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RULED MINIMAL LAGRANGIAN SUBMANIFOLDS OF COMPLEX PROJECTIVE 3-SPACE*

J. BOLTON† AND L. VRANCKEN‡

Abstract. We show how a ruled minimal Lagrangian submanifold of complex projective 3-space may be used to construct two related minimal surfaces in the 5-sphere.

Key words. Complex projective space, Lagrangian submanifold, sphere, minimal surface

AMS subject classifications. 53B25, 53B20

1. Introduction. In previous papers [1], [2] we showed how a Lagrangian submanifold M of complex projective 3-space $\mathbb{C}P^3(4)$ satisfying Chen's equality [7] but having no totally geodesic points may be used to construct a minimal surface in the unit 5-sphere $S^5(1)$ with ellipse of curvature a circle.

In this paper, we replace the assumption concerning Chen's equality with the assumption that M is minimal and admits a foliation by asymptotic curves, that is to say curves with vanishing normal curvature. In fact, these curves turn out to be geodesics of $\mathbb{C}P^3(4)$ (hence our description of M as a *ruled* submanifold of $\mathbb{C}P^3(4)$), and we show that the local construction referred to above may be applied to M to give two minimal surfaces in $S^5(1)$ whose ellipses of curvature are not circles. We also show that these minimal surfaces are related by a transform which generalises that of the polar (see [3], [9]) for linearly full minimal surfaces in $S^5(1)$ whose ellipses of curvature are circles. In a forthcoming paper [4], we will show that this transform may be defined for all non totally geodesic minimal surfaces in $S^5(1)$.

2. Ruled minimal Lagrangian submanifolds. Let M be a Lagrangian submanifold of $\mathbb{C}P^3(4)$. That is to say, if J is the complex structure of $\mathbb{C}P^3(4)$, then J maps the tangent bundle of M onto the normal bundle. Let $\tilde{\nabla}$ denote the Riemannian connection on $\mathbb{C}P^3(4)$, and ∇, $\nabla^{\perp}$ the induced connections on M and the normal bundle of M. Let $h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$ denote the second fundamental form of M, and, if N is a normal vector field, let $A_N(X) = -\tilde{\nabla}_X N + \nabla^{\perp}_X N$ denote the corresponding shape operator. If $\langle\ ,\ \rangle$ denotes the Fubini-Study metric on $\mathbb{C}P^3(4)$, then [5, 8], the cubic form

$$C(X,Y,Z) = \langle h(X,Y), JZ\rangle = \langle A_{JZ}(X), Y\rangle \tag{1}$$

is symmetric in X, Y and Z. In particular,

$$A_{JX}(Y) = A_{JY}(X) = -Jh(X,Y). \tag{2}$$

We now assume that M admits a smooth unit length vector field $\mathbf{e}_1$ whose integral curves are asymptotic curves in M, that is to say they have zero normal curvature, so that

$$h(\mathbf{e}_1, \mathbf{e}_1) = 0. \tag{3}$$

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†Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, U.K. (john.bolton@durham.ac.uk).

‡LAMATH, ISTV2, Université de Valenciennes, Campus du Mont Houy, 59313 Valenciennes Cedex 9, France (luc.vrancken@univ-valenciennes.fr).

If $A_{J\mathbf{e}_1}$ vanishes identically at some point $p \in M$, then M satisfies Chen's equality at p (see [7]), and the situation in which this holds on an open subset of M has been discussed in [1] and [2]. Since we are dealing with a local theory here, we will from now on assume that M does not satisfy Chen's equality at any point.

It follows from (2) and (3) that $A_{J\mathbf{e}_1}\mathbf{e}_1 = 0$, so we may choose eigenvectors $\mathbf{e}_2$ and $\mathbf{e}_3$ of $A_{J\mathbf{e}_1}$ such that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of the tangent space of M. Let λ_2, λ_3 be the eigenvalues corresponding to $\mathbf{e}_2$, $\mathbf{e}_3$ respectively.

We now assume that M is minimal, so that

$$0 = \langle h(\mathbf{e}_2, \mathbf{e}_2) + h(\mathbf{e}_3, \mathbf{e}_3), J\mathbf{e}_1\rangle = \langle A_{J\mathbf{e}_1}\mathbf{e}_2, \mathbf{e}_2\rangle + \langle A_{J\mathbf{e}_1}\mathbf{e}_3, \mathbf{e}_3\rangle = \lambda_2 + \lambda_3. \tag{4}$$

Thus $\lambda_2 = -\lambda_3 = \lambda$, where we may assume that λ is a strictly positive function on M and $\mathbf{e}_2$, $\mathbf{e}_3$ are smooth unit vector fields.

