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Symmetric calorons

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Abstract

Calorons (periodic instantons) interpolate between monopoles and instantons, and their holonomy gives approximate Skyrmion configurations. We show that, for each caloron charge $N \leq 4$, there exists a one-parameter family of calorons which are symmetric under subgroups of the three-dimensional rotation group. In each family, the corresponding symmetric monopoles and symmetric instantons occur as limiting cases. Symmetric calorons therefore provide a connection between symmetric monopoles, symmetric instantons and Skyrmsions.

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1. Introduction

Calorons are finite-action self-dual gauge fields in four dimensions, which are periodic in one of the four coordinates. Call the periodic coordinate t , with period β . Special cases include instantons on \mathbf{R}^4 (where $\beta \rightarrow \infty$) and BPS monopoles (where the gauge field is independent of t). The holonomy Ω of the gauge field in the t -direction is a map from \mathbf{R}^3 to the gauge group, and as such can serve as an approximation to Skyrmsions [1]. Calorons therefore provide a link between monopoles, instantons and Skyrmsions.

Skyrmsions resemble polyhedral shells, invariant under appropriate subgroups of the three-dimensional rotation group $O(3)$ [2,3]. The idea of producing approximate Skyrmsion configurations as instanton holonomy has motivated several studies of instantons invariant under such groups [4–7]. Finally, there are symmetric monopoles [8] which have the same polyhedral shape as the Skyrmsions of corresponding charge, suggesting a kinship between Skyrmsions and monopoles [9]. So symmetric calorons, namely calorons invariant under subgroups G of $O(3)$ (rotations about the t -axis), are relevant in this context. This Letter demonstrates the existence of symmetric calorons of charge N , for $N \leq 4$; they include, as limiting cases, symmetric monopoles and symmetric instantons.

Large classes of calorons were described some years ago [10–13]; of these, only the $N = 1$ case admits the relevant symmetry. So one needs more general solutions. There is a construction (the ADHMN construction) which

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generates caloron solutions [14], possibly all of them (see [15] for a recent analysis). In the last few years, this construction has been used to investigate and interpret caloron solutions, especially those for which the holonomy Ω is non-trivial at spatial infinity [16–18]; but this recent work was not concerned with symmetric solutions as such. In this Letter, we shall see how symmetric calorons arise from the ADHMN construction.

2. Calorons, monopoles and Skyrmions

We take the gauge group to be $SU(2)$ throughout. The standard coordinates on \mathbf{R}^4 are denoted $x^\mu = (x^1, x^2, x^3, x^4) = (x^j, t)$; let r be the quantity defined by $r^2 = x^j x^j$. The gauge potential A_μ is anti-Hermitian, and the corresponding gauge field is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. A gauge transformation acts as $A_\mu \mapsto \Lambda^{-1} A_\mu \Lambda + \Lambda^{-1} \partial_\mu \Lambda$. A *caloron* [10–12] is a gauge field with the following properties:

- $A_\mu(x^\alpha)$ is periodic in $x^4 = t$, with period β (in some gauge);
- $A_\mu(x^\alpha)$ is smooth everywhere (in some gauge);
- $F_{\mu\nu}$ is self-dual: $F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$;
- $\text{tr}(F_{\mu\nu} F_{\mu\nu}) = O(1/r^4)$ as $r \rightarrow \infty$.

A special case of this is where A_μ is independent of $x^4 = t$; this is a *monopole*, where we make the usual interpretation of A_t as a Higgs field Φ . The holonomy (or Wilson loop)

$$\Omega(x^j) = \mathcal{P} \exp \left[- \int_0^\beta A_t(x^j, t) dt \right] \quad (1)$$

in the t -direction takes values in the gauge group; under a periodic gauge transformation, it transforms as

$$\Omega(x^j) \mapsto \Lambda(x^j, 0)^{-1} \Omega(x^j) \Lambda(x^j, 0). \quad (2)$$

