 1)^t 0^t$ which remain to be dealt with by 3 further permutations.

As $2(2v - 1) < 4v$ Lemma 5 allows us to achieve the non-zero separations in three permutations with the consequent gaps providing the zero separations.

(c) $(t, s) = (4v + 1, 2v - 3)$: Let $b = v - 3$ (we can assume here that $v > 2$ as otherwise we have $s = 1$ which has been dealt with above) and apply Lemma 6(b). This provides $4v - 10$ permutations and gives all separations except $(2v - 2)^{3v+1} 3^v 2^t 0^t$ which remain to be dealt with by 5 further permutations.

Lemma 5 can deliver $(2v - 2)^{3v+1}$ in two permutations with two odd and one even gap. We are then left to find the rest from 6 permutations, 5 of odd length and one of even length. t is large enough for Lemma 4 to finish off this case.

(d) $(t, s) = (4v + 1, 2v - 2)$: Since $s \geq 3$, $v \geq 3$. Let $b = v - 2$ and apply Lemma 6(b). This provides $4v - 6$ permutations and gives all separations except $(2v - 1)^{3v+1} 2^v 0^t$ which remain to be dealt with by 3 further permutations. Lemma 5 gives 2 permutations for the separations $(2v - 1)^{3v+1}$ with 3 gaps, 2 odd and one even. Lemma 4 will then complete the construction of the separations $2^v 0^t$ by 4 permutations, 3 odd and one even as t is large enough.

(e) $(t, s) = (4v + 1, 2v - 1)$: Let $b = v - 2$ and apply Lemma 6(b). This provides $4v - 6$ permutations and gives all separations except $(2v)^t (2v - 1)^{3v+1} 2^v 0^t$ which remain to be dealt with by 5 further permutations. This is similar to Case (4) with the separations $(2v)^t$ dealt with by 3 permutations with 4 gaps; 3 odd and one even. Lemma 4 will then complete the construction of the separations $2^v 0^t$ by 7 permutations, 5 odd and 2 even as t is at least 9 and so is large enough.

(f) $(t, s) = (4v + 2, 2v - 2)$: Let $b = v - 2$ and apply Lemma 6(c). This provides $4v - 6$

permutations and gives all separations except $(2v-1)^{2v}2^{2v+2}0^t$ which remain to be dealt with by 3 further permutations. Here the separations $(2v-1)^{2v}$ require 2 permutations from Lemma 5 with 3 even gaps. Again t is large enough to deal with the remaining separations via Lemma 4.
(g) $(t,s)=(4v+2,2v-1)$: Let $b=v-2$ and apply Lemma 6(c). This provides $4v-6$ permutations and gives all separations except $(2v)^t(2v-1)^{2v}2^{2v+2}0^t$ which remain to be dealt with by 5 further permutations. The separations $(2v)^t$ can be dealt with in 3 permutations with 4 gaps, 2 even and 2 odd, by Lemma 5. Using Lemma 5 and two other permutations to deal with the separations $(2v-1)^{2v}$ as in Case (6), we then need to apply Lemma 4 with 7 permutations, 5 even and 2 odd, to complete the construction. This requires $t \geq 14$ which is the case when $v \geq 3$. Guess $(10,3)$ is solved with 3 copies of (03)(14)(25)(68)(79), one copy of (02)(14)(35)(68)(79), 2 copies of (04)(15)(26)(37) and one copy of (04)(15).
(h) $(t,s)=(4v+3,2v-2)$: Since $s \geq 3$, $v \geq 3$. Let $b=v-2$ and apply Lemma 6(d). This provides $4v-6$ permutations and gives all separations except $(2v-1)^{v+1}2^{3v+2}0^t$ which remain to be dealt with by 3 further permutations. Lemma 5 shows we can achieve the separations $(2v-1)^{v+1}$ with a single permutation with one odd and one even gap. Then t is large enough to apply Lemma 4 to what remains.
(i) $(t,s)=(4v+3,2v-1)$: Let $b=v-2$ and apply Lemma 6(d). This provides $4v-6$ permutations and gives all separations except $(2v)^t(2v-1)^{v+1}2^{3v+2}0^t$ which remain to be dealt with by 5 further permutations. As in subcase (h) the separations $(2v-1)^{v+1}$ are achieved with a single permutation with one odd and one even gap. $(2v)^t$ requires 3 permutations with 3 odd and 1 even gap. So we have to achieve the remaining separations with 7 permutations of which 2 are even. This can be done if $t \geq 9$ which is the case here.
(j) $(t,s)=(4v+3,2v)$: Let $b=v-2$ and apply Lemma 6(d). This provides $4v-6$ permutations and gives all separations except $(2v+1)^t(2v)^t(2v-1)^{v+1}2^{3v+2}0^t$ which remain to be dealt with by 7 further permutations. The addition from Case (9) is the additional permutations $(2v+1)^t$ in which Lemma 5 deals with in 3 permutations with 3 odd and one even gap. So we need to complete the remaining separations in 10 permutations where 3 are even. This requires $t \geq 13$. But $v=2$ gives $t=11$. So Guess $(11,4)$ requires an ad hoc solution: two copies of (05)(16)(27)(38)(49) and of (04)(15)(26)(37)(8 10), 3 copies of (03)(14)(25)(68)(79), and one copy of (04)(13)(26)(59)(8 10) and of (05)(14)(36)(79).