If we put $a = \langle A_{J\mathbf{e}_2}\mathbf{e}_2, \mathbf{e}_2\rangle$, $b = \langle A_{J\mathbf{e}_2}\mathbf{e}_2, \mathbf{e}_3\rangle$ then it is easy to check using (2), (3) and (4) that, with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we have the following matrix expressions.

$$A_{J\mathbf{e}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \tag{5}$$

$$A_{J\mathbf{e}_2} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & a & b \\ 0 & b & -a \end{pmatrix}, \tag{6}$$

$$A_{J\mathbf{e}_3} = \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & b & -a \\ -\lambda & -a & -b \end{pmatrix}. \tag{7}$$

Let z^i_j be the connection 1-forms on M defined by

$$\nabla \mathbf{e}_j = z^i_j \mathbf{e}_i, \tag{8}$$

and define the connection coefficients z^i_{kj} by

$$z^i_j(\mathbf{e}_k) = z^i_{kj}, \tag{9}$$

so that

$$z^i_{kj} = \langle \nabla_{\mathbf{e}_k}\mathbf{e}_j, \mathbf{e}_i\rangle = -z^j_{ki}. \tag{10}$$

We use the fundamental equations of submanifold theory, namely the Gauss, Codazzi and Ricci equations, to find relations between z^i_{jk}, a, b and λ. However, for Lagrangian submanifolds, the Gauss and Ricci equations are equivalent.

First consider the Codazzi equations, namely,

$$\nabla^\perp_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = \nabla^\perp_Y (h(X,Z)) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z).$$

If we apply J to this expression and take (2) into account, we see that that the Codazzi equations are equivalent to

$$\nabla_X(A_{JY}Z) - A_{J(\nabla_X Y)}Z - A_{JY}(\nabla_X Z) = \nabla_Y(A_{JX}Z) - A_{J(\nabla_Y X)}Z - A_{JX}(\nabla_Y Z). \tag{11}$$

Equations (5), (6) and (7) may be used to show that (11) is equivalent to the following system (12)-(19).

$$z_{11}^2 = z_{11}^3 = 0, \quad z_{12}^3 = z_{21}^3 = -z_{31}^2, \quad z_{21}^2 = z_{31}^3, \tag{12}$$

$$\mathbf{e}_1(\lambda) = -2z_{21}^2\lambda, \tag{13}$$

$$\mathbf{e}_2(\lambda) = -2z_{32}^3\lambda, \tag{14}$$

$$\mathbf{e}_3(\lambda) = 2z_{22}^3\lambda, \tag{15}$$

$$\mathbf{e}_1(a) = 2bz_{12}^3 - az_{21}^2 - 2z_{32}^3\lambda, \tag{16}$$

$$\mathbf{e}_1(b) = -2az_{12}^3 - bz_{21}^2 + 2z_{22}^3\lambda, \tag{17}$$

$$\mathbf{e}_3(a) - \mathbf{e}_2(b) = 3az_{22}^3 + 3bz_{32}^3 + 4z_{12}^3\lambda, \tag{18}$$

$$\mathbf{e}_3(b) + \mathbf{e}_2(a) = 3bz_{22}^3 - 3az_{32}^3 - 2z_{21}^2\lambda. \tag{19}$$

In particular, we note from (12) that $\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0$, so that, by (3), the integral curves of $\mathbf{e}_1$ are geodesics in $CP^3(4)$. We have thus proved the following lemma.

Lemma 1. *Let M be a minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$. If M admits a foliation by asymptotic curves then these curves are geodesics of $\mathbb{C}P^3(4)$, so that M is a ruled submanifold.*

We next investigate the Gauss curvature equation, which states that the curvature tensor R of ∇ is given by

$$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y + (A_{h(Y,Z)}X - A_{h(X,Z)}Y).$$

Taking (2) into account, the above equation is equivalent to

$$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y + [A_{JX}, A_{JY}]Z. \tag{20}$$

Using (5), (6), (7) and (12), we find that (20) is equivalent to the following system (21)-(24).

$$\mathbf{e}_1(z_{12}^3) = -2z_{12}^3 z_{21}^2, \qquad \mathbf{e}_1(z_{21}^2) = -1 + (z_{12}^3)^2 - (z_{21}^2)^2 + \lambda^2, \tag{21}$$

$$\mathbf{e}_1(z_{22}^3) = -\mathbf{e}_3(z_{21}^2) - z_{21}^2 z_{22}^3, \qquad \mathbf{e}_1(z_{32}^3) = \mathbf{e}_2(z_{21}^2) - z_{21}^2 z_{32}^3, \tag{22}$$

$$\mathbf{e}_2(z_{12}^3) = -\mathbf{e}_3(z_{21}^2) + 2b\lambda, \qquad \mathbf{e}_3(z_{12}^3) = \mathbf{e}_2(z_{21}^2) - 2a\lambda, \tag{23}$$

$$\mathbf{e}_2(z_{32}^3) = \mathbf{e}_3(z_{22}^3) + 2a^2 + 2b^2 - 1 - 3(z_{12}^3)^2 - (z_{21}^2)^2 - (z_{22}^3)^2 - (z_{32}^3)^2 + \lambda^2. \tag{24}$$

The Gauss, Codazzi and Ricci equations provide a full set of integrability conditions, so we have the following theorem.

