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CONSTRAINT SATISFACTION PROBLEMS ON INTERVALS AND LENGTHS*

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Abstract. We study interval-valued constraint satisfaction problems (CSPs), in which the aim is to find an assignment of intervals to a given set of variables subject to constraints on the relative positions of intervals. Many well-known problems such as INTERVAL GRAPH RECOGNITION and INTERVAL SATISFIABILITY can be considered as examples of such CSPs. One interesting question concerning such problems is to determine exactly how the complexity of an interval-valued CSP depends on the set of constraints allowed in instances. For the framework known as Allen's interval algebra this question was completely answered earlier by the authors, by giving a complete description of the tractable cases and showing that all remaining cases are NP-complete.

Here we extend the qualitative framework of Allen's algebra with additional constraints on the lengths of intervals. We allow these length constraints to be expressed as Horn disjunctive linear relations, a well-known tractable and sufficiently expressive form of constraints. The class of problems we consider contains, in particular, problems that are very closely related to the previously studied UNIT INTERVAL GRAPH SANDWICH problem. We completely characterize sets of qualitative relations for which the CSP augmented with arbitrary length constraints of the above form is tractable. We also show that, again, all the remaining cases are NP-complete.

Key words. Allen's interval algebra, interval satisfiability, computational complexity, tractable cases, dichotomy theorem

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1. Introduction and summary of results. A wide range of combinatorial search problems encountered in computer science and artificial intelligence can be naturally expressed as "constraint satisfaction problems" [29], in which the aim is to find an assignment of *values* to a given set of *variables* subject to specified *constraints*. For example, the standard propositional SATISFIABILITY problem [11] may be viewed as a constraint satisfaction problem (CSP) where the variables must be assigned Boolean values, and the constraints are specified by clauses. Further examples include GRAPH COLORABILITY, CLIQUE, and BANDWIDTH problems, scheduling problems, and many others (see [2, 19]).

Constraints are usually specified by means of relations. Hence the general CSP can be parameterized according to the relations allowed in an instance. For any set of relations $\mathcal{F}$, the class of CSP instances where the constraint relations are all members of $\mathcal{F}$ is denoted $CSP(\mathcal{F})$. The most well-known examples of such parameterized

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problems are Generalized Satisfiability [34], where the parameter is the set of allowed logical relations, and Graph H-coloring [17], where the parameter is the single graph H.

In studying CSPs over infinite sets of values, arguably the most important type of problem is when the constraints are specified by binary relations and the set of possible values for the variables is the set of intervals on the real line. Such problems arise, for example, in many forms of temporal reasoning [1, 16, 25, 30], where an event is identified with the interval during which it occurs. They also arise in computational biology, where various problems connected with physical mapping of DNA lead to interval-valued constraints [4, 12, 13, 21]. Interval-valued CSPs can be naturally augmented with constraints on the lengths of the intervals, and the complexity of such extended problems will be our main interest in this paper.

Before we describe our new results, we first discuss four closely related families of problems involving intervals which have previously been studied.

The prototypical problem from the first family is the Interval Graph Recognition problem [18]. An *interval graph* is an undirected graph such that there is an assignment of intervals to the nodes with two nodes adjacent if and only if the two corresponding intervals intersect. Given an arbitrary graph G, the question of deciding whether G is an interval graph is rarely viewed as a CSP, but in fact it is easily formulated as such a problem in the following way: every pair of adjacent nodes is constrained by the relation $r =$"intersect" over pairs of intervals, and every pair of nonadjacent nodes is constrained by the complementary relation $\bar{r} =$"disjoint." This fundamental Interval Graph Recognition problem is tractable, and it also remains tractable if we impose additional constraints on the lengths of the intervals which require all intervals to be of the same length (the Unit Interval Graph Recognition problem [5]). In contrast, it was shown in [32] that if we allow boundaries to be specified for the lengths of intervals, or even exact lengths (which are not necessarily all equal), then the corresponding problems (called Bounded Interval Graph Recognition and Measured Interval Graph Recognition, respectively) are NP-complete.

A number of other problems are closely related to the Interval Graph Recognition problem, including the Circle Graph Recognition problem and the Containment Graph Recognition problem [10, 15]. These problems can also be formulated as CSPs in a similar way by simply using a different constraint relation.

A typical problem from the second family is the Interval Graph Sandwich problem [13, 16]. Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, the question is whether there is an interval graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$. Clearly, this is a generalization of the corresponding recognition problem (the case when $E_1 = E_2$). The Interval Graph Sandwich problem can be represented as a CSP as follows: to any $e \in E_1$ assign the constraint $r =$"intersect," to any $e \notin E_2$ assign the constraint $\bar{r} =$"disjoint," and leave all pairs of variables corresponding to edges from $E_2 \setminus E_1$ unrelated. This problem was shown to be NP-complete along with the Unit Interval Graph Sandwich problem, where all intervals are required to be of the same length [13].

Graph Sandwich problems for a variety of other graph properties have also been considered [14]. For example, the Circle Graph Sandwich problem is obtained from the Interval Graph Sandwich problem by changing "interval graph G" to "circle graph G." This problem was shown to be NP-complete in [14]; it can be formulated as a CSP in the same way as above using the constraint relation

$r =$ "overlap."

The third family of problems we mention is the so-called INTERVAL SATISFIABILITY problems [16, 33, 35]. In these problems every pair of interval variables is again constrained in some way, but the constraints this time are chosen from a given set $\mathcal{F}$ of relations. In [16, 33, 35] only a small number of possibilities for $\mathcal{F}$ are considered. It is shown there that for some choices of $\mathcal{F}$ the resulting problem is tractable, while for others it is NP-complete. The complexity of INTERVAL SATISFIABILITY with all intervals of the same length is also studied in [33].

The fourth type of problem we mention is the satisfiability problem for *Allen's interval algebra* [1], denoted $\mathcal{A}$-SAT. Allen's algebra contains 13 basic relations (corresponding to the 13 ways two given intervals can be related from a "qualitative" point of view). The set $\mathcal{A}$ contains not just these basic relations but all $2^{13} = 8192$ possible unions of them. The problems $\mathcal{A}$-SAT($\mathcal{F}$) are similar to problems of the third type above, except that not every pair of pair of variables has to be constrained. They can also be represented as INTERVAL SATISFIABILITY with $\mathcal{F}$ being an arbitrary subset of $\mathcal{A}$ containing the total relation. The complexity of problems of the form $\mathcal{A}$-SAT($\mathcal{F}$) has been intensively studied in the artificial intelligence community (see, e.g., [7, 8, 30]), and a complete classification of the complexity of such problems was obtained in [25]. In that paper it is shown that there are exactly 18 maximal tractable fragments of $\mathcal{A}$, and for any subset $\mathcal{F}$ not entirely contained in one of those the problem $\mathcal{A}$-SAT($\mathcal{F}$) is NP-complete.

Many variants of $\mathcal{A}$-SAT($\mathcal{F}$) where additional constraints are allowed have been considered in the literature; cf. [3, 22, 28]. For instance, certain scheduling problems can conveniently be expressed as $\mathcal{A}$-SAT($\mathcal{F}$) with additional constraints on the lengths of the intervals. Moreover, in [2] it was suggested that many important forms of constraints on lengths can be expressed in the form of Horn disjunctive linear relations. This class of relations is known to be tractable [20] and at the same time allows us to express all elementary constraints, such as fixing the length, bounding the length of an interval by a given number, or comparing the lengths of two intervals. It was proved in [2] that only three out of the 18 maximal tractable fragments for $\mathcal{A}$-SAT($\mathcal{F}$) preserve tractability when extended with Horn disjunctive linear constraints on lengths; the other 15 become NP-complete. In this paper we study how we need to further restrict those 15 fragments to obtain tractable cases. The main result is a complete classification of complexity for $\mathcal{A}$-SAT($\mathcal{F}$) with additional constraints on lengths. We show that such problems are either tractable or strongly NP-complete. Moreover, we give a complete description of the tractable cases, which allows one to easily determine whether a given set $\mathcal{F}$ falls into one of the tractable cases.

As well as giving a complete classification, our result also establishes a new dichotomy theorem for complexity. Dichotomy theorems are results concerning a class of related problems (with some parameter) which assert that, for some values of the parameter, the problems in the class are tractable while for all other values they are NP-complete. Such theorems are of interest because it is well known [26] that if P$\neq$NP, then, within NP, there are infinitely many pairwise inequivalent problems of intermediate complexity. Dichotomy results rule out such a possibility within certain classes of problems.

Dichotomy theorems have previously been established for the GENERALIZED SATISFIABILITY [34] and GRAPH H-COLORING [17] problems mentioned above as well as the DIRECTED SUBGRAPH HOMEOMORPHISM problem [9].

CSPs have been a fruitful source of dichotomy results (see, e.g., [6, 23]). For

TABLE 1

The 13 basic relations of Allen's interval algebra. (The endpoint relations $x^- < x^+$ and $y^- < y^+$ that are valid for all relations have been omitted.)

Basic relation		Example	Endpoints
x precedes y	p	xxx	$x^+ < y^-$
y preceded by x	p^{-1}	yyy	
x meets y	m	xxxx	$x^+ = y^-$
y met by x	m^{-1}	yyyy	
x overlaps y	o	xxxx	$x^- < y^- < x^+$,
y overlapped by x	o^{-1}	yyyy	$x^+ < y^+$
x during y	d	xxx	$x^- > y^-$,
y includes x	d^{-1}	yyyyyyy	$x^+ < y^+$
x starts y	s	xxx	$x^- = y^-$,
y started by x	s^{-1}	yyyyyyy	$x^+ < y^+$
x finishes y	f	xxx	$x^+ = y^+$,
y finished by x	f^{-1}	yyyyyyy	$x^- > y^-$
x equals y	$\equiv$	xxxx	$x^- = y^-$,
		yyyy	$x^+ = y^+$

CSPs, the relevant parameter is usually the set of relations, $\mathcal{F}$, specifying the allowed constraints. This parameter usually runs over an infinite set of values. In the case of Allen's algebra, even though the number of different values for $\mathcal{F}$ is finite, it is astronomical ($2^{8192} \approx 10^{2466}$), which excludes the possibility of computer-aided exhaustive case analysis.

The usual tool for proving dichotomy theorems is reducibility via expressibility. This is done by showing that one set of relations expresses another so that one problem can be reduced to the other. This is the method used in [6, 25, 34], and a similar method is used here. After identifying certain tractable fragments, we find some NP-complete fragments and then show how any subset not entirely contained in one of the tractable sets can express some already known NP-complete fragment.

2. Preliminaries and background. Allen's interval algebra [1], denoted $\mathcal{A}$, is a formalism for expressing *qualitative* binary relations between intervals on the real line. By "qualitative" we mean "invariant under all continuous injective monotone transformations of the real line." An interval x is represented as a pair $[x^-, x^+]$ of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively. The qualitative relations between intervals are the $2^{13} = 8192$ possible unions of the 13 *basic interval relations*, which are shown in Table 1. It is easy to see that the basic relations are jointly exhaustive and pairwise disjoint in the sense that any two given intervals are related by exactly one basic relation. For the sake of brevity, relations between intervals will be written as collections of basic relations, omitting the sign of union. So, for instance, we write (pmf^{-1}) instead of $\mathsf{p} \cup \mathsf{m} \cup \mathsf{f}^{-1}$.

The problem of *satisfiability* ($\mathcal{A}$-SAT) in Allen's algebra is defined as follows.

DEFINITION 2.1. *Let $\mathcal{F} \subseteq \mathcal{A}$ be a set of interval relations. An instance I of $\mathcal{A}$-SAT$(\mathcal{F})$ over a set, V, of variables is a set of constraints of the form xry, where $x, y \in V$ and $r \in \mathcal{F}$. The question is whether I is* satisfiable, *i.e., whether there exists a function, f, from V to the set of all intervals such that $f(x)\, r\, f(y)$ holds for every constraint xry in I. Any such function f is called a* model *of I.*

Example 2.1. The instance $\{x(\mathsf{m})y, y(\mathsf{m})z, x(\mathsf{m})z\}$ is not satisfiable because the first two constraints imply that interval x must precede interval z, which contradicts the third constraint.

Example 2.2. The instance $I = \{x(\mathsf{mo})y, y(\mathsf{df}^{-1})z, z(\equiv \mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})x\}$ is

TABLE 2
Composition table for the basic relations in Allen's algebra.

$\circ$	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
$\equiv$	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
p	p	p	$\top$	p	ρ	p	ρ	ρ	p	p	p	ρ	p
p^{-1}	p^{-1}	$\top$	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	p^{-1}	p^{-1}
m	m	p	ρ^{-1}	p	θ	p	β	β	p	m	m	β	p
m^{-1}	m^{-1}	λ	p^{-1}	σ	p^{-1}	γ^{-1}	p^{-1}	γ^{-1}	p^{-1}	γ^{-1}	p^{-1}	m^{-1}	m^{-1}
o	o	p	ρ^{-1}	p	β^{-1}	α	ν	β	λ	o	γ	β	α
o^{-1}	o^{-1}	λ	p^{-1}	γ	p^{-1}	ν	α^{-1}	γ^{-1}	ρ^{-1}	γ^{-1}	α^{-1}	o^{-1}	β^{-1}
d	d	p	p^{-1}	p	p^{-1}	ρ	λ^{-1}	d	$\top$	d	λ^{-1}	d	ρ
d^{-1}	d^{-1}	λ	ρ^{-1}	γ	β^{-1}	γ	β^{-1}	ν	d^{-1}	γ	d^{-1}	β^{-1}	d^{-1}
s	s	p	p^{-1}	p	m^{-1}	α	γ^{-1}	d	λ	s	σ	d	α
s^{-1}	s^{-1}	λ	p^{-1}	γ	m^{-1}	γ	o^{-1}	γ^{-1}	d^{-1}	σ	s^{-1}	o^{-1}	d^{-1}
f	f	p	p^{-1}	m	p^{-1}	β	α^{-1}	d	ρ^{-1}	d	α^{-1}	f	θ
f^{-1}	f^{-1}	p	ρ^{-1}	m	β^{-1}	o	β^{-1}	β	d^{-1}	o	d^{-1}	θ	f^{-1}

$\alpha = (\mathsf{pmo})$ $\quad\beta = (\mathsf{ods})$ $\quad\gamma = (\mathsf{od}^{-1}\mathsf{f}^{-1})$ $\quad\sigma = (\equiv \mathsf{ss}^{-1})$ $\quad\theta = (\equiv \mathsf{ff}^{-1})$
$\rho = (\mathsf{pmods})$ $\quad\lambda = (\mathsf{pmod}^{-1}\mathsf{f}^{-1})$ $\quad\nu = (\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$
$\top = (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$

satisfiable. The function f given by $f(x) = [0, 2]$, $f(y) = [1, 3]$, and $f(z) = [0, 4]$ is a model of I.

