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The O(N) σ-model Laplacian

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Abstract

For fields that vary slowly on the scale of the lightest mass the logarithm of the vacuum functional of a massive quantum field theory can be expanded in terms of local functionals satisfying a form of the Schrödinger equation, the principal ingredient of which is a regulated functional Laplacian. We construct to leading order a Laplacian for the O(N) σ-model that acts on such local functionals. It is determined by imposing rotational invariance in the internal space together with closure of the Poincaré algebra.

1 The O(N) σ-model Laplacian

The O(N) σ-model shares many features with Yang-Mills theory. They are both classically conformally invariant but generate mass quantum mechanically. Both are renormalisable [1], asymptotically free, and have large-N expansions [2]. However the O(N) σ-model, has the advantage of tractability so that, for example, mass generation can be explicitly demonstrated within the large-N expansion. This makes it a useful laboratory to test techniques that are ultimately intended for the study of Yang-Mills theory. Recently one of us proposed a new approach to the eigenvalue problem for the Hamiltonians of massive quantum field theories that should be applicable to theories that are classically massless [3]. This is based on a version of the functional Schrödinger equation that acts directly on a local expansion of the vacuum functional. (The Schrödinger representation approach to field theory is discussed in [4]-[11]). This local expansion is applicable when the vacuum functional is evaluated for fields that vary slowly on the scale of the inverse of the mass. The purpose of this paper is to construct to leading order the functional Laplacian for the O(N) σ-model that acts on local functionals, which is the principal ingredient in this approach. We will show that this operator is determined by the symmetries of the model, namely Poincaré invariance and rotational symmetry in the internal space.

Consider the quantum mechanics of a non-relativistic particle of mass, m, moving on the N-dimensional sphere with co-ordinates $z^\mu(\tau)$ at time τ. The action is $S = \frac{m}{2}\int d\tau g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu$ where $g_{\mu\nu}$ is the metric on the sphere which we take to have radius a. The Hamiltonian in the Schrödinger representation is $H = -\frac{1}{2m}\Delta$, where Δ is the Laplacian, and the eigenfunctions are the spherical harmonics. The O(N) σ-model can be thought of as a natural generalisation in which we replace the particle by a curve parametrised by σ, say, and choose an action that is relativistically invariant in the space-time (σ, τ) so that

$$S = \frac{1}{2\alpha}\int d\sigma d\tau\, g_{\mu\nu}\left(\dot{z}^\mu\dot{z}^\nu - z'^\mu z'^\nu\right) \tag{1}$$

where $'$ and $\cdot$ denote differentiation with respect to σ and τ, and α is a coupling constant. Formally, the Hamiltonian in the Schrödinger representation would be

$$H = -\frac{\alpha}{2}\Delta + \frac{1}{2\alpha}\int d\sigma g_{\mu\nu}\, z'^\mu z'^\nu \tag{2}$$

where Δ is now the Laplacian constructed from an inner product on variations of the co-ordinates

$$(\delta z, \delta z) = \int d\sigma\, g_{\mu\nu}\,\delta z^\mu \delta z^\nu \tag{3}$$