THEOREM 1. *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal moving frame on a simply-connected Riemannian manifold M, and let $\{z^i_{kj}\}$ be the connection coefficients of the corresponding Riemannian connection. If there exist functions $\lambda > 0$, a, b on M satisfying (12)-(19) and (21)-(24) then M may be isometrically immersed as a ruled minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$ with shape operator A given by (5), (6) and (7). Moreover the immersion is unique up to holomorphic isometries of $\mathbb{C}P^3(4)$.*

We now show the existence of such submanifolds M of $\mathbb{C}P^3(4)$. In fact, we will show in a forthcoming paper [4] that a solution $f(x, y)$ to the sinh-Gordon equation

$$f_{xx} + f_{yy} + 4 \sinh f = 0$$

determines a ruled minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$ with the property that the distribution orthogonal to the rulings is integrable. Indeed, let f be such a function and let $\mu(t, x, y) = \cos t \sinh f + \cosh f$. Then define a Riemannian metric on a suitable open subset of $\mathbb{R}^3$ by taking $\mathbf{e}_1 = -2(\partial/\partial t)$, $\mathbf{e}_2 = \mu^{-1/2}(\partial/\partial x)$, $\mathbf{e}_3 = \mu^{-1/2}(\partial/\partial y)$ to be an orthonormal moving frame. It follows easily from the Koszul formula that the non-zero connection coefficients of the corresponding Riemannian connection are given by

$$z^2_{21} = -z^1_{22} = z^3_{31} = -z^1_{33} = -\frac{\mu_t}{\mu}, \quad z^3_{32} = -z^2_{33} = \frac{\mu_x}{2\mu^{3/2}}, \quad z^3_{22} = -z^2_{23} = -\frac{\mu_y}{2\mu^{3/2}},$$

and, in particular, $\{\mathbf{e}_2, \mathbf{e}_3\}$ span an integrable distribution. Then taking

$$\lambda = 1/\mu, \quad a = \frac{f_x \sin t}{2\mu^{3/2}}, \quad b = \frac{f_y \sin t}{2\mu^{3/2}},$$

it may be checked that (12)-(19) and (21)-(24) are all satisfied, so we may apply Theorem 1 to prove the existence of a corresponding ruled minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$.

Returning now to the general situation, let $\mathbf{E}_0$ be a local horizontal lift of a ruled minimal Lagrangian submanifold M of $\mathbb{C}P^3(4)$ to the total space of the Hopf fibration $\pi : S^7(1) \to \mathbb{C}P^3(4)$, where $S^7(1)$ is the unit sphere in $\mathbb{R}^8 = \mathbb{C}^4$. The existence of such a lift follows from a result of Reckziegel [10], and any two such lifts $\mathbf{E}_0$ $\tilde{\mathbf{E}}_0$ are related by $\tilde{\mathbf{E}}_0 = e^{i\theta}\mathbf{E}_0$, where θ is a constant.

For $j = 1, 2, 3$, let $\mathbf{E}_j$ be the image under the derivative $d\mathbf{E}_0$ of $\mathbf{e}_j$, and let $\mathcal{E} = (\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ be the map from M to the unitary group $U(4)$ so constructed.

We now write down the moving frame equations of $\mathcal{E}$. In fact, if $\omega_1, \omega_2, \omega_3$ is the dual frame to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, a routine calculation using (5)-(9) and (12) shows that

$$d\mathcal{E} = \mathcal{E}(\alpha + i\beta) \tag{25}$$

where

$$\alpha = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -z^2_{21}\omega_2 + z^3_{12}\omega_3 & -z^3_{12}\omega_2 - z^2_{21}\omega_3 \\ \omega_2 & z^2_{21}\omega_2 - z^3_{12}\omega_3 & 0 & -z^3_{12}\omega_1 - z^3_{22}\omega_2 - z^3_{32}\omega_3 \\ \omega_3 & z^3_{12}\omega_2 + z^2_{21}\omega_3 & z^3_{12}\omega_1 + z^3_{22}\omega_2 + z^3_{32}\omega_3 & 0 \end{pmatrix} \tag{26}$$

and

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\omega_2 & -\lambda\omega_3 \\ 0 & \lambda\omega_2 & \lambda\omega_1 + a\omega_2 + b\omega_3 & b\omega_2 - a\omega_3 \\ 0 & -\lambda\omega_3 & b\omega_2 - a\omega_3 & -\lambda\omega_1 - a\omega_2 - b\omega_3 \end{pmatrix}. \tag{27}$$

Note that taking a different horizontal lift $\mathbf{E}_0$ would imply that we multiply $\mathbf{E}_0$ (and thus also $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$) by a factor $e^{i\theta}$, where θ is a constant. Thus we may choose a lift for which $\mathcal{E}$ lies in $SU(4)$ at some point. It then follows from (26) and (27) that $\mathcal{E}$ always lies in $SU(4)$ so, by choosing a suitable horizontal lift E_0, we may assume that

$$\mathcal{E} : M \to SU(4). \tag{28}$$

We now compose $\mathcal{E}$ with a suitably chosen standard double-cover of $SO(6)$ by $SU(4)$ to obtain a map $\mathcal{U} : M \to SO(6)$. In fact, if we let V be the 6-dimensional real subspace of the second exterior power $\wedge^2\mathbb{C}^4$ of $\mathbb{C}^4$ spanned by

$$\mathbf{U}_1 = \frac{1}{\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_3 + \mathbf{E}_1 \wedge \mathbf{E}_2), \qquad \mathbf{U}_2 = \frac{1}{\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_1 + \mathbf{E}_2 \wedge \mathbf{E}_3), \tag{29}$$

$$\mathbf{U}_3 = \frac{1}{\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_2 + \mathbf{E}_3 \wedge \mathbf{E}_1), \qquad \mathbf{U}_4 = \frac{1}{i\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_1 - \mathbf{E}_2 \wedge \mathbf{E}_3), \tag{30}$$

$$\mathbf{U}_5 = \frac{1}{i\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_2 - \mathbf{E}_3 \wedge \mathbf{E}_1), \qquad \mathbf{U}_6 = \frac{1}{i\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_3 - \mathbf{E}_1 \wedge \mathbf{E}_2), \tag{31}$$

then V is a constant subspace. If we extend the standard inner product on $\mathbb{C}^4$ to $\wedge^2\mathbb{C}^4$ and identify V with $\mathbb{E}^6$ by choosing an orthonormal basis of V, then we obtain our required map $\mathcal{U} = (U_1, \dots, U_6) : M \to SO(6)$.