An instance of $\mathcal{A}$-SAT($\mathcal{F}$) can also be represented, in an obvious way, as a labelled digraph, where the nodes are the variables from V, and the labelled arcs correspond to the constraints. This way of representing instances is sometimes more transparent.

Allen's interval algebra $\mathcal{A}$ consists of the 8192 possible relations between intervals together with three standard operations on binary relations: *converse* $\cdot^{-1}$*, intersection* $\cap$*, and composition* $\circ$. It is easy to see that the converse of $r = (b_1 \cdots b_n)$ is equal to $(b_1^{-1} \cdots b_n^{-1})$. Using the definition of composition, it can be shown that

$$(b_1 \cdots b_n) \circ (b'_1 \cdots b'_m) = \bigcup \{b_i \circ b'_j \mid 1 \le i \le n,\ 1 \le j \le m\}.$$

Hence the composition of two relations $r_1, r_2 \in \mathcal{A}$ is determined by the compositions of the basic relations they contain. The compositions of all possible pairs of basic relations are given in Table 2.

Subsets of $\mathcal{A}$ that are closed under the operations of converse, intersection, and composition are said to be *subalgebras*. For a given subset $\mathcal{F}$ of $\mathcal{A}$, the smallest subalgebra containing $\mathcal{F}$ is called the subalgebra *generated* by $\mathcal{F}$ and is denoted by $\langle \mathcal{F} \rangle$. It is easy to see that $\langle \mathcal{F} \rangle$ is obtained from $\mathcal{F}$ by adding all relations that can be obtained from the relations in $\mathcal{F}$ by using the three operations of the algebra $\mathcal{A}$.

It is known [30] and easy to prove that, for every $\mathcal{F} \subseteq \mathcal{A}$, the problem $\mathcal{A}$-SAT($\langle \mathcal{F} \rangle$) is polynomially equivalent to $\mathcal{A}$-SAT($\mathcal{F}$). Therefore, to classify the complexity of $\mathcal{A}$-SAT($\mathcal{F}$) it is sufficient to consider *subalgebras* of $\mathcal{A}$. Throughout the paper, $\mathcal{S}$ denotes a subalgebra of $\mathcal{A}$.

In the following we shall use the symbol $\pm$, which should be interpreted as follows. A condition involving $\pm$ means the conjunction of two conditions: one corresponding

TABLE 3
The 18 maximal tractable subalgebras of Allen's algebra.

$\mathcal{S}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$
$\mathcal{S}_{\mathsf{d}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d}^{-1})^{\pm 1} \subseteq r\}$
$\mathcal{S}_{\mathsf{o}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}$
$\mathcal{A}_1 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s}^{-1})^{\pm 1} \subseteq r\}$
$\mathcal{A}_2 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$
$\mathcal{A}_3 = \{r \mid r \cap (\mathsf{pmodf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$
$\mathcal{A}_4 = \{r \mid r \cap (\mathsf{pmodf}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$

$\mathcal{E}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$
$\mathcal{E}_{\mathsf{d}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r\}$
$\mathcal{E}_{\mathsf{o}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}$
$\mathcal{B}_1 = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$
$\mathcal{B}_2 = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f})^{\pm 1} \subseteq r\}$
$\mathcal{B}_3 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$
$\mathcal{B}_4 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$

$$\mathcal{E}^* = \left\{ r \,\middle|\, \begin{array}{l} 1)\ r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ff}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{S}^* = \left\{ r \,\middle|\, \begin{array}{l} 1)\ r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ss}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{H} = \left\{ r \,\middle|\, \begin{array}{l} 1)\ r \cap (\mathsf{os})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{o}^{-1}\mathsf{f})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{d}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pm})^{\pm 1} \neq \emptyset \ \&\ r \not\subseteq (\mathsf{pm})^{\pm 1} \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r \end{array} \right\}$$

$\mathcal{A}_{\equiv} = \{r \mid r \neq \emptyset \Rightarrow (\equiv) \subseteq r\}$

to $+$ and one corresponding to $-$. For example, the condition

$$r \cap (\mathsf{dsf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r$$

means that both of the following conditions hold:

$$r \cap (\mathsf{dsf}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r,$$
$$r \cap (\mathsf{d}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}) \neq \emptyset \Rightarrow (\mathsf{d}^{-1}) \subseteq r.$$

The main advantage of using the $\pm$ symbol is conciseness: in any subalgebra of $\mathcal{A}$, the "+" and the "−" conditions are satisfied (or not satisfied) simultaneously, and therefore only one of them needs to be verified.

A complete classification of the complexity of problems of the form $\mathcal{A}$-SAT$(\mathcal{F})$ was obtained in [25].

THEOREM 2.2 (see [25]). *For any subset $\mathcal{F}$ of $\mathcal{A}$, either $\mathcal{A}$-SAT$(\mathcal{F})$ is NP-complete or $\mathcal{F}$ is included in $\mathcal{S}$, where $\mathcal{S}$ is one of the* 18 *subalgebras listed in Table* 3*, for which $\mathcal{A}$-SAT$(\mathcal{S})$ is tractable.*

In this paper we present a complete complexity classification for a more general problem, namely, for $\mathcal{A}$-SAT$(\mathcal{F})$ extended with constraints on the lengths of intervals. Now we define the exact form of constraints on lengths we shall allow.

DEFINITION 2.3. *Let V be a set of real-valued variables and α, β linear polynomials over V with rational coefficients. A* linear relation *over V is an expression of the form $\alpha R \beta$, where $R \in \{<, \leq, =, \neq, \geq, >\}$.*

A disjunctive linear relation *(DLR) over V is a disjunction of a nonempty finite set of linear relations. A DLR is said to be* Horn *if and only if at most one of its disjuncts is not of the form $\alpha \neq \beta$.*

Example 2.3. The expression

$$x + 2y \leq 3z + 42.3$$

is a linear relation,

$$(x + 2y \leq 3z + 42.3) \vee (x + z < 4y - 8) \vee \left(x > \frac{3}{12}\right)$$

is a disjunctive linear relation, and

$$(x + 2y \leq 3z + 42.3) \vee (x + z \neq 4y - 8) \vee \left(x \neq \frac{3}{12}\right)$$

is a Horn disjunctive linear relation.

DEFINITION 2.4. *The problem of* satisfiability *for finite sets, D, of DLRs, denoted* DLRSAT*, is that of checking whether there exists an assignment f from variables in V to real numbers such that all DLRs in D are satisfied. Such an f is said to be a* model *of D. The satisfiability problem for finite sets of Horn DLRs is denoted* HORNDLRSAT.

THEOREM 2.5 (see [20, 24]). *The problem* DLRSAT *is NP-complete, but the problem* HORNDLRSAT *is solvable in polynomial time.*

We are interested in how the complexity of a problem depends on the value of the parameter $\mathcal{F}$ which, in our case, is a set of qualitative relations. Therefore we shall allow only those constraints on lengths which can be expressed by Horn DLRs and thus are tractable. This class of constraints subsumes all forms of constraints on lengths which have been considered in [32, 33].

We can now define the general interval satisfiability problem with constraints on lengths.

DEFINITION 2.6. *An instance of the problem of* interval satisfiability with constraints on lengths *for a set $\mathcal{F} \subseteq \mathcal{A}$, denoted $\mathcal{A}^l$-SAT$(\mathcal{F})$, is a pair $Q = (I, D)$, where*

(i) *I is an instance of $\mathcal{A}$-SAT$(\mathcal{F})$ over a set V of variables and*

(ii) *D is an instance of* HORNDLRSAT *over the set of variables $\{l(v) \mid v \in V\}$.*

The question is whether Q is satisfiable*, i.e., whether there exists a model f of I such that the DLRs in D are satisfied with $l(v)$ equal to the length of $f(v)$ for all $v \in V$.*

Example 2.4. Consider the instance $Q = (I, D)$, where $I = \{x(\mathsf{mo})y, y(\mathsf{df}^{-1})z, z(\equiv \mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})x\}$, as in Example 2.2, and $D = \{l(x) > l(y) + l(z)\}$. This instance is not satisfiable: any set of intervals satisfying the constraints in I must have $z^- \leq x^- < x^+ < y^+$ and $y \cap z$ nonempty and thus cannot satisfy the length constraint in D.

PROPOSITION 2.7. *$\mathcal{A}^l$-SAT$(\mathcal{F}) \in NP$ for every $\mathcal{F} \subseteq \mathcal{A}$.*

Proof. Every instance of $\mathcal{A}^l$-SAT*$(\mathcal{F})$* over a set of variables V can be translated in a straightforward way into an instance of DLRSAT over the set of variables $\{v^-, v^+ \mid v \in V\}$. Now the result follows from Theorem 2.5. $\square$

Example 2.5. The instance $Q = (I, D)$ defined in Example 2.4 corresponds to the instance D' of DLRSAT containing the following constraints:

$$
\begin{aligned}
&(x^- < x^+),\\
&(y^- < y^+),\\
&(z^- < z^+),\\
&\left.\begin{aligned}&(x^+ = y^-) \vee (x^- < y^-),\\ &(x^+ = y^-) \vee (y^- < x^+),\\ &(x^+ = y^-) \vee (x^+ < y^+),\end{aligned}\right\}\text{corresponding to } x(\mathsf{mo})y\\
&\left.\begin{aligned}&(y^- > z^-) \vee (y^+ = z^+),\\ &(y^- > z^-) \vee (y^- < z^-),\\ &(y^+ \le z^+),\\ &(y^+ < z^+) \vee (y^- < z^-),\end{aligned}\right\}\text{corresponding to } y(\mathsf{df}^{-1})z\\
&z^- \le x^-, \Big\}\text{corresponding to } z(\equiv \mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})x\\
&(x^+ - x^-) > (y^+ - y^-) + (z^+ - z^-).
\end{aligned}
$$

The complexity of $\mathcal{A}^l$-SAT*(*$\mathcal{S}$*)* has already been determined for each subalgebra $\mathcal{S}$ identified in Theorem 2.2.

PROPOSITION 2.8 (see [2]). *The problem* $\mathcal{A}^l$-SAT$(\mathcal{S})$ *is tractable for* $\mathcal{S} \in \{\mathcal{S}_\mathsf{p}, \mathcal{E}_\mathsf{p}, \mathcal{H}\}$ *and is NP-complete for the other* 15 *subalgebras listed in Table* 3.

In the next section, we determine the complexity of $\mathcal{A}^l$-SAT*(*$\mathcal{F}$*)* for every possible subset $\mathcal{F} \subseteq \mathcal{A}$.

3. Main result.

THEOREM 3.1. *For any subset* $\mathcal{F}$ *of* $\mathcal{A}$*, either* $\mathcal{A}^l$-SAT*(*$\mathcal{F}$*) is strongly NP-complete or* $\mathcal{F}$ *is included in* $\mathcal{S}$*, where* $\mathcal{S}$ *is one of the* 10 *subalgebras listed in Table* 4*, for which* $\mathcal{A}^l$-SAT*(*$\mathcal{S}$*) is tractable.*

In section 3.1, we discuss polynomial-time algorithms for the 10 subalgebras listed in Table 4, and in section 3.2 we give the NP-completeness results we need. (Strong NP-completeness of the NP-complete cases follows from the fact that the biggest number used in these NP-completeness proofs is 5.) Finally, in section 3.3, we give the classification proof.

The following notation is used throughout the proofs: if f is a model of an instance over a set V of variables and $v \in V$, then we denote the left and right endpoints of $f(v)$ by $f(v^-)$ and $f(v^+)$, respectively.

We shall say that a relation is *nontrivial* if it is not equal to the empty relation or the relation $(\equiv)$. Given a relation $r \in \mathcal{A}$, we write r^* to denote the relation $r \cap r^{-1}$. Evidently, every subalgebra of $\mathcal{A}$ is closed under the operation $\cdot^*$ (of taking the symmetric part of a relation).

Now we introduce the notion of *derivation with lengths* which will be used frequently in the proofs below. This notion is an extension of the notion of derivation in Allen's algebra used in [25].