The quantity $\Omega(x^j)$ is, in general, non-trivial at spatial infinity [11]; but for the examples below, $\Omega(x^j)$ tends to a constant group element (in fact the identity) as $r \rightarrow \infty$. Such a field may be viewed as an approximate *Skyrmion* configuration; the Skyrmion number is the degree of Ω , and the normalized Skyrme energy is

$$E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \text{tr}(L_j L_j) - \frac{1}{16} \text{tr}([L_i, L_j][L_i, L_j]) \right\} d^3x, \quad (3)$$

where $L_j = \Omega^{-1} \partial_j \Omega$. Provided Ω is asymptotically trivial, the topological charge (caloron number)

$$N = -\frac{1}{32\pi^2} \int_0^\beta dt \int d^3x \text{tr}(\varepsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}) \quad (4)$$

is an integer, and is equal to the Skyrmion number of Ω [11]. In the t -independent (monopole) case, it is also the monopole number, provided we take β to be related to the asymptotic norm of the Higgs field by

$$-\frac{1}{2} \text{tr}(\Phi_\infty)^2 = \left(\frac{\pi}{\beta} \right)^2. \quad (5)$$

A large number of caloron solutions can be generated [10] by the Corrigan–Fairlie–’t Hooft [19] or Jackiw–Nohl–Rebbi [20] ansatz. These express the gauge potential in terms of a solution ϕ (periodic, in the caloron case)

of the four-dimensional Laplace equation. For example, the component A_t is given by

$$A_t = \frac{i}{2}(\partial_j \log \phi)\sigma_j, \tag{6}$$

where σ_j are the Pauli matrices. For the JNR solutions one has $\phi \rightarrow 0$ as $r \rightarrow \infty$, whereas for the CF'tH solutions one has $\phi \rightarrow 1$ as $r \rightarrow \infty$. In the case of instantons on \mathbf{R}^4 , one regards the CF'tH solutions as being limiting cases of the JNR solutions, but for calorons it is the other way round: to produce an N -caloron in JNR form, one uses a ϕ with N poles (not $N + 1$ as for instantons), and this is a limiting case of the CF'tH form with N poles.

To illustrate this, let us review the $N = 1$ case. The 1-caloron (with trivial holonomy at infinity) is generated [10] by the 1-pole function

$$\phi = 1 + \frac{W^2 \sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu t)]}, \tag{7}$$

where $\mu = 2\pi/\beta$, and $W > 0$ is a constant. This caloron is spherically-symmetric; it depends on the period β and on the parameter W . The gauge field is not affected by an overall scale factor in ϕ , so the $W \rightarrow \infty$ limit of (7) gives, in effect, the JNR-type solution with

$$\phi = \frac{\sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu t)]}; \tag{8}$$

this corresponds to a 1-caloron which is in fact gauge-equivalent to the 1-monopole [21]. Another way of viewing things is to use the dimensionless combination $\theta = \beta/W^2$: for $\theta = 0$ (or $W \rightarrow \infty$) we get the 1-monopole, while for $\theta \rightarrow \infty$ (or $\beta \rightarrow \infty$) we get the 1-instanton on \mathbf{R}^4 . In other words, we have a one-parameter family of spherically-symmetric calorons, with the 1-monopole at one end and the 1-instanton at the other end. The holonomy $\Omega(x^j)$ can be computed exactly in this case [22,23]; if one restricts to spherically-symmetric gauges, then Ω is actually gauge-invariant. The Skyrme energy (3) of this configuration Ω attains a minimum for $\theta \approx 7$; this minimum is only slightly less [22] than the value obtained from 1-instanton holonomy.