We now write down the moving frame equations of $\mathcal{U}$. In fact, if

$$d\mathcal{U} = \mathcal{U}\Omega \tag{32}$$

for a 6×6 matrix Ω of 1-forms on M, then a calculation using (26) and (27) shows that

$$\Omega = \begin{pmatrix} 0 & (z^3_{12}-1)\omega_2 + z^2_{21}\omega_3 & (z^3_{12}+1)\omega_1 + z^3_{22}\omega_2 + z^3_{32}\omega_3 & -\lambda\omega_3 & b\omega_2 - a\omega_3 & -\lambda\omega_1 - a\omega_2 - b\omega_3 \\ (1-z^3_{12})\omega_2 - z^2_{21}\omega_3 & 0 & -z^2_{21}\omega_2 + (z^3_{12}-1)\omega_3 & 0 & \lambda\omega_2 & -\lambda\omega_3 \\ -(z^3_{12}+1)\omega_1 - z^3_{22}\omega_2 - z^3_{32}\omega_3 & z^2_{21}\omega_2 + (1-z^3_{12})\omega_3 & 0 & \lambda\omega_2 & \lambda\omega_1 + a\omega_2 + b\omega_3 & b\omega_2 - a\omega_3 \\ \lambda\omega_3 & 0 & -\lambda\omega_2 & 0 & -z^2_{21}\omega_2 + (z^3_{12}+1)\omega_3 & -(z^3_{12}+1)\omega_2 - z^2_{21}\omega_3 \\ -b\omega_2 + a\omega_3 & -\lambda\omega_2 & -\lambda\omega_1 - a\omega_2 - b\omega_3 & z^2_{21}\omega_2 - (z^3_{12}+1)\omega_3 & 0 & (1-z^3_{12})\omega_1 - z^3_{22}\omega_2 - z^3_{32}\omega_3 \\ \lambda\omega_1 + a\omega_2 + b\omega_3 & \lambda\omega_3 & -b\omega_2 + a\omega_3 & (z^3_{12}+1)\omega_2 + z^2_{21}\omega_3 & (z^3_{12}-1)\omega_1 + z^3_{22}\omega_2 + z^3_{32}\omega_3 & 0 \end{pmatrix}. \tag{33}$$

It is clear from the above that $d\mathbf{U}_2(\mathbf{e}_1) = 0$, while

$$d\mathbf{U}_2(\mathbf{e}_2) = (-1 + z_{12}^3)\mathbf{U}_1 + z_{21}^2\mathbf{U}_3 - \lambda\mathbf{U}_5, \tag{34}$$
$$d\mathbf{U}_2(\mathbf{e}_3) = z_{21}^2\mathbf{U}_1 + (1 - z_{12}^3)\mathbf{U}_3 + \lambda\mathbf{U}_6. \tag{35}$$

It follows that the image of $\mathbf{U}_2$ is a surface S in $S^5(1)$, and we now show that this is a minimal surface.

LEMMA 2. *The vectors $X = d\mathbf{U}_2(\mathbf{e}_2)$ and $Y = d\mathbf{U}_2(\mathbf{e}_3)$ are perpendicular and have the same (non-zero) length.*

THEOREM 2. *The image S of $\mathbf{U}_2$ is a minimal surface in $S^5(1)$.*

Proof. Let II denote the second fundamental form of S in $S^5(1)$. It follows from Lemma 2 that we need only check that $II(X,X)+II(Y,Y) = 0$, or, equivalently, that $dX(\mathbf{e}_2)+dY(\mathbf{e}_3)$ is a linear combination of $\mathbf{U}_2$, X and Y. In fact, a calculation using (14), (15), (23) and (33) shows that the component of $dX(\mathbf{e}_2)+dY(\mathbf{e}_3)$ perpendicular to $\mathbf{U}_2$ is equal to $-z_{32}^3X + z_{22}^3Y$, from which the result follows. □

We now investigate the *ellipse of curvature* E of S. Recall that the ellipse of curvature at a point p of a minimal surface is that (possibly degenerate) ellipse in the first normal space given by

$$E = \{II(Z,Z) \mid Z \text{ is a unit tangent vector to } S \text{ at } p\}.$$

LEMMA 3. *The ellipse of curvature at any point of S is not a circle. The direction of the minor axis is given by $II(X,X)$ and that of the major axis by $II(X,Y)$.*