Suppose $\mathcal{F} \subseteq \mathcal{A}$ and $Q = (I, D)$ is an instance of $\mathcal{A}^l$-SAT$(\mathcal{F})$. Let variables x, y be involved in I. Suppose a relation $r \in \mathcal{A}$ satisfies the following condition: Q is satisfiable if and only if xry. Then we say that r is *derived (with lengths)* from $\mathcal{F}$. It can easily be checked that the problems $\mathcal{A}^l$-SAT$(\mathcal{F})$ and $\mathcal{A}^l$-SAT$(\mathcal{F} \cup \{r\})$ are polynomially equivalent because, in any instance of the second problem, any constraint

TABLE 4
The 10 tractable cases of $\mathcal{A}^l$-SAT.

$$\mathcal{S}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$$

$$\mathcal{E}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$$

$$\mathcal{H} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{os})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{o}^{-1}\mathsf{f})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{d}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pm})^{\pm 1} \neq \emptyset \ \&\ r \not\subseteq (\mathsf{pm})^{\pm 1} \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r \end{array} \right\}$$

$$\mathcal{C}_{\mathsf{o}} = \{r \mid r \neq \emptyset \Rightarrow (\mathsf{oo}^{-1}) \subseteq r\}$$

$$\mathcal{C}_{\mathsf{m}} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \neq \emptyset \Rightarrow (\mathsf{mm}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{D}_{\mathsf{s}} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{dsf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset \Rightarrow (\equiv \mathsf{ss}^{-1}) \subseteq r \end{array} \right\}$$

$$\mathcal{D}_{\mathsf{f}} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{dsf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset \Rightarrow (\equiv \mathsf{ff}^{-1}) \subseteq r \end{array} \right\}$$

$$\mathcal{D}_{\mathsf{d}} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{dsf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset \Rightarrow (\equiv \mathsf{dd}^{-1}) \subseteq r \end{array} \right\}$$

$$\mathcal{D}'_{\mathsf{d}} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{dsf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{pmo})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{odd}^{-1})^{\pm 1} \subseteq r \end{array} \right\}$$

$$\mathcal{D}''_{\mathsf{d}} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{dsf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}) \neq \emptyset \Rightarrow (\mathsf{oo}^{-1}\mathsf{dd}^{-1}) \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}) \neq \emptyset \Rightarrow (\equiv \mathsf{dd}^{-1}) \subseteq r \end{array} \right\}$$

involving r can be replaced by the set of constraints in Q (introducing fresh variables as needed), and this can be done in polynomial time. It follows that it is sufficient to classify the complexity of problems $\mathcal{A}^l$-SAT$(\mathcal{S})$, where $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ closed under derivations with lengths.

Note that if we prove that the 10 sets shown in Table 4 are the only maximal sets $\mathcal{F}$ for which $\mathcal{A}^l$-SAT$(\mathcal{F})$ is tractable, then it will follow that they are all subalgebras closed under derivation with lengths; that is, we do not have to give a separate proof of this fact.

We will also use the following principle of *duality* to reduce the number of cases to be considered in the forthcoming proofs. We make use of a function reverse which is defined on the basic relations of $\mathcal{A}$ by the following table:

b	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
reverse(b)	$\equiv$	p^{-1}	p	m^{-1}	m	o^{-1}	o	d	d^{-1}	f	f^{-1}	s	s^{-1}

It is also defined for all other elements of $\mathcal{A}$ by setting $\mathsf{reverse}(r) = \bigcup_{b \subseteq r} \mathsf{reverse}(b)$.

Let $Q = (I, D)$ be any instance of $\mathcal{A}^l$-SAT with set of variables V, and let $Q' =$

(I', D) be the instance obtained from Q by replacing every r in I with $\mathsf{reverse}(r)$. It is easy to check that Q has a model f if and only if Q' has a model f' given by

$$f'(v) = [-f(v^+), -f(v^-)] \text{ for all } v \in V.$$

In other words, f' is obtained from f by redirecting the real line and leaving all intervals (as geometric objects) in their places. This observation leads to the following lemma.

LEMMA 3.2. *Let $\mathcal{F} = \{r_1, \ldots, r_n\} \subseteq \mathcal{A}$ and $\mathcal{F}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{A}$ be such that, for all $1 \leq k \leq n$, $r'_k = \mathsf{reverse}(r_k)$. Then $\mathcal{A}^l$-SAT$(\mathcal{F})$ is tractable (NP-complete) if and only if $\mathcal{A}^l$-SAT$(\mathcal{F}')$ is tractable (NP-complete).*

3.1. Tractability results.

PROPOSITION 3.3. *The problem $\mathcal{A}^l$-SAT$(\mathcal{S})$ is tractable whenever $\mathcal{S}$ is one of $\mathcal{S}_\mathsf{p}$, $\mathcal{E}_\mathsf{p}$, $\mathcal{H}$, $\mathcal{C}_\mathsf{o}$, $\mathcal{C}_\mathsf{m}$, $\mathcal{D}_\mathsf{s}$, $\mathcal{D}_\mathsf{f}$, $\mathcal{D}_\mathsf{d}$, $\mathcal{D}'_\mathsf{d}$, or $\mathcal{D}''_\mathsf{d}$.*

Polynomial-time algorithms solving $\mathcal{A}^l$-SAT$(\mathcal{S})$ for $\mathcal{S} \in \{\mathcal{S}_\mathsf{p}, \mathcal{E}_\mathsf{p}, \mathcal{H}\}$ are given in [2]. The remaining cases are dealt with below.

LEMMA 3.4. *Let $Q = (I, D)$ be an instance of $\mathcal{A}^l$-SAT$(\mathcal{C}_\mathsf{o})$. Then Q is satisfiable if and only if D is satisfiable.*

Proof. Let $V = \{x_1, \ldots, x_n\}$. If D is not satisfiable, then, obviously, the whole instance Q is not satisfiable. Suppose D is satisfiable, and $l(x_1) = a_1, \ldots, l(x_n) = a_n$ is a solution of D. Then reorder variables in V so that $a_1 \leq \cdots \leq a_n$. Let $\epsilon = a_1/n$, and let, for $1 \leq i \leq n$, $f(x_i) = [\epsilon \cdot i, \epsilon \cdot i + a_i]$. It is easy to check that this f satisfies all constraints in Q. □

It follows that the problem $\mathcal{A}^l$-SAT$(\mathcal{C}_\mathsf{o})$ has exactly the same complexity as the problem HORNDLRSAT, and hence is tractable (see Theorem 2.5).

Algorithms for the remaining 6 subalgebras are given in Figure 1, and in the remainder of this subsection we prove that these algorithms are correct. (Checking that they are polynomial-time is straightforward and is left to the reader.)

Algorithms A_i, $1 \leq i \leq 4$, and Procedure P take an instance $Q = (I, D)$ over a set of variables V as input. We shall assume that D always contains all constraints of the form $l(v) > 0$, $v \in V$. We will also assume that I does not contain a constraint vrw, where $r = \emptyset$. This trivial necessary condition for satisfiability can obviously be checked in polynomial time.

The following lemma from [7] is crucial in our proofs of correctness.

LEMMA 3.5 (see [7]). *Let D be a satisfiable set of Horn DLRs, and let $x_1, \ldots, x_n$ be the variables used in D. If $\tilde{D} = \{x_i \neq x_j \mid D \cup \{x_i \neq x_j\}$ is satisfiable$\}$, then $D \cup \tilde{D}$ is satisfiable.*

Using this lemma we can always divide the set of variables V into classes such that, in every model of an instance, variables from the same class must be assigned intervals of the same length while any variables from different classes can be assigned intervals of different lengths all at the same time.

LEMMA 3.6. *Algorithm A_1 correctly solves $\mathcal{A}^l$-SAT$(\mathcal{C}_\mathsf{m})$.*

Proof. Obviously, if A_1 rejects in line 1, then Q is not satisfiable.

Suppose A_1 rejects in line 3. Then G contains a simple cycle of odd length, $x_1, \ldots, x_{2t+1}, x_1$. Then, in any model f of Q, all of the intervals $f(x_1), \ldots, f(x_{2t+1})$ must have the same length, and hence, by definition of $\mathcal{C}_\mathsf{m}$, for all $1 \leq i \leq 2t$ we have $f(x_i)$ (mm^{-1}) $f(x_{i+1})$. These conditions imply that $f(x_1)$ $(\equiv \mathsf{pp}^{-1})$ $f(x_{2t+1})$. Therefore, it is impossible that $f(x_1)$ (mm^{-1}) $f(x_{2t+1})$, so Q is not satisfiable.

Suppose now that the algorithm accepts. We will show how to construct a model of Q. Note that in this case D is satisfiable. Let $V = \{x_1, \ldots, x_n\}$, and let

Input: Instance $Q = (I, D)$ of $\mathcal{A}^l$-SAT$(\mathcal{S})$ with set of variables V

Algorithm A_1 for $\mathcal{S} = \mathcal{C}_{\mathsf{m}}$

(1) If D is not satisfiable, then reject;

(2) Construct a graph $G = (V, E)$, where $(v, w) \in E$ if and only if
  – $D \cup \{l(v) \neq l(w)\}$ is not satisfiable, and
  – $vrw \in I$ for some r such that $(\equiv) \not\subseteq r$;

(3) If G is 2-colorable, then accept; else reject.

Procedure P

(1) Let $D' = D$;

(2) For each $vrw \in I$ such that $r \subseteq (\mathsf{dsf})$ or $r \subseteq (\mathsf{d}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$,
  add the constraint $l(v) < l(w)$ or $l(v) > l(w)$, respectively, to D';

(3) For each $vrw \in I$ such that $(\equiv) \subseteq r \subseteq (\equiv \mathsf{dsf})$ or $(\equiv) \subseteq r \subseteq (\equiv \mathsf{d}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$,
  add the constraint $l(v) \leq l(w)$ or $l(v) \geq l(w)$, respectively, to D';

(4) If D' is not satisfiable, then reject.

Algorithm A_2 for $\mathcal{S} \in \{\mathcal{D}_{\mathsf{s}}, \mathcal{D}_{\mathsf{f}}, \mathcal{D}_{\mathsf{d}}\}$

(1) Call procedure P;

(2) Accept.

Algorithm A_3 for $\mathcal{S} = \mathcal{D}'_{\mathsf{d}}$

(1) Call procedure P;

(2) Construct a graph $G = (V, E)$, where $(v, w) \in E$ if and only if
  $D' \cup \{l(v) \neq l(w)\}$ is not satisfiable;

(3) Identify the connected components $S_1, \ldots, S_k$ of G;

(4) For each S_j, let $I_j = I|_{S_j} = \{vrw \in I \mid v, w \in S_j\}$
  and $I'_j = \{v\ r \cap (\equiv \mathsf{oo}^{-1})\ w \mid vrw \in I_j\}$;

(5) Solve I'_j, $1 \leq j \leq k$, as instances of $\mathcal{A}$-SAT$(\mathcal{S}_{\mathsf{o}})$;

(6) If every I'_j is satisfiable, then accept; else reject.

Algorithm A_4 for $\mathcal{S} = \mathcal{D}''_{\mathsf{d}}$

(1) Call procedure P;

(2) Construct a graph $G = (V, E)$, where $(v, w) \in E$ if and only if
  – $D' \cup \{l(v) \neq l(w)\}$ is not satisfiable, and
  – $vrw \in I$ for some r such that $(\equiv) \subseteq r \cap (\equiv \mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}) \subseteq (\equiv \mathsf{mm}^{-1})$;

(3) Identify the connected components $S_1, \ldots, S_k$ of G;

(4) For each S_j, let $I_j = I|_{S_j} = \{vrw \in I \mid v, w \in S_j\}$
  and $I'_j = \{v\ r\ w \mid vrw \in I_j \text{ and } (\equiv) \not\subseteq r\}$;

(5) If every I'_j is empty, then accept; else reject.

FIG. 1. *Polynomial-time algorithms for the tractable cases of* $\mathcal{A}^l$-SAT.

$\tilde{D} = \{l(x_i) \neq l(x_j) \mid D \cup \{l(x_i) \neq l(x_j)\}$ is satisfiable$\}$. Then, by Lemma 3.5, $D \cup \tilde{D}$ is satisfiable. Let $l(x_1) = a_1, \ldots, l(x_n) = a_n$ be a solution of $D \cup \tilde{D}$. We know that G can be colored with two colors, say black and white. Now if x_i is black let $f(x_i) = [0, a_i]$; otherwise let $f(x_i) = [-a_i, 0]$. Obviously, this satisfies all constraints containing $(\equiv)$ because all constraints in I already allow $(\mathsf{mm}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Suppose that $x_i r x_j \in I$ for some r such that $(\equiv) \not\subseteq r$. If $(x_i, x_j) \in E$ then x_i and x_j are of different colors, and we have $f(x_i)\ (\mathsf{mm}^{-1})\ f(x_j)$. Otherwise we know, by Lemma 3.5, that the lengths of $f(x_i)$ and $f(x_j)$ are different, which means that $f(x_i)\ (\mathsf{mm}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\ f(x_j)$, as required. ∎

The next three algorithms use preprocessing Procedure P (see Figure 1). This procedure can obviously be performed in polynomial time. It is also easy to see that P does not change the set of solutions to an input.

LEMMA 3.7. *Algorithm A_2 correctly solves problems $\mathcal{A}^l$-SAT($\mathcal{S}$), where $\mathcal{S} \in \{\mathcal{D}_{\mathsf{s}}, \mathcal{D}_{\mathsf{f}}, \mathcal{D}_{\mathsf{d}}\}$.*

Proof. Obviously, if the algorithm rejects (in P), then the instance is not satisfiable.

Suppose now the algorithm accepts. Let $l(x_1) = a_1, \ldots, l(x_n) = a_n$ be a solution of D' and order the variables in V so that $a_1 \leq \cdots \leq a_n$.

If $\mathcal{F} = \mathcal{D}_{\mathsf{s}}$, then let $f(x_i) = [0, a_i]$ for all i. The constraints added to D during preprocessing P ensure that this f is a model of Q. Similarly, if $\mathcal{F} = \mathcal{D}_{\mathsf{f}}$, then the mapping given by $f(x_i) = [-a_i, 0]$ for all i satisfies all constraints in Q. Finally, if $\mathcal{F} = \mathcal{D}_{\mathsf{d}}$, then the mapping $f(x_i) = [-a_i/2, a_i/2]$ for all i satisfies all constraints in Q. □

LEMMA 3.8. *Algorithm A_3 correctly solves $\mathcal{A}^l$-SAT($\mathcal{D}'_{\mathsf{d}}$).*

Proof. Suppose first that A_3 accepts on an input Q. We construct a model of Q as follows. Let $v_{j,l}$, $1 \leq l \leq |S_j|$, be the members of S_j, $1 \leq j \leq k$. Let $\tilde{D} = \{l(v) \neq l(w) \mid D' \cup \{l(v) \neq l(w)\}$ is satisfiable$\}$. By Lemma 3.5, $D' \cup \tilde{D}$ is satisfiable.