It is straightforward to produce spherically-symmetric calorons of higher charge in this way: for example, the function

$$\phi = 1 + \frac{W^2 \sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu t)]} + \frac{\widehat{W}^2 \sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu(t - t_0))]} \tag{9}$$

generates a spherically-symmetric 2-caloron, for any $t_0 \in (0, \beta)$ and $W, \widehat{W} > 0$. The holonomy of this is a spherically-symmetric (hedgehog) 2-Skyrmion configuration (cf. [1,4]). The limits $\beta \rightarrow \infty$ and $W, \widehat{W} \rightarrow \infty$ are both regular; the former is a 2-instanton, but the latter is not a 2-monopole (since, unlike in the $N = 1$ case, the t -dependence cannot be gauged away). It seems very unlikely that the CF'tH ansatz can yield any examples (other than for $N = 1$) of symmetric calorons having symmetric monopoles as a limiting case—for that, one needs more general solutions. A way of generating such solutions is described in the next section.

3. The ADHMN construction for calorons

There is a construction which produces caloron solutions [14]; for gauge group $SU(2)$, and for calorons which have trivial holonomy at infinity, it is as follows. As before, N is a positive integer which will turn out to be the caloron charge, and β is a positive number which will turn out to be the caloron period. It is convenient to use quaternion notation, with a quaternion q being represented by the 2×2 matrix $q^4 + iq^j \sigma^j$; in particular, x^μ corresponds to the quaternion $x = t + ix^j \sigma^j$. The unit quaternion ($q^4 = 1, q^j = 0$) is denoted $\mathbf{1}$.

The Nahm data consists of four Hermitian $N \times N$ matrix functions $T_\mu(s)$, and an N -row-vector W of quaternions, such that $T_\mu(s)$ is periodic in the real variable s with period $2\pi/\beta$, and the Nahm equation

$$\frac{d}{ds}T_j - i[T_4, T_j] - \frac{i}{2}\varepsilon_{jkl}[T_k, T_l] = \frac{1}{2}\text{tr}_2(\sigma_j W^\dagger W)\delta(s - \pi/\beta) \quad (10)$$

is satisfied. The trace is over quaternions, so the right-hand side is an $N \times N$ Hermitian matrix (as is the left-hand side). Given such data, we construct a caloron as follows. Let $U(s, x)$ be an N -column-vector of quaternions, and $V(x)$ a single quaternion, such that

1. $U(s, x)$ is periodic in s with period $2\pi/\beta$;
2. $U(s, x + \beta) = U(s, x)\exp(i\beta s)$;
3. $V(x + \beta) = V(x)$;
4. $\int_{-\pi/\beta}^{\pi/\beta} U(s, x)^\dagger U(s, x) ds + V(x)^\dagger V(x) = \mathbf{1}$;
5. U and V satisfy the linear equation

$$\frac{d}{ds}U - [i(T_4 + tI_n) \otimes \mathbf{1} + I_n \otimes x^j \sigma^j + T_j \otimes \sigma^j]U = iW^\dagger V\delta(s - \pi/\beta). \quad (11)$$

Note that both T_j and U are periodic in s , and have jump discontinuities at one value of s , which we have taken to be $s = \pi/\beta$. The discontinuities could equally well be located anywhere else; the choice in (10) and (11) is for later convenience. Note also that the overall quaternionic phase of the N -vector $W = [W_1 \dots W_N]$ is irrelevant; so we may, without loss of generality, take W_1 to be real.

The pair (U, V) determines the caloron gauge potential according to

$$A_\mu = V(x)^\dagger \partial_\mu V(x) + \int_{-\pi/\beta}^{\pi/\beta} U(s, x)^\dagger \partial_\mu U(s, x) ds. \quad (12)$$

The freedom in (U, V) is $U \mapsto U\Lambda$, $V \mapsto V\Lambda$, where Λ is a quaternion satisfying $\Lambda^\dagger \Lambda = 1$; this corresponds exactly to the gauge freedom in A_μ .