This is a metric with an infinite number of components labelled by σ and also μ, ν. If we use bold-face type to denote infinite component tensors and ordinary type to denote tensors on S^N then $\mathbf{g}_{\mu_1\mu_2}(\sigma_1,\sigma_2) = g_{\mu_1\mu_2}(z(\sigma_1))\,\delta(\sigma_1-\sigma_2)$. The pair (μ_1, σ_1) should be treated as a single index, as should (μ_2, σ_2), so that $\mathbf{g}$ is a two-index tensor, as is usual. This has an inverse, $\mathbf{g}^{\mu_1\mu_2}(\sigma_1,\sigma_2) = g^{\mu_1\mu_2}(z(\sigma_1))\,\delta(\sigma_1-\sigma_2)$. The infinite dimensional Kronecker delta $\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2) = \delta^{\mu_1}_{\mu_2}\delta(\sigma_1-\sigma_2)$, and is equal to the functional derivative of $z^{\mu_1}(\sigma_1)$ with respect to $z^{\mu_2}(\sigma_2)$. It is tempting to try to generalise Riemannian geometry to this infinite dimensional case, so that $\mathbf{g}$ and $\mathbf{I}$ transform covariantly under general co-ordinate transformations in which $z^\mu(\sigma) \to \tilde{z}^\mu(\sigma)$ and $\tilde{z}$ is a functional of z i.e. it depends on the entire curve $z = z(\sigma)$. However, the utility of this for quantum field theory is not clear. What is important is that the theory be invariant under rotations in the internal space, that is to say the isometries of the finite dimensional metric $g_{\mu\nu}$, since these underpin the renormalisability of the theory [1]. These are rigid co-ordinate transformations, so we will restrict our attention to co-ordinate transformations in which $\tilde{z}^\mu(\sigma)$ is a function of the co-ordinates $z^\nu(\sigma)$ i.e. just at the point σ on the curve. Under this restricted class of transformations a finite-dimensional vector $V^\mu(z)$ on S^N, e.g. z'^μ, may be thought of as an infinite dimensional vector $\mathbf{V}^\mu(\sigma) = V^\mu(z(\sigma))$. To construct the Laplacian we need a covariant derivative. Given the infinite dimensional metric we can follow the usual construction of the Levi-Civita connection, $\mathbf{D}$, which will transform covariantly under general co-ordinate transformations and therefore under our restricted transformations. Thus

$$\mathbf{D}_{\mu_2}(\sigma_2)\mathbf{V}^{\mu_1}(\sigma_1) = \frac{\delta \mathbf{V}^{\mu_1}(\sigma_1)}{\delta z^{\mu_2}(\sigma_2)} + \int d\sigma_3\, \mathbf{\Gamma}^{\mu_1}_{\mu_2\mu_3}(\sigma_1,\sigma_2,\sigma_3)\mathbf{V}^{\mu_3}(\sigma_3) \tag{4}$$

where the infinite dimensional Christoffel symbol is related to that on S^N by

$$\mathbf{\Gamma}^{\mu_1}_{\mu_2\mu_3}(\sigma_1,\sigma_2,\sigma_3) = \delta(\sigma_1-\sigma_2)\,\delta(\sigma_2-\sigma_3)\,\Gamma^{\mu_1}_{\mu_2\mu_3}(z(\sigma_1)) \tag{5}$$

If we apply this to a vector that depends on σ and $z(\sigma)$ but not on its derivatives (we will call this property ultra-locality) then it is straightforward to compute

$$\mathbf{D}_{\mu_2}(\sigma_2)\mathbf{V}^{\mu_1}(\sigma_1) = (D_{\mu_2}V^{\mu_1})|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2), \tag{6}$$

where D is the covariant derivative on S^N. Similarly we can easily compute the covariant derivative of $\mathbf{z}'^\mu$ as

$$\mathbf{D}_{\mu_2}(\sigma_2)\,\mathbf{z}'^{\mu_1}(\sigma_1) = \delta^{\mu_1}_{\mu_2}\,\delta'(\sigma_1-\sigma_2) + \left(\Gamma^{\mu_1}_{\mu_2\rho}z'^\rho\right)|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) \tag{7}$$

with Γ the finite dimensional Christoffel symbol. Now the finite dimensional intrinsic derivative $\mathcal{D} = \frac{\partial}{\partial\sigma} + z'^\mu D_\mu$ maps finite dimensional vectors to finite dimensional vectors, so we can use it to define new infinite component vectors as $\mathcal{D}|_\sigma \mathbf{V}^\mu(\sigma) \equiv (\mathcal{D}V)^\mu|_{z(\sigma)}$. The infinite dimensional tensor $\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2)$, thought of in terms of finite dimensional tensors, is an element of the product of the tangent space at $z(\sigma_1)$ and the co-tangent space at $z(\sigma_2)$. If we apply the intrinsic derivative with respect to σ_1 then it acts only on the μ_1 index to give

$$\mathcal{D}|_{\sigma_1}\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2) = \delta^{\mu_1}_{\mu_2}\,\delta'(\sigma_1-\sigma_2) + \left(\Gamma^{\mu_1}_{\mu_2\rho}z'^\rho\right)|_{z(\sigma_2)}\,\delta(\sigma_1-\sigma_2) = \mathbf{D}_{\mu_2}(\sigma_2)\,\mathbf{z}'^{\mu_1}(\sigma_1). \tag{8}$$