Case 6. $s=3$. The only cases not dealt with so far are $6 \leq t \leq 8$. These are solved as:
Guess $(6,3)$: 6 copies of (04)(13)(25) and the identity.
Guess $(7,3)$: 7 copies of (04)(13)(25).
Guess $(8,3)$: 2 copies of (04)(15)(26)(37) and of (02)(13)(46)(57) and of (03)(14)(25) together with (03)(14).

Case 7. $t=2u \leq 2s+2$. Let $s+1=u+a$ so that $0 \leq a \leq u-2$ (the extremes being when $t=2s+2$ and $t=s+3$ respectively). Applying Lemma 7 (using Claim (u,a)) delivers $2a+1$ permutations with aggregate separations $\prod_{x=u-a}^{u+a}(x)^u$. We take two copies of each together with a solution to Guess $(t,u-a-2)$ (which exists by induction, since $u-a-2=s-(2a+1)$) to achieve all the required separations.

In the case that $t = s+3$, then $a = u-2$ which means that the inductive step is an application of Guess $(t,0)$ which is the identity as required. Note that this effectively provides a direct proof of part (b) of the Theorem.

Finally, as in Theorem 9, we have to deal with the remaining cases where t is odd, which is where the most complication occurs. Note that, as we have dealt with t even, $t-s=3$, $s \leq 3$, and $t \geq 2s+2$ we must have $11 \leq t \leq 2s+1$ odd. Let $t = 2u+1$.

Sublemma 11.1. *There is a set of* $4(s+1-u)$ *permutations delivering the separations* $0^6(2u-s-1)^{u-3}(2u-s)^{u+1}\prod_{x=2u-s+1}^{s+1}(x)^t$. *The inductive step then proceeds so long as* $t \leq \min\{2s-1, \frac{1}{3}(4s+17)\}$ *and in particular when* $t = s+4$.

Proof. Apply Lemma 8 with $a = s+1-u$ to get the set of permutations. This leaves us requiring $4u-2s-3$ permutations to deliver the remaining separations, which amount to $(2u-s)^u(2u-s-1)^{u+4}0^{t-6}\prod_{x=2}^{2u-s+1}(x)^t$. Note that, as $t \geq s+4$, $2u-s-2 \geq 2$ with equality only if $t = s+4$.

Apply Lemma 4. All the permutations are of odd length. If $t > s+4$ so that $2u-s-2 \geq 3$, then to apply Lemma 4 we require $t-6 \geq 4u-2s-3$ which simplifies to $t \leq 2s-1$ and $t \geq (4u-2s-3)+(2u-s)-1-2$ which simplifies to $t \leq \frac{1}{2}(3s+9)$ and $u+4 \geq (4u-2s-3)+(2u-s)-1-(2u-s-1)$ which simplifies to $t \leq \frac{1}{3}(4s+17)$. On the other hand, for positive s, $\frac{1}{2}(3s+9) \geq \frac{1}{3}(4s+17)$.

If $t = s+4$, then $2u-s-1 = 2$ and we are looking for 3 odd permutations with separations $3^u 2^{u+4} 0^{t-6}$. Lemma 4 applies so long as $t-6 \geq 3$ and $u+4 \geq 3$, i.e. $t \geq 9$ which is the case here. □

Case 8. $s = 4$. The only case not covered above is Guess $(9,4)$. A solution is: 2 copies each of $(0\ 5)(1\ 6)(2\ 7)(3\ 8)$, $(0\ 4)(1\ 5)(2\ 6)(3\ 7)$ and $(0\ 2)(1\ 3)(4\ 6)(5\ 7)$ and one copy each of $(0\ 3)(1\ 4)(2\ 5)(6\ 8)$, $(0\ 3)(1\ 4)(2\ 6)(5\ 8)$ and $(0\ 3)(1\ 6)(2\ 5)(4\ 7)$.