By contrast, the usual formulation of the ADHMN construction for monopoles involves three matrices $T_j(s)$, satisfying

$$\frac{d}{ds}T_j - \frac{i}{2}\varepsilon_{jkl}[T_k, T_l] = 0. \quad (13)$$

In this case, the $T_j(s)$ are not periodic in s , but rather are smooth on the open interval $|s| < 1$, with poles at the endpoints $s = \pm 1$. (The length of this interval sets the scale of the monopole.) In addition, the T_j satisfy

$$T_j(-s) = T_j(s)^t. \quad (14)$$

The idea here is that given a solution of the monopole Nahm equation (13), one may re-interpret it as a solution of the caloron Nahm equation (10), with $T_4 = 0$ and with a suitable choice of W , namely such that

$$T_j(-\pi/\beta) - T_j(\pi/\beta) = \frac{1}{2}\text{tr}_2(\sigma_j W^\dagger W). \quad (15)$$

We need to take $\beta > \pi$, so that the T_j are bounded for $|s| \leq \pi/\beta$. The symmetric part of T_j can, because of (14), be regarded as a continuous periodic function on $[-\pi/\beta, \pi/\beta]$; while the antisymmetric part of T_j has a jump discontinuity as in (15).

The limit $\beta \rightarrow \pi$ is the original monopole, while the limit $\beta \rightarrow \infty$ gives an instanton on \mathbf{R}^4 . This instanton limit works as follows. For $\beta \gg \pi$, we are solving (11) on the small interval $|s| \leq \pi/\beta$, so we may approximate

the solution as $U(s) = U_0 + U_1 s$. Eq. (15) then gives

$$U_1 = (it + x^j \sigma^j + T_j \otimes \sigma^j) U_0 = -\frac{i\beta}{2\pi} W^\dagger V, \tag{16}$$

where $T_j = T_j(0)$, and where U_0 and V satisfy the constraint

$$U_0^\dagger U_0 + V(x)^\dagger V(x) = \mathbf{1}. \tag{17}$$

If we write $\Lambda = \sqrt{\beta/2\pi} W$, then this is exactly the ADHM construction [24] for instantons, with the ADHM matrix Δ being given by

$$\Delta = \begin{bmatrix} \Lambda \\ x + iT_j \otimes \sigma^j \end{bmatrix}. \tag{18}$$

This Δ is an $(n + 1) \times n$ matrix of quaternions, satisfying the condition that $\Delta^\dagger \Delta$ is an $n \times n$ real matrix.

Let us now consider calorons which are symmetric under subgroups of the three-dimensional rotation group acting on x^j . For any rotation R , let $R_2 \in SU(2)$ denote the image of R in the 2-dimensional irreducible representation of $SO(3)$; in other words, R acts on the quaternion x according to $x \mapsto R_2^{-1} x R_2$. Similarly, let R_N denote the image of R in the N -dimensional irreducible representation of $SO(3)$, and write $\Theta_R = R_N \otimes R_2$. A monopole is invariant [8] under the group $G \subseteq SO(3)$ iff

$$\Theta_R^{-1} (T_j \otimes \sigma^j) \Theta_R = T_j \otimes \sigma^j \tag{19}$$

for all $R \in G$. For the corresponding caloron to be G -invariant, we need an additional condition on W , and this is easily seen (from (10) and (11)) to be

$$\Theta_R W^\dagger = W^\dagger \tau_R, \tag{20}$$

where τ_R , for each $R \in G$, is some quaternionic phase (namely a quaternion with $\tau_R^\dagger \tau_R = \mathbf{1}$). So given a symmetric monopole, there is a family of symmetric calorons parametrized by the solutions W (if there are any) of (15) and (20). In the $N = 1$ case, for example, we have $G = SO(3)$ (spherical symmetry) and $T_j = 0$; and W is an arbitrary positive constant, which is precisely the parameter appearing in the expression (7). In the next section, we shall see that analogous one-parameter families of symmetric calorons exist for $N = 2, 3$ and 4.