Similarly the intrinsic derivative with respect to σ_2 acts only on the μ_2 index to give

$$\mathcal{D}|_{\sigma_2}\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2) = -\mathbf{D}_{\mu_2}(\sigma_2)\,\mathbf{z}'^{\mu_1}(\sigma_1) \tag{9}$$

Since $z'^\mu = \mathcal{D}z^\mu(\sigma)$ this implies that $[\mathbf{D},\mathcal{D}]z^\mu = 0$, so that this commutator also annihilates any ultra-local scalar. It will prove useful to know the value of this commutator when it acts on vectors. If V^μ is ultra-local then

$$[\mathbf{D}_{\mu_1}(\sigma_1),\,\mathcal{D}|_{\sigma_2}]\,\mathbf{V}^{\mu_2}(\sigma_2) = z'^\rho(\sigma_1)\left(R_{\rho\mu_1\lambda}{}^{\mu_2}V^\lambda\right)|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) \tag{10}$$

where R is the finite dimensional Riemann tensor given by $[D_\mu, D_\lambda]\, V^\beta = R_{\lambda\mu\rho}{}^{\beta} V^\rho$, so that on the sphere $R_{\alpha\beta\gamma\delta} = (g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta})/a^2$. (This computation is essentially the same as for the finite dimensional case, decorated by delta-functions so we omit the details). For z'^μ we obtain

$$[\mathbf{D}_{\mu_1}(\sigma_1), \mathcal{D}|_{\sigma_2}]\, \mathbf{z}'^{\mu_2}(\sigma_2) = z'^\rho(\sigma_1) z'^\lambda(\sigma_1)\, R_{\mu_1\rho\lambda}{}^{\mu_2}|_{z(\sigma_1)}\, \delta(\sigma_1 - \sigma_2). \tag{11}$$

The obvious definition of the Laplacian as the second covariant derivative with indices contracted using $\mathbf{g}$ does not exist because the two functional derivatives act at the same value of σ and also because the determinant of the infinite dimensional metric $\mathbf{g}$ is ill-defined. Instead we will look for a regulated expression of the form

$$\Delta_s = \int d\sigma_1 d\sigma_2\, \mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2; s)\, \mathbf{D}_{\mu_1}(\sigma_1)\mathbf{D}_{\mu_2}(\sigma_2). \tag{12}$$

The kernel, $\mathbf{G}$, is constrained by a number of physical requirements. We will see that these are sufficent to determine its form, at least to leading order, when the Laplacian acts on local functionals. Firstly we require that it is a regularisation of the inverse metric, so we will assume that it depends on a cut-off parameter, s, with the dimensions of squared length and takes the form$\mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2; s) = \mathcal{G}_s(\sigma_1 - \sigma_2)\, K^{\mu_1\mu_2}(\sigma_1, \sigma_2)$, where $\mathcal{G}_s(\sigma_1 - \sigma_2) \sim \delta(\sigma_1 - \sigma_2)$ as $s \downarrow 0$, and K is a non-singular functional of z^μ included so that $\mathbf{G}$ transforms as a tensor under the restricted class of co-ordinate transformations. Our problem is to construct K. For $\mathbf{G}$ to be a regularisation of $\mathbf{g}$ we require that $K^{\mu_1\mu_2}(\sigma, \sigma) = g^{\mu_1\mu_2}(z(\sigma))$. Finally, since we work in a Hamiltonian formalism, Poincaré invariance is not manifest and must be imposed by demanding that the generators of these transformations satisfy the Poincaré algebra. This is the integrability condition for the action of the generators on wave-functionals, Ψ. Ignoring regularisation the Poincaré generators are the Hamiltonian, (2), the momentum $P = \int d\sigma z'^\mu \mathbf{D}_\lambda(\sigma)$ and the Lorentz generator $L = -\alpha M + \alpha^{-1} N$, where

$$M = \frac{1}{2}\int d\sigma\, \sigma g^{\mu_1\mu_2}\, \mathbf{D}_{\mu_1}(\sigma)\mathbf{D}_{\mu_2}(\sigma), \quad N = \frac{1}{2}\int d\sigma\, \sigma g_{\mu_1\mu_2} z'^{\mu_1} z'^{\mu_2} \tag{13}$$