Let $l(v_{j,l}) = a_{j,l}$, where $1 \leq j \leq k$ and $1 \leq l \leq |S_j|$, be a solution of $D' \cup \tilde{D}$. Note that, for every $1 \leq j \leq |S_j|$, we have $a_{j,1} = \cdots = a_{j,|S_j|}$. Reorder the S_j's so that $a_{1,1} < a_{2,1} < \cdots < a_{k,1}$ holds. Let

$$\epsilon = \begin{cases} \min\{\frac{a_{i+1,1} - a_{i,1}}{3} \mid 1 \leq i < k\} & \text{if } k > 1, \\ 1 & \text{if } k = 1. \end{cases}$$

For all $1 \leq j \leq k$, let f_j be a model of I'_j (and then of I_j as well) and assume without loss of generality that the variables in I_j are ordered so that $f_j(v^-_{j,1}) \leq f_j(v^-_{j,2}) \leq \cdots \leq f_j(v^-_{j,|S_j|})$. By applying an appropriate translation and scaling, all models f_j can be chosen so that $0 < f_j(v^-_{j,1}) \leq \cdots \leq f_j(v^-_{j,|S_j|}) < \epsilon$.

Now we combine the models f_j of I_j into one model f of $Q = (I, D)$: let $f(v^-_{j,l}) = -j \cdot \epsilon + f_j(v^-_{j,l})$ and $f(v^+_{j,l}) = f(v^-_{j,l}) + a_{j,l}$ (see Figure 2).

We immediately see that f satisfies all length constraints and all constraints within each I_j. It is also easy to check that we have $f(v_{i,l})$ (d) $f(v_{i',l'})$ whenever $i < i'$. Due to the check in Procedure P, this satisfies all constraints between variables from different I_j's.

Assume now that algorithm A_3 rejects. We will show that Q is not satisfiable. The result holds trivially if A_3 rejects on line 1 (that is, in P). Assume to the contrary that some I'_j is not satisfiable but Q is satisfiable. Clearly, if Q is satisfiable, then the instance I_j has a model f with all intervals of the same length a. Then f is also a model of $I''_j = \{v\ r \cap (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})\ w \mid vrw \in I_j\}$.

Reorder the variables in I_j so that $f(v^-_{j,1}) \leq f(v^-_{j,2}) \leq \cdots \leq f(v^-_{j,|S_j|})$, and suppose that $\{f(v^-_{j,l}) \mid 1 \leq l \leq |S_j|\} = \{b_1, \ldots, b_t\}$, where $1 \leq t \leq |S_j|$ and $b_1 < \cdots < b_t$.

By definition of $\mathcal{D}'_{\mathsf{d}}$, every constraint allowing (pm) allows (o) as well. Therefore the function g defined by

$$g(v_{j,l}) = [a \cdot s/|S_j|, a \cdot s/|S_j| + a] \qquad \text{when } f(v^-_{j,l}) = b_s$$

is a model of I_j. Moreover, it is also a model of I'_j, a contradiction. □

LEMMA 3.9. *Algorithm A_4 correctly solves $\mathcal{A}^l$-SAT($\mathcal{D}''_{\mathsf{d}}$).*

Proof. If A_4 rejects in line 1 (that is, in P), then Q is obviously not satisfiable.

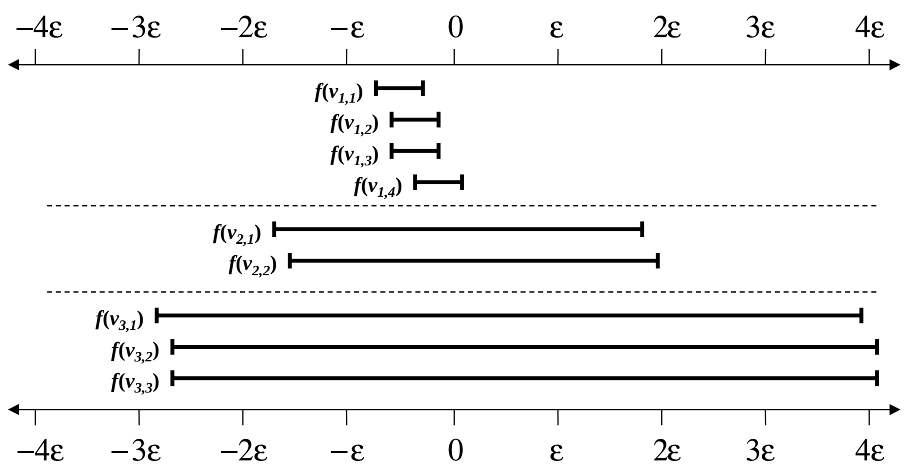

FIG. 2. *A combined model for an instance of* $\mathcal{A}^l$-SAT$(\mathcal{D}'_{\mathsf{d}})$ *(Lemma* 3.8*).*

Suppose A_4 rejects in line 6. It follows that there are variables $x_1, \dots, x_q \in V$ such that, in any model f of Q,

(i) intervals $f(x_1), \dots, f(x_q)$ have the same length, and

(ii) $f(x_i)$ $(\equiv \mathsf{mm}^{-1})$ $f(x_{i+1})$ for all $1 \le i \le q-1$, and

(iii) (by definition I'_j and $\mathcal{D}''_{\mathsf{d}}$) the intervals $f(x_1)$ and $f(x_q)$ are related by $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$.

It is clear that these three conditions cannot be satisfied simultaneously. Therefore Q is not satisfiable.

Suppose that the algorithm accepts. We will show how to construct a model of Q. Let $v_{j,l}$, $1 \le l \le |S_j|$, be the members of S_j, $1 \le j \le k$. Let $\tilde{D} = \{l(v) \ne l(w) \mid D' \cup \{l(v) \ne l(w)\}$ is satisfiable$\}$. By Lemma 3.5, $D' \cup \tilde{D}$ is satisfiable.

Let $l(v_{j,l}) = a_{j,l}$, where $1 \le j \le k$ and $1 \le l \le |S_j|$, be a solution of $\tilde{D}$. Note that, for every $1 \le j \le |S_j|$, we have $a_{j,1} = \cdots = a_{j,|S_j|}$. Reorder the S_j's so that $a_{1,1} \le a_{2,1} \le \cdots \le a_{k,1}$ holds (note that some of the $a_{j,1}$'s may coincide). Let $\{a_{1,1}, \dots, a_{k,1}\} = \{b_1, \dots, b_t\}$, where $b_1 < \cdots < b_t$, and let

$$\epsilon = \begin{cases} \min\{b_1, \frac{b_{i+1}-b_i}{3} \mid 1 \le i < t\} & \text{if } t > 1, \\ 1 & \text{if } t = 1. \end{cases}$$

Further, let $f(v^-_{j,l}) = -s \cdot \epsilon + \frac{j}{|V|} \cdot \epsilon$, where s is such that $b_s = a_{j,l}$, and let $f(v^+_{j,l}) = f(v^-_{j,l}) + a_{j,l}$ (see Figure 3). We will show that f is a model of Q. By the choice of $a_{j,l}$, f satisfies all length constraints.

Suppose $v_{j,l}\ r\ v_{j',l'} \in I$ and check that $f(v_{j,l})\ r\ f(v_{j',l'})$.

Case 1. $j = j'$.

If the variables are from the same connected component of G, then we have that $(\equiv) \subseteq r$. Indeed, we have $f(v_{j,l})$ $(\equiv)$ $f(v_{j',l'})$ by the definition of f.

Case 2. $j \ne j'$, but $a_{j,l} = a_{j',l'}$.

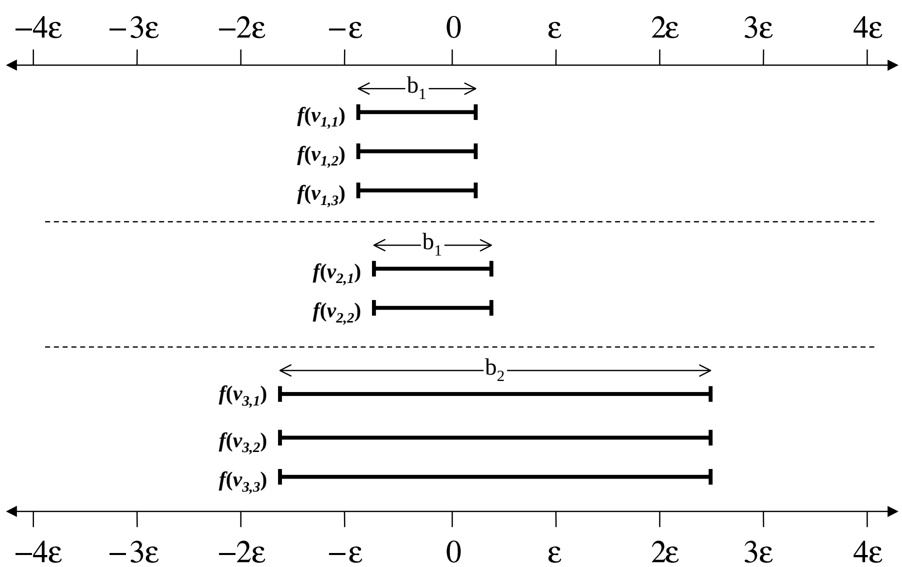

FIG. 3. *A combined model for an instance of* $\mathcal{A}^l$-SAT$(\mathcal{D}''_{\mathsf{d}})$ *(Lemma* 3.9*).*

By definition of G, we have either $r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$ or $(\equiv) \not\subseteq r$. In the former case we immediately get $(\mathsf{oo}^{-1}) \subseteq r$ by the definition of $\mathcal{D}''_{\mathsf{d}}$. Suppose that $(\equiv) \not\subseteq r$. Then $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}) = \emptyset$. Due to the check in P, the equality $a_{j,l} = a_{j',l'}$ is necessary. It follows from this fact and from the definition of $\mathcal{D}''_{\mathsf{d}}$ that we have $(\mathsf{oo}^{-1}) \subseteq r$ again. Indeed, it is easy to check that $f(v_{j,l})$ (o) $f(v_{j',l'})$ if $j < j'$, by the definition of f.

Case 3. $a_{j,l} \neq a_{j',l'}$.

Assume without loss of generality that $a_{j,l} < a_{j',l'}$. It follows from the definition of $\mathcal{D}''_{\mathsf{d}}$ that either we have $(\mathsf{dd}^{-1}) \subseteq r$ or (due to the check in P) $(\mathsf{d}) \subseteq r \subseteq (\mathsf{dsf})$. It is not hard to verify that, indeed, $f(v_{j,l})$ (d) $f(v_{j',l'})$, by the definition of f. □

3.2. NP-completeness results. First let us mention the obvious fact that, for any $\mathcal{F} \subseteq \mathcal{A}$, NP-completeness of $\mathcal{A}$-SAT$(\mathcal{F})$ implies NP-completeness of $\mathcal{A}^l$-SAT$(\mathcal{F})$.

LEMMA 3.10. *Suppose that* $r_1, \ldots, r_n \in \mathcal{A}$ *are relations such that the problem* $\mathcal{A}$-SAT$(\{r_1, \ldots, r_n\})$ *is NP-complete.*

1. *If, for every* $1 \leq i \leq n$, $r'_i \in \{r_i, r_i \cup (\equiv)\}$, *then* $\mathcal{A}^l$-SAT$(\{r'_1, \ldots, r'_n\})$ *is NP-complete.*
2. *If* $\emptyset \neq r_1 \subseteq (\mathsf{pmo})$ *and* r'_1 *satisfies* $r_1 \subseteq r'_1 \subseteq r_1 \cup (\equiv \mathsf{dsf})$, *then the problem* $\mathcal{A}^l$-SAT$(\{r'_1, r_2, \ldots, r_n\})$ *is NP-complete.*

Proof.

1. The proof is by polynomial-time reduction from $\mathcal{A}$-SAT($\{r_1, \dots, r_n\}$) to $\mathcal{A}^l$-SAT($\{r'_1, \dots, r'_n\}$). Let I be an instance of $\mathcal{A}$-SAT($\{r_1, \dots, r_n\}$) over a set V of variables. Construct an instance (I', D') of $\mathcal{A}^l$-SAT($\{r'_1, \dots, r'_n\}$) as follows:

(i) for every constraint urv in I such that $(\equiv) \subseteq r$ add urv to I';

(ii) for every constraint urv in I such that $(\equiv) \not\subseteq r$ add urv to I' and $l(u) \neq l(v)$ to D'.

Obviously, every solution to (I', D') is also a solution to I. Let f be a model of I, and let $\{x_1, \dots, x_m\}$ be the set of all endpoints of intervals $f(x)$, $x \in V$. We may without loss of generality assume that $0 < x_1 < \cdots < x_m$. Set $x'_1 = x_1$, $x'_2 = x_2$, and, for every $i > 2$, set $x'_i = 2x'_{i-1} + 1$. It is easy to check that the function f' such that $f'(v) = [x'_i, x'_j]$ if $f(v) = [x_i, x_j]$ is a model of (I', D').

2. Modify the previous construction as follows:

(i) for every constraint ur_1v in I add constraints ur'_1v to I' and $l(u) > l(v)$ to D';

(ii) for every constraint ur_iv, $i > 1$, in I add ur_iv to I'.