At this point we again have to separate into two strands depending whether u is even or odd.

Sublemma 11.2. *(a) If* $u = 2v \geq 6$ *and* $3v \geq s+2$, *then there is a set of* $6u-4s-6$ *permutations delivering separations* $(2u-s-1)^v(s-u+2)^{3v+1}\prod_{x=s-u+3}^{2u-s-2}(x)^t$. *In this case the problem reduces to finding* $2s-2u+3$ *permutations to deliver separations* $(2u-s)^{2v}(2u-s-1)^{v+4}(s-u+2)^v 0^{t-6}\prod_{x=2}^{s-u+1}(x)^t$.
(b) If $u = 2v+1 \geq 5$ *and* $3v \geq s+1$ *then there is a set of* $6u-4s-8$ *permutations delivering separations* $(2u-s-2)^{3v+2}(s-u+2)^{v+1}\prod_{x=s-2v+3}^{2u-s-3}(x)^t$. *In this case the problem reduces to finding* $2s-2u+5$ *permutations to deliver separations*

$$(2u-s)^{2v+1}(2u-s-1)^{2v+5}(2s-u+1)^{v+1}(s-u+2)^{3v+2}0^{t-6}\prod_{x=1}^{s-u+1}(x)^t.$$

Proof. These are applications of Lemma 6(b) with $b = 3v-s-2$ and of Lemma 6(d) with $b = 3v-s-1$ respectively. □

Case 9. $t = 2s+1$ and $t \neq 11$. First suppose $u = s = 2v$ is even. Then $t = 4v+1$ and as $s > 5$, $v \geq 3$. From Sublemma 11.2 we need 3 permutations to achieve the separations $u^u(u-1)^{v+4}2^v0^{t-6}$. From Lemma 5, there is one permutation with separations u^u0^1.

Again, using Lemma 5, we can achieve the separations $(u-1)^{v+4}$ with either one or two permutations. If $v > 5$ this needs 1 permutation with one odd and one even gap. If $v < 5$ this needs 2 permutations with two odd and one even gap. If $v = 5$ we do it with one permutation and one odd gap.

Thus we are left to achieve the separations 2^v0^{t-7} using Lemma 4. In terms of the inequalities required, the worst case arises when $v = 3$ when $k = 3$, $m = 1$, $n_0 = t-7 = 6$ and the inequalities are clearly satisfied.

Next suppose $u = s = 2v+1$. Then $t = 4v+3$ and as $t \neq 11$, $v \geq 3$. From Sublemma 11.2 we need 5 permutations to achieve the separations $u^u(u-1)^{u+4}(u-2)^{v+1}2^{3v+2}0^{t-6}$. As above, Lemma 5 gives one permutation with separations u^u0^1.

As $v \geq 3$, $u \geq 7$ and so $u+4 \leq 2(u-1)$ and $v+1 < u-2$. Then Lemma 5 gives two permutations with separations $(u-1)^{u+4}$ and three gaps, two odd and one even. It also gives one permutation with separations $(u-2)^{v+1}$ and two gaps one odd and one even. So we apply Lemma 4 to find the remaining separations $2^{3v+2}0^{t-7}$ from 6 permutations, two of which have even length. The worst case situation arises when $v = 3$ and $t = 15$ when it is clear that the inequalities of Lemma 4 are satisfied.

Case 10. $t = 11$ and $s = 5$. After applying Sublemma 11.1, we need to look for 7 permutations delivering $5^54^93^{11}2^{11}0^5$. The following achieve this:

$(0\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)$ gives 5^50^1.

$(0\ 4\ 1\ 5\ 2\ 6\ 3)(7\ 9)(8\ 10)$ and its inverse give $4^33^42^4$.

This leaves us requiring separations $4^63^72^70^4$ in 4 11-permutations. Lemma 4 says this can be done.