Again ignoring problems of regularisation, these satisfy the Poincaré algebra

$$[P, H] = 0, \quad [L, P] = H, \quad [L, H] = P. \tag{14}$$

We require that this algebra still holds when regulators are in place. The momentum operator does not need to be regulated. We regulate the Laplacian as above to yield a cut-off Hamiltonian H_s. In scalar φ^4 theory it was shown by Symanzik [4] that in the Schrödinger representation wave-functionals expressed in terms of renormalised fields have a finite limit as the regulator is removed, and since the Hamiltonian generates displacements in τ it has a finite action on these wave-functionals. (The field undergoes an additional renormalisation for φ^4 due to boundary effects but these are presumably absent for the $O(N)$ sigma-model due to rotational invariance in the internal space). Thus the limit as $s \downarrow 0$ of $H_s\Psi$ exists and is what we mean by the Hamiltonian applied to Ψ. We assume that this carries over to the O(N) sigma-model. Similarly we introduce a cut-off into L to obtain L_s which should have a finite limit when applied to wave-functionals.

The commutator $[L, P] = H$ implies that L should be regulated with the same kernel as H, so we replace the operator M by

$$\int d\sigma_1 d\sigma_2 \, \frac{\sigma_1 + \sigma_2}{2} \, \mathbf{G}_s^{\mu_1\mu_2}(\sigma_1, \sigma_2) \, \mathbf{D}_{\mu_1}(\sigma_1)\mathbf{D}_{\mu_2}(\sigma_2) \equiv M_s \tag{15}$$

Ultimately we are interested in constructing the Schrödinger equation for an expansion of the vacuum-functional in terms of local functionals, i.e. integrals of functions of $z(\sigma)$ and a finite number of its derivatives at the point σ. So we consider the conditions on the kernel $\mathbf{G}$ that arise from applying (14) to such test functionals. It is convenient to order such test functionals according to the powers of $\mathcal{D}$ for the following reason. If we act with Δ_s on a functional of the form $\int d\sigma \, f(z(\sigma), \sigma)_{\mu_1..\mu_n} z'^{\mu_1}..z'^{\mu_n} \equiv F_n$, where f is ultra-local, then the two functional derivatives in the Laplacian will act on the $z'^{\mu_1}..z'^{\mu_n}$ to generate a second order differential operator acting on $\delta(\sigma_1 - \sigma_2)$. Integrating by parts allows this operator to act on one of the σ arguments of the kernel, whilst the delta-function sets both arguments equal. The consequence of this is that $\Delta_s F_n$ depends on the second derivative of the kernel evaluated at co-incident points. Demanding the closure of the Lorentz algebra acting on F_n will constrain this quantity, whereas if we consider the Laplacian applied to a test-functional containing higher derivatives we obtain information about the higher derivatives of the kernel. F_n is the lowest order functional that gives a constraint. We treat F_n as a scalar so that

$$\mathbf{D}_\mu(\sigma) F_n = \frac{\delta F_n}{\delta z^\mu(\sigma)} = \left(D_\mu \, f_{\mu_1..\mu_n} z'^{\mu_1}..z'^{\mu_n} - n\mathcal{D}(f_{\mu\mu_2..\mu_n} z'^{\mu_2}..z'^{\mu_n})\right)|_\sigma \tag{16}$$

which is an infinite component co-vector. Using the commutators of $\mathbf{D}$ and $\mathcal{D}$ worked out above it is easy to show that

$$\begin{aligned} \Delta_s F_n = \quad & \int d\sigma \, \mathbf{G}^{\mu\nu}(\sigma, \sigma) \left(D_\mu D_\nu f_{\rho_1..\rho_n} - n R_{\mu\rho_1\nu}{}^{\lambda} \, f_{\lambda\rho_2..\rho_n} \right) z'^{\rho_1}..z'^{\rho_n} \\ & + n \int d\sigma \, \left((\mathcal{D}|_\sigma + \mathcal{D}|_{\sigma'})\mathbf{G}^{\mu\nu}(\sigma, \sigma')\right)|_{\sigma=\sigma'} D_\mu f_{\nu\rho_2..\rho_n} z'^{\rho_2}..z'^{\rho_n} \\ & + n(n-1) \int d\sigma \, (\mathcal{D}|_\sigma \mathcal{D}|_{\sigma'} \, \mathbf{G}^{\mu\nu}(\sigma, \sigma'))_{\sigma=\sigma'} \, f_{\mu\nu\rho_3..\rho_n} z'^{\rho_3}..z'^{\rho_n} \end{aligned} \tag{17}$$