Every solution to (I', D') is also a solution to I because ur'_1v and $l(u) > l(v)$ imply ur_1v. Let f be a model of I, and let $\{x_1, \dots, x_m\}$ be the set of all endpoints of intervals $f(x)$ for some $x \in V$. We may without loss of generality assume that $x_1 < \cdots < x_m < 0$. Set $x'_m = x_m$, $x'_{m-1} = x_{m-1}$, and, for every $1 \leq i < m - 1$, set $x'_i = 2x'_{i+1} - 1$. It is easy to check that the function f' such that $f'(v) = [x'_i, x'_j]$ if $f(v) = [x_i, x_j]$ is a model of (I', D'). ◻

Example 3.1. It follows from Theorem 2.2 that $\mathcal{A}$-SAT($\{(\mathsf{mm}^{-1})\}$) is NP-complete. Using Lemma 3.10(1) we conclude that $\mathcal{A}^l$-SAT($\{(\equiv \mathsf{mm}^{-1})\}$) is also NP-complete.

LEMMA 3.11. *$\mathcal{A}^l$-SAT($\mathcal{F}$) is NP-complete if $\mathcal{F}$ is* $\{(\mathsf{oo}^{-1}), (\mathsf{s})\}$, $\{(\mathsf{oss}^{-1}\mathsf{ff}^{-1})\}$, *or* $\{(\mathsf{sf}), (\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\}$.

Proof. First let $\mathcal{F} = \{(\mathsf{oo}^{-1}), (\mathsf{s})\}$. The constraints

$$\{x(\mathsf{oo}^{-1})y,\ x(\mathsf{oo}^{-1})z,\ y(\mathsf{s})z;\ l(z) > l(x) + l(y)\}$$

are satisfiable if and only if $x(\mathsf{o})y$. Further, the constraints $\{x(\mathsf{o})z, z(\mathsf{o})y; l(z) > l(x) + l(y)\}$ are satisfiable if and only if $x(\mathsf{p})y$. It follows from Theorem 2.2 that $\mathcal{A}$-SAT($\{(\mathsf{oo}^{-1}), (\mathsf{p})\}$) is NP-complete. The above constructions show how to reduce $\mathcal{A}$-SAT($\{(\mathsf{oo}^{-1}), (\mathsf{p})\}$) to $\mathcal{A}^l$-SAT($\{(\mathsf{oo}^{-1}), (\mathsf{s})\}$) in polynomial time.

Now let $\mathcal{F} = \{(\mathsf{oss}^{-1}\mathsf{ff}^{-1})\}$. Note that in this case we can also make use of the relation $(\mathsf{ss}^{-1}\mathsf{ff}^{-1})$, which is equal to $(\mathsf{oss}^{-1}\mathsf{ff}^{-1})^*$.

We give a polynomial-time reduction from the NP-complete problem UNNEGATED ONE-IN-THREE 3SAT (Problem [LO4] in [11]) to $\mathcal{A}^l$-SAT($\{(\mathsf{oss}^{-1}\mathsf{ff}^{-1})\}$); let (X, C) be an arbitrary instance of UNNEGATED ONE-IN-THREE 3SAT. Consider the following set of constraints over the variables a, b, c, c':

$$\begin{array}{lll} a(\mathsf{oss}^{-1}\mathsf{ff}^{-1})b, & & l(a) = l(b) = 2, \\ c(\mathsf{ss}^{-1}\mathsf{ff}^{-1})a, & c(\mathsf{ss}^{-1}\mathsf{ff}^{-1})b, & l(c) = 1, \\ c'(\mathsf{ss}^{-1}\mathsf{ff}^{-1})a, & c'(\mathsf{ss}^{-1}\mathsf{ff}^{-1})b, & l(c') = 3. \end{array}$$

We impose the constraints $x(\mathsf{ss}^{-1}\mathsf{ff}^{-1})a$, $x(\mathsf{ss}^{-1}\mathsf{ff}^{-1})b$ on every $x \in X$ and note that this implies $l(x) \in \{1, 3\}$. To complete the reduction, we add the constraint $l(x) + l(y) + l(z) = 5$ for each $\{x, y, z\} \in C$. It is easy to show that the resulting set of constraints is satisfiable if and only if (X, C) has a solution.

Finally, let $\mathcal{F} = \{(\mathsf{sf}), (\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\}$. The constraints $\{x(\mathsf{sf})z, y(\mathsf{sf})z; l(z) > l(x)+l(y)\}$ are satisfiable if and only if $x(\equiv \mathsf{pp}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})y$. Hence, we can obtain the relation $(\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \cap (\equiv \mathsf{pp}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. To show NP-completeness, use the same construction as above but replace $(\mathsf{oss}^{-1}\mathsf{ff}^{-1})$ with $(\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. ∎

LEMMA 3.12. *$\mathcal{A}^l$-SAT$(\{r\})$ is NP-complete whenever*

$$(\mathsf{mm}^{-1}) \subseteq r \subseteq (\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) \text{ or } (\mathsf{mm}^{-1}) \subseteq r \subseteq (\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1}).$$

Proof. We consider only r with $(\mathsf{mm}^{-1}) \subseteq r \subseteq (\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1})$; the other case is dual. We may without loss of generality assume that $r = r^*$.

Case 1. $r = (\mathsf{mm}^{-1})$.

It follows from Theorem 2.2 that $\mathcal{A}$-SAT$(\{(\mathsf{mm}^{-1})\})$ is NP-complete.

Case 2. $r = (\mathsf{mm}^{-1}\mathsf{dd}^{-1})$.

The constraints

$$\begin{aligned} &x(\mathsf{dd}^{-1}\mathsf{mm}^{-1})y, \quad l(x) < l(y),\\ &a(\mathsf{dd}^{-1}\mathsf{mm}^{-1})x, \quad l(a) < l(x),\\ &a(\mathsf{dd}^{-1}\mathsf{mm}^{-1})y, \quad l(a) < l(y), \end{aligned}$$

are satisfiable if and only if $x(\mathsf{d})y$. Furthermore, the constraints

$$u(\mathsf{mm}^{-1}\mathsf{dd}^{-1})v, \quad x(\mathsf{d})u, \quad y(\mathsf{d})v, \quad l(u) = l(v),$$

are satisfiable if and only if $x(\mathsf{pp}^{-1})y$. It follows from Theorem 2.2 that the problem $\mathcal{A}$-SAT$(\{(\mathsf{d}), (\mathsf{pp}^{-1})\})$ is NP-complete. We have derived (d) and (pp^{-1}) from $(\mathsf{mm}^{-1}\mathsf{dd}^{-1})$, and hence $\mathcal{A}^l$-SAT$(\{(\mathsf{mm}^{-1}\mathsf{dd}^{-1})\})$ is also NP-complete.

Case 3. $r = (\mathsf{mm}^{-1}\mathsf{ss}^{-1})$.

The constraints

$$\begin{aligned} &a(\mathsf{mm}^{-1}\mathsf{ss}^{-1})x, \quad a(\mathsf{mm}^{-1}\mathsf{ss}^{-1})y, \quad l(x) > l(a),\\ &b(\mathsf{mm}^{-1}\mathsf{ss}^{-1})x, \quad b(\mathsf{mm}^{-1}\mathsf{ss}^{-1})y, \quad l(y) > l(b),\\ &x(\mathsf{mm}^{-1}\mathsf{ss}^{-1})y, \quad l(x) = l(y), \end{aligned}$$

are satisfiable if and only if $a(\equiv \mathsf{ss}^{-1})b$, so we can derive the relation $(\equiv \mathsf{ss}^{-1})$. Furthermore, the constraints

$$\begin{aligned} &a(\equiv \mathsf{ss}^{-1})x, \quad b(\equiv \mathsf{ss}^{-1})x, \quad l(x) > l(a),\\ &a(\equiv \mathsf{ss}^{-1})y, \quad b(\equiv \mathsf{ss}^{-1})y, \quad l(y) > l(b),\\ &x(\mathsf{mm}^{-1}\mathsf{ss}^{-1})y, \quad l(x) = l(y), \end{aligned}$$

are satisfiable if and only if $a(\mathsf{pp}^{-1})b$. Now NP-completeness follows from Theorem 2.2.

Case 4. $r = (\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1})$.

Replace $(\mathsf{mm}^{-1}\mathsf{ss}^{-1})$ with $(\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1})$ in the previous case. ∎

LEMMA 3.13. *$\mathcal{A}^l$-SAT$(\{r_1, r_2\})$ is NP-complete whenever $(\equiv) \not\subseteq r_2$ and*

$$r_1 \cap (\equiv \mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}) = (\mathsf{mm}^{-1}) \subsetneq r_2 \cap (\equiv \mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}).$$

Proof. Let us assume that all intervals have length one and prove that the problem $\mathcal{A}^l$-SAT$(\{r_1, r_2\})$ is NP-complete even under this assumption. This assumption reduces the number of cases to be considered because, in this case, we have $r_1 = (\mathsf{mm}^{-1})$ and $(\mathsf{mm}^{-1}) \subset r_2 \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$. Moreover, we may without loss of generality assume that either $r_2^* = (\mathsf{mm}^{-1})$ or $r_2^* = r_2$.

Case 1. $\{(\mathsf{mm}^{-1}), (\mathsf{pmm}^{-1})\}$.

Let $G = (V, E)$ and $H = (V', E')$ denote two directed graphs. A homomorphism from G to H is a function $h : V \to V'$ such that $(v, w) \in E$ implies $(f(v), f(w)) \in E'$.

Let H be the graph $(V', E') = (\{0, 1, 2\}, \{(0, 1)(0, 2), (1, 0), (1, 2), (2, 1)\})$. Deciding whether there exists a homomorphism from an arbitrary graph to H is NP-complete, as follows from Theorem 4.4 in [27]. We denote this problem Graph Homomorphism(H).

We prove that $\{(\mathsf{mm}^{-1}), (\mathsf{pmm}^{-1})\}$ is NP-complete by a polynomial-time reduction from Graph Homomorphism(H). Arbitrarily choose a directed graph $G = (V, E)$.

The relations (pm) and (m) can be derived as follows. The constraints

$$\{x(\mathsf{mm}^{-1})x', y(\mathsf{pmm}^{-1})x, y(\mathsf{pmm}^{-1})x'\}$$

are satisfiable if and only if $y(\mathsf{pm})x$, and we have $(\mathsf{m}) = (\mathsf{pm}) \cap (\mathsf{mm}^{-1})$.

Introduce five fresh variables and the constraints $a(\mathsf{m})b(\mathsf{m})c(\mathsf{m})d(\mathsf{m})e$. For each node $v \in V$, add the constraints $a(\mathsf{pm})v(\mathsf{pm})e$. For each edge $(v, w) \in E$, add the constraint $v(\mathsf{pmm}^{-1})w$.

We show that the resulting set I of constraints are satisfiable if and only if there exists a homomorphism from G to H.

Only-if: Assume without loss of generality that f is a model of I such that $f(a) = [-1, 0]$. Construct a function $h : V \to V'$ as follows: $h(v) = \lfloor f(v^-) \rfloor$. To see that h is a homomorphism from G to H, arbitrarily choose an edge $(v, w) \in E$. We consider three cases:

(i) $h(v) = 0$. This implies that $0 \leq f(v^-) < 1$. Since $v(\mathsf{pmm}^{-1})w \in I$ and $f(w^+) \leq 3$, we know that $1 \leq f(w^-) \leq 2$ and $h(w) \in \{1, 2\}$. Hence, $(h(v), h(w)) \in E'$.

(ii) $h(v) = 1$. Either $0 \leq f(w^-) < 1$ (corresponding to $v(\mathsf{m}^{-1})w$) or $f(w^-) = 2$ (corresponding to $v(\mathsf{m})w$), so $h(w) \in \{0, 2\}$ and $(h(v), h(w)) \in E'$.

(iii) $h(v) = 2$. Then $f(w^-) = 1$ (corresponding to $v(\mathsf{m}^{-1})w$), $h(w) = 1$, and $(h(v), h(w)) \in E'$.

If: Assume $h : V \to V'$ is a homomorphism from G to H. Then f (as defined below) is a model of I:

$$f(a) = [-1, 0],\ f(b) = [0, 1],\ f(c) = [1, 2],\ f(d) = [2, 3],\ f(e) = [3, 4],$$

and for every $v \in V$ let $f(v) = [h(v), h(v) + 1]$.

Case 2. $\{(\mathsf{mm}^{-1}), (\mathsf{pp}^{-1}\mathsf{mm}^{-1})\}$.

The proof is by polynomial-time reduction from the NP-complete problem Graph 3-colorability (Problem [GT4] in [11]). Let $G = (V, E)$ be an arbitrary instance. Fix a fresh interval variable x. Introduce two interval variables v, v' for each $v \in V$ together with the constraints $v(\mathsf{mm}^{-1})v'(\mathsf{mm}^{-1})x$. Finally, add the constraint $v(\mathsf{pp}^{-1}\mathsf{mm}^{-1})w$ for every $(v, w) \in E$. It is easy to check that the resulting set of constraints is satisfiable if and only if G is 3-colorable. For example, if $f(x) = [3, 4]$, then constraints of the first type imply that $f(v) \in \{[1, 2], [3, 4], [5, 6]\}$ for any $v \in V$, while the constraints of the second type ensure that the values for "adjacent" variables are distinct.

Case 3. $\{(\mathsf{mm}^{-1}), (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})\}$.

Use $(\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$ instead of $(\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ in Case 2.

Case 4. $\{(\mathsf{mm}^{-1}), (\mathsf{mm}^{-1}\mathsf{oo}^{-1})\}$.

The proof is by polynomial-time reduction from the NP-complete problem BETWEENNESS[1] (Problem [MS1] in [11]), which is defined as follows:

INSTANCE: A finite set A, a collection T of ordered triples (a,b,c) of distinct elements from A.