Sublemma 11.3. *If $u = 2v \geq 6$ then the inductive step proceeds when $2s+1 > t \geq \max\{s+8, \frac{1}{3}(4s+11)\}$.*

Proof. Let $r = 2u - s - 1$ then $t \geq \frac{1}{3}(4s+11)$ corresponds to $r \geq (u+2)/2 = v+1$ and $3v \geq s+2$. So we can apply Sublemma 11.2(a) and we are left searching for $s-r+2$ permutations delivering separations $(r+1)^{2v}r^{v+4}(u-r+1)^v0^{t-6}\prod_{x=2}^{u-r}(x)^t$. As $t \leq 2s$ it follows that $t \geq 2r+4 > 2(r+1)$. We now apply Lemma 5 to deal with the separations of size r and $r+1$.

As $2s+1 > t > 2(r+1)$, $2v \geq r+1$. Now Lemma 5 implies that we can obtain the separations $(r+1)^{2v}$ in two permutations with 3 gaps of fixed points two of odd length and one of even length. We use Lemma 5 also to obtain the separations r^{v+4} in two permutations with 4 gaps, two odd and two even (if r is large enough relative to v this might be possible in a single permutation with fewer gaps, but we deliberately choose otherwise to cut down the case analysis).

Now we are searching for $2s-2u+6$ permutations, three of them of even length to find separations $(s+1-u)^v0^{t-6}\prod_{x=2}^{s-u}(x)^t$. We test the inequalities of Lemma 4.

We require $(s+1-u)+(2s-2u+6)-1-2 \leq t$. This is equivalent to $5t \geq 6s+11$ which is certainly true from the hypothesis.

We require $t - 6 \geq (2s - 2u + 6) + 3$ which is equivalent to $t \geq s + 8$ which is again true by hypothesis. The third inequality involving just n_0 is true *a fortiori* and we are done. □

Combining Sublemmas 11.1 and 11.3 leads to

Case 11. $t = 4v + 1$. When $t \geq 13$, then, from Sublemma 11.1 we are done if $t \leq \min\{2s - 1, \frac{1}{3}(4s + 17)\}$. From Sublemma 11.3 we are done if $t \geq \max\{s + 8, \frac{1}{3}(4s + 11)\}$. Clearly we are done unless either $2s - 1$ or $s + 8$ are the preferred limits. The latter case only occurs if $s < 13$, and the former case can occur when $s < 10$.

The remaining case is $(t, s) = (9, 5)$, a solution for which is:

Three copies each of $(0\ \ 6\ \ \ 1\ \ 7\ \ 2\ \ 8\ \ 4)(3\ \ 5)$ and its inverse which together give separations $6^9 5^6 4^6 2^6$.

$(0\ \ 5)(1\ \ 6)(2\ \ 7)(4\ \ 8)$ giving separations $5^3 4^1 0^1$.

$(0\ \ 2)(1\ \ 5)(3\ \ 7)(4\ \ 6)$ giving separations $4^2 2^2 0^1$.

Two copies of $(0\ \ 3)(1\ \ 4)(2\ \ 5)$ giving separations $3^6 0^6$.

One copy of $(0\ \ 3)(1\ \ 4)(2\ \ 5)(6\ \ 8)$ giving separations $3^3 2^1 0^1$.

We finally have to deal with the $t = 4v + 3$ case.

Sublemma 11.4. *If $u = 2v + 1 \geq 5$ then the inductive step proceeds when $2s + 1 > t \geq \max\{\frac{1}{2}(2s + 19), \frac{1}{3}(4s + 13)\}$.*

Proof. Let $r = 2u - s - 1$ then $t \geq \frac{1}{3}(4s + 13)$ corresponds to $r \geq (u + 3)/2 = v + 2$ and $3v \geq s + 1$. We apply Sublemma 11.2(b) which leaves us searching for $s - r + 3$ permutations delivering separations $(r+1)^{2v+1} r^{2v+5} (r-1)^{v+1} (u-r+1)^{3v+2} 0^{t-6} \prod_{x=2}^{u-r} (x)^t$. As $t \leq 2s$ it again follows that $t \geq 2r + 4 > 2r + 3$. We now apply Lemma 5 to deal with the separations of size $r - 1$, r and $r + 1$. As $2s + 1 > t > 2r + 3$, $2v + 1 > r + 1$. Now Lemma 5 implies that we can obtain the separations $(r+1)^{2v+1}$ in two permutations with 3 gaps of fixed points two of odd length and one of even length. An exactly similar argument deals with the separations r^{2v+5}.