Given the dimension of $\mathbf{G}$, (inverse length), and its transformation properties we can set

$$\mathbf{G}^{\mu\nu}(\sigma, \sigma) = \frac{1}{\sqrt{s}} \left(b_0^0 \, g^{\mu\nu} \right) \tag{18}$$

so that

$$((\mathcal{D}|_\sigma + \mathcal{D}|_{\sigma'})\mathbf{G}^{\mu\nu}(\sigma, \sigma'))_{\sigma=\sigma'} = \mathcal{D}\left(\frac{1}{\sqrt{s}} \left(b_0^0 \, g^{\mu\nu} \right) \right) = 0 \tag{19}$$

and

$$(\mathcal{D}|_\sigma \mathcal{D}|_{\sigma'} \, \mathbf{G}^{\mu\nu}(\sigma, \sigma'))_{\sigma=\sigma'} = -\frac{1}{\sqrt{s}^3} \left(b_0^1 \, g^{\mu\nu} + s b_1^1 \, g_{\lambda\rho} z'^\lambda z'^\rho \, g^{\mu\nu} + s b_2^1 \, z'^\mu z'^\nu \right) \tag{20}$$

where $b_1^0, b_0^1, b_1^1, b_2^1..$ are dimensionless constants. b_0^0 and b_0^1 are determined by our choice of regularisation of the delta-function, $\mathcal{G}_s$

$$b_0^0 = \sqrt{s} \, \mathcal{G}_s(0), \quad b_0^1 = \sqrt{s}^3 \, \mathcal{G}_s''(0), \tag{21}$$

our problem is to relate them to the remaining coefficents by imposing the closure of the Poincaré algebra. Using these expressions we can write $\Delta_s F_n$ as

$$\Delta_s F_n = -\frac{n(n-1)\, b_0^1}{\sqrt{s}^3} \int d\sigma\, g^{\mu\nu} f_{\mu\nu\rho_3..\rho_n}\, z'^{\rho_3}..z'^{\rho_n} + \frac{1}{\sqrt{s}} \int d\sigma\, (J_n f)_{\rho_1..\rho_n}\, z'^{\rho_1}..z'^{\rho_n} \tag{22}$$

where

$$\begin{aligned}(J_n f)_{\rho_1..\rho_n} &= b_0^0 g^{\mu\nu} \left(D_\mu D_\nu\, f_{\rho_1..\rho_n} + n\, f_{\lambda(\rho_2..\rho_n} R_{\rho_1)\mu\nu}{}^{\lambda} \right) \\ &\quad -n(n-1) \left(b_1^1 g^{\mu\nu}\, f_{\mu\nu(\rho_3..\rho_n} g_{\rho_1\rho_2)} + b_2^1\, f_{\rho_1..\rho_n} \right),\end{aligned} \tag{23}$$

and bracketed indices are symmetrised. The calculation of $M_s F_n$ is essentially the same, but with $\frac{\sigma_1+\sigma_2}{2}\mathbf{G}$ replacing $\mathbf{G}$, so that there is an additional piece coming from the second integral on the right-hand side of (17)

$$\begin{aligned} M_s F_n &= -\frac{n(n-1)\, b_0^1}{\sqrt{s}^3} \int d\sigma\, \sigma\, g^{\mu\nu} f_{\mu\nu\rho_3..\rho_n}\, z'^{\rho_3}..z'^{\rho_n} \\ &\quad + \frac{1}{\sqrt{s}} \int d\sigma\, \sigma\, (J_n f)_{\rho_1..\rho_n}\, z'^{\rho_1}..z'^{\rho_n} + \frac{n\, b_0^0}{\sqrt{s}} \int d\sigma\, D^\mu f_{\mu\rho_2..\rho_n} z'^{\rho_2}..z'^{\rho_n} \end{aligned} \tag{24}$$