QUESTION: Is there a total ordering $<$ on A such that for each $(a,b,c) \in T$ we have either $a<b<c$ or $c<b<a$?

Let (A,T) be an arbitrary instance of BETWEENNESS and note that the constraints $\{x(\mathsf{mm}^{-1})x',\ y(\mathsf{mm}^{-1}\mathsf{oo}^{-1})x, y(\mathsf{mm}^{-1}\mathsf{oo}^{-1})x'\}$ are satisfiable if and only if $x(\mathsf{oo}^{-1})y$. We construct an instance I over $\{(\mathsf{mm}^{-1}),(\mathsf{oo}^{-1})\}$ as follows:

(i) for each pair of distinct elements $a,b \in A$, add the constraint $a(\mathsf{oo}^{-1})b$ to I;

(ii) for each triple $(a,b,c) \in T$, introduce two fresh variables x,y and add the constraints $\{x(\mathsf{mm}^{-1})a,\ x(\mathsf{oo}^{-1})b,\ x(\mathsf{oo}^{-1})c,\ y(\mathsf{oo}^{-1})a,\ y(\mathsf{oo}^{-1})b,\ y(\mathsf{mm}^{-1})c\}$.

We will henceforth refer to the variables in I that correspond to the set A as "basic" variables and the other variables as "auxiliary" variables.

Assume that I has a model f. Then, due to the constraints added in step (i), the intervals $f(a), a \in A$, are pairwise distinct. Moreover, the relation (o) induces a total order on the set $\{f(a) \mid a \in A\}$. Suppose that there is a triple $(a,b,c) \in T$ such that the model f satisfies $f(b)\,(\mathsf{o})\,f(a)\,(\mathsf{o})\,f(c)$ and consider the constraints over the auxiliary variables x and y introduced in step (ii) for the triple (a,b,c). The variable x has to satisfy $x(\mathsf{mm}^{-1})a$, which implies that either $x(\mathsf{p})c$ or $x(\mathsf{p}^{-1})b$, a contradiction. We can analogously rule out all orderings of $f(a), f(b), f(c)$ except $f(a)\,(\mathsf{o})\,f(b)\,(\mathsf{o})\,f(c)$ and $f(c)\,(\mathsf{o})\,f(b)\,(\mathsf{o})\,f(a)$. Hence there is a solution to the instance (A,T): for all $a,b \in A$, set $a<b$ if and only if $f(a)\,(\mathsf{o})\,f(b)$.

Conversely, assume that there exists a total order $<$ on A that is a solution to the instance (A,T). We will show how to construct a model f of I. For all $a,b \in A$, set $f(a)\,(\mathsf{o})\,f(b)$ if and only if $a<b$. Clearly, this satisfies all constraints added in step (i). To show that there exists consistent values for all auxiliary variables, arbitrarily pick one triple $(a,b,c) \in T$ (corresponding to the auxiliary variables x and y) and assume without loss of generality that $a<b<c$. Let $a(\mathsf{m})x$, i.e., $f(x) = [f(a^-), f(a^-)+1]$ and $y(\mathsf{m})c$; i.e., $f(y) = [f(c^-)-1, f(c^-)]$. It is straightforward to verify that this construction satisfies all constraints.

Case 5. $\{(\mathsf{mm}^{-1}),(\mathsf{mm}^{-1}\mathsf{o})\}$.

The constraints $x(\mathsf{mm}^{-1})x',\ y(\mathsf{omm}^{-1})x,\ y(\mathsf{o}^{-1}\mathsf{mm}^{-1})x'$ are satisfiable if and only if $x(\mathsf{o}^{-1})y$. The constraints $x(\mathsf{o})x',\ y(\mathsf{o})x'$ are satisfiable if and only if $x(\equiv \mathsf{oo}^{-1})y$. The constraints $x(\equiv \mathsf{oo}^{-1})y,\ x'(\equiv \mathsf{oo}^{-1})x,\ x'(\mathsf{mm}^{-1})y$ are satisfiable if and only if $x(\mathsf{oo}^{-1})y$. Consequently, we can derive (mm^{-1}) and (oo^{-1}); continue as in Case 4.

Case 6. $\{(\mathsf{mm}^{-1}),(\mathsf{pmm}^{-1}\mathsf{o})\}$.

The constraints $x(\mathsf{mm}^{-1})x',\ y(\mathsf{pmm}^{-1}\mathsf{o})x,\ y(\mathsf{p}^{-1}\mathsf{mm}^{-1}\mathsf{o}^{-1})x'$ are satisfiable if and only if $x(\mathsf{o}^{-1})y$. Continue as in Case 5.

Case 7. $\{(\mathsf{mm}^{-1}),(\mathsf{pmm}^{-1}\mathsf{o}^{-1})\}$.

The constraints $x(\mathsf{mm}^{-1})x',\ y(\mathsf{pmm}^{-1}\mathsf{o}^{-1})x,\ y(\mathsf{pmm}^{-1}\mathsf{o}^{-1})x'$ are satisfiable if and only if $y(\mathsf{pm})x$. The relation $(\mathsf{m}) = (\mathsf{pm}) \cap (\mathsf{p}^{-1}\mathsf{mm}^{-1}\mathsf{o})$.

The constraints $a(\mathsf{m})b(\mathsf{m})c(\mathsf{m})d,\ a(\mathsf{pm})x,\ y(\mathsf{pm})d$ are satisfiable if and only if $x(\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1})y$. Hence, we can derive the relation $(\mathsf{pmm}^{-1}\mathsf{o}^{-1}) \cap (\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1}) = (\mathsf{mm}^{-1}\mathsf{o}^{-1})$, and NP-completeness follows from Case 5. ☐

3.3. Classification of complexity. The classification proof splits into 8 lemmas. In each lemma, it is proved that if a subalgebra $\mathcal{S}$ which is closed under deriva-

[1]This problem is also known as the TOTAL ORDERING PROBLEM [31].

tions with lengths satisfies a certain condition, then either $\mathcal{S}$ is contained in one of the 10 tractable subalgebras, or some lemma from section 3.2 can be applied to some subset of $\mathcal{S}$, or $\mathcal{S}$ satisfies the conditions of one of the previous lemmas. It is easy to verify that the assumptions of these 8 lemmas are exhaustive (note that, due to closedness under derivations with lengths, a subalgebra containing $r \cup (\equiv)$, where $r \subseteq (\mathsf{dsf})$, also contains r itself).

We can assume without loss of generality that each subalgebra $\mathcal{S}$ contains the total relation (the union of all basic relations), since we always allow pairs of variables to be unrelated. For each basic relation b of $\mathcal{A}$, we will write r_b to denote the least relation $r \in \mathcal{S}$ such that $(b) \subseteq r$, i.e., the intersection of all $r \in \mathcal{S}$ with this property. (Obviously, the relations r_b depend on $\mathcal{S}$; however, $\mathcal{S}$ will always be clear from the context.)

We use the relations of the form r_b in the algebraic proofs below to show that S is contained in one or another subalgebra. For example, suppose we know that the relation (p) is contained in r_{o}. Then any relation $r \in \mathcal{S}$ such that $(\mathsf{o}) \subseteq r$ satisfies also $(\mathsf{p}) \subseteq r$. To see this, note that if there is $r_1 \in \mathcal{S}$ such that $(\mathsf{o}) \subseteq r$, but $(\mathsf{p}) \not\subseteq r$, then $(\mathsf{o}) \subseteq r_1 \cap r_{\mathsf{o}}$ and $r_1 \cap r_{\mathsf{o}}$ is strictly contained in r_{o}, which contradicts the definition of r_{o}. By a similar argument, if we know that (p) is contained in all of r_{m}, r_{o}, r_{d}, and r_{s}, then we can conclude that, for every $r \in \mathcal{S}$, $(\mathsf{p}) \subseteq r$ whenever $r \cap (\mathsf{pmods}) \neq \emptyset$, which means that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{p}}$.

LEMMA 3.14. *Suppose $\mathcal{S}$ contains a nontrivial relation $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$. Then either $\mathcal{S}$ is contained in one of $\mathcal{C}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{p}}$, and $\mathcal{H}$ or else $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete.*

Proof.

Case 1. $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1})$.

If $\mathcal{S}$ is contained in one of $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{p}}$, and $\mathcal{H}$, then $\mathcal{A}^l$-SAT$(\mathcal{S})$ is tractable by Proposition 3.3. Otherwise let $\mathcal{S} = \{r_1, \ldots, r_{n-1}\}$ and $r_n = r \setminus (\equiv)$ and apply Theorem 2.2 and Lemma 3.10(1) with $r_1, \ldots, r_n$ to obtain NP-completeness of $\mathcal{A}^l$-SAT$(\mathcal{S})$.

Case 2. $r \cap (\mathsf{oo}^{-1}) = (\mathsf{o})$.

If $r^* \not\subseteq (\equiv)$, then the previous case applies. Assume that $r^* \subseteq (\equiv)$. If $r \not\subseteq (\equiv \mathsf{pmo})$, then using Lemma 3.10(1) one can show that $\mathcal{A}^l$-SAT$(\{r\})$ is NP-complete. If $(\mathsf{o}) \subseteq r \subseteq (\equiv \mathsf{pmo})$, then the constraints $\{xrz, zry; l(x)+l(y) < l(z)\}$ are satisfiable if and only if $x(\mathsf{p})y$. Therefore $(\mathsf{p}) \in \mathcal{S}$, and we go back to the first case.

Case 3. $(\mathsf{oo}^{-1}) \subseteq r$.

We may now assume that r is symmetric. We shall prove that either $\mathcal{S}$ is contained in one of $\mathcal{C}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, and $\mathcal{E}_{\mathsf{p}}$ or else $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete. Assume that $\mathcal{S} \not\subseteq \mathcal{C}_{\mathsf{o}}$; that is, there is $r' \in \mathcal{S}$ such that $(\mathsf{oo}^{-1}) \not\subseteq r'$. If $r \cap r' \not\subseteq (\equiv)$, then we obtain the required result by Cases 1 and 2. Therefore we may assume that $r \cap r'$ is either $\emptyset$ or $(\equiv)$ for every $r' \in \mathcal{S}$ such that $(\mathsf{oo}^{-1}) \not\subseteq r'$.

It now follows from Theorem 2.2 and Lemma 3.10(1) that if $\mathcal{S}$ is not contained in one of $\mathcal{S}_{\mathsf{o}}$, $\mathcal{E}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, and $\mathcal{E}_{\mathsf{p}}$, then $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete. If $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{p}}$ or in $\mathcal{E}_{\mathsf{p}}$, then, by Proposition 3.3, $\mathcal{A}^l$-SAT$(\mathcal{S})$ is tractable. Suppose $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{o}}$ or in $\mathcal{E}_{\mathsf{o}}$ but neither in $\mathcal{S}_{\mathsf{p}}$ nor in $\mathcal{E}_{\mathsf{p}}$. Then $\mathcal{S}$ contains a nontrivial symmetric relation r'' such that $(\mathsf{oo}^{-1}) \subseteq r'' \subseteq (\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1})$. Also, r' must be a nontrivial subrelation of $(\equiv \mathsf{ss}^{-1})$ or of $(\equiv \mathsf{ff}^{-1})$. We consider only the first case; the second is dual. Assume without loss of generality that $(\mathsf{s}) \subseteq r'$. Then the constraints $\{xr'y; l(x) < l(y)\}$ are satisfiable if and only if $x(\mathsf{s})y$. Therefore $(\mathsf{s}) \in \mathcal{S}$. Since $(r'' \circ (\mathsf{s}))^* = (\mathsf{oo}^{-1}) \in \mathcal{S}$, the problem $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete by Lemma 3.11. ◻

LEMMA 3.15. *Suppose $\mathcal{S}$ contains a nontrivial relation r such that $r^* \subseteq (\equiv)$ and* *neither r nor r^{-1} is contained in* $(\equiv \mathsf{dsf})$*. Then either $\mathcal{S}$ is contained in one of $\mathcal{C}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{p}}$, and $\mathcal{H}$ or else $\mathcal{A}^l$-*SAT*$(\mathcal{S})$ is NP-complete.*

Proof. If neither $r \setminus (\equiv)$ nor $r^{-1} \setminus (\equiv)$ is contained in one of $(\mathsf{pmod}^{-1}\mathsf{sf}^{-1})$, $(\mathsf{pmod}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$, (pmodsf), or (pmodsf^{-1}), then $\mathcal{A}$-SAT$(\{r \setminus (\equiv)\})$ is NP-complete by Theorem 2.2, and we get the required result by Lemma 3.10(1).

Suppose now that $r \setminus (\equiv)$ is contained in one of the four relations above. Then (taking $r \circ r \circ r$ instead of r if needed) r can be chosen so that it satisfies one of the following conditions:

1. $r \subseteq (\equiv \mathsf{pmos})$;
2. $r \subseteq (\equiv \mathsf{pmof}^{-1})$;
3. $(\mathsf{pmosf}^{-1}) \subseteq r \subseteq (\equiv \mathsf{pmosf}^{-1})$;
4. $(\mathsf{pmods}) \subseteq r$;
5. $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \subseteq r$.

Note that conditions 1 and 2 and conditions 4 and 5 are dual. Therefore it is sufficient to consider only conditions 1, 3, and 4.

Suppose condition 1 holds. Then, by assumption, $r \not\subseteq (\equiv \mathsf{s})$. Now it can be checked that the constraints $\{xrz, zry; l(x) > l(z)\}$ are satisfiable if and only if $xr'y$ for some nontrivial $r' \in \mathcal{A}$ such that $r' \subseteq (\mathsf{pmo})$. Then we apply Lemma 3.14.

Suppose condition 3 holds. Then the constraints $\{xrz, zry; l(x) < l(z), l(z) > l(y)\}$ are satisfiable if and only if $x(\mathsf{pmo})y$. Therefore we again apply Lemma 3.14.