We use Lemma 5 also to obtain the separations $(r - 1)^{v+1}$ in two permutations with 4 gaps, two odd and two even (if r is large enough relative to v this might be possible in a single permutation with fewer gaps, but we again deliberately choose otherwise to cut down the case analysis).

This leaves us trying to find $2s - 2u + 8$ gaps, four of them of even length to find separations $(s + 1 - u)^{3v+2} 0^{t-6} \prod_{x=2}^{s-u} (x)^t$. We test the inequalities of Lemma 4.

We require $(s + 1 - u) + (2s - 2u + 8) - 1 - 2 \leq t$. This is equivalent to $5t \geq 6s + 15$ which is certainly true from the hypothesis.

We require $t - 6 \geq (2s - 2u + 8) + 4$ which is equivalent to $2t \geq 2s + 19$ which is true as $t \geq s + 10$ as is $t - 6 \geq (2s - 2u + 8) - 4$ *a fortiori*. Thus these are all satisfied and we are done. □

Combining Sublemmas 11.1 and 11.4 leads to

Case 12. $t = 4v + 3$. As we have $s > 3$, we can certainly assume that $t \geq 11$ and $u \geq 5$. From Sublemma 11.1 we are done if $t \leq \min\{2s - 1, \frac{1}{3}(4s + 17)\}$. From Sublemma 11.4 we are done if $2s+1 > t \geq \max\{\frac{1}{2}(2s+19), \frac{1}{3}(4s+13)\}$. Clearly we are done unless either $2s-1$ or $\frac{1}{2}(2s+19)$ are the preferred limits. The latter case only occurs if $s < 16$. The former case can only occur when $s < 10$. A detailed analysis of the small number cases shows that the only difficulty occurs when $(t, s) = (11, 5)$ which was covered explicitly in Case 10. $\square$

6 The case $q = p + 4$

Theorems 9 and 11 above solve the c-value problem except for the case $q = p+4$ for which by Lemma 10 the method used above of splitting the problem into halves was doomed to fail. For this last case we need the full strength of Lemma 1. We will construct a sequence $(\sigma_j)_{j=1}^{2s+1}$ of permutations of $\{0, \ldots, t-1\}$ and $(\pi_j)_{j=1}^{2s+1}$ of $\{0, \ldots, t\}$ such that in aggregate the σ_j and π_j achieve the separations $(s+1)^{2t+1} s^{2t+1} \cdots 2^{2t+1} 0^{2t+1}$. These constructions can be handled easily enough, however we will need to do the cases t even and t odd separately.

Proposition 12. *We can solve the c-value problem for $q = 4m+3$ and $p = 4m-1$.*

Proof. Recall that $t = 2m+1$ and $s = 2m-1$. Let $(\tau_j)_{j=1}^{2m-1}$ be a solution to Claim $(m, m-1)$ and define σ_j by $\sigma_j(i) = m+1+\tau_j(i)$ for $0 \leq i \leq m-1$, $\sigma_j(m) = m$, and $\sigma_j(i) = \tau_j(i-m-1)$ for $m+1 \leq i \leq 2m$. Then in aggregate $\sigma_j(i) - i$ achieves separations $(2m)^m (2m-1)^m \cdots 2^m 0^{2m-1}$. For the full list of $4m-1$ σ's take two copies of each of the σ_j and one copy of $(0 \;\; 2m)(1 \;\; 2m-1) \cdots (m-1 \;\; m+1)$. In total these achieve separations $(2m)^{2m+1} (2m-1)^{2m} (2m-2)^{2m+1} \cdots 3^{2m} 2^{2m+1} 0^{4m-1}$.

For the π's, we start with the solution to Guess $(2m+2, 2m-1)$ given by Theorem 11 which achieves separations $(2m)^{2m+2} \cdots 2^{2m+2} 0^{2m+2}$ and includes the identity. We replace the identity with the permutation $(0 \;\; 2m-1)(1 \;\; 2m-2) \cdots (m-2 \;\; m+1)$. This permutation achieves separations $(2m-1)^1 (2m-3)^1 \cdots 3^1 0^4$, thus in total we achieve $(2m)^{2m+2} (2m-1)^{2m+3} \cdots 3^{2m+3} 2^{2m+2} 0^4$. Combining the σ defined above and these π gives every separation $q = 4m+3$ times, as required. $\square$