In [3] it was shown that for scalar field theory the expansion of the vacuum functional in terms of local functionals, which is valid for slowly varying fields, does not satisfy the obvious Schrödinger equation because expanding in terms of local functionals does not commute with removing the cut-off. However, by studying the analyticity properties of the Laplacian it was shown how to re-sum the cut-off dependence of $\Delta_s \Psi$ so as to be able to remove the cut-off correctly. This re-summation is accomplished by performing a contour integral over s and has the effect of replacing s^{-n} in $\Delta_s \Psi$ by $\lambda^n c(n)$ where $c(n)$ is just a numerical function of n and λ is a new cut-off that is to be taken to infinity. We shall assume that such a re-summation may be performed here.

We will now consider the conditions placed on the kernel by demanding that the Poincaré algebra close when acting on the local functionals that we hope to construct the wave functionals from. We require that $([L, H] - P)F_n = 0$. Introducing regulators into the generators we require that as $s_1, s_2 \downarrow 0$

$$\frac{1}{4}[-\alpha M_{s_1} + \alpha^{-1} N,\, -\alpha \Delta_{s_2} + \alpha^{-1} V] F_n = P F_n \tag{25}$$

Contributing to this equation there will be a number of terms in which the removal of the cut-off simply replaces the kernel $\mathbf{G}$ by the metric $\mathbf{g}$ without any singularity arising. By themselves the sum of such terms satisfies the equation since they are just what would occur if we ignored the problem of regularisation altogether. The remaining terms have to cancel against each other to satisfy (25). These are the terms that, in the absence of a regulator, involve two functional derivatives at the same point acting on a single local functional. Thus we require $[M_{s_1}, \Delta_{s_2}]F_n = 0$ as well as $M_{s_1} V = 0$ and $\Delta_{s_2} N = 0$. Using the results above we compute

$$
\begin{aligned}
&[M_{s_1}, \Delta_{s_2}]F_n = \\
&(s_2^{-1/2} s_1^{-3/2} - s_2^{-3/2} s_1^{-1/2})\, 2n(n-1)\, b_0^1 \left(kb_0^0 + (N-1)b_1^1 + (4n-6)(b_1^1 + b_2^1)\right) \times \\
&\qquad \int d\sigma\, \sigma\, (tr\, f)_{\rho_3..\rho_n} z'^{\rho_3}..z'^{\rho_n} \\
&-2n(n-1)\, b_0^0 (s_1 s_2)^{-1/2} \int d\sigma \left(\left(b_1^1 + \frac{kb_0^0}{N-1}\right) D_{(\rho_2} (tr\, f)_{\rho_3..\rho_n)} \right. \\
&\qquad \left. + \left(b_2^1 - \frac{kb_0^0}{N-1}\right) D^\mu f_{\mu\rho_2..\rho_n} \right) z'^{\rho_2}..z'^{\rho_n}
\end{aligned} \tag{26}
$$

where $(tr\, f)_{\rho_3..\rho_n} = g^{\mu\nu} f_{\mu\nu\rho_3..\rho_n}$. This will vanish for all n by taking

$$
b_1^1 = -b_2^1 = -\frac{kb_0^0}{N-1} = -\frac{b_0^0}{a^2} \tag{27}
$$

Finally, consider the conditions $M_s V = 0$ and $\Delta_s N = 0$. Using (22) and (24) we obtain

$$
M_s\, V = \Delta_s\, N = -\frac{2Nb_0^1}{\sqrt{s}^3} \int d\sigma\, \sigma - \frac{2}{\sqrt{s}} (kb_0^0 + Nb_1^1 + b_2^1) \int d\sigma\, \sigma\, g_{\mu\nu}\, z'^\mu z'^\nu, \tag{28}
$$

so that these conditions are also satisfied by (27) if we take the ill-defined integral $\int d\sigma\, \sigma$ to vanish on the grounds that the integrand is odd. (We note in passing that $\Delta_s V$ is just a constant.)