Suppose condition 4 holds. If $(\equiv) \subseteq r$, then the constraints $\{xrz, zry; l(x) > l(z)\}$ are satisfiable if and only if $x(\equiv \mathsf{pmods})y$. Similarly, if $(\equiv) \not\subseteq r$, then the constraints $\{xrz, zry; l(x) > l(z)\}$ are satisfiable if and only if $x(\mathsf{pmods})y$. Therefore a relation $r_1 \in \mathcal{A}$ with $(\mathsf{pmods}) \subseteq r_1 \subseteq (\equiv \mathsf{pmods})$ belongs to $\mathcal{S}$.

If $\mathcal{S}$ contains a nontrivial relation $r_2 \subseteq r_1$ such that $(\mathsf{d}) \not\subseteq r_2$, then either r_2 satisfies condition 1 (and then we get the required result) or r_2 is one of (s), $(\equiv \mathsf{s})$. In the latter case the constraints $\{xr_2y; l(x) < l(y)\}$ are satisfiable if and only if $x(\mathsf{s})y$. So we have $(\mathsf{s}) \in \mathcal{S}$. Then the constraints $\{x(\mathsf{s})z, zr_1y; l(x) + l(y) = l(z)\}$ are satisfiable if and only if $x(\mathsf{p})y$. So we have $(\mathsf{p}) \in \mathcal{S}$, and we can apply Lemma 3.14.

From now on in this proof we assume that every nontrivial $r_2 \in \mathcal{S}$ such that $r_2 \subseteq r_1$ satisfies $(\mathsf{d}) \subseteq r_2$. It now follows that, for every $r \in \mathcal{S}$, $r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset$ implies $(\mathsf{d})^{\pm 1} \subseteq r$. In other words, we have $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

If $(\mathsf{p}) \subseteq r_{\mathsf{d}}$, then, for every $r \in \mathcal{S}$, $r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset$ also implies $(\mathsf{pd})^{\pm 1} \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{p}}$, and we get the required result. If $(\mathsf{o}) \subseteq r_{\mathsf{d}}$, then, for every $r \in \mathcal{S}$, $r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset$ also implies $(\mathsf{od})^{\pm 1} \subseteq r$, and then it is easy to check that $\mathcal{S} \subseteq \mathcal{H}$.

Assume that $r_{\mathsf{d}} \subseteq (\equiv \mathsf{mds})$. If $(\mathsf{m}) \subseteq r_{\mathsf{d}}$, then it can be checked that the constraints $\{zr_{\mathsf{d}}x, zr_{\mathsf{d}}y; l(z) > l(y) > l(x)\}$ are satisfiable if and only if $x(\mathsf{s})y$. It is proved above that, in the presence of r_1 and (s), the required result holds.

Now we may assume that $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\equiv \mathsf{ds})$. Then r_{d} is either (d) or (ds) because $(\equiv)$ can be removed by adding the constraint $l(x) < l(y)$.

Assume now that $\mathcal{S} \not\subseteq \mathcal{H}$. It is easy to see that every relation in $\mathcal{S}$ satisfies condition 1 of $\mathcal{H}$. If there is $r_3 \in \mathcal{S}$ failing to satisfy condition 3 of $\mathcal{H}$, then $r_4 = r_3 \cap r_1$ satisfies $r_4 \subseteq (\mathsf{pmds})$ and $r_4 \cap (\mathsf{pm}) \neq \emptyset$. Then the constraints $\{xr_{\mathsf{d}}z, zr_4y; l(z) > l(y)\}$ are satisfiable if and only if $x(\mathsf{p})y$. Hence we have $(\mathsf{p}) \in \mathcal{S}$, and we can apply Lemma 3.14.

Finally, assume that every $r \in \mathcal{S}$ satisfies conditions 1 and 3 of $\mathcal{H}$, but some $r_5 \in \mathcal{S}$ fails to satisfy condition 2 of $\mathcal{H}$. We can assume that $r_5 \cap (\mathsf{ds}) \neq \emptyset$ and $r_5 \cap (\mathsf{d}^{-1}\mathsf{f}^{-1}) \neq \emptyset$, but $(\mathsf{o}) \not\subseteq r_5$. Let $\nu = (\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Since $\nu = r_{\mathsf{d}}^{-1} \circ r_{\mathsf{d}}$

belongs to $\mathcal{S}$, we may assume that $r_5 \subseteq \nu$; otherwise replace r_5 by $r_5 \cap \nu$. Note that $(\mathsf{d}) \subseteq r_5$.

If $(\mathsf{o}^{-1}) \subseteq r_5$, then $(\mathsf{d}^{-1}) \subseteq r_5$ because $(\mathsf{d}) \subseteq r_{\mathsf{o}}$. Then the constraints in the set $\{xr_5^* y; l(x) \neq l(y)\}$ are satisfiable if and only if $xr_6 y$, where $r_6 \in \mathcal{A}$ is a symmetric relation such that $(\mathsf{dd}^{-1}) \subseteq r_6 \subseteq (\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. We have $(\mathsf{pmods}) = r_1 \circ r_{\mathsf{d}} \in \mathcal{S}$. It follows from Theorem 2.2 that $\mathcal{A}$-SAT$(\{r_6, (\mathsf{pmo})\})$ is NP-complete. Then $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete by Lemma 3.10(2).

Let $(\mathsf{o}^{-1}) \not\subseteq r_5$. If $(\mathsf{d}^{-1}) \subseteq r_5$, then the argument is as above. Otherwise we have $(\mathsf{df}^{-1}) \subseteq r_5 \subseteq (\equiv \mathsf{dsff}^{-1})$ (note that $(\mathsf{s}^{-1}) \not\subseteq r_5$ because $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$). Then the constraints in the set $\{xr_5 y; l(x) > l(y)\}$ are satisfiable if and only if $x(\mathsf{f}^{-1})y$. We may then assume that $(\mathsf{f}) \in \mathcal{S}$. It follows that the relations $(\mathsf{ods}) = (\mathsf{f}^{-1}) \circ r_{\mathsf{d}}$ and $(\equiv \mathsf{dff}^{-1}) = (\mathsf{f}) \circ r_5$ both belong to $\mathcal{S}$, and therefore $(\mathsf{d}) = r_{\mathsf{d}} \cap (\equiv \mathsf{dff}^{-1}) \in \mathcal{S}$. It follows from Theorem 2.2 that $\mathcal{A}$-SAT$(\{(\mathsf{o}), (\mathsf{d}), (\mathsf{f}), (\equiv \mathsf{dff}^{-1})\})$ is NP-complete. Since $(\mathsf{ods}) \in \mathcal{S}$, we conclude that $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete by Lemma 3.10(2). ∎

LEMMA 3.16. *If $\mathcal{S}$ contains two nontrivial relations r_1 and r_2 such that $r_1 \cap r_2 \subseteq (\equiv)$ and $r_1, r_2 \subseteq (\equiv \mathsf{dsf})$, then either $\mathcal{S} \subseteq \mathcal{H}$ or else $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete.*

Proof. We may assume that $(\equiv) \not\subseteq r_1, r_2$ because it can be removed by adding the constraint $l(x) < l(y)$. If $r_1 = (\mathsf{d})$ and $r_2 = (\mathsf{sf})$, then $\mathcal{A}$-SAT$(\{r_1, r_2\})$ is NP-complete by Theorem 2.2.

In all other cases $r_1 \circ r_2^{-1}$ (or its converse) satisfies the assumptions of Lemma 3.15 or Lemma 3.16. It remains to notice that $\{r_1, r_2\}$ is not contained in one of $\mathcal{C}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{p}}$. ∎

LEMMA 3.17. *If $\mathcal{S}$ contains two nontrivial symmetric relations r_1 and r_2 such that $r_1 \cap r_2 \subseteq (\equiv)$, then either $\mathcal{S}$ is contained in one of $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{p}}$, $\mathcal{H}$ or else $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete.*

Proof. We may assume that r_1 and r_2 are minimal (with respect to inclusion) among nontrivial symmetric relations.

It follows from Theorem 2.2 that if none of r_1, r_2 is contained in one of $(\equiv \mathsf{ss}^{-1})$, $(\equiv \mathsf{ff}^{-1})$, then $\mathcal{A}$-SAT$(\mathcal{S})$ (and, consequently, $\mathcal{A}^l$-SAT$(\mathcal{S})$) is NP-complete.

We shall consider only the case $r_1 \subseteq (\equiv \mathsf{ss}^{-1})$; the case $r_1 \subseteq (\equiv \mathsf{ff}^{-1})$ is dual. Then we may assume that $(\mathsf{ss}^{-1}) \in \mathcal{S}$ and $(\mathsf{s}) \in \mathcal{S}$ because these constraints are equivalent to $\{xr_1 y; l(x) \neq l(y)\}$ and $\{xr_1 y; l(x) < l(y)\}$, respectively. If $r_2 \subseteq (\equiv \mathsf{dd}^{-1}\mathsf{ff}^{-1})$, then, by imposing the constraint $l(x) < l(y)$, we can obtain a nonempty subrelation of (df), and we can apply Lemma 3.16. We therefore may assume that $r_2 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$. Now it follows from minimality of r_2 and from Theorem 2.2 that if $\mathcal{A}$-SAT$(\mathcal{S})$ is not NP-complete, then either $\mathcal{S} \subseteq \mathcal{H}$ or every relation $r \in \mathcal{S}$ such that $r^* \not\subseteq (\equiv \mathsf{ss}^{-1})$ satisfies $r_2 \subseteq r$.

It can be easily checked that if $(\mathsf{dd}^{-1}) \not\subseteq r_2$, then either $r_2 \subseteq (\equiv \mathsf{mm}^{-1})$ or $r_3 = ((\mathsf{s}) \circ r_2)^*$ is nonempty and satisfies $r_3 \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$. In the former case $\mathcal{A}^l$-SAT$(\{r_2\})$ is NP-complete by Lemma 3.10(1). In the latter case we apply Lemma 3.14.

Further, let $(\mathsf{dd}^{-1}) \subseteq r_2$. Suppose some nontrivial relation $r_3 \in \mathcal{S}$ is strictly contained in r_2. Then, by the choice of r_2, we have $r_3^* \subseteq (\equiv)$, and, since $\mathcal{S} \not\subseteq \mathcal{C}_{\mathsf{o}}$, we can apply Lemma 3.15 or Lemma 3.16.

Now we may assume that, for every $r \in \mathcal{S}$ such that $r \cap r_2 \neq \emptyset$, we have $(\mathsf{dd}^{-1}) \subseteq r_2 \subseteq r$.

It can now be checked using Theorem 2.2 that if $\mathcal{A}$-SAT$(\mathcal{S})$ is not NP-complete, then $\mathcal{S}$ is contained in one of $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{d}}$, and $\mathcal{H}$. Suppose that $\mathcal{S} \not\subseteq \mathcal{S}_{\mathsf{p}}$ and $\mathcal{S} \not\subseteq \mathcal{H}$, since otherwise there is nothing to prove. Then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$, and for every relation $r \in \mathcal{S}$ such that $r \not\subseteq (\equiv \mathsf{ss}^{-1})$, we have $(\mathsf{dd}^{-1}) \subseteq r_2 \subseteq r$. If r_2 contains (pp^{-1}) or (oo^{-1}), then $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{p}}$ or $\mathcal{H}$, which contradicts the assumptions just made. Otherwise we have $(\mathsf{mm}^{-1}\mathsf{dd}^{-1}) \subseteq r_2 \subseteq (\equiv \mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1})$. Hence $((\mathsf{s}) \circ r_2)^* = (\mathsf{mm}^{-1}\mathsf{dd}^{-1}) \in \mathcal{S}$. By minimality, it follows that $r_2 = (\mathsf{mm}^{-1}\mathsf{dd}^{-1})$. Then $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete by Lemma 3.12. ∎

LEMMA 3.18. *If* $(\mathsf{s}) \in \mathcal{S}$ *or* $(\mathsf{f}) \in \mathcal{S}$ *then either* $\mathcal{S}$ *is contained in one of the* 10 *subalgebras listed in Theorem* 3.1 *or else* $\mathcal{A}^l$-SAT$(\mathcal{S})$ *is NP-complete.*

Proof. We consider only the case $(\mathsf{s}) \in \mathcal{S}$; the other case is dual.

By Lemmas 3.15 and 3.16, we may assume that, for every nontrivial $r \in \mathcal{S}$ such that $r^* \subseteq (\equiv)$, we have either $(\mathsf{s}) \subseteq r \subseteq (\mathsf{dsf})$ or $(\mathsf{s}) \subseteq r^{-1} \subseteq (\mathsf{dsf})$. We may also assume that $(\mathsf{ss}^{-1}) \in \mathcal{S}$ because the constraints $\{x(\mathsf{s})z, z(\mathsf{s})y; l(x) \neq l(y)\}$ are satisfiable if and only if $x(\mathsf{ss}^{-1})y$.