Proof. We first note that $2II(X,X)$ (*resp.* $2II(X,Y)$) is equal to the component of $dX(\mathbf{e}_2)-dY(\mathbf{e}_3)$ (*resp.* $dX(\mathbf{e}_3))+dY(\mathbf{e}_2)$) perpendicular to S. In order to facilitate the calculations, which we carried out using Mathematica, we let

$$\mathbf{K}_1 = dX(\mathbf{e}_2) - dY(\mathbf{e}_3) + 3(z_{22}^3Y + z_{32}^3X),$$

and

$$\mathbf{K}_2 = dX(\mathbf{e}_3) + dY(\mathbf{e}_2) + 3(z_{32}^3Y - z_{22}^3X),$$

so that $2II(X,X)$ and $2II(X,Y)$ are the components of $\mathbf{K}_1$ and $\mathbf{K}_2$ perpendicular to S. A calculation shows that

$$\mathbf{K}_1 = \mu_2\mathbf{U}_1 + \mu_1\mathbf{U}_3 + \mu_3\mathbf{U}_5 + \mu_4\mathbf{U}_6, \tag{36}$$
$$\mathbf{K}_2 = \mu_1\mathbf{U}_1 - \mu_2\mathbf{U}_3 - 4\lambda\mathbf{U}_4 + \mu_4\mathbf{U}_5 - \mu_3\mathbf{U}_6, \tag{37}$$

where

$$\mu_1 = 4(z_{22}^3 - z_{12}^3z_{22}^3 + z_{21}^2z_{32}^3) + \mathbf{e}_3(z_{12}^3) + \mathbf{e}_2(z_{21}^2), \tag{38}$$
$$\mu_2 = 4(z_{21}^2z_{22}^3 - z_{32}^3 + z_{12}^3z_{32}^3) + \mathbf{e}_2(z_{12}^3) - \mathbf{e}_3(z_{21}^2), \tag{39}$$
$$\mu_3 = 2(b(1 - z_{12}^3) - az_{21}^2), \tag{40}$$
$$\mu_4 = 2(a(z_{12}^3 - 1) - bz_{21}^2). \tag{41}$$

It is clear from (36) and (37) that $\mathbf{K}_1$ and $\mathbf{K}_2$ are orthogonal vectors, and from (34) and (35) that

$$(\mathbf{K}_1, X) = (\mathbf{K}_2, Y) \quad \text{and} \quad (\mathbf{K}_1, Y) = -(\mathbf{K}_2, X),$$

where (,) denotes the standard inner product on $\mathbb{R}^6$. Hence the components of $\mathbf{K}_1$ and $\mathbf{K}_2$ tangential to S are orthogonal and have the same length. It now follows that $II(X,X)$ and $II(X,Y)$ are also orthogonal, so that $II(X,X)$ and $II(X,Y)$ lie along the axes of the ellipse of curvature. Also, it is clear from (36) and (37) that $|\mathbf{K}_2|^2 - |\mathbf{K}_1|^2 = 16\lambda^2$, implying that

$$|II(X,Y)|^2 - |II(X,X)|^2 = 4\lambda^2. \tag{42}$$

Hence the ellipse of curvature is not a circle since $\lambda \neq 0$. □

We note from (42) that there is a positive function ϕ such that

$$|II(X,Y)| = 2\lambda \cosh\phi \quad \text{and} \quad |II(X,X)| = 2\lambda \sinh\phi. \tag{43}$$

We may express the eccentricity e of the ellipse of curvature in terms of ϕ. In fact,

$$\begin{aligned} e &= \sqrt{1 - \tfrac{|II(X,X)|^2}{|II(X,Y)|^2}} \\ &= \operatorname{sech}\phi. \end{aligned}$$

We have seen that X and Y determine geometrically significant directions on S, so we would therefore expect that $dX(\mathbf{e}_1)$ and $dY(\mathbf{e}_1)$ are scalar multiples of X and Y respectively. In fact, it follows from (21), (13) and (33) that $dX(\mathbf{e}_1) = -z^2_{21}X$ and $dY(\mathbf{e}_1) = -z^2_{21}Y$.

We now determine conditions on M in order that S lies in a totally geodesic $S^3(1)$ in $S^5(1)$.

THEOREM 3. *Let S be the minimal surface in $S^5(1)$ determined by $\mathbf{U}_2$. Then the following conditions are equivalent.*

- *(i) The surface S is contained in a totally geodesic $S^3(1)$ in $S^5(1)$.*
- *(ii) The ellipse of curvature of S is degenerate at each point of S.*
- *(iii) The vector field $\mathbf{e}_1$ on M is a Killing vector field.*
- *(iv) $\langle \nabla_{\mathbf{e}_2}\mathbf{e}_1, \mathbf{e}_2\rangle = 0$.*
- *(v) $\langle \nabla_{\mathbf{e}_3}\mathbf{e}_1, \mathbf{e}_3\rangle = 0$.*

REMARK. Minimal Lagrangian submanifolds M admitting a unit length Killing vector field whose integral curves are geodesics in $\mathbb{C}P^3(4)$ are investigated in [6]. In particular, explicit examples of minimal Lagrangian tori admitting such a vector field are constructed.

Proof. The equivalence of (*iv*) and (*v*) is immediate from (12).

Minimality of S, together with the Codazzi equations for S show that (*i*) holds if and only if $II(X,X) \equiv 0$ on an open subset of S or, equivalently, (*ii*) holds.