4. Symmetric examples for $N = 2, 3, 4$

We begin with the $N = 2$ case, taking $G = SO(2)$ (corresponding to rotations about the x^2 -axis). The solution of (13) which generates the axially-symmetric $N = 2$ monopole is $T_j(s) = f_j(s) \sigma_j$ (not summed over j), where

$$f_1 = f_3 = \frac{\pi}{4} \sec(\pi s/2), \quad f_2 = -\frac{\pi}{4} \tan(\pi s/2). \tag{21}$$

Then (15) and (20) have a solution W which is unique (given that W_1 is real), namely

$$W = \lambda [\mathbf{1} \quad -i\sigma_2], \quad \text{where } \lambda = \sqrt{\frac{\pi}{2} \tan\left(\frac{\pi^2}{2\beta}\right)}. \tag{22}$$

So we get a family of $N = 2$ axially-symmetric caloron solutions, depending on the parameter $\beta > \pi$. It is possible to solve (11) analytically, and hence obtain exact expressions for the caloron (cf. [25] for the monopole case), although the expressions are rather complicated. The limit $\beta \rightarrow \pi$ is the 2-monopole, and $\beta \rightarrow \infty$ is a 2-instanton on \mathbf{R}^4 , generated by the ADHM matrix

$$\Delta = \frac{\pi}{4} \begin{bmatrix} \sqrt{2} & -i\sqrt{2}\sigma_2 \\ i\sigma_3 & i\sigma_1 \\ i\sigma_1 & -i\sigma_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x & 0 \\ 0 & x \end{bmatrix}. \tag{23}$$

This axially-symmetric 2-instanton can be obtained in the JNR form, and its holonomy was used to approximate the minimum-energy 2-Skyrmion [4,26]. The holonomy Ω of the caloron gives a one-parameter family of axially-symmetric 2-Skyrmion configurations; as in the $N = 1$ case, this gives an approximation to the true Skyrmion which is better than the instanton one, but only marginally so.

Let us now consider the $N = 3$ case. There is a 3-monopole with tetrahedral symmetry [8,27], corresponding to the following Nahm data. (Note that the T_j in [8,27] have to be multiplied by a factor of $-i$ to agree with the conventions used here.) Define

$$\Sigma_1 = 2i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Sigma_2 = 2i \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \Sigma_3 = 2i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (24)$$

and

$$S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

Then $T_j(s) = x(s)\Sigma_j + y(s)S_j$, where

$$x(s) = -\frac{\omega\wp'(u)}{12\wp(u)}, \quad y(s) = -\frac{\omega}{\sqrt{3}\wp(u)}, \quad (26)$$

with $u = \omega(s + 3)/3$ and $\omega = \Gamma(1/6)\Gamma(1/3)/(4\sqrt{\pi})$. Here \wp is the Weierstrass p-function satisfying $\wp'(u)^2 = 4\wp(u)^3 - 4$. The unique solution of (15), with $W_1 > 0$, is

$$W = \lambda[\mathbf{1} \quad i\sigma_3 \quad -i\sigma_2], \quad \text{where } \lambda = 2\sqrt{x(\pi/\beta)}. \quad (27)$$

Explicit calculation then verifies that (20) is satisfied for each of the elements of the tetrahedral group. So we have a one-parameter family of tetrahedrally-symmetric 3-calorons, interpolating between the tetrahedral 3-monopole and a tetrahedrally-symmetric 3-instanton. The latter is generated by the ADHM matrix

$$\Delta = \frac{\omega}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & i\sigma_3 & -i\sigma_2 \\ 0 & i\sigma_3 & i\sigma_2 \\ i\sigma_3 & 0 & i\sigma_1 \\ i\sigma_2 & i\sigma_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}. \quad (28)$$

A tetrahedrally-symmetric 3-instanton can also be obtained in JNR form, and its holonomy was used to approximate the minimum-energy 3-Skyrmion [5].