Substituting our results back into $\mathbf{G}$ we obtain the following information

$$
\mathbf{G}^{\mu\nu}(\sigma, \sigma) = \mathcal{G}_s(0)\, g^{\mu\nu} \tag{29}
$$

and

$$
(\mathcal{D}|_\sigma \mathcal{D}|_{\sigma'} \mathbf{G}^{\mu\nu}(\sigma, \sigma'))\,|_{\sigma=\sigma'} = -\mathcal{G}_s''(0)\, g^{\mu\nu} + \mathcal{G}_s(0)\, R^{\mu\ \nu}_{\ \lambda\ \rho} z'^\lambda z'^\rho \tag{30}
$$

To these we can add the condition $(\mathcal{D}|_\sigma \mathbf{G}^{\mu\nu}(\sigma, \sigma'))\,|_{\sigma=\sigma'} = 0$, which follows from dimensional analysis and rotational invariance. These results can be used to re-construct $\mathbf{G}$ at non-coincident points in a Taylor expansion. To do this covariantly we introduce $\mathbf{W}^{\mu_1}_{\mu_2}(\sigma_1, \sigma_2)$ defined by $\mathcal{D}|_{\sigma_1} \mathbf{W}^{\mu_1}_{\mu_2}(\sigma_1, \sigma_2) = 0$, and $\mathbf{W}^{\mu_1}_{\mu_2}(\sigma, \sigma) = \delta^{\mu_1}_{\mu_2}$, so that $\mathbf{W}$ is the path-ordered exponential integral of the Christoffel symbols. It is invertible as an $N \times N$ matrix, so we can set $\mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2) = \mathbf{W}^{\mu_1}_\nu(\sigma_1, \sigma_2)\, T^{\nu\mu_2}(\sigma_1, \sigma_2)$ where T is a finite dimensional tensor at $z(\sigma_2)$ but depends on σ_1. Now $(\mathcal{D}|_{\sigma_1})^n \mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2) = \mathbf{W}^{\mu_1}_\nu(\sigma_1, \sigma_2) (\frac{\partial}{\partial\sigma_1})^n T^{\nu\mu_2}(\sigma_1, \sigma_2)$, so that when we set σ_1 and σ_2 equal the intrinsic derivatives of $\mathbf{G}$ reduce to the ordinary derivatives of T, enabling us to use the usual Taylor expansion to find the σ_1-dependence of T.

In conclusion, we have sought a Laplacian for the O(N) σ-model of the form

$$
\Delta_s = \int d\sigma_1 d\sigma_2\, \mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2)\, \mathbf{D}_{\mu_1}(\sigma_1) \mathbf{D}_{\mu_2}(\sigma_2). \tag{31}
$$

where the kernel $\mathbf{G}$ is a regularisation of the infinite-dimensional metric $\mathbf{g}^{\mu_1\mu_2}(\sigma_1,\sigma_2) = g^{\mu_1\mu_2}\delta(\sigma_1-\sigma_2)$. It takes the form $\mathcal{G}_s(\sigma_1-\sigma_2)\,K^{\mu_1\mu_2}(\sigma_1,\sigma_2)$ where $\mathcal{G}$ is a regularisation of a delta-function. By demanding manifest invariance of Δ_s under rotations in internal space and the closure of the Poincaré algebra acting on local functionals of the form $\int d\sigma\, f(z(\sigma),\sigma)_{\mu_1..\mu_n} z'^{\mu_1}..z'^{\mu_n} \equiv F_n$ we obtained a number of conditions on $\mathbf{G}$ that were all satisfied by (27). These translate into the statement that up to terms of $O\left((\sigma_1-\sigma_2)^3\right)$

$$K^{\mu_1\mu_2}(\sigma_1,\sigma_2) = \mathbf{W}^{\mu_1}_{\nu}(\sigma_1,\sigma_2)\left(g^{\nu\mu_2}|_{z(\sigma_2)} - \frac{1}{2}(\sigma_1-\sigma_2)^2 R^{\nu}{}_{\lambda}{}^{\mu_2}{}_{\rho} z'^{\lambda} z'^{\rho}|_{z(\sigma_2)}\right) \qquad (32)$$

Given a choice of $\mathcal{G}_s$ these conditions are sufficent to fix the action of Δ_s on F_n. By considering the closure of the Poincaré algebra acting on test functionals containing higher derivatives of the co-ordinates we would obtain information about the higher terms in the Taylor expansion of K.

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