On the other hand, (*iii*) holds if and only if $\mathbf{e}_1$ satisfies the Killing equations, namely $\langle \nabla_U \mathbf{e}_1, V\rangle + \langle \nabla_V \mathbf{e}_1, U\rangle = 0$ for all vectors U, V tangential to M. It follows from (12) that this holds if and only if (*iv*) holds.

Hence, we may prove the theorem by showing that the vector $\mathbf{K}_1$ given by (36) is a linear combination of X and Y if and only if $z^2_{21} = 0$.

We first assume that $z^2_{21} = 0$. In this case, using (23) we see that

$$\mathbf{K}_1 = 2(2z^3_{32}(z^3_{12}-1)+b\lambda)\mathbf{U}_1-2(2z^3_{22}(z^3_{12}-1)+a\lambda)\mathbf{U}_3+2b(1-z^3_{12})\mathbf{U}_5+2a(z^3_{12}-1)\mathbf{U}_6.$$

This is a linear combination of X and Y if and only if both the following equations hold.

$$a(z^3_{12}-1)^2 = 2z^3_{22}\lambda(z^3_{12}-1) + a\lambda^2, \tag{44}$$

and

$$b(z^3_{12}-1)^2 = 2z^3_{32}\lambda(z^3_{12}-1) + b\lambda^2. \tag{45}$$

Using (21), these equations simplify to

$$az^3_{12} = \lambda z^3_{22}, \tag{46}$$

and

$$bz^3_{12} = \lambda z^3_{32}. \tag{47}$$

However, it follows from (12) that $[\mathbf{e}_1, \mathbf{e}_2] = 0$, and, applying this to z^3_{12} using (13), (17), (21) and (23), we obtain (46). Similarly, $[\mathbf{e}_1, \mathbf{e}_3] = 0$, and, applying this to z^3_{12} , we obtain (47). Thus $\mathbf{K}_1$ is a linear combination of X and Y as required.

Conversely, assume that $\mathbf{K}_1$ is a linear combination of X and Y. It then follows from (34), (35) and (36) that

$$\lambda\mu_1 = -\mu_3 z^2_{21} + \mu_4(1 - z^3_{12}), \tag{48}$$

and

$$\lambda\mu_2 = \mu_3(1 - z^3_{12}) + \mu_4 z^2_{21}. \tag{49}$$

We may use the above two equations, together with (23) to obtain the following algebraic expressions for $\mathbf{e}_2(z^2_{21})$, $\mathbf{e}_3(z^2_{21})$, $\mathbf{e}_2(z^3_{12})$ and $\mathbf{e}_3(z^3_{12})$.

$$\lambda\mathbf{e}_2(z^2_{21}) = -(z^3_{12}-1)^2a + 2(z^3_{12}-1)(bz^2_{21} + \lambda z^3_{22}) + a((z^2_{21})^2 + \lambda^2) - 2\lambda z^2_{21}z^3_{32}, \tag{50}$$

$$\lambda\mathbf{e}_3(z^2_{21}) = -(z^3_{12}-1)^2b + 2(z^3_{12}-1)(-az^2_{21} + \lambda z^3_{32}) + b((z^2_{21})^2 + \lambda^2) + 2\lambda z^2_{21}z^3_{22}, \tag{51}$$

$$\lambda\mathbf{e}_2(z^3_{12}) = (z^3_{12}-1)^2b + 2(z^3_{12}-1)(az^2_{21} - \lambda z^3_{32}) + b(-(z^2_{21})^2 + \lambda^2) - 2\lambda z^2_{21}z^3_{22}, \tag{52}$$

$$\lambda\mathbf{e}_3(z^3_{12}) = -(z^3_{12}-1)^2a + 2(z^3_{12}-1)(bz^2_{21} + \lambda z^3_{22}) + a((z^2_{21})^2 - \lambda^2) - 2\lambda z^2_{21}z^3_{32}. \tag{53}$$

We now consider the integrability conditions for λ. In fact, the only one we will need is obtained by applying $\nabla_{\mathbf{e}_2}\mathbf{e}_3 - \nabla_{\mathbf{e}_3}\mathbf{e}_2 - [\mathbf{e}_2, \mathbf{e}_3]$ to λ and equating the answer to zero. Carrying out this process using (12)-(15), we obtain

$$\mathbf{e}_2(z^3_{22}) + \mathbf{e}_3(z^3_{32}) = 2z^3_{12}z^2_{21}. \tag{54}$$

Using the above equations and (12)-(19), (21)-(24), a calculation (for which we used Mathematica) shows that the integrability condition obtained by applying $\nabla_{\mathbf{e}_2}\mathbf{e}_3 - \nabla_{\mathbf{e}_3}\mathbf{e}_2 - [\mathbf{e}_2, \mathbf{e}_3]$ to z^3_{12} and equating the answer to zero reduces to $z^2_{21} = 0$. This completes the proof of the theorem. ∎

We now return to the general situation governed by the ruled minimal Lagrangian submanifold M of $\mathbb{C}P^3(4)$. We note that the arguments applied to $\mathbf{U}_2$ may be used to show that the image $\hat{S}$ of $\mathbf{U}_4$ is also a minimal surface in $S^5(1)$. We now investigate the relation between the two minimal surfaces S and $\hat{S}$.