For the final example, we consider 4-calorons with cubic symmetry (so G is the 24-element octahedral group). The Nahm data in [8,27] do not satisfy (14), and so we have to change to a basis in which (14) holds. Define

$$\begin{aligned} \Sigma_1 &= \begin{bmatrix} -\sqrt{3} & 0 & -i & -1 \\ 0 & \sqrt{3} & -1 & i \\ i & -1 & -\sqrt{3} & 0 \\ -1 & -i & 0 & \sqrt{3} \end{bmatrix}, & \Sigma_2 &= \begin{bmatrix} 0 & \sqrt{3} & 1 & -i \\ \sqrt{3} & 0 & -i & -1 \\ 1 & i & 0 & \sqrt{3} \\ i & -1 & \sqrt{3} & 0 \end{bmatrix}, \\ \Sigma_3 &= \begin{bmatrix} 2 & -i & 0 & 0 \\ i & 2 & 0 & 0 \\ 0 & 0 & -2 & -i \\ 0 & 0 & i & -2 \end{bmatrix}, & S_1 &= 2 \begin{bmatrix} \sqrt{3} & 0 & -4i & 1 \\ 0 & -\sqrt{3} & 1 & 4i \\ 4i & 1 & \sqrt{3} & 0 \\ 1 & -4i & 0 & -\sqrt{3} \end{bmatrix}, \\ S_2 &= 2 \begin{bmatrix} 0 & -\sqrt{3} & -1 & -4i \\ -\sqrt{3} & 0 & -4i & 1 \\ -1 & 4i & 0 & -\sqrt{3} \\ 4i & 1 & -\sqrt{3} & 0 \end{bmatrix}, & S_3 &= 4 \begin{bmatrix} -1 & -2i & 0 & 0 \\ 2i & -1 & 0 & 0 \\ 0 & 0 & 1 & -2i \\ 0 & 0 & 2i & 1 \end{bmatrix}. \end{aligned}$$

Then $T_j(s) = x(s)\Sigma_j + y(s)S_j$, where

$$y = \frac{\omega_2}{10\wp'(u)}, \quad x = [5\wp(u)^2 - 3]y, \tag{29}$$

with $\omega_2 = (1 + i)\Gamma(1/4)^2/(4\sqrt{2\pi})$ and $u = \omega_2(s + 1)/2$. Here \wp is the Weierstrass p-function satisfying $\wp'(u)^2 = 4\wp(u)^3 - 4\wp(u)$. The condition (14) follows from the relations

$$\begin{bmatrix} x(-s) \\ y(-s) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -16 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}. \tag{30}$$

Then, as before, (15) has a unique solution

$$W = \lambda[\mathbf{1} \quad i\sigma_3 \quad i\sigma_1 \quad i\sigma_2], \quad \text{where } \lambda = \sqrt{2x(\pi/\beta) + 16y(\pi/\beta)}; \tag{31}$$

and one may check explicitly that (20) is satisfied for each element of the octahedral group. So here we have a one-parameter family of octahedrally-symmetric 4-calorons, interpolating between the cubic (octahedrally-symmetric) 4-monopole and an octahedrally-symmetric 4-instanton. This instanton is generated by the ADHM matrix

$$\Delta = \frac{|\omega_2|}{\sqrt{2}} \begin{bmatrix} \mathbf{1} & i\sigma_3 & i\sigma_1 & i\sigma_2 \\ \frac{\sqrt{3}}{2}i\sigma_1 - i\sigma_3 & -\frac{\sqrt{3}}{2}i\sigma_2 & -\frac{1}{2}i\sigma_2 & \frac{1}{2}i\sigma_1 \\ -\frac{\sqrt{3}}{2}i\sigma_2 & -\frac{\sqrt{3}}{2}i\sigma_1 - i\sigma_3 & \frac{1}{2}i\sigma_1 & \frac{1}{2}i\sigma_2 \\ -\frac{1}{2}i\sigma_2 & \frac{1}{2}i\sigma_1 & \frac{\sqrt{3}}{2}i\sigma_1 + i\sigma_3 & -\frac{\sqrt{3}}{2}i\sigma_2 \\ \frac{1}{2}i\sigma_1 & \frac{1}{2}i\sigma_2 & -\frac{\sqrt{3}}{2}i\sigma_2 & -\frac{\sqrt{3}}{2}i\sigma_1 + i\sigma_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix}, \tag{32}$$

which may be compared with the symmetric 4-instanton example described in [5].