Suppose that $\mathcal{S} \not\subseteq \mathcal{D}_{\mathsf{s}}$. Then there exists a relation $r_1 \in \mathcal{S}$ such that $r_1 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$, but $(\equiv \mathsf{ss}^{-1}) \not\subseteq r_1$. If $(\equiv) \subseteq r_1$, then we can apply either Lemma 3.15 with r_1 or Lemma 3.17 with $\{r_1^*, (\mathsf{ss}^{-1})\}$. So we may now assume that $(\equiv) \not\subseteq r_1$. It can be checked that there is a nontrivial $r_2 \in \mathcal{A}$ such that $\{ur_1v, u(\mathsf{s})x, v(\mathsf{s})y; l(u) = l(v)\}$ is satisfiable if and only if xr_2y. Then $r_2 \in \mathcal{S}$. Moreover, we have $(\equiv \mathsf{ss}^{-1}) \cap r_2 = \emptyset$. If r_2 satisfies $r_2^* \subseteq (\equiv)$, then we apply Lemma 3.15 or Lemma 3.16. Otherwise $\{r_2^*, (\mathsf{ss}^{-1})\} \subseteq \mathcal{S}$, and we get the required result by Lemma 3.17. ∎

LEMMA 3.19. *If* $(\mathsf{sf}) \in \mathcal{S}$, *then either* $\mathcal{S} \subseteq \mathcal{D}_{\mathsf{s}}$ *or* $\mathcal{S} \subseteq \mathcal{D}_{\mathsf{f}}$ *or else* $\mathcal{A}^l$-SAT$(\mathcal{S})$ *is NP-complete.*

Proof. We have $(\mathsf{dsf}) = (\mathsf{sf}) \circ (\mathsf{sf}) \in \mathcal{S}$. We may assume that neither (s) nor (f) belong to $\mathcal{S}$; otherwise we obtain the result by Lemma 3.18, since, out of the 10 subalgebras, (sf) is contained only in $\mathcal{D}_{\mathsf{s}}$ and in $\mathcal{D}_{\mathsf{f}}$. It now follows that $(\mathsf{dsf})^{\pm 1} \cap r \neq \emptyset$ implies $(\mathsf{sf})^{\pm 1} \subseteq r$ for any $r \in \mathcal{S}$.

Suppose that $\mathcal{S}$ is not contained in $\mathcal{D}_{\mathsf{s}}$. Then there is $r_1 \in \mathcal{S}$ such that $(\equiv \mathsf{ss}^{-1}) \not\subseteq r_1$ and $r_1 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$. Assume that $(\equiv) \subseteq r_1$. If $(\mathsf{ss}^{-1}) \cap r_1 = \emptyset$, then, by the previous paragraph, we have $r_1 \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$, and we apply Lemma 3.14. Assume now that $(\mathsf{ss}^{-1}) \cap r_1 = (\mathsf{s})$. Then $r_1 \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dsf})$. Now we apply Lemma 3.14 if $r_1^* \not\subseteq (\equiv)$ and Lemma 3.15 otherwise.

Now assume that $(\equiv) \not\subseteq r_1$ and $r_1 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$. If there is such an r_1 with the additional property that $r_1 \cap (\mathsf{oo}^{-1}) = \emptyset$, then the set of constraints $\{x(\mathsf{sf})u, ur_1v, y(\mathsf{sf})v; l(u) = l(v)\}$ is satisfiable if and only if $xr'y$, where $r' \in \mathcal{A}$ is some nontrivial relation such that $r' \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$. Then we can apply Lemma 3.14.

Suppose $r_1 \cap (\mathsf{oo}^{-1}) \neq \emptyset$. We have $(\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{s}^{-1}\mathsf{f}^{-1}) \circ (\mathsf{sf}) \in \mathcal{S}$. Consider $r_2 = r_1 \cap (\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. If $r_2 \subseteq (\mathsf{oo}^{-1})$, then $\mathcal{A}$-SAT$(\{(\mathsf{sf}), r_2\})$ is NP-complete by Theorem 2.2. Otherwise $(\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \subseteq r_2$, and we have either $r_2 = (\mathsf{oss}^{-1}\mathsf{ff}^{-1})$ or $r_2 = (\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. In both cases $\mathcal{A}^l$-SAT$(\{(\mathsf{sf}), r_2\})$ is NP-complete by Lemma 3.11. ∎

LEMMA 3.20. *If there is* $r \in \mathcal{S}$ *such that* $(\mathsf{d}) \subseteq r \subseteq (\mathsf{dsf})$, *then either* $\mathcal{S}$ *is contained in one of the* 10 *subalgebras listed in Theorem* 3.1 *or else* $\mathcal{A}^l$-SAT$(\mathcal{S})$ *is NP-complete.*

Proof. Note that $\nu = (\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = r^{-1} \circ r \in \mathcal{S}$.

We may assume that every $r \in \mathcal{S}$ such that $r^* \subseteq (\equiv)$ satisfies $(\mathsf{d}) \subseteq r \subseteq (\equiv \mathsf{dsf})$ or $(\mathsf{d}) \subseteq r^{-1} \subseteq (\equiv \mathsf{dsf})$; otherwise we apply Lemmas 3.15, 3.18, or 3.19. It follows, in particular, that no nontrivial subrelation of $(\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$ belongs to $\mathcal{S}$.

Suppose that there exists $r_1 \in \mathcal{S}$ such that $r_1 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$, but $(\mathsf{dd}^{-1}) \not\subseteq r_1$. If $r_1^* \subseteq (\equiv)$, then we can apply Lemma 3.15 with r_1. Otherwise r_1^* is a symmetric nontrivial relation satisfying $(\mathsf{dd}^{-1}) \cap r_1^* = \emptyset$. If $r_1 \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1})$, then we can apply Lemma 3.14. Otherwise the relation $r_2 = \nu \cap r_1^* \in \mathcal{S}$ is nontrivial and satisfies $r_2 \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. We have $(\mathsf{oo}^{-1}) \subseteq r_2$ and $r_2 \cap r = \emptyset$, since no subrelation of (sf) belongs to $\mathcal{S}$. Now it is easy to verify that $\mathcal{A}$-SAT$(\{r_2, r\})$ is NP-complete, by Theorem 2.2.

From now on (in this proof) we may assume that, for every $r \in \mathcal{S}$, whenever $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$ we have $(\mathsf{dd}^{-1}) \subseteq r$. It now follows that condition 1 of $\mathcal{D}_{\mathsf{d}}$ and $\mathcal{D}'_{\mathsf{d}}$ is satisfied in $\mathcal{S}$.

Suppose there is $r_2 \in \mathcal{S}$ such that $r_2 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}) \neq \emptyset$, but $r_2 \cap (\equiv \mathsf{oo}^{-1}) = \emptyset$. It is easy to check that there exists a nontrivial $r_3 \in \mathcal{A}$ with $r_3 \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ such that $\{ur_2v, xru, yrv; l(u) = l(v)\}$ is satisfiable if and only if xr_3y (the relation r_3 depends on r and r_2). Then $r_3 \in \mathcal{S}$, and we can apply Lemma 3.14.

From now on (in this proof) we may also assume that, for every $r \in \mathcal{S}$, $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}) \neq \emptyset$ implies $r \cap (\equiv \mathsf{oo}^{-1}) \neq \emptyset$.

We know that $r_{\mathsf{o}} \subseteq \nu$. If $(\equiv) \subseteq r_{\mathsf{o}}$, then it is easy to check that $\mathcal{S} \subseteq \mathcal{D}_{\mathsf{d}}$.

Suppose $r_{\mathsf{o}} \cap (\equiv \mathsf{oo}^{-1}) = (\mathsf{o})$ and $\mathcal{S} \not\subseteq \mathcal{D}'_{\mathsf{d}}$. Then there is $r_4 \in \mathcal{S}$ such that $r_4 \cap (\mathsf{pm}) \neq \emptyset$, but $(\mathsf{o}) \not\subseteq r_4$. Then there exists a nontrivial $r_5 \in \mathcal{A}$ with $r_5 \subseteq (\mathsf{pm})$ such that the constraints

$$\{ur_{\mathsf{o}}z,\ zr_{\mathsf{o}}v,\ ur_4v,\ xru,\ yrv;\ l(u) = l(z) = l(v)\}$$

are satisfiable if and only if xr_5y. Then $r_5 \in \mathcal{S}$, and we can apply Lemma 3.14.

It remains to consider the case $r_{\mathsf{o}} \cap (\equiv \mathsf{oo}^{-1}) = (\mathsf{oo}^{-1})$. Then every $r_6 \in \mathcal{S}$ such that $r_6 \cap (\mathsf{oo}^{-1}) = \emptyset$, but $r_6 \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}) \neq \emptyset$ satisfies $(\equiv) \subseteq r_6$.

If there is such r_6 with $r_6 \cap (\mathsf{pp}^{-1}) \neq \emptyset$, then there exists a nontrivial $r_7 \in \mathcal{A}$ with $r_7 \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ such that the constraints

$$\{ur_{\mathsf{o}}z,\ zr_6v,\ ur_6v,\ xru,\ yrv;\ l(u) = l(z) = l(v)\}$$

are satisfiable if and only if xr_7y. Then $r_7 \in \mathcal{S}$, and we can apply Lemma 3.14.

Now we may assume that every $r \in \mathcal{S}$ with $r \cap (\mathsf{pp}^{-1}) \neq \emptyset$ also satisfies $(\mathsf{oo}^{-1}) \subseteq r$. Suppose $\mathcal{S} \not\subseteq \mathcal{D}'_{\mathsf{d}}$. Then there is $r_8 \in \mathcal{S}$ such that $(\mathsf{m}) \subseteq r_8$ and $r_8 \cap (\equiv \mathsf{pp}^{-1}\mathsf{oo}^{-1}) = (\equiv)$. Moreover, every $r \in \mathcal{S}$ such that $r \cap (\mathsf{mm}^{-1}) \neq \emptyset$ satisfies $(\equiv) \subseteq r$, since otherwise we can obtain a relation $r_9 (= r \cap r_8)$ such that $r_9 \cap (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$ is nonempty and is contained in (mm^{-1}), a contradiction.

Now either $\mathcal{S} \subseteq \mathcal{D}''_{\mathsf{d}}$ or else there is $r_{10} \in \mathcal{S}$ such that $(\mathsf{poo}^{-1}) \subseteq r_{10}$ and $(\equiv) \not\subseteq r_{10}$. In the latter case, again, there exists a relation $r_{11} \in \mathcal{A}$ with $r_{11} \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ such that the constraints

$$\{ur_{10}z,\ zr_8^{-1}v,\ ur_8v,\ xru,\ yrv;\ l(u) = l(z) = l(v)\}$$

are satisfiable if and only if $xr_{11}y$. Then $r_{11} \in \mathcal{S}$ and we can apply Lemma 3.14. $\square$

LEMMA 3.21. *If there is a symmetric nontrivial relation $r' \in \mathcal{S}$ such that every nontrivial $r \in \mathcal{S}$ satisfies $r' \subseteq r$, then either $\mathcal{S}$ is contained in one of the* 10 *subalgebras listed in Theorem* 3.1 *or else $\mathcal{A}^l$-SAT$(\mathcal{S})$ is NP-complete.*

Proof. If r' contains (pp^{-1}), or (oo^{-1}), or $(\equiv \mathsf{dd}^{-1})$, or $(\equiv \mathsf{ss}^{-1})$, or $(\equiv \mathsf{ff}^{-1})$, then $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{p}}$, or $\mathcal{C}_{\mathsf{o}}$, or $\mathcal{D}_{\mathsf{d}}$, or $\mathcal{D}_{\mathsf{s}}$, or $\mathcal{D}_{\mathsf{f}}$, respectively. If $r' \subseteq (\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$, then we can obtain an asymmetric relation in $\mathcal{S}$ which contradicts the assumption of this step. If $r' = (\equiv \mathsf{mm}^{-1})$, then $\mathcal{A}^l$-SAT$(\{r'\})$ is NP-complete by Example 3.1.

From now on (in this proof) we assume that all nontrivial $r \in \mathcal{S}$ satisfy the condition that $(\mathsf{mm}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \subseteq r$; otherwise one of the earlier cases applies. If every nontrivial $r \in \mathcal{S}$ satisfies $(\equiv \mathsf{mm}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \subseteq r$, then $\mathcal{S} \subseteq \mathcal{D}_{\mathsf{s}}$. Suppose that there is $r_1 \in \mathcal{S}$ such that $(\equiv) \not\subseteq r_1$. Then $r_1 \cap (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$. If, for all $r \in \mathcal{S}$, $(\mathsf{pp}^{-1}) \subseteq r$ or, for all $r \in \mathcal{S}$, $(\mathsf{oo}^{-1}) \subseteq r$, then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{p}}$ or $\mathcal{S} \subseteq \mathcal{C}_{\mathsf{o}}$, respectively. Else, we can choose r_1 so that $r_1 \cap (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) = (\mathsf{mm}^{-1})$. Now it is not hard to check that either $\mathcal{S} \subseteq \mathcal{C}_{\mathsf{m}}$ or else there is $r_2 \in \mathcal{S}$ such that the system $\{r_1, r_2\}$ satisfies the conditions of Lemma 3.13. $\square$

Classification is complete. Theorem 3.1 is proved.

4. Conclusion. In this paper we have given a complete classification of the complexity of interval satisfiability problems with very general length restrictions. Our main result, Theorem 3.1, determines the complexity of $\mathcal{A}^l$-SAT*($\mathcal{F}$)* for every possible subset $\mathcal{F} \subseteq \mathcal{A}$.

To conclude, we note that our NP-completeness proofs use only a very restricted subset of the allowable length constraints. In fact, we use constraints on lengths only of the following forms:

(i) comparing $l(x) + l(y)$ with $l(z)$,
(ii) comparing $l(x)$ and $l(y)$,
(iii) comparing $l(x)$ with a given number.

It follows that the NP-complete fragments of $\mathcal{A}^l$-SAT remain NP-complete even if we allow only these very limited forms of Horn DLRs to specify length constraints. This prompts us to make the following conjecture.

CONJECTURE 4.1. *All NP-complete cases of $\mathcal{A}^l$-SAT remain NP-complete if we allow fixing individual interval lengths as the only form of constraints on lengths.*

In fact, we suggest that an even stronger result may be true: it may be that in all cases where imposing restrictions on interval lengths causes intractability, simply requiring all intervals to have the same length will already be intractable.

PROBLEM 4.1. *Do all NP-complete cases of $\mathcal{A}^l$-SAT remain NP-complete if we search only for models with all intervals of the same length?*

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