LEMMA 4. *If S is contained in a totally geodesic $S^3(1)$ then $\hat{S}$ is the polar of S in the sense of Lawson [9].*

Proof. In this situation, $II(X,X) = 0$, so that $\mathbf{K}_1$ is a linear combination of X and Y. We also have from Theorem 3 that $z_{21}^2 = 0$, so it follows from (34), (35), (36) and (37) that $2II(X,Y)$, the component of $\mathbf{K}_2$ perpendicular to X and Y, is equal to $-4\lambda\mathbf{U}_4$. In particular, $\hat{S}$ is in the totally geodesic $S^3(1)$ containing S and at each point is orthogonal to S and the tangent space to S. Thus $\hat{S}$ is the polar of S. □

We now assume that S is not contained in a totally geodesic $S^3(1)$. Theorem 3, together with real analyticity of minimal surfaces imply that, by restricting to an open dense subset of M, we may assume that z_{21}^2 is a nowhere vanishing function. Therefore, by replacing $\mathbf{e}_1$ with $-\mathbf{e}_1$ (and interchanging $\mathbf{e}_2$ and $\mathbf{e}_3$ in order to keep λ positive) if necessary, we may also assume that z_{21}^2 is a strictly positive function. We now let $\mathbf{N}$ be the unit vector in $\mathbb{R}^6$ such that

$$\{\mathbf{U}_2, X, Y, II(X,X), II(X,Y), \mathbf{N}\} \text{ is a positively oriented orthogonal frame of } \mathbb{R}^6. \tag{55}$$

It follows from (34), (35) and (36) that $\mathbf{U}_4$ is orthogonal to $\mathbf{U}_2$, X, Y, and $II(X,X)$, and so is a linear combination of $II(X,Y)$ and $\mathbf{N}$. Also, it follows from (37) that $(\mathbf{U}_4, II(X,Y)) = -2\lambda$ so that, using the positive function ϕ introduced in (43), we see that

$$\mathbf{U}_4 = -\operatorname{sech}\phi\,\tfrac{II(X,Y)}{|II(X,Y)|} + \epsilon\tanh\phi\,\mathbf{N},$$

where $\epsilon = \pm 1$.

In order to determine the sign of ϵ, we compute the determinant of $(\mathbf{U}_2, X, Y, II(X,X), II(X,Y), \mathbf{U}_4)$. In fact,

$$\begin{aligned}
4\det(\mathbf{U}_2, X, Y, II(X,X), II(X,Y), \mathbf{U}_4) &= \det(\mathbf{U}_2, X, Y, \mathbf{K}_1, \mathbf{K}_2, \mathbf{U}_4)\\
&= -({\mu_1}^2 + {\mu_2}^2)\lambda^2 + 2\left((z_{12}^3 - 1)(\mu_1\mu_4 + \mu_2\mu_3) + z_{21}^2(\mu_1\mu_3 - \mu_2\mu_4)\right)\lambda\\
&\quad - ({\mu_3}^2 + {\mu_4}^2)((z_{12}^3 - 1)^2 + (z_{21}^2)^2).
\end{aligned}$$

Regarding this as a quadratic equation in λ, we see that its discriminant is given by

$$-\left((\mu_4\mu_2 + \mu_3\mu_1)(z_{12}^3 - 1) + z_{21}^2(\mu_2\mu_3 - \mu_1\mu_4)\right)^2.$$

This is always non-positive, implying that

$$\det(\mathbf{U}_2, X, Y, II(X,X), II(X,Y), \mathbf{U}_4) \leq 0,$$

from which it follows that $\epsilon = -1$.

We remark that the minimal surface $\hat{S}$ determined by $\mathbf{U}_4$ may be determined directly from S together with a choice of direction along the major axis of the ellipse of curvature E of S. In fact, the unit vector $\mathbf{N}$ determined by (55) does not depend on the choice of basis $\{X, Y\}$ of the tangent space of S, so if V is the unit vector in the direction determined by $II(X,Y)$ along the major axis of E then we may define $\hat{}: S \to \hat{S}$, where

$$\hat{p} = (-\operatorname{sech}\phi\ V - \tanh\phi\ \mathbf{N})(p), \quad p \in S.$$

We will call this the $(-)$transform of S. This, together with a related construction called the $(+)$transform, may be described geometrically as follows. Let E be the ellipse of curvature of S at a point $p \in S$ and let P be the 3-plane orthogonal to $\mathbf{U}_2$ and its tangent space. Let R_θ be the rotation of P about the minor axis of E through an angle θ, $0 \le \theta \le \pi/2$, such that $R_\theta(V)$ makes an acute angle with $\mathbf{N}$ and having the property that the orthogonal projection of $R_\theta(E)$ onto the plane containing E is a circle. Then the inverse rotation $R_{-\theta}$ has a similar geometric effect on E, and we define the $(+)$transform and $(-)$transform of S by setting

$$p^+ = R_\theta(\mathbf{N}) = (-\operatorname{sech}\phi\ V + \tanh\phi\ \mathbf{N})(p), \quad p \in S, \tag{56}$$

and

$$p^- = R_{-\theta}(-\mathbf{N}) = (-\operatorname{sech}\phi\ V - \tanh\phi\ \mathbf{N})(p), \quad p \in S. \tag{57}$$