In conclusion, we have seen that, at least for charge $N \leq 4$, there is an intimate connection between symmetric monopoles, symmetric calorons, symmetric instantons, and (via holonomy) Skyrmions. Many open questions remain, of which the following are a few.

- Several more symmetric monopoles (of higher charge) are known—do all of these arise as limiting cases of calorons with the same symmetry? More generally, is it true that any symmetric monopole has to be a special case of a symmetric caloron?
- Similarly, does every symmetric instanton [6] extend to a family of symmetric calorons? Note that such families are much more general, in that there may not be a symmetric monopole at the ‘other end’;
- What is the role of harmonic maps, which are known to be related to symmetric monopoles and Skyrmions [9]? Does this involve the interpretation of calorons as monopoles with a loop group as their gauge group [28,29]?

References

[1] M.F. Atiyah, N.S. Manton, Phys. Lett. B 222 (1989) 438.
 [2] E. Braaten, S. Townsend, L. Carson, Phys. Lett. B 235 (1990) 147.
 [3] R. Battye, P.M. Sutcliffe, Rev. Math. Phys. 14 (2002) 29.
 [4] M.F. Atiyah, N.S. Manton, Commun. Math. Phys. 152 (1993) 391.
 [5] R.A. Leese, N.S. Manton, Nucl. Phys. A 572 (1995) 575.
 [6] M.A. Singer, P.M. Sutcliffe, Nonlinearity 12 (1999) 987.
 [7] P.M. Sutcliffe, hep-th/0309157.
 [8] N.J. Hitchin, N.S. Manton, M.K. Murray, Nonlinearity 8 (1995) 661.
 [9] C.J. Houghton, N.S. Manton, P.M. Sutcliffe, Nucl. Phys. B 510 (1998) 507.
 [10] B.J. Harrington, H.K. Shepard, Phys. Rev. D 17 (1978) 2122.
 [11] D.J. Gross, R.D. Pisarski, L.G. Yaffe, Rev. Mod. Phys. 53 (1981) 43.

- [12] A. Actor, *Ann. Phys.* 148 (1983) 32.
- [13] A. Chakrabarti, *Phys. Rev. D* 35 (1987) 696.
- [14] W. Nahm, Self-Dual Monopoles and Calorons, in: *Springer Lecture Notes in Phys.*, vol. 201, 1984, pp. 189–200.
- [15] T.M.W. Nye, hep-th/0311215.
- [16] T.C. Kraan, P. van Baal, *Nucl. Phys. B* 533 (1998) 627.
- [17] K. Lee, C. Lu, *Phys. Rev. D* 58 (1998) 025011.
- [18] F. Bruckmann, P. van Baal, *Nucl. Phys. B* 645 (2002) 105.
- [19] E. Corrigan, D.B. Fairlie, *Phys. Lett. B* 67 (1977) 69.
- [20] R. Jackiw, C. Nohl, C. Rebbi, *Phys. Rev. D* 15 (1977) 1642.
- [21] P. Rossi, *Nucl. Phys. B* 149 (1979) 170.
- [22] K.J. Eskola, K. Kajantie, *Z. Phys. C* 44 (1989) 347.
- [23] M.A. Novak, I. Zahed, *Phys. Lett. B* 230 (1989) 108.
- [24] M.F. Atiyah, V.G. Drinfeld, N.J. Hitchin, Yu.I. Manin, *Phys. Lett. A* 65 (1978) 185.
- [25] H. Panagopoulos, *Phys. Rev. D* 28 (1989) 380.
- [26] A. Hosaka, S.M. Griffies, M. Oka, R.D. Amado, *Phys. Lett. B* 251 (1990) 1.
- [27] C.J. Houghton, P.M. Sutcliffe, *Commun. Math. Phys.* 180 (1996) 343.
- [28] H. Garland, M.K. Murray, *Commun. Math. Phys.* 120 (1988) 335.
- [29] P. Norbury, *Commun. Math. Phys.* 212 (2000) 557.