Lemma 13. *For any $m \geq 2$ there exists a sequence $(\tau_1, \ldots, \tau_{m-1})$ of permutations of $\{0, 1, \ldots, m-1\}$ such that $\tau_j(0) \neq m-1$ and for all $0 \leq k \leq m-2$ the total number of solutions to $\tau_j(i) - i = k$ plus the total number of solutions to $\tau_j(i) - i = -k-1$ is m.*

Proof. Define $(\phi_j)_{j=1}^{m-1}$ by $\phi_j = (0 \;\; j)$ and $\tau_j(i) = m-1-\phi_j(i)$. We will use the geometric description of permutations introduced at the beginning of Section 3. Consider the amalgamation of all pairs $(i, \phi_j(i))$. Since each $1 \leq i \leq m-1$ is fixed by all but one of the ϕ_j we see we get pairs (i, i) for $i \neq 0$ exactly $m-2$ times. Also we get each pair $(0, i)$ or $(i, 0)$ with $i \neq 0$ exactly once. Thus amalgamating all pairs $(i, \tau_j(i))$ gives us every anti-diagonal square $(i, m-i-1)$ for $i \neq 0$ exactly $m-2$ times and every square $(0, i)$ for $i \neq m-1$ and every square $(i, m-1)$ for $i \neq 0$ exactly once. The anti-diagonal squares contribute $m-2$ solutions to $\tau_j(i) - i = k$ for $1-m \leq k < m-1$ and $k \equiv m-1 \pmod 2$. Since exactly one out of each pair $\{k, -k-1\}$ with $0 \leq k \leq m-2$ has this form, the anti-diagonal squares contribute $m-2$ solutions to each such pair. The squares $(0,i)$ for $i \neq m-1$ and $(i,m-1)$ for $i \neq 0$ contribute 2 solutions to $\tau_j(i)-i=k$ for $0 \leq k \leq m-2$. Since exactly one out of each pair $\{k,-k-1\}$ with $0 \leq k \leq m-2$ has this form, these squares contribute 2 to each such pair. Thus each pair gets m solutions, as desired. □

Proposition 14. *We can solve the c-value problem for $q=4m+1$ and $p=4m-3$.*

Proof. As usual let $t=2m$, $s=2m-2$ and let $(\tau_j)_{j=1}^{m-1}$ be as given by Lemma 13. Define permutations σ_j of $\{0,\ldots,t-1\}$ by $\sigma_j(i)=\tau_j^{-1}(i)+m$ for $0 \leq i \leq m-1$ and $\sigma_j(i)=\tau_j(i-m)$ for $m \leq i \leq 2m-1$. Choose $4m-3$ permutations of $\{0,\ldots,t-1\}$ by taking four copies of each σ_j and one copy of

$$(0\ \ 2m-1)(1\ \ 2m-2)\cdots(m-2\ \ m+1).$$

Define permutations π_j of $\{0,\ldots,t\}$ by $\pi_j(i)=\tau_j(i)+m+1$ for $0 \leq i \leq m-1$, $\pi_j(m)=m$, and $\pi_j(i)=\tau_j^{-1}(i-m-1)$ for $m+1 \leq i \leq 2m$. Choose $4m-3$ permutations of $\{0,\ldots,t\}$ by taking four copies of each π_j and one copy of

$$(1\ \ 2m-1)(2\ \ 2m-2)\cdots(m-1\ \ m+1).$$

We claim that this collection of permutations works. To see this note that all the permutations built are value-symmetric, so it suffices to look at non-negative values. The number of solutions to $\sigma_j(i)-i=k$ for $2 \leq k \leq 2m-1$ is exactly the number of solutions to $\tau_j(i)-i=m-k$. The number of solutions to $\pi_j(i)-i=k$ for $2 \leq k \leq 2m-1$ is exactly the number of solutions to $\tau_j(i)-i=k-m-1$. Thus by Lemma 13 we get exactly $4m$ separations of k from the four copies of σ_j and π_j. The remaining permutation for t gives every odd separation $2 \leq k \leq 2m-1$ once and the remaining permutation for $t+1$ gives every even separation $2 \leq k \leq 2m-1$ once. Thus we get all nonzero separations exactly $q=4m+1$ times. By subtraction or direct count (the σ_j have no fixed points, each π_j has one fixed point, and the two extra permutations have two and three fixed points respectively) we see we get $4m+1$ fixed points. Thus we have the desired solution to the c-value problem. □

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