Thus $\mathbf{U}_4$ is obtained by applying the $(-)$transform to the minimal surface determined by $\mathbf{U}_2$, and we now show that $\mathbf{U}_2$ is obtained by applying the $(+)$transform to the minimal surface determined by $\mathbf{U}_4$. We begin by noting that if, in our construction, we replace $\mathbf{e}_3$ by $-\mathbf{e}_3$ then a suitable lift to $SU(4)$ gives the map $\tilde{\mathcal{U}} = (\tilde{\mathbf{U}}_1, \dots, \tilde{\mathbf{U}}_6) : M \to SO(6)$ where

$$\tilde{\mathbf{U}}_1 = \mathbf{U}_6, \quad \tilde{\mathbf{U}}_2 = -\mathbf{U}_4, \quad \tilde{\mathbf{U}}_3 = -\mathbf{U}_5, \quad \tilde{\mathbf{U}}_4 = \mathbf{U}_2, \quad \tilde{\mathbf{U}}_5 = \mathbf{U}_3, \quad \tilde{\mathbf{U}}_6 = -\mathbf{U}_1. \tag{58}$$

Thus, from Theorem 3, $\hat{S}$ is not contained in a totally geodesic $S^3(1)$. Also,

$$\tilde{\mathbf{U}}_4 = -\operatorname{sech}\tilde{\phi}\frac{\tilde{II}(\tilde{X},\tilde{Y})}{|\tilde{II}(\tilde{X},\tilde{Y})|} - \tanh\tilde{\phi}\ \tilde{\mathbf{N}}, \tag{59}$$

where $\tilde{\phi} > 0$ is such that $\operatorname{sech}\tilde{\phi}$ is the eccentricity of the ellipse of curvature of the surface $\tilde{S}$ determined by $\tilde{\mathbf{U}}_2$, $\tilde{X} = d\tilde{\mathbf{U}}_2(\mathbf{e}_2)$, $\tilde{Y} = d\tilde{\mathbf{U}}_2(-\mathbf{e}_3)$, $\tilde{II}$ is the second fundamental form of $\tilde{S}$, and $\tilde{\mathbf{N}}$ is the unit vector in $\mathbb{R}^6$ such that $\{\tilde{\mathbf{U}}_2, \tilde{X}, \tilde{Y}, \tilde{II}(\tilde{X},\tilde{X}), \tilde{II}(\tilde{X},\tilde{Y}), \tilde{\mathbf{N}}\}$ is a positively oriented orthogonal frame of $\mathbb{R}^6$.

Now let $\hat{X} = d\mathbf{U}_4(\mathbf{e}_2)$, $\hat{Y} = d\mathbf{U}_4(\mathbf{e}_3)$, and let $\hat{II}$ be the second fundamental form of $\hat{S}$. Then $\hat{II}(\hat{X},\hat{X}) = -\tilde{II}(\tilde{X},\tilde{X})$ is along the minor axis of the ellipse of curvature of $\hat{S}$, while $\hat{II}(\hat{X},\hat{Y}) = \tilde{II}(\tilde{X},\tilde{Y})$ is along the major axis. Now let $\hat{\mathbf{N}}$ be the unit vector in $\mathbb{R}^6$ such that $\{\mathbf{U}_4, \hat{X}, \hat{Y}, \hat{II}(\hat{X},\hat{X}), \hat{II}(\hat{X},\hat{Y}), \hat{\mathbf{N}}\}$ is a positively oriented orthogonal basis of $\mathbb{R}^6$. It then follows from (58) that $\tilde{\mathbf{N}} = -\hat{\mathbf{N}}$, and from (58) and (59) that if $\hat{\phi} > 0$ is such that $\operatorname{sech}\hat{\phi}$ is the eccentricity of the ellipse of curvature of $\hat{S}$, then

$$\mathbf{U}_2 = -\operatorname{sech}\hat{\phi}\frac{\hat{II}(\hat{X},\hat{Y})}{|\hat{II}(\hat{X},\hat{Y})|} + \tanh\hat{\phi}\ \hat{\mathbf{N}}.$$

Thus, taking the direction along the major axis of the ellipse of curvature of $\hat{S}$ to be that determined by $\hat{II}(\hat{X},\hat{Y})$, then applying the $(+)$transform to $\hat{S}$ gives us S.

The following theorem summarises the results of the paper.

THEOREM 4. *A ruled minimal Lagrangian submanifold M of $\mathbb{C}P^3(4)$ defines two minimal surfaces S and $\hat{S}$ in $S^5(1)$. These surfaces are related geometrically in that $\hat{S}$ is obtained from S by the $(-)$transform and S is obtained from $\hat{S}$ by the $(+)$transform.*

Thus, ruled minimal Lagrangian submanifolds of $\mathbb{C}P^3(4)$ induce two constructions on what could be a special class of minimal surfaces in $S^5(1)$, namely a $(-)$transform, given by (57), producing $\hat{S}$ from S and a $(+)$transform, given by (56), producing S from $\hat{S}$.

In a forthcoming paper [4] we shall show that if we apply either of these constructions to an arbitrary minimal surface with non-circular non-degenerate ellipse of curvature in $S^5(1)$ then we obtain another minimal surface in $S^5(1)$. As a consequence of this we will show that every such minimal surface in $S^5(1)$ may be constructed locally in the manner described in the present paper from a ruled minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